On the numerical solution of Lane-Emden type equations

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Abstract- In this paper, a reliable algorithm is employed to investigate the differential equations of Lane-Emden type. The algorithm rest mainly on the power series method with an alternate framework designed to overcome the difficulty of the singular point. The proposed framework is applied to a generalization of Lane-Emden equations so that it can be used in differential equations of the same type.

Keywords- Lane-Emden type equations; power series method; Chandrasekhar equation; Taylor series method.

1 Introduction

Many problems in the literature of mathematical physics can be distinctively formulated as equations of Lane-Emden type [1,2] defined in the form

\[ y'' + \frac{2}{x} y' + f(y) = 0, \]

subject to the boundary conditions

\[ y(0) = \alpha, y'(0) = 0. \]

Where the prime denotes the differentiation with respect to \(x\), \(\alpha\) is a constant, \(f(y)\) is some given function of \(y\). For example, it modes the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [2-4] when \(f(y) = y^m\), the gravitational potential of the degenerate white-dwarf stars [1] when \(f(y) = (y^2 - c)^{\frac{3}{2}}\).

More over, the other nonlinear forms of \(f(y)\) are the exponential functions

\[ f(y) = e^{y(x)}, \] (3)

and

\[ f(y) = e^{-y(x)}. \]

Substituting (3) into (1) results in a model that describes the isothermal gas spheres where the temperature remains constant and the index \(m\) is infinite. On the other hand, combining (1) and (4) gives a model that appears in the theory of thermionic currents thoroughly investigated by Richardson [4].

Furthermore, the function \(f(y)\) appears in eight additional cases [2], namely, four triangular forms defined by

\[ f(y) = \pm \sin y, \pm \cos y, \]

and four hyperbolic functions defined by

\[ f(y) = \pm \sinh y, \pm \cosh y, \]

A thorough discussion of the formulation of these models and the physical structure of the solutions can be found in [1,2,4].
The equation (1) cannot have solution by Adomian decomposition method when the coefficient of $\frac{y'}{x}$ is negative real number. For example, $y'' - \frac{y'}{2x} + 1 = 0$. It is the aim of this paper to find the series solution of equation (1) by power series method. Our next aim is to test the proposed algorithm in handling a generalization of this type of problems.

A difficult element in the analysis of this type of equations is the singularity behavior occurring at $x = 0$.

2 The method

We assumed the solution of (1) can be

$$y = y(0) + y'(0)x + ax^2.$$  \hspace{1cm} (5)

Substitute (5) into (1) and neglect higher order term. We have the linear equation of $a$ in the form

$$Aa = B,$$  \hspace{1cm} (6)

where $A$ and $B$ are a constant. Solving this equation (6), the coefficients of $x^2$ in (5) can be determined. Repeating the above procedure for higher order terms we can get the arbitrary order power series of the solution for (1).

Let step size of $x$ to be $h$ and substitute it into the power series of $y$, $y'$ and $y''$, we have $y, y'$ and $y''$ at $x = x_0 + h$. If we repeat above procedure, we have numerical solution of differential equation in (1)[5,6]. This method has been used to obtained approximate numerical and theoretical solutions of a large class of differential-algebraic equation[7,8] references therein.

3 Applications

3.1 Lane-Emden equation of index $m$

The Lane-Emden equation of index $m$ is a basic equation in the theory of stellar structure[9]. The equation describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of thermodynamics [1,2,9]. The Lane-Emden equation of index $m$ is of the form

$$y'' + \frac{2}{x}y' + y^m = 0$$

we rewrite it in the form

$$y'' + \frac{2}{x}y' = -y^m,$$  \hspace{1cm} (7)

which has been the object of much study [1,4,6,8]. The boundary conditions, which are of most interest[2], are the following

$$y(0) = 1, y'(0) = 0.$$  

The general solution for Eq.(7) is to be constructed for all possible values of the index $m$, $m \geq 0$. Notice that Eq.(7) is linear for $m = 0$ and $m = 1$, and nonlinear otherwise. From boundary condition the solution of (7) can be supposed as

$$y = y(0) + y'(0)x + ax^2 = 1 + ax^2.$$  \hspace{1cm} (8)

Substitute (8) into (7) and neglect higher order terms, we have

$$2a + 4a = -(1 + ax^2),$$

$$6a = -(1 + ax^2)^m.$$  

Here we will use Taylor series for $(1 + ax^2)^m$,

$$(1 + ax^2)^m = 1 + amx^2 + ...$$

$$6a = -(1 + amx^2 + ...),$$

$$6a = -1 - amx^2 - ..., $$

$$6a = -1 + 0(x^2),$$

neglect higher order terms $0(x^2)$

$$6a = -1$$

$$a = \frac{-1}{6}.$$  

Then

$$y = 1 - \frac{1}{6}x^2.$$  \hspace{1cm} (9)

From (9) the solution of (7) can be supposed as

$$y = 1 - \frac{1}{6}x^2 + ax^3.$$  \hspace{1cm} (10)
In this manner, substitute (10) into (7) and neglect higher order terms, we have

\[-1 + 12ax = -(1 - \frac{1}{6}x^2 + ax^3)^m.\]

By Taylor series \((1 - \frac{1}{6}x^2 + ax^3)^m = 1 - \frac{m}{6}x^2 + amx^3 + ...\)

then

\[-1 + 12ax = -1 + mx^2 - amx^3 + ...\]

neglect higher order terms \(0(x^2)\).

\[12ax = 0\]

\[a = 0.\]

therefore

\[y = 1 - \frac{1}{6}x^2.\]

the solution of (7) can be supposed as

\[y = 1 - \frac{1}{6}x^2 + ax^4.\]  (11)

