A Modified Algorithm for Solving
the Split Feasibility Problem\textsuperscript{1}

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Abstract

In this paper a modified algorithm for solving the split feasibility problem (SFP) is presented. This algorithm uses the generalized Armijo line search in computing predictor step size and gives a correction step rule in the iterative process, which makes an accelerated convergence to the solution of SFP. Meanwhile, it needs not to compute the matrix inverses and the large eigenvalue of the matrix.

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1. Introduction

Let $C \subset R^n$ and $Q \subset R^m$ are nonempty closed convex sets and $A$ is a $m \times n$ real matrix. The split feasibility problem (SFP) is to find with \(x \in C\) if such $x$ exists. There are many applications, especially in the field of image restoration to deal with this problem [3].

The split feasibility problem (SFP) was first introduced by Censor and Elfving [4] and further studied by Byrne. In [2], Byrne gave a CQ algorithm

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to solve SFP:
\[ x^{k+1} = P_C(x^k + \lambda A^T(P_Q - I)Ax^k), \quad (1.1) \]
where \( \lambda \in (0, 2/L) \) and \( L \) is the large eigenvalue of the matrix \( A^T A \). The algorithm (1.1), it was assumed that the project could be easily computed. However, it often cost a great deal of time in accurately computing the project on a closed convex set and sometimes the results were not ideal. So in this paper a modified algorithm for solving the split feasibility problem (SFP) is presented. We define half space \( C_k \) and \( Q_k \) to replace the sets \( C \) and \( Q \). The orthogonal projection onto \( C_k \) and \( Q_k \) can be computed efficiently. Furthermore, we notice that step size plays an important role in convergence of the algorithms. So in this paper the modified algorithm uses the generalized Armijo line search to compute predictor step size and gives a correction step rule in the iterative process, which makes an accelerated convergence to the solution of SFP. We also show convergence of the algorithm under some mild assumptions.

The rest of this paper is organized as follows. In the section 2, we review some concepts and exiting results. In the Section 3, a modified projection algorithm is given and its convergence is shown.

2. Preliminaries

In this section, we review some definitions and basic results which will be used later on.

Definition 2.1. Let \( F \) be a mapping from set \( C \subset \mathbb{R}^n \) into \( \mathbb{R}^n \). Then
(a) \( F \) is said to be Lipschitz continuous on \( C \) with constant \( \lambda > 0 \), if
\[ \|F(x) - F(y)\| \leq \lambda \|x - y\|, \forall x, y \in C; \]
(b) \( F \) is said to be uniformly monotone on \( C \) with modulus \( \delta > 0 \), if
\[ \langle F(x) - F(y), x - y \rangle \geq \delta \|F(x) - F(y)\|^2, \forall x, y \in C. \]

Definition 2.2. Let \( g \) be a convex function defined on \( \mathbb{R}^n \). The set
\[ \partial g(x) = \{\xi \in \mathbb{R}^n | g(z) \geq g(x) + \langle \xi, z - x \rangle, \forall z \in \mathbb{R}^n \} \]
is said to be the subdifferential of \( g(x) \) at \( x \).

Definition 2.3. Let \( C \) be a nonempty closed convex subset in \( \mathbb{R}^n \). The set
\[ P_C(y) = \text{arg min}\{\|x - y\|, x \in C\}, y \in \mathbb{R}^n. \]
is said to be the orthogonal projection from $\mathbb{R}^n$ onto $C$.

It has the following well-known properties.

**Lemma 2.1** (see [9]). Let $C$ be a nonempty closed convex subset in $\mathbb{R}^n$.

Then for any $y,z \in \mathbb{R}^n$ and $x \in C$, we have that

(a) $\langle y - P_C(y), P_C(y) - x \rangle \geq 0$;

(b) $\|P_C(y) - P_C(z)\|^2 \leq \langle P_C(y) - P_C(z), y - z \rangle$;

(c) $\|P_C(z) - x\|^2 \leq \|z - x\|^2 - \|P_C(z) - z\|^2$.

