

## FRACTIONAL MAXWELL FLUID WITH FRACTIONAL DERIVATIVE WITHOUT SINGULAR KERNEL

by

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*In this paper we propose a new model for the fractional Maxwell fluid within fractional Caputo-Fabrizio derivative operator. We present the fractional Maxwell fluid in the differential form for the first time. The analytical results for the proposed model with the fractional Losada-Nieto integral operator are given to illustrate the efficiency of the fractional order operators to the line viscoelasticity.*

*Key words: fractional Maxwell fluid, fractional Caputo-Fabrizio derivative, fractional Losada-Nieto integral, analytical solution, line viscoelasticity*

### Introduction

Fractional-order viscoelasticity used to describe the laws of deformation for modeling the viscoelastic behavior of real materials had firstly studied by Scott-Blair [1]. The Scott-Blair presented the constitutive relationship of fractional mechanical element, given by [2, 3]:

$$\sigma(\tau) = \Lambda \mu^\alpha \frac{d^\alpha \varepsilon(\tau)}{d\tau^\alpha} \quad (1a)$$

where  $\Lambda$ ,  $\mu$ , and  $\alpha$  are material-dependant constants and  $\partial^\alpha / \partial \tau^\alpha$  is the Riemann-Liouville fractional derivative operator.

A generalized version of eq. (1a), suggested by Gerasimov [4], was written:

$$\sigma(\tau) = E_0 \frac{d^\beta \varepsilon(\tau)}{d\tau^\beta} \quad (1b)$$

where  $E_0$  is the generalized viscosity of the material and  $\partial^\beta / \partial \tau^\beta$  is the fractional derivative [3, 4].

When  $\alpha = 1$ , from eq. (1a) we have:

$$\sigma(\tau) = \Lambda \mu \frac{d\varepsilon(\tau)}{d\tau} \quad (1c)$$

which is used to describe the Newton's law of Newtonian fluids [5].

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When  $\alpha = 0$ , from eq. (1a) we have:

$$\sigma(\tau) = \Lambda \varepsilon(\tau) \quad (1d)$$

which is used to explain the Hooke's law of elastic body [5].

Another version of eq. (1a), structured by Caputo and Mainardi [6, 7], was written:

$$\sigma(\tau) = \Lambda \mu^\chi \frac{d^\chi \varepsilon(\tau)}{d\tau^\chi} \quad (1e)$$

where  $\Lambda$ ,  $\mu$ , and  $\chi$  are material-dependant constants and  $\partial^\chi / \partial \tau^\chi$  is the Caputo fractional derivative [8-10].

Recently, fractional Caputo-Fabrizio derivative [10, 11] and fractional Losada-Nieto integral [12] were proposed and developed to model the fractional electrical circuit [13, 14] and mass-spring-damper system [15].

In view of the previously mentioned results, the aim of the paper is to propose the fractional Maxwell fluid within fractional Caputo-Fabrizio derivative.

### Mathematical tools

We recall the fractional Caputo-Fabrizio (CF) derivative operator [10, 11], given by:

$${}^{CF}D_\tau^{(\vartheta)} \phi(\tau) = \frac{(2-\vartheta)\wp(\vartheta)}{2(1-\vartheta)} \int_0^\tau \exp\left[-\frac{\vartheta}{1-\vartheta}(\tau-\mu)\right] \phi^{(1)}(\mu) d\mu \quad (2a)$$

and fractional Losada-Nieto (LN) integral operator (inverse operator of fractional Caputo-Fabrizio derivative) [12], given by:

$${}^{LN}I_\tau^\vartheta \phi(\tau) = \frac{2(1-\vartheta)}{(2-\vartheta)\wp(\vartheta)} \phi(\tau) + \frac{2\vartheta}{(2-\vartheta)\wp(\vartheta)} \int_0^\tau \phi(\mu) d\mu \quad (2b)$$

where  $\wp(\vartheta)$  is a normalization constant depending on  $\vartheta$ .