Substitute (11) into (7) and neglect higher order terms, we have

\[-1 + 20ax^2 = -(1 - \frac{1}{6}x^2 + ax^3)^m,\]

by Taylor series

\[(1 - \frac{1}{6}x^2 + ax^3)^m = 1 - \frac{m}{6}x^2 + amx^3 + (\frac{-m}{72} + \frac{m^2}{72})x^4 + ... - 1 + 20ax^2\]

\[= -(1 - \frac{m}{6}x^2 + amx^3 + (\frac{-m}{72} + \frac{m^2}{72})x^4 + ...)\]

\[-1 + 20ax^2 = -1 + \frac{m}{6}x^2 + 0(x^3),\]

\[20a = \frac{m}{6};\]

\[a = \frac{m}{120}.\]

Then

\[y = 1 - \frac{1}{6}x^2 + \frac{m}{120}x^4.\]

Repeating above procedure we have

\[y = 1 - \frac{1}{6}x^2 + \frac{m}{120}x^4 + \frac{m(8m - 5)}{3 	imes 7!}x^6 + \ldots + \frac{m(70 - 183m + 122m^2}{9 	imes 9!}x^{10} + \ldots.\]  (12)

Substituting \(m = 0, 1\) and 5 into (12) leads to the exact solution

\[y(x) = 1 - \frac{1}{3!}x^2, \quad y(x) = \frac{\sin(x)}{x}\]

and

\[y(x) = (1 + \frac{x^2}{3})^{\frac{1}{3}},\]

respectively.

3.2. The white-dwarf equation

In this model we consider the "white-dwarf" equation

\[y'' + \frac{2}{x}y' = -(y^2 - c)^{\frac{3}{2}},\]  (13)

introduced by Davis[3] and Chandrasekhar[1] in his study of the gravitational potential of the degenerate white-dwarf stars. The boundary conditions of (13) are

\[y(0) = 1, \quad y'(0) = 0.\]  (14)

It is clear that Eq.(13) is of Lane-Emden type where

\[f(y) = (y^2 - c)^{\frac{3}{2}},\]  (15)

If \(c = 0\), Eq.(13) reduces to Lane-Emden equation of index \(m = 3\). For a thorough discussion of the "white-dwarf" formula (13), see [1]. For our purposes, it is sufficient to handle the problem by using Approach as introduced above.

The solution of (13) can be supposed as

\[y = y(0) + y'(0)x + ax^2 = 1 + ax^2.\]  (16)

Substitute (16) into (13) and neglect higher order terms. We have

\[6a = -(1 + ax^2)^{\frac{3}{2}},\]

by Taylor series

\[((1 + ax^2)^{\frac{3}{2}} = (1 - c)^{\frac{3}{2}} + 3a\sqrt{1 - cx} + \ldots,\]

\[6a = -(1 - c)^{\frac{3}{2}} + 0(x),\]
\[ a = \frac{-(1 - c)^{\frac{3}{2}}}{6}. \]
\[ y = 1 - \frac{(1 - c)^{\frac{3}{2}}}{6} x^2. \]  
(17)

From (17) the solution of (13) can be supposed as
\[ y = 1 - \frac{(1 - c)^{\frac{3}{2}}}{6} x^2 + ax^3, \]  
(18)

substitute (18) into (13) and neglect higher order terms, we have
\[-(1 - c)^{\frac{3}{2}} + 12ax = -(1 - c)^{\frac{3}{2}} + \frac{1}{2} (1 - c)x^2 + 3a\sqrt{1 - cx^2} + ..., \]
\[-(1 - c)^{\frac{3}{2}} + 12ax = -(1 - c)^{\frac{3}{2}} + 0(x^2), \]
\[ 12a = 0, \]
\[ a = 0, \]
\[ y = 1 - \frac{1}{6} (1 - c)^{\frac{3}{2}} x^2. \]

Supposed
\[ y = 1 - \frac{1}{6} (1 - c)^{\frac{3}{2}} x^2 + ax^4, \]  
(19)

substitute (19) into (13) and neglect higher order terms, we have
\[-(1 - c)^{\frac{3}{2}} + 20ax^2 = -(1 - c)^{\frac{3}{2}} + \frac{1}{2} (1 - c)^2 x^2 + 0(x^3), \]
\[-(1 - c)^{\frac{3}{2}} + 20ax^2 = -(1 - c)^{\frac{3}{2}} + \frac{1}{2} (1 - c)^2 x^2 + 0(x^3), \]
\[ 20a = \frac{1}{2} (1 - c)^2, \]
\[ a = \frac{1}{40} (1 - c)^2. \]

Therefore
\[ y = 1 - \frac{1}{6} (1 - c)^{\frac{3}{2}} x^2 + \frac{1}{40} (1 - c)^2 x^4. \]

Repeating above procedure. The series solution is given by
\[ y(x) = 1 - \frac{1}{6} (1 - c)^{\frac{3}{2}} x^2 + \frac{1}{40} (1 - c)^2 x^4 \]

\[ -\frac{1}{n!} (5(c - 1) + 14)(c - 1)^{\frac{5}{2}} x^n + ..., \]

3.3. Isothermal gas spheres equation

In the following, we consider the differential equation of isothermal gas spheres
\[ y'' + \frac{2}{x} y' = -e^y, \]  
(20)

subject to the boundary conditions
\[ y(0) = y'(0) = 0. \]

The solution of (20) can be supposed as
\[ y = y(0) + y'(0) x + ax^2, \]  
(21)

substitute (21) into (20) and neglect higher order terms, we have
\[ 6a = -e^{ax^2}, \]

by Taylor series \( e^{ax^2} = 1 + