NOTE

A NOTE ON PACKING TREES INTO COMPLETE BIPARTITE GRAPHS AND ON FISHBURN’S CONJECTURE

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In this note we improve significantly the result appeared in [4] by showing that any sequence of trees \( \{T_2, T_3, \ldots, T_t\} \) can be packed into the complete bipartite graph \( K_{n-1,n-2} \) (n even) for \( t = 0.3n \). Furthermore we support Fishburn’s Conjecture [2] by showing that any sequence \( \{T_2, T_4, \ldots, T_t\} \) can be packed into \( H_n \) (n even) where \( t = 0.23n \).

The sequence of graphs \( \{G_1, G_2, \ldots, G_t\} \) is said to be packed into a graph \( G \) if \( G \) has edge disjoint subgraphs \( H_1, H_2, \ldots, H_t \) such that \( H_j \cong G_j, j = 1, \ldots, t \). We say that a graph \( G \) is \( m \)-panarboreal if each tree on \( m \) vertices, \( T_m \), can be packed into \( G \).

Gyarfas and Lehel [3] conjectured that if \( T_i \) is any tree of order \( i \) then the sequence of trees \( \{T_2, T_3, \ldots, T_t\} \) can be packed into the complete graph \( K_n \). (For references about the known results one may look at [5]).

We mention here only the result due to Bollobás [1]:

**Theorem 1.** Suppose that \( 3 \leq t < \sqrt{n/2} \). Then any sequence of trees \( \{T_2, T_3, \ldots, T_t\} \) can be packed into \( K_n \).

The proof is based upon the following lemma which is also our main tool in obtaining our results.

**Lemma 2.** Let \( k \) be a positive integer. Suppose \( H \) is a graph of order \( n \geq k + 1 \). If

\[
e(H) \geq (k-1)(n-k) + \binom{k}{2} + 1,
\]

then \( H \) contains a subgraph \( F \) such that \( \delta(F) \geq k \).

It is easy to see that such a graph \( F \) contains every tree \( T_{k+1} \).

To the sequel we assume that \( n \) is an even integer.

Hobbs, Bourgeois and Kasiraj [4] conjectured that any sequence of trees
\{T_2, T_3, \ldots, T_n\} can be packed into the complete bipartite graph \(K_{n-1,n/2}\). They showed that for \(t = 2\sqrt{3n}\) any sequence of trees \(\{T_2, T_3, \ldots, T_t\}\) is packed into \(K_{n-1,n/2}\).

Their proof is based upon a straightforward construction using some difficult impressive lemma about packing any two different trees into \(K_{n-1,n/2}\).

In this note we improve their result by showing:

**Theorem 3.** Let \(t = 0.3n\) (actually \((\sqrt{13} - 3)n/2\)). Then any sequence of trees \(\{T_2, T_3, \ldots, T_t\}\) is packed into \(K_{n-1,n/2}\).

**Proof.** First we show that \(K_{n-1,n/2}\) is \(n\)-panarboreal. But this is obviously true since any tree is a bipartite graph and if its order is \(n\) then its smallest part does not exceed \(n/2\). So we may start induction. Consider the packing of \(T_{k+1}, T_{k+2}, \ldots, T_t\) into \(K_{n-1,n/2}\). Put, \(H = K_{n-1,n/2} \setminus \bigcup_{j=k+1}^t T_j\). The graph \(H\) has order \((3n/2) - 1\) and size

\[
e(H) = \frac{n(n-1)}{2} - \frac{(t-k)(t-k)}{2}. \tag{1}
\]

\(T_k\) can be packed into \(H\) since \(H\) has a subgraph \(F\) with \(\delta(F) \geq k - 1\), indeed otherwise, by Lemma 2,

\[
e(H) \leq \left(\frac{k-1}{2}\right) + (k-2)\left(\frac{3n}{2} - k\right). \tag{2}
\]

Relations (1) and (2) imply

\(t^2 - t - n^2 - 5n + 3kn - 2k^2 + 2k \geq 0\).

Contradicting the choice of \(t\). \(\square\)

Next we shall consider the problem of packing sequences

\(\{T_2, T_3, \ldots, T_{2m}\}\) \(\tag{3}\)

and

\(\{T_3, T_5, \ldots, T_{2m+1}\}\) \(\tag{4}\)

into graphs \(H_{2m}\) and \(H_{2m+1}\), respectively, as introduced by Fishburn [2]. He called them **half complete graphs** and defined them as follows:

For each \(m \geq 1\), \(H_{2m}\) is the graph with \(2m\) vertices and degree sequence \(\{2m - 1, 2m - 2, \ldots, m + 1, m, m, m - 1, \ldots, 2, 1\}\), and \(H_{2m+1}\) is a graph with \(2m + 1\) vertices and degree sequence \(\{2m, 2m - 1, \ldots, m + 1, m, m, m - 1, \ldots, 2, 1\}\).