**Lemma 2.2** (see [1]). Let $C$ be a nonempty closed convex subset in $\mathbb{R}^n$.

For some certain $x \in C$ and $d \in \mathbb{R}^n$, we define

$$H(a) = P_C(x - ad), a \geq 0.$$ 

Then we have that

(a) $\langle H(a) - x + ad, y - H(a) \rangle \geq 0, \forall y \in C$;

(b) $\langle d, x - H(a) \rangle \geq \frac{\|x - H(a)\|^2}{a}$.

**Lemma 2.3** (see [5,7]). Let $C$ be a nonempty closed convex subset in $\mathbb{R}^n$.

For any $x \in C$ and $d \in \mathbb{R}^n$, we have that

(a) $\|x - H(a)\|$ on $a \geq 0$ is nondecreasing;

(b) $\frac{\|x - H(a)\|}{a}$ on $a > 0$ is nonincreasing.

From Lemma 2.1, we know that $P_C$ is Lipschitz continuous (with rank 1) (i.e., $\|P_C(y) - P_C(z)\| \leq \|y - z\|$) (see [9]) and uniformly monotone (with modulus 1).

Let $F$ be a mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $a > 0$, we define

$$x(a) = P_Q(x - aF(x)), e(x, a) = x - x(a), r(x, a) = \|e(x, a)\|.$$ 

**Remark 2.1.** $\|e(x, a)\|$ on $a \geq 0$ is nondecreasing and $\frac{\|e(x, a)\|}{a}$ on $a > 0$ is nonincreasing.

**Lemma 2.4.** Let $F$ be a mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $a > 0$, we have

$$\min\{1, a\} \|e(x, 1)\| \leq \|e(x, a)\| \leq \max\{1, a\} \|e(x, 1)\|.$$ 

3. Algorithm and Its Convergence

In this section, a modified algorithm is established. Firstly, we assume the following conditions.

The convex set $C$ and $Q$ are given by:

$$C = \{x \in \mathbb{R}^n | f(x) \leq 0\}, \quad Q = \{y \in \mathbb{R}^n | g(y) \leq 0\},$$
where \( f, g : \mathbb{R}^n \rightarrow \mathbb{R} \) are convex and \( C, Q \) are nonempty. Then the split feasibility problem is to find elements \( x \), which satisfy:

\[
x \in C, \ Ax \in Q,
\]

if such \( x \) exists.

For any \( x, y \in \mathbb{R}^n \), each of \( \xi \in \partial f(x) \) and \( \psi \in \partial g(x) \) have at least one subgradient that can be computed. So we define the convex set \( C_k \) and \( Q_k \), which have the following form:

\[
C_k = \{ x \in \mathbb{R}^n | f(x^k) + \langle \xi^k, x - x^k \rangle \leq 0 \}, \xi^k \in \partial f(x).
\]

\[
Q_k = \{ y \in \mathbb{R}^n | g(y^k) + \langle \psi^k, y - y^k \rangle \leq 0 \}, \psi^k \in \partial g(x).
\]

By the definition of subdifferential in (3.2) and (3.3), it is clear that half space \( C \subset C_k \) and \( Q \subset Q_k \). From the expression of \( C_k \) and \( Q_k \), the orthogonal projection onto \( C_k \) and \( Q_k \) can be computed efficiently.

For every \( k \), using \( Q_k \) we define the function \( F_k : \mathbb{R}^n \rightarrow \mathbb{R}^n \) by

\[
F_k(x) = A^T (I - P_{Q_k}) Ax.
\]

**Remark 3.1.** \( F_k \) is Lipschitz continuous on \( \mathbb{R}^n \) with constant 1 and uniformly monotone on \( \mathbb{R}^n \) with modulus 1, where denote the identity operator.

