On induced subgraphs of trees, with restricted degrees

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Abstract
It is proved that every tree $T$ on $n \geq 2$ vertices contains an induced subgraph $F$ such that all its degrees are odd and $|F| \geq \lceil n/2 \rceil$.

1. Introduction

More than 30 years ago, Gallai proved the following theorem (see [4, Problem 5.17] for a simple proof, or [3]).

**Theorem 1.1.** Let $G$ be an arbitrary graph.

1. There exists a partition $V(G) = A \cup B$, $A \cap B = \emptyset$, such that in the induced subgraph on $\langle A \rangle$ and $\langle B \rangle$ all the degrees are even.

2. There exists a partition $V(G) = A \cup B$, $A \cap B = \emptyset$, such that in the induced subgraph $\langle A \rangle$ all the degrees are even and in the induced subgraph $\langle B \rangle$ all the degrees are odd.

Clearly, from Theorem 1.1 we infer that every graph $G$ contains an induced subgraph $H$ such that $|H| \geq |G|/2$ and all the degrees in $H$ are even.

The following related conjecture seems to be surprisingly hard.

**Conjecture 1.2.** There exists a positive constant $c$ such that every graph $G$ with $\delta(G) \geq 1$ contains an induced subgraph $H$ such that $|H| \geq c|G|$ and all the degrees in $H$ are odd.

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Recently, some results were obtained (see [2]), of which we choose to mention here the following theorem.

**Theorem 1.3.** Let $G$ be a graph on $n$ vertices and suppose $\delta(G) \geq 1$. Then $G$ contains an induced subgraph $H$ in which all the degrees are odd and $|H| \geq \sqrt{(n-\sqrt{n})/6}$.

**Theorem 1.4.** Let $G$ be a self-complementary graph on $n$ vertices. Then $G$ contains an induced subgraph $H$ in which all the degrees are odd and $|H| \geq \lceil (n-1)/2 \rceil$.

Our main object in this paper is to prove Conjecture 1.2 in the case of trees and to show that $c = \frac{1}{2}$ is permitted. Some related problems will be considered.

The notations used in this article are standard following [1]. In particular, $\delta(G)$ and $\Delta(G)$ denote the minimal and maximal degrees of $G$, respectively. If $A \subseteq V(G)$ then $\langle A \rangle$ is the induced subgraph of $G$ on the vertex set $A$. Finally, for $i = 0, 1$ let the function $f_i(G)$ denote the maximum cardinality of an induced subgraph $H$ of $G$ such that all the degrees in $H$ are congruent to $i \pmod{2}$. Hence, by Theorem 1.1, $f_0(G) \geq |G|/2$.

## 2. Results and proofs

We start with one of the main results.

**Theorem 2.1.** Let $T$ be a tree on $n \geq 2$ vertices. Then $f_1(T) \geq \lceil (n+1)/2 \rceil$ unless $T = P_4$, in which case $f_1(P_4) = 2 = n/2$.

**Proof.** We use induction on $n$. For $2 \leq n \leq 9$ the assertion of the theorem is easy to verify by direct checking (see [2] for the list of trees). Let $t_i$ denote the number of vertices of degree $i$ in $T$. Consider the following cases.

**Case 1:** Each vertex of even degree (even-vertex) is adjacent to an end-vertex.

Then by deleting an end-vertex from each even-vertex, we are done since we can proceed as follows:

1. By the assumption, $t_1 \geq \sum_{2} t_i$. If $t_i \neq 0$ for some odd $i \geq 3$ then $n = \sum t_i \geq 2 \sum t_i + 1$. Hence, $\sum t_i \leq (n-1)/2$ and after deleting one end-vertex from each even-vertex, we are left with at least $\lceil n-(n-1)/2 \rceil$ vertices. Moreover, if $t_1 > \sum t_i$ then again $\sum t_i \leq (n-1)/2$ and the same argument works.

2. We may assume that $t_1 = \sum t_i$ and, as $n \geq 5$, $t_i > 0$ for some $i \geq 4$.

Let $v$ be an even-vertex such that $\deg(v) = d \geq 4$. Consider the components of $T \setminus v$, $T_1, T_2, \ldots, T_d$. Clearly, $|T_i| = 1$ for the adjacent end-vertex, but for $2 \leq k \leq d$, $T_k$ is neither an isolated vertex nor a $P_4$ (because of the assumption of Case 1 and (1)).
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By induction,

\[
f_1(T) \geq \sum_{k=2}^{d} f_1(T_k) \geq \sum_{k=2}^{d} \left\lceil \frac{|T_k| + 1}{2} \right\rceil \geq \sum_{k=2}^{d} \frac{|T_k| + 1}{2} = \sum_{k=2}^{d} \frac{|T_k|}{2} + \frac{d-1}{2} \geq n - 2 + \frac{3}{2} = n + 1.
\]

Hence, \( f_1(T) \geq \lceil (n+1)/2 \rceil \).

Case 2: Suppose there is an even vertex \( v \) which is adjacent to no end-vertex.

(1) Suppose first that \( \text{deg}(v) = 2 \). Then \( T \setminus v \) has two components \( T_1, T_2 \) of orders \( m_1, m_2 \). If \( T_1, T_2 \neq P_4 \), then by induction,

\[
f_1(T) = f_1(T_1) + f_1(T_2) \geq \frac{m_1 + 1}{2} + \frac{m_2 + 1}{2} = \frac{m_1 + 1}{2} + \frac{m_2 + 1}{2} = \frac{n + 1}{2}.
\]

Hence, \( f_1(T) \geq \lceil (n+1)/2 \rceil \).

If now \( T_1 = P_4 \) and \( T_2 \neq P_4 \) then there are exactly two cases as indicated in Fig. 1. In both cases we can delete vertex \( u \) instead of \( v \), and apply induction, since \( n > 9 \), to obtain the required result.

Finally, suppose \( T_1 = T_2 = P_4 \). Then \( T \) must be one of the trees in Fig. 2. In each case we may consider the induced graph on the vertex set \( \{a, b, c, u, d, e\} \) and find that \( f_1(T) = 6 \geq \lceil (n+1)/2 \rceil \).

(2) Assume now that \( \text{deg}(v) = 2k \geq 4 \). Consider the \( 2k \) components of \( T \setminus v \), say \( T_1, T_2, \ldots, T_{2k} \), of corresponding orders \( m_1, m_2, \ldots, m_{2k} \). Assume \( t \) of the components are \( P_4 \). By induction we have

\[
f_1(T) \geq \sum_{i=1}^{2k} f_1(T_i) \geq \sum_{i=1}^{2k} \frac{m_i + 1}{2} = \sum_{i=1}^{2k} \frac{m_i}{2} + k - \frac{t}{2}.
\]

Now if \( t \leq 2k - 2 \) then \( k - t/2 \geq 1 \) and we obtain

\[
f_1(T) \geq \left\lceil \sum_{i=1}^{2k} \frac{m_i}{2} + 1 \right\rceil = \left\lceil \frac{n + 1}{2} \right\rceil.
\]

Fig. 1.
Hence, we may assume that $2k - 1 \leq t \leq 2k$. From each of the components $T_i = P_4$, $1 \leq i \leq 2k - 2$, take an edge whose vertices are not adjacent to $v$ and consider the last component $T = T_{2k-1} \cup T_{2k} \cup \{v\}$. By induction and the construction above (as $T \neq P_4$), we have

$$f_1(T) \geq (2k - 2)f_1(P_4) + f_1(T') \geq 4k - 4 + \frac{1}{2} \left[ \frac{m_{2k-1} + m_{2k} + 1}{2} \right]$$

$$\geq 4k - 4 + \frac{m_{2k-1} + m_{2k}}{2} + 1 = \frac{n + 1}{2}.$$

Hence, $f_1(T) \geq \lceil (n + 1)/2 \rceil$. This completes the proof of the theorem. □

We conjecture that the following stronger result holds.

**Conjecture 2.2.** For every tree $T$ on $n \geq 2$ vertices $f_1(T) \geq (2n - 2)/3$.

One may see that Conjecture 2.2 is sharp for paths and some spiders, but for forests the lower bound $n/2$ is best possible, as one may choose a forest consisting of $P_4$ trees only.

The proof technique of Theorem 2.1 can be used to obtain a sharp estimate to the following problem:

**Estimate** $f(k, T)$ := the largest order of an induced subgraph of a tree $T$ in which $\deg(v) \equiv 0 \pmod{k}$ for every vertex $v$.

**Theorem 2.3.** Let $T$ be a tree on $n \geq 2$ vertices. Let $k \geq 3$ be an integer. Then

$$f(k, T) \geq \frac{(k-2)n + 2}{k-1}.$$

This bound is the best possible.
Proof. Apply induction on \( n \). For \( n = 2 \) it is easy to check. Suppose we have proved it for \( 2 \leq m \leq n - 1 \). We prove it for \( m = n \). Consider two cases.

Case 1: There is a vertex \( v \), \( \deg(v) \equiv 0 \pmod{k} \), which is not adjacent to any end-vertex.

