Fractional Variational Optical Flow Model for Motion Estimation

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Abstract: This paper introduces a new fractional variational optical flow model, which combines the fractional derivative with the variational optical flow method, for motion estimation. In the model, the data term is a weighted least-squares fit of local brightness constraint function and the prior term is a fractional-order optical flow model. We use the 2-D discrete Fourier transform to implement the numerical algorithm and propose an iterative scheme based on the coarse-to-fine warping strategy. Finally, we list various numerical results on six typical image sequences. Experiments demonstrate that the novel method is fairly insensitive to fractional-order variations and gives smaller angular errors for optical flow estimation.

Keywords: Optical flow, fractional derivative, variational calculus, Fourier transform, multi-scale strategy.

1. INTRODUCTION

Optical flow is the 2D velocity field which is generated by the moving objects in the scene or the observer motion. Through the analysis of optical flow, we can get much useful motion information, such as speed, direction, and number of objects. There is no doubt that optical flow estimation is one of the key problems in computer vision. And it has been widely applied in many fields, such as automatic driving, medical diagnosis and intelligent monitoring. Since the original approaches of Horn and Schunck (1981) and Lucas and Kanade (1981) were proposed, the number and quality of optical flow estimation methods have increased dramatically. Many new technologies were developed to improve the original models. Barron et al. (1994) used a weight function, which gave more influence to constraint at the center of the neighborhood than those at the periphery, to bound the optical flow constraint. The method is more robust because it took full advantages of the information between the pixels in the neighborhood. Anandan (1989) and Black and Anandan (1996) applied the coarse-to-fine strategy in optical flow estimation. The main idea is that we can estimate the optical flow in different layers and make the coarse solution as the initial value for solving a refined version until optical flow is calculated. So, the coarse-to-fine strategy can increase the accuracy of the algorithm. Variational formulations of the optical flow estimation problem have many advantages. Firstly, they lead to a sound model where one can clearly describe the model assumptions. Secondly, some sophisticated variational method can be used to optimize the model. Brox et al. (2004) proposed a novel variational approach which had a good accuracy and robustness.

Fractional calculus is the branch of calculus that generalizes the derivative of a function to a non-integer order. Because of some of its characteristics superior to the integer order, the concepts of fractional calculus have been investigated extensively in many engineering applications and science, such as physics [Baleanu (2009)], quantum mechanics [Laskin (2000)], control theory [Li and Chen (2006)], electrochemical [Chen et al. (2008)], signal processing [Sheng and Chen (2010)] and so on. Particularly, in the field of image processing, fractional differentiations are typical tools and have been applied to many fields such as noise removal [Liu and Chang (1997); Bai and Feng (2007)], edge detection [Mathieu et al. (2003)] and optical flow estimation [Kashu et al. (2009)]. There are many methods that can define the fractional derivative. The usual definitions among them involve: Riemann-Liouville fractional derivative [Momanli and Odbat (2007)], Grünwald-Letnikov definition fractional derivative [Murio (2009); Gorenflo and Abdel-Rehim (2007)] and frequency-domain (Fourier-domain) fractional derivative [Davis et al. (2001); Tseng et al. (2000)].

In this paper we propose a novel fractional variational optical flow model that integrates several of the above mentioned concepts and which can be minimized with a solid numerical method. The outline of this paper is as follows. In Section 2, we give a simple introduction of the fractional derivative and show that the time evolution of the proposed models seeks to minimize an energy functional of the weighted sum of the weighted brightness constancy constraint and fractional-order smoothness constraint. An iterative solution of the proposed models is obtained. In Section 3, we use the 2-D discrete Fourier transform (DFT) to implement the numerical algorithm and briefly describe the procedure of the algorithm. Numerical examples are presented in Section 4 and the paper is concluded in Section 5.
2. FRACTIONAL VARIATIONAL OPTICAL FLOW MODEL