Now we give a modified algorithm for the split feasibility problem (SFP) as follow:

**Algorithm 3.1**

Let \( x^0 \) be arbitrary, \( \gamma_1, \gamma_2 \in (0, +\infty) \) and \( u_1, u_2 \in (0, 1) \) be given;

Let

\[
\begin{align*}
\bar{x}^k &= P_{C_k} [x^k - a_k F_k(x^k)] \\
x^{k+1} &= P_{C_k} [x^k - \beta_k (\alpha_k F_k(\bar{x}^k))]
\end{align*}
\]

where the predictor step size \( \alpha_k \) satisfies,

\[
\| F_k(x^k) - F_k(x^k_{\alpha_k}) \| \leq \mu_1 \frac{\| x^k - \bar{x}^k \|}{a_k},
\]

\[
\alpha_k \geq \gamma_1 \text{ or } \alpha_k \geq \gamma_2 \bar{a}_k > 0.
\]

\( \bar{a}_k \) satisfies

\[
\| F_k(x^k) - F_k(x^k_{(\bar{a}_k)}) \| > \mu_2 \frac{\| x^k - \bar{x}^k \|}{\bar{a}_k},
\]
the corrector step size $\beta_k$ satisfies,

$$\beta_k^{(a)} = \frac{g(x_k, \alpha_k) e(x_k, \alpha_k)}{\|g(x_k, \alpha_k)\|^2}, \quad g(x_k, \alpha_k) = \alpha_k F(x).$$

The feasibility of $\alpha_k$ and $\beta_k$ can be found in [8].

Lemma 3.1 (see [6]). Let $x^k$ is given by the predictor step size Algorithm (3.1). We have

$$(x^k - x^*)^T F_k(x^k) \geq (x^k - x^*)^T F_k(x^k) \geq (1 - u_1) \frac{\beta}{\alpha_k} \|x^k - x^*\|^2.$$ 

Theorem 3.1. Let $\{x^k\}$ be a sequence generated by Algorithm 3.1. If the solution set for solving the split feasibility problem of (3.1) is nonempty, then $\{x^k\}$ converges to a solution of the SFP.

Proof. Let $x^*$ be a solution of the SFP and $g^k = g(x_k, \alpha_k), e_k = e(x_k, a_k)$. By Algorithm 3.1, Lemma 3.1 and the (3.4), it is obtained that

$$\|x^{k+1} - x^*\|^2 = \|P_{C_k}(x^k - \beta_k^{(a)} g^k - x^*)\|^2$$

$$\leq \|x^k - \beta_k^{(a)} g^k - x^*\|^2$$

$$= \|x^k - x^*\|^2 + (\beta_k^{(a)})^2 \|g^k\|^2 - 2\beta_k^{(a)} (x^k - x^*)^T g^k$$

$$\leq \|x^k - x^*\|^2 + (\beta_k^{(a)})^2 \|g^k\|^2 - 2\beta_k^{(a)} g_k$$

$$= \|x^k - x^*\|^2 - \|e_k g^k\|^2.$$ 

It is true

$$\sum_{k=0}^{\infty} \left\|e_k g^k\right\|^2 \leq \sum_{k=0}^{\infty} \left\|x^k - x^*\right\|^2 - \|x^{k+1} - x^*\|^2 < +\infty. \quad (3.6)$$

The (3.6) yields that the sequence is monotonically decreasing. Therefore is bounded. By deducing, we have

$$\|F_k(x^k)\| = \|F_k(x^k) + F_k(x^k) - F_k(x^k)\| \leq \|F_k(x^k)\| + u_1 \frac{\|x^k - x^k\|}{\alpha_k}$$

$$\leq \|F_k(x^k)\| + u_1 \|F_k(x^k)\| = (1 + u_1) \|F_k(x^k)\|.$$ 

For all $k = 0, 1, \ldots$ we can conclude

$$\frac{(e_k^T g^k)^2}{\|g^k\|^2} \geq \frac{\alpha_k^2 (1 - u_1)^2 (e_k^T F_k)^2}{\alpha_k^2 \|F_k(x^k)\|^2} \geq m((e_k^T F_k)^2), m > 0. \quad (3.7)$$

The (3.7) yields

$$e_k^T F_k \to 0, k \to \infty.$$ \quad (3.8)
We assume that $x^\theta$ is an accumulation point of $\{x^k\}$ and $\lim_{k_i \to \infty} x^{k_i} = x^\theta$, where is a subsequence of $\{x^{k_i}\}$.