Consider \( F = T \setminus \{v\} \). Let \( n_1, n_2, \ldots, n_k \) be the orders of the components of \( F \). Obviously, we have \( \sum_{i=1}^{k} n_i = n - 1 \). By the induction hypothesis we get

\[
\begin{align*}
\sum_{i=1}^{k} \frac{(k-2)n_i + 2}{k-1} &\geq \frac{1}{k-1} \left\{ (k-2) \sum_{i=1}^{k} n_i + 2tk \right\} \\
&= \frac{1}{k-1} \left\{ (k-2)(n-1) + 2tk \right\} \geq \frac{(k-2)n + 2}{k-1}.
\end{align*}
\]

Case 2: Each vertex of degree \( 0 \pmod{k} \) is adjacent to at least one end-vertex.

For each such vertex choose one end-vertex and delete it. Then in the resulting tree \( T' \) there are no vertices of degree \( 0 \pmod{k} \). Let \( t_i \) denote the number of vertices in \( T \) with degree \( i \). Then \( \sum_{i=1}^{n} it_i = 2n - 2 \) and \( \sum_{i=1}^{n} 2t_i = 2n \).

By subtracting the last two equalities we obtain \( \sum_{i=1}^{n} (i-2)t_i = -2 \) and, hence, \( t_1 - 2 = \sum_{i=1}^{n} (i-2)t_i = \sum_{k|i} (k-2)t_i \). Adding \( 2 + \sum_{k|i} t_i \) to both sides, we get \( (k-1) \sum_{k|i} t_i + 2 \leq t_1 + \sum_{k|i} t_i \leq n \). Hence \( \sum_{k|i} t_i \leq (n-2)/(k-1) \). Thus, the number of end-vertices to be deleted is at most \( (n-2)/(k-1) \) and the resulting forest is of order at least

\[
\frac{n - 2}{k - 1} \geq \frac{(k-2)n + 2}{k - 1}.
\]

In order to see that the above bound is in general optimal, consider the trees, each of degree \( k \), of the form shown in Fig. 3.

We leave the easy details to the reader. \( \square \)

Finally, let

\[
f_{1,k}(T) := \text{the largest order of an induced subgraph of a tree } T \text{ in which } \deg(v) \equiv 1 \pmod{k} \text{ for every vertex } v.
\]

The problem of estimating the lower bound of \( f_{1,k}(T) \) is a natural generalization of the problem concerning \( f_1(T) \).

Fig. 3.
Theorem 2.4. Let \( k \geq 2 \) be an integer and let \( T \) be a tree on \( n \geq 2 \) vertices. Then
\[
f_{1,k}(T) \geq \frac{2(n-1)}{3k}.
\]

Proof. For \( k = 2 \) we already know the stronger result of Theorem 2.1. Suppose \( k \geq 3 \). Consider the tree as rooted at a vertex \( v \). For \( j = 1, 2, 3 \), let \( E_j \) be the set of all edges at distance \( j \) (mod 3) from \( v \). Then \( \sum_{i=1}^{3} E_i = n-1 \). Without loss of generality, assume that \( E_1 > (n-1)/3 \). The edges of \( E_2 \) induce a forest \( F \), whose components are stars. Let \( t_i \) denote the number of stars in \( F \) having \( i \) edges. Then
\[
|E_1| = \sum_{i=1}^{n} it_i \geq \frac{n-1}{3} \quad \text{and} \quad |F| = \sum_{i=1}^{n} (i+1)t_i.
\]

From each star on \( i \) edges, we have to delete \((i-1)(\text{mod} \ k)\) edges (and hence \((i-1)(\text{mod} \ k)\) vertices), in order to obtain a star whose degrees are \((1 \text{mod} \ k)\). Hence,
\[
f_{1,k}(T) \geq \sum_{i=1}^{n} t_i((i+1)-(i-1)(\text{mod} \ k)) \geq \sum_{1 \leq i < k} 2t_i + \sum_{i \geq k} t_i(i+1-(k-1))
\]
\[
> \frac{2 \sum_{i=1}^{k-1} it_i}{k} + \frac{2 \sum_{i=k}^{n} it_i}{k} = \frac{2 \sum_{i=1}^{n} it_i}{k} \geq \frac{2(n-1)}{3k}.
\]

We conjecture that the following stronger lower bound holds.

Conjecture 2.5. Let \( T \) be a tree on \( n \geq 2 \) vertices and \( k \geq 3 \) be an integer. Then
\[
f_{1,k}(T) \geq \frac{n+2k-4}{k-1}.
\]

Note added in proof. We have recently learned to know that Conjecture 2.2 was proved by A.J. Radcliffe and A.D. Scott.

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References