2.1 Fractional-order Operator

Before introducing the fractional variational optical flow model, we give a brief description of the fractional derivative. The fractional derivative can be seen as the generalization of the integer derivative. Recently, many methods have been proposed to fit the concept of the fractional derivative. The most famous of these methods that have been used widely are the Riemann-Liouville and Grunwald-Letnikov definitions. However, in this paper, we solve the fractional partial differential equation using the frequency domain definition because it is easy to implement. For any \( f(x, y) \in L^2(\mathbb{R}^2) \), the corresponding 2-D Fourier transform pairs are

\[
F(w_1, w_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(x, y) e^{-i(xw_1 + yw_2)} \, dx \, dy, \tag{1}
\]

\[
f(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} F(w_1, w_2) e^{i(xw_1 + yw_2)} \, dw_1 \, dw_2. \tag{2}
\]

Thus, the fractional partial derivatives are

\[
D_x^\alpha f(x, y) = F^{-1}\{(jw_1)^\alpha F(w_1, w_2)\}, \tag{3}
\]

\[
D_y^\alpha f(x, y) = F^{-1}\{(jw_2)^\alpha F(w_1, w_2)\}, \tag{4}
\]

where \( F^{-1} \) is the inverse 2-D continuous Fourier transform operator, and \( \alpha \) can be any real number.

Using the 2-D fractional-order derivative operator defined by Kashu et al. (2009), we can get

\[
D^\alpha f(x, y) = F^{-1}\{((\sqrt{w_1^2 + w_2^2})^\alpha F(w_1, w_2)\}. \tag{5}
\]

From (3) to (5), we can get the relation as follows:

\[
D^{2\alpha} = (-\Delta)^\alpha, \tag{6}
\]

where \( \Delta \) is the Laplacian operator defined by

\[
\nabla f(x, y) = D^2_x f(x, y) + D^2_y f(x, y). \tag{7}
\]

So we can transform the computation of fractional derivative into the computation of Laplacian operator using (6). It is easy to implement by numerical method which will be introduced in detail in the next section.

2.2 Fractional Variational Optical Flow Model

We now briefly describe the fractional variational optical flow model. The variational optical flow estimation method can be seen as an improvement of the typical optical flow method. They have many advantages. Firstly, they lead to a sound model where one can clearly formulate the model assumptions. Secondly, the models can be driven by the Euler–Lagrange equation easily. In this paper, we propose a fractional variational optical flow model which integrates several concepts that are important for accurate optic flow estimation, such as fractional-order smoothness constraint, neighborhood weight method, and coarse-to-fine strategy.

Brightness Constancy Assumption. For a spatiotemporal image, let us suppose that the image intensity is given by \( f(x, t) \), where the intensity is now a function of time \( t \) and the position \( x = (x, y)^T \). The optical flow vector is defined by

\[
u = \dot{x} = (u, v)^T, \tag{8}
\]

where \( \dot{x} = u(x, y) \) and \( \dot{y} = v(x, y) \), which are the speeds that the object is moving in the \( x \) and \( y \) directions, respectively. Furthermore, let us suppose that the intensity of a pixel is not changed by the displacement. Then, we have

\[
f(x + u, t + 1) = f(x, t). \tag{9}
\]

Linearizing (9) by applying a Taylor expansion to the left hand side yields

\[
f_x u + f_v v + f_t = \nabla f^T \nu + f_t = 0, \tag{10}
\]

which is called the optical flow constraint equation. Here, \( \nabla \) is a gradient operator defined by

\[
\nabla f = (f_x, f_v)^T. \tag{11}
\]

Although the optical flow constraint equation can be used to calculate optical flow field \( u(x, y) \) and \( v(x, y) \), it is sensitive to the uncertain information because it does not take advantages of the information between the pixels in the neighborhood. In order to solve this problem, we use a weighted constraint equation defined by

\[
\sum_{x \in \Omega} (W^2(x)(\nabla f^T \nu + f_t)^2) = 0, \tag{12}
\]

where \( \Omega \) is a small spatial neighborhood of each pixel \( x \). \( W(x) \) denotes a window function which satisfies the Gaussian distribution and gives more influence to constraint at the center of the neighborhood than those at the periphery. So it has good robustness.

Then the global deviations from the brightness constancy assumption can be measured by the energy function

\[
E_{data}(u, v) = \int_{\mathbb{R}^2} \sum_{x \in \Omega} (W^2(x)(\nabla f^T \nu + f_t)^2) \, dx \, dy, \tag{13}
\]

which is called data term.