First we show that $x^\theta \in C$. By $x^{k_i+1} \in C_k$ and $C_k$ is defined as the follows,

$$f(x^{k_i}) + \langle \xi^{k_i}, x^{k_i+1} - x^{k_i} \rangle \leq 0, \forall i = 1, 2, \ldots. \quad (3.9)$$

Passing the limit in (3.9) and by (3.8) and lemma (3.1), it follows that $f(x^\theta) \leq 0$, that is $x^\theta \in C$.

In what follows we prove $x^\theta \in Q$.

(a) If $\{\alpha_k\} \geq \alpha_{\min} > 0$, by Lemma 1.4 Algorithm 2.1 and (3.8) then, we have that

$$e_k^T F_k \geq \frac{\|e_k\|^2}{\alpha_k} = \frac{\alpha_k \|e_k\|^2}{\alpha^2_k} \geq \alpha_{\min} \left(\frac{\|e_k\|}{\alpha_k}\right)^2. \quad \text{By deducing with (3.8), we have } \frac{\|e_k\|}{\alpha_k} \to 0, (k \to \infty).$$

As

$$\alpha_k (F(x^k)^T (x^\theta - x) \leq (x^k - x^\theta)^T (x - x^\theta) \leq (x^k - x^\theta)^T (x - x^\theta), \forall x \in Q. \quad \text{then, it follows}$$

$$F(x^k)^T (x^k - x) = F(x^k)^T (x - x^\theta) + F(x^k)^T (x^\theta - x) \leq F(x^k)^T (x - x^\theta) + \frac{\|x^k - x^\theta\|}{\alpha_k} \|x - x^\theta\| \to 0. \quad \text{Passing } k_i \to \infty, \text{ we get } F(x^\theta)^T (x^\theta - x) \leq 0. \quad \text{Moreover, we can obtain}$$

$$\lim_{k_i \to \infty} [(I - P_{Q_k}) A x^{k_i}] = 0.$$

As $P_{Q_k} (A x^{k_i}) \in Q_k$, we get

$$g(A x^{k_i}) + \langle A x^{k_i}, P_{Q_k} (A x^{k_i}) - A x^{k_i} \rangle \leq 0. \quad (3.10)$$

Passing $k_i \to \infty$ in (3.11) leads to $g(A x^{k_i}) \leq 0$. This yields to $A x^\theta \in Q$.

(b) If $\{\alpha_k\} \geq \alpha_{\min} > 0$ does not hold, there exists a subsequence $\{\alpha_{\theta_k}\} \to 0$.

As $x^k$ is bounded and $\{\alpha_{\theta_k}\} \to 0$, it follows

$$\|x^{k_i} - x^k(\alpha_{\theta_k})\| \leq \alpha_{\theta_k} \|F_{k_i}(x^{k_i})\| \to 0, k_i \to \infty.$$

by (3.5) and (b) in Lemma 1.3, we conclude that

$$u_2 \|e(x_{\theta_k}, 1)\| \leq u_2 \frac{\|e(x_{\theta_k}, \alpha_{\theta_k})\|}{\alpha_{\theta_k}} < \|F_{k_i}(x^{k_i}) - F_{k_i}(x^{k_i}(\alpha_{\theta_k}))\| \to 0.$$
Then, it follows
\[
\lim_{k_i \to \infty} \left\| e(x^{k_i}, 1) \right\| = 0.
\]

Similar to the method in (a), we can get \( x^\theta \in Q \). Consequently, \( x^k \) converges to a solution of the SFP.

4. Concluding Remarks

In this paper, a modified algorithm for solving the split feasibility problem has been presented and it uses the generalized Armijo line search in computing the predictor step size, which makes an accelerated convergence to the solution of SFP. We also find that the corrector step size \( \beta_k \) plays an important role in the algorithm, which is worth future study.

References


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