More details about those graphs can be found in [2].

Fishburn [2] conjectured that (3) can be packed into \(H_{2m}\) and (4) can be packed
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into $H_{2m+1}$. He himself proved his conjectures for certain classes of trees (leading to an affirmative answer to Gyarfas–Lehel Conjecture for $m \leq 9$).

We prove here

**Theorem 4.** Let $t < 0.23n$ be an even integer. Then any sequence of trees $\{T_2, T_4, \ldots, T_t\}$ is packed into $H_n$.

In the proof of Theorem 4 we use the same arguments as in the proof of Theorem 3 so that we have to show first that $H_m$ ($m$ even or odd) is $m$-panarboreal. To do so let us first give some definitions and notation.

We orient $H_m$ as follows: Let $V(H_m) = K \cup L$ where $|K| = \lfloor m/2 \rfloor$, and let $[x]$ denotes the largest integer not exceeding $x$. The degree sequence of the vertices of $K$ is $(m-1, m-2, \ldots, \lfloor (m+1)/2 \rfloor)$. Denote the vertices of $K$ by the numbers $\{1, 2, \ldots, \lfloor m/2 \rfloor\}$ respectively, i.e. $d(v_1) = m - 1$, $d(v_2) = m - 2$, $\ldots$, $d(v_{\lfloor m/2 \rfloor}) = \lfloor (m+1)/2 \rfloor$. The set $L$ is the set of the remaining vertices of $H_m$. One can see that for each $y \in L$, $\exists x \in K$ such that $(x, y) \in E(H_m)$ and $L$ is an independent set of vertices of $H_m$. The vertices of $L$ are labeled by $\{\lfloor m/2 \rfloor + 1, \ldots, m\}$ in a way that $d(v_{\lfloor m/2 \rfloor + 1}) = [m + 1/2] + 1, \ldots, d(v_m) = 1$. Denote by $n(x)$ the number given to $x$ in the labeling. The orientation is given as follows:

$$\forall x \in K, \ x \to y, \ \text{if} \ (x, y) \in E(H_m) \ \text{and} \ n(x) < n(y).$$

The directed graph $H_m$ is denoted by $\tilde{H}_m$ and its adjacency matrix by $A(\tilde{H}_m)$.

Notice that $G$ is a spanning subgraph of $F$ iff $A(F) \geq A(G)$ (i.e. $A(F) - A(G)$ is a nonnegative matrix). Let $T$ be any tree. A cutvertex $v$ of $T$ is called end cutvertex if $T \setminus v$ has at most one non-trivial component. (It is easy to see that at least one endcutvertex exist if $T$ is not an edge).

A standard orientation $\tilde{T}$ of $T$ is defined as follows: $T$ has a unique representation as a bipartite graph with parts $X, Y, X \cup Y = V(T)$, and choose $|X| \leq |Y|$. Then we orient the edges of $T$ from $X$ to $Y$.

**Lemma 5.** The graph $H_m$ ($m$ even or odd) is $m$-panarboreal.

**Proof.** It is sufficient to prove that $A(\tilde{T}_m) \leq A(\tilde{H}_m)$. The proof is by induction on $m$. For $m \leq 3$ it is obvious. Let now $v \in T_m$ be any end-cutvertex and $u$ any endvertex joint with $v$. If $v \in X$ ($X \cup Y = V(T)$, $|X| \leq |Y|$), then by the induction hypothesis $A(\tilde{T}_m \setminus u \setminus v) < A(\tilde{H}_{m-2})$. By assigning the number 1 to $v$ and $m$ to $u$, we obtain $A(\tilde{T}_m) \leq A(\tilde{H}_m)$. If $v \in Y$ then we assign the number $|X| + 1$ to $v$ and $|X|$ to $u$. This completes the proof. \Box

**Proof of Theorem 4.** Once the panarboriality of $H_n$ is established, by Lemma 5, one can start induction as in the proof of Theorem 3, defining $H' = H_n \setminus \bigcup_{j=k+1} T_j$, where the index in the union “jumps” by two each time. Evaluating $e(H')$ and using Lemma 2 applied to $H'$ proves the theorem as well. \Box
Final Remarks

The proofs of Theorem 3 and 4 for $n$ odd are similar and left for the reader.

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References