Fractional-order Smoothness Constraint. Unfortunately, the brightness constancy assumption gives us only one equation per pixel for two unknowns, so it is necessary to introduce one more assumption. An intuitive idea is that except the area near the edges of moving objects, the motion in the neighborhood should be quite similar. From this smoothness assumption, we can obtain the first-order smoothness energy function defined by

\[
E_{prior}(u, v) = \int_{\mathbb{R}^2} (|D_u|^2 + |D_v|^2) \, dx \, dy. \tag{14}
\]

Of course, there is no reason why the order must be one, it can be any real number. Thus, we define the fractional-order smoothness energy function

\[
E_{prior}(u, v) = \int_{\mathbb{R}^2} (|D^\alpha u|^2 + |D^\alpha v|^2) \, dx \, dy, \tag{15}
\]

which is called prior term.

Multi-scale Strategy. Multi-scale strategy has been widely used to solve the aliasing problem in optical flow estimation, because the aliasing is less severe at the coarse scales. So we apply the multi-scale strategy in the our model to improve the accuracy of our model. The optical flow fields are estimated at coarse scales firstly, and then the estimations are used as the initial values to warp finer scales to compensate for the larger displacements. At different scales, the data term and the prior term are respectively defined by

\[
E_{data}^l(u^l, v^l) = \int_{\mathbb{R}^2} \sum_{x \in \Omega} (W^2(x)(\nabla f^T l \nu^l + f_t^l)^2) \, dx \, dy, \tag{16}
\]

and

\[
E_{prior}^l(u^l, v^l) = \int_{\mathbb{R}^2} (|D^\alpha u^l|^2 + |D^\alpha v^l|^2) \, dx \, dy, \tag{17}
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Furthermore, let us suppose that the intensity of a pixel is not changed by the displacement. Then, we have

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\]

and

\[
E_{prior}^l(u^l, v^l) = \int_{\mathbb{R}^2} (|D^\alpha u^l|^2 + |D^\alpha v^l|^2) \, dx \, dy, \tag{17}
\]
where $\alpha \in \mathbb{R}^+$ and the $l$ denotes the scale levels.

### 2.3 Model Derivation

With the description above, we can calculate the total energy using the weighted sum between the data term and the prior term at different scales

$$
E^l = E_{\text{data}} + \lambda E_{\text{prior}},
$$

(18)

where $\lambda$ is weight parameter.

So the optical flow fields $u(x, y)$ and $v(x, y)$ can be estimated by minimizing the total energy function. We can formally compute the Euler–Lagrange equation for this minimization problem as follows.

Take any two test functions $\eta(x, y), \varepsilon(x, y) \in L^2(\mathbb{R}^2)$. Define

$$
\Psi(\varepsilon) = \int_{\mathbb{R}^2} E(u + \varepsilon, v + \varepsilon) \, dx \, dy.
$$

(19)

We obtain

$$
\frac{\partial \Psi}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \int_{\mathbb{R}^2} \left\{ \left( \sum_{x \in \Omega} \left( W^2(x) (\nabla f \cdot f \cdot f) + \lambda D^{2\alpha} u \right) \eta + \left( \sum_{x \in \Omega} (W^2(x) (\nabla f \cdot f \cdot f) + \lambda D^{2\alpha} v) \right) \varepsilon \right) \, dx \, dy \right\}
$$

(20)

Thus, the Euler–Lagrange equation is

$$
\sum_{x \in \Omega} \left( (\nabla f^T \cdot f \cdot f) + \lambda D^{2\alpha} u \right) \eta \, dx = \sum_{x \in \Omega} \left( (\nabla f^T \cdot f \cdot f) + \lambda D^{2\alpha} v \right) \varepsilon \, dx.
$$

(21)

From (21), we can derive the iteration form

$$
M \mathbf{u}^{k+1} = -\lambda D^{2\alpha} \mathbf{u}^k - B,
$$

(22)

where $M$ and $B$ are defined by

$$
M = \left( \sum_{x \in \Omega} f_x^2 \sum_{x \in \Omega} f_y \right),
$$

(23)

and

$$
B = \left( \sum_{x \in \Omega} f_x f_y \right).
$$

(24)

### 3. NUMERICAL ALGORITHM

We can calculate the optical flow field $\mathbf{u}$ using iteration equation (22). The equation includes three unknown terms $M$, $B$ and $D^{2\alpha} \mathbf{u}$. With the observed image as the initial condition, we can calculate the $M$ and $B$ directly. Unfortunately, $D^{2\alpha} \mathbf{u}$ cannot be solved directly because of the fractional-order $\alpha$. We introduce a numerical algorithm [Kashu et al. (2009)] to compute the fractional difference.