In eqs. (2a) and (2b),  $\wp(\vartheta)$  is an unknown constant and it is inconvenient to compute the physical parameters. In this case, Losada and Nieto suggested that there be  $\wp(\vartheta) = 2/(2-\vartheta)$ , which leads to the new fractional Caputo-Fabrizio derivative operator [10, 11];

$${}^{CF}D_\tau^{(\vartheta)} \phi(\tau) = \frac{1}{1-\vartheta} \int_0^\tau \exp\left[-\frac{\vartheta}{1-\vartheta}(\tau-\mu)\right] \phi^{(1)}(\mu) d\mu \quad (2c)$$

and fractional Losada-Nieto integral operator:

$${}^{LN}I_\tau^\vartheta \phi(\tau) = (1-\vartheta)\phi(\tau) + \vartheta \int_0^\tau \phi(\mu) d\mu \quad (2d)$$

The Laplace transforms of eq. (2a) and (2b) are [12]:

$$L[{}^{CF}D_\tau^{(\vartheta)} \phi(\tau)] = \frac{(2-\vartheta)\wp(\vartheta)}{2[s+\vartheta(1-s)]} \{sL[\phi(\tau)] - \phi(0)\} \quad (2e)$$

$$L[{}^{LN}I_\tau^\vartheta \phi(\tau)] = \frac{2(1-\vartheta)}{(2-\vartheta)\wp(\vartheta)} L[\phi(\tau)] + \frac{2\alpha L[\phi(\tau)]}{(2-\vartheta)\wp(\vartheta)s} \quad (2f)$$

respectively, where  $L[m(\tau)] = m(s)$  represents the Laplace transform of the function  $m(\tau)$ .

## Fractional Maxwell fluid in fractional Caputo-Fabrizio derivative

### Mathematical models

Following Scott-Blair's view [1], we can suggest the constitutive relationship of fractional mechanical element:

$$\sigma(\tau) = \Lambda \mu^{\vartheta} \frac{d^{\vartheta} \varepsilon(\tau)}{d\tau^{\vartheta}} \quad (3a)$$

where  $\Lambda$ ,  $\mu$ , and  $\vartheta$  are material-dependant constants and  $\partial^{\vartheta}/\partial\tau^{\vartheta}$  is the fractional Caputo-Fabrizio derivative operator.

In this case, eq. (3a) is the Newton's law of Newtonian fluids of fractional dashpot and the element of the corresponding mechanical model is shown in fig. 1(a).

When  $\vartheta = 0$ , from eq. (3a) we obtain the Hooke's law of elastic body in the form:

$$\sigma(\tau) = \Lambda \varepsilon(\tau) \quad (3b)$$

where  $\Lambda$  is the viscosity of the material, and the element of the corresponding mechanical model is shown in fig. 1(b).

The dashpot, fig. 1(a), is the viscous (or dissipative) element. The spring, fig. 1(b), is the elastic (or storage) element.

The mechanical model of the fractional Maxwell fluid within fractional Caputo-Fabrizio derivative is illustrated in fig. 2.



(a)



(b)

Figure 1. (a) Newton's law, (b) Hooke's law



Figure 2. Fractional Maxwell fluid within fractional Caputo-Fabrizio derivative

The branch of the spring and fractional dashpot is in series. The shear stress-strain relationship in the fractional Maxwell fluid within fractional Caputo-Fabrizio derivative is written in the form:

$$\sigma(\tau) + \frac{1}{\mu^{\vartheta}} \frac{d^{\vartheta} \sigma(\tau)}{d\tau^{\vartheta}} = \Lambda \frac{d^{\vartheta} \varepsilon(\tau)}{d\tau^{\vartheta}} \quad (3c)$$

where  $\tau_0 = 1/\mu^{\vartheta}$  is called to be the relaxation time, and  $\Lambda \mu^{\vartheta}$  is the shear modulus.

By using the Boltzmann superposition principle and under the assumption of causal histories, the constitutive equation of the stress and strain in eqs. (3a), (3b), and (3c) can be expressed through the equation of the Volterra type:

$$\varepsilon(\tau) = \frac{\sigma(\tau)}{E_J} + \int_0^{\tau} J(\tau - \mu) \frac{d^{\vartheta} \sigma(\mu)}{d\tau^{\vartheta}} d\mu \quad (3d)$$

or

$$\sigma(\tau) = \varepsilon(\tau) E_G + \int_0^{\tau} G(\tau - \mu) \frac{d^{\vartheta} \varepsilon(\mu)}{d\tau^{\vartheta}} d\mu \quad (3e)$$

where  $[d^{\vartheta} \sigma(\mu)]/d\tau^{\vartheta}$  and  $[d^{\vartheta} \varepsilon(\mu)]/d\tau^{\vartheta}$  involve a delta function  $\delta(\tau)$ ,  $J(\tau)$  is the creep compliance function of the material,  $G(\tau)$  is the relaxation modulus function of the material, and  $E_I$  and  $E_G$  are instantaneous modulus functions of creep and relaxation in the material, respectively.

With the help of the Laplace transforms of eqs. (3d) and (3e) yields:

$$\tilde{\varepsilon}(s) = \frac{(2-\vartheta)\wp(\vartheta)}{2[s+\vartheta(1-s)]} s\tilde{J}(s)\tilde{\sigma}(s), \quad \tilde{\sigma}(s) = \frac{(2-\vartheta)\wp(\vartheta)}{2[s+\vartheta(1-s)]} s\tilde{G}(s)\tilde{\varepsilon}(s) \quad (3f,g)$$

where  $L[J(\tau)] = \tilde{J}(s)$ ,  $L[G(\tau)] = \tilde{G}(s)$ ,  $L[\varepsilon(\tau)] = \tilde{\varepsilon}(s)$ , and  $L[\sigma(\tau)] = \tilde{\sigma}(s)$ .

From eqs. (3f) and (3g) we present the reciprocity relation in the Laplace domain, namely:

$$\frac{(2-\vartheta)\wp(\vartheta)s}{2[s+\vartheta(1-s)]} \tilde{J}(s) = \frac{1}{\frac{(2-\vartheta)\wp(\vartheta)s}{2[s+\vartheta(1-s)]} \tilde{G}(s)} \quad (3h)$$

or

$$\tilde{J}(s)\tilde{G}(s) = \frac{1}{\left\{ \frac{(2-\vartheta)\wp(\vartheta)s}{2[s+\vartheta(1-s)]} \right\}^2} \quad (3i)$$

Then, from eq. (3i) we rewrite:

$$\begin{aligned} \tilde{J}(s)\tilde{G}(s) &= \frac{4(1-\vartheta)^2 + 4\vartheta^2 + 8(1-\vartheta)\vartheta s}{[(2-\vartheta)\wp(\vartheta)]^2 s^2} = \\ &= \frac{4(1-\vartheta)^2}{[(2-\vartheta)\wp(\vartheta)]^2} + \frac{8(1-\vartheta)\vartheta}{[(2-\vartheta)\wp(\vartheta)]^2 s} + \frac{4\vartheta^2}{[(2-\vartheta)\wp(\vartheta)]^2 s^2} \end{aligned} \quad (3j)$$

By setting  $J(\tau) \circ G(\tau) := \int_0^{\tau} J(\tau-\mu)G(\mu)d\mu$ , we have:

$$J(\tau) \circ G(\tau) = \frac{8(1-\vartheta)\vartheta}{[(2-\vartheta)\wp(\vartheta)]^2} + \frac{4\vartheta^2 \tau}{[(2-\vartheta)\wp(\vartheta)]^2} \quad (3k)$$

### The creep and relaxation processes

The creep representation with fractional Caputo-Fabrizio derivative operator is a slow continuous deformation of a material under constant stress  $\sigma(\tau) = \sigma_0(\tau)$ .

In view of eq. (3c), we obtain:

$$\Lambda \frac{d^{\vartheta} \varepsilon(\tau)}{d\tau^{\vartheta}} = \sigma_0(\tau) \quad (4a)$$

Thus, we obtain, from the result [12], the solution of eq. (4), namely:

$$\varepsilon(\tau) = \frac{2(1-\vartheta)}{(2-\vartheta)\wp(\vartheta)} \frac{\sigma_0(\tau)}{\Lambda} + \frac{2\vartheta}{(2-\vartheta)\wp(\vartheta)} \frac{\sigma_0(\tau)\tau}{\Lambda} + \Pi_c \quad (4b)$$

where  $\Pi_c$  is a constant.