Assume that $u$ is the digital image which has been sampled from its continuous version at uniformly spaced points, and $u(x, y)$ denotes the grayscale value at the location $(x, y)$. We use the 2-D discrete Fourier transform (2-D DFT) to calculate the fractional difference. The 2-D DFT pair of $u(x, y)$ are

$$
u(x, y) = \frac{1}{N} \sum_{m,n=0}^{N-1} u(x, y) e^{-2\pi i \frac{m x + n y}{N}},
$$

(25)

$$
u(x, y) = \frac{1}{N} \sum_{w_1,w_2=0}^{N-1} \nu(w_1, w_2) e^{2\pi i \frac{m x + n y}{N}}.
$$

(26)

The translation property of the 2-D DFT is

$$
u(x - x_0, y - y_0) \leftrightarrow e^{-2\pi i \frac{m x_0 + n y_0}{N}} \nu(x, y), \quad (27)
$$

where $\leftrightarrow$ denotes the Fourier transform pair. Then, we can obtain

$$
D_x u = u(x + \frac{1}{2}, y) - u(x - \frac{1}{2}, y)
$$

(28)

and

$$
D_y u = u(x + \frac{1}{2}, y) - u(x, y - \frac{1}{2})
$$

(29)

where $F^{-1}$ is the inverse 2-D DFT operator. From equations (6), (7), (28) and (29), we can obtain

$$
D^{2\alpha} \mathbf{u} = (-\Delta)^{\alpha} \nu
$$

(30)

$D^{2\alpha} \nu$ can be computed in the same way. Now, we can calculate the $D^{2\alpha} \mathbf{u}$ directly given the initial optical flow field.

To summarize, our fractional variational optical flow estimation approach for two frames can be done in the following steps.

1) Let $f_1$ and $f_2$ be the input images, and initialize the parameters: optical flow field $\mathbf{u}$, weight parameter $\lambda$, weight window $W$, and scale levels $l$.

2) Down sample $f_1$, $f_2$ and $\mathbf{u}$. Let $f_1^l$, $f_2^l$ and $\mathbf{u}^l$ denote the results at the $l$th scale level.

3) Calculate the $f_1^l$, $f_2^l$, $f_1^l$, and then obtain $M$, $B$ and $D^{2\alpha} \mathbf{u}^l$ using equations (23), (24) and (30), respectively.

4) Estimate the optical flow field $\mathbf{u}^l$ using equation (22).

5) If $l = 1$, stop; else, set $l = l - 1$, interpolate $f_1^l$, $f_2^l$ and $\mathbf{u}^l$ and then goto step 3.

### 4. NUMERICAL EXPERIMENTS

#### 4.1 Evaluation Database

In this section, we present numerical results obtained by applying our proposed new optical flow model to gray image sequences. We test the proposed algorithm on Venus, Hydrangea, RubberWhale, Grove, Urban2 and Urban3 image sequences which are obtained from a famous database [Baker et al. (2007)]. As is shown in Fig. 1, these image sequences respectively include two frames. Venus is a stereo sequences; Hydrangea and RubberWhale are hidden texture sequences; Grove, Urban2 and Urban3 are synthetic sequences. The proposed optical flow model can be evaluated comprehensively using these sequences.