Considering the initial condition  $\varepsilon(0) = \sigma_0(\tau)/\Lambda$ , and substituting it into eq. (4b), we have:

$$\frac{\sigma_0(\tau)}{\Lambda} = \frac{2(1-\vartheta)}{(2-\vartheta)\wp(\vartheta)} \frac{\sigma_0(\tau)}{\Lambda} + \Pi_c \quad (4c)$$

which leads to:

$$\Pi_c = \frac{\sigma_0(\tau)}{\Lambda} - \frac{2(1-\vartheta)}{(2-\vartheta)\wp(\vartheta)} \frac{\sigma_0(\tau)}{\Lambda} \quad (4d)$$

Thus, substituting eq. (4d) into eq. (4b), we obtain the solution of eq. (4a), namely:

$$\varepsilon(\tau) = \frac{\sigma_0(\tau)}{\Lambda} + \frac{2\vartheta}{(2-\vartheta)\wp(\vartheta)} \frac{\sigma_0(\tau)\tau}{\Lambda} \quad (4e)$$

Making  $\wp(\vartheta) = 2/(2-\vartheta)$  [12], eq. (4e) can be written in the form:

$$\varepsilon(\tau) = \frac{\sigma_0(\tau)}{\Lambda} + \vartheta \frac{\sigma_0(\tau)\tau}{\Lambda} \quad (4f)$$

When  $\vartheta = 1$ , eq. (4f) becomes [16]:

$$\varepsilon(\tau) = \frac{\sigma_0(\tau)}{\Lambda} + \frac{\sigma_0(\tau)\tau}{\Lambda} \quad (4g)$$

The relaxation representation with fractional Caputo-Fabrizio derivative operator is that the model is subjected to a constant strain  $\varepsilon(\tau) = \varepsilon_0(\tau)$ .

We obtain from eq. (3c):

$$\sigma(\tau) + \frac{1}{\mu^\vartheta} \frac{d^\vartheta \sigma(\tau)}{d\tau^\vartheta} = 0 \quad (5a)$$

Taking the Laplace transform of eq. (5a) gives:

$$\frac{1}{\mu^\vartheta} \frac{(2-\vartheta)\wp(\vartheta)}{2[s+\vartheta(1-s)]} \{sL[\sigma(\tau)] - \sigma(0)\} = L[\sigma(\tau)] \quad (5b)$$

which reduces to:

$$L[\sigma(\tau)] = \frac{\frac{(2-\vartheta)\wp(\vartheta)\sigma(0)}{[(2-\vartheta)\wp(\vartheta) - (2\mu^\vartheta - 2\mu^\vartheta\vartheta)]}}{s + \frac{2\mu^\vartheta\vartheta}{[(2-\vartheta)\wp(\vartheta) - (2\mu^\vartheta - 2\mu^\vartheta\vartheta)]}} \quad (5c)$$

Thus, the solution of eq. (5a) is given by:

$$\sigma(\tau) = \frac{(2-\vartheta)\wp(\vartheta)\sigma(0)}{[(2-\vartheta)\wp(\vartheta) - (2\mu^\vartheta - 2\mu^\vartheta\vartheta)]} e^{-\frac{2\mu^\vartheta\vartheta\tau}{[(2-\vartheta)\wp(\vartheta) - (2\mu^\vartheta - 2\mu^\vartheta\vartheta)]}} \quad (5d)$$

Taking the initial condition  $\sigma(0) = \Lambda \varepsilon_0(\tau)$ , eq. (5d) can be written:

$$\Lambda \varepsilon_0(\tau) = \frac{(2 - \vartheta)\wp(\vartheta)\sigma(0)}{[(2 - \vartheta)\wp(\vartheta) - (2\mu^\vartheta - 2\mu^\vartheta\vartheta)]} \quad (5e)$$

Thus, eq. (5d) can be written in the form:

$$\sigma(\tau) = \Lambda \varepsilon_0(\tau) e^{-\frac{2\mu^\vartheta\vartheta\tau}{[(2-\vartheta)\wp(\vartheta)-(2\mu^\vartheta-2\mu^\vartheta\vartheta)]}} \quad (5f)$$

Making  $\wp(\vartheta) = 2/(2 - \vartheta)$  [12], eq. (5f) can be written in the form:

$$\sigma(\tau) = \Lambda \varepsilon_0(\tau) e^{-\frac{\mu^\vartheta\vartheta\tau}{1-\mu^\vartheta+\mu^\vartheta\vartheta}} \quad (6a)$$

When  $\vartheta = 1$ , eq. (6a) becomes [16]:

$$\sigma(\tau) = \Lambda \varepsilon_0(\tau) e^{-\mu^\vartheta\tau} = \Lambda \varepsilon_0(\tau) e^{-\frac{\tau}{\tau_0}} \quad (6b)$$

## Conclusion

In this paper, we proposed the mathematical model for the fractional Maxwell fluid within fractional Caputo-Fabrizio derivative operator. The analytical results for the creep and relaxation processes with the fractional Losada-Nieto integral operator were discussed. The mathematical problems for the line viscoelasticity via fractional Caputo-Fabrizio derivative operator is open topics.

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## Nomenclature

$\varepsilon(\tau)$  – strain, [-]  
 $\sigma(\tau)$  – stress, [-]

$\tau$  – time, [s]  
 $\vartheta$  – fractional dimension, [-]

## References

- [1] Blair, G. S., The Role of Psychophysics in Rheology, *Journal of Colloid Science*, 2 (1947), 1, pp. 21-32
- [2] Debnath, L., Recent Applications of Fractional Calculus to Science and Engineering, *International Journal of Mathematics and Mathematical Sciences*, 2003 (2003), 54, pp. 3413-3442
- [3] Rogosin, S., Mainardi, F., George William Scott Blair – the Pioneer of Fractional Calculus in Rheology, *Communications in Applied and Industrial Mathematics*, 6 (2014), 1, e681
- [4] Gerasimov, A. N., A Generalization of Linear Laws of Deformation and Its Application to Problems of Internal Friction (in Russian), *Akad. Nauk SSSR. Prikl. Mat. Meh.*, 12 (1948), pp. 251-260
- [5] Kai-Xin, H., et al., Mechanical Analogies of Fractional Elements, *Chinese Physics Letters*, 26 (2009), 10, ID 108301
- [6] Caputo, M., Mainardi, F., Linear Models of Dissipation in Anelastic Solids, *Rivista del Nuovo Cimento (Ser. II)*, 1 (1971), 2, pp. 161-198
- [7] Mainardi, F., An Historical Perspective on Fractional Calculus in Linear Viscoelasticity, *Fractional Calculus and Applied Analysis*, 15 (2012), 4, pp. 712-717
- [8] Carpinteri, A., Mainardi, F., *Fractals and Fractional Calculus in Continuum Mechanics*, Springer Verlag, New York, USA, 1997

- [9] Kilbas, A. A., et al., *Theory and Applications of Fractional Differential Equations*, Elsevier Science Limited, Amsterdam, The Netherlands, 2006
- [10] Yang, X.-J., et al., *Local Fractional Integral Transforms and Their Applications*, Academic Press, New York, USA, 2015
- [11] Caputo, M., Fabrizio, M., A New Definition of Fractional Derivative without Singular Kernel, *Progress in Fractional Differentiation and Applications*, 1 (2015), 2, pp. 73-85
- [12] Lozada, J., Nieto, J. J. Properties of a New Fractional Derivative without Singular Kernel, *Progress in Fractional Differentiation and Applications*, 1 (2015), 2, pp. 87-92
- [13] Alsaedi, A., et al., Fractional Electrical Circuits, *Advances in Mechanical Engineering*, 7 (2015), 12, pp. 1-7
- [14] Atangana, A., Nieto, J. J., Numerical Solution for the Model of RLC Circuit via the Fractional Derivative without Singular Kernel, *Advances in Mechanical Engineering*, 7 (2015), 10, pp. 1-7
- [15] Gomez-Aguilar, J. F., et al., Modeling of a Mass-Spring-Damper System by Fractional Derivatives with and without a Singular Kernel, *Entropy*, 17 (2015), 9, pp. 6289-6303
- [16] Mainardi, F., *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, World Scientific, Singapore, 2010