#### 4.2 Evaluation Methodology

Recently, more and more methods have been proposed to evaluate the optical flow estimation. However, the most commonly used evaluation methodology is the angular error (AE). The AE between the flow vector $(u, v)^T$ and the corresponding ground-truth flow vector $(u_{GT}, v_{GT})^T$ is the angle in 3D space between $(u, v, 1)^T$ and $(u_{GT}, v_{GT}, 1)^T$. The definition is given by

$$
AE = \arccos\left(\frac{u \times u_{GT} + v \times v_{GT} + 1}{\sqrt{u^2 + v^2 + 1} \sqrt{u_{GT}^2 + v_{GT}^2 + 1}}\right).
$$

(31)

Statistics is an effective evaluation tool. Combining AE with the statistics, we can obtain average of AE (AVAE) and standard
deviations of AE (SDAE) [Barron et al. (1994)] which can be used to evaluate the optical flow estimation intuitively.

In order to obtain convincing results, we use AVAE and SDAE to evaluate the optical flow estimation simultaneously in the following numerical experiments.

4.3 Parameter Setting

There are $l$, $W$, $\lambda$, and $\alpha$ parameters in our proposed model. We now briefly describe how to set these parameters. Parameter $l$ denotes the scale level in the multi-scale strategy. Considering the factors such as real-time and accuracy comprehensively, we can set $l$ to 4. Parameter $W$ is the weight window function. In order to take full advantage of the information between the pixels, we use the famous Gaussian filter with a standard deviation of 3 as the window function. The window’s size is $30 \times 30$ pixels.

Parameter $\lambda$ is the weight of the prior term, and parameter $\alpha$ is the fractional-order in the prior term. So these two parameters have a close link. For the purpose of setting $\lambda$ and $\alpha$, we perform the following experiment. In the experiment, we firstly change the value of $\lambda$ from 0 to 1.5 by the same interval, and then we change the value of $\alpha$ from 0.1 to 2 by the same interval. For every parameter pair $(\alpha, \lambda)$, we calculate the AVAE of $u$. Fig. 2 shows the experiment results on six image sequences respectively.

In the figures, the blue regions are available for the parameter setting because AVAE is small in these regions. Meanwhile, it is interesting to find that these available regions are consistent for all these types of image sequences. According to this phenomenon, we can get the conclusion that $\alpha$ and $\lambda$ have good robustness for these different types of image sequences. Therefore, we do not need to optimize $\alpha$ and $\lambda$. This will greatly save program execution time. In this paper, we set $\alpha$ to 0.55 and set $\lambda$ to 1.2. And the following experiment results will show that it will not have a greater impact on the accuracy of estimation.

4.4 Algorithm Experiment

Table 1 shows a comparison of our results for all sequences to the results of the modern Horn-Schunck algorithm which is implemented by Sun (2007). In the table, better result is changed to bold. As one can see, the AVAE results of our fractional variational algorithm outperform the modern Horn-Schunck algorithm’s for all these types of image sequences. In particular, for the Urban3 image sequences, our algorithm is much better than the modern H-S algorithm. Although there are three types of image sequences where the SDAE results of our algorithm are not better than the modern H-S algorithm’s, the

![Image](image_url)
difference between the results is very small. This shows that the overall performance of our algorithm is superior to the modern H-S algorithm.

Fig. 3 shows the visual comparison between the modern H-S algorithm and our algorithm on one of the sequences along with the ground truth flow. From the visual comparison between the ground truth and the estimated flow, we can see that both algorithms are able to accurately estimate the motion flow of sequence. However, it is obvious that our approach is closer to the true value, because our algorithm exhibits the more clear motion boundaries.

5. CONCLUSION

We proposed a class of fractional variational optical flow model which includes weighted brightness constancy constraint and fractional-order smoothness constraint. The model was derived as a process that seeks to minimize an energy function. For the purpose of application, we introduced a numerical fractional differentiation method which was performed by 2-D discrete Fourier transform. In addition, we also outlined the process of the algorithm based on the coarse-to-fine warping strategy. Finally, many numerical experiments were performed to evaluate our proposed algorithm. The main contribution is that we apply the fractional differential method to optical flow estimation and provide a new way for optical flow estimation. Secondly, through experiments we found that the optimal order is often not an integer, so it is necessary to apply the fractional-order instead of the integer-order in the model. In addition, we also give a reasonable order setting method which is shown to be robust. Finally, the experiments demonstrate that our algorithm has better performance for many types of image sequences. Future research efforts involve applying the proposed model to the UAV auto-landing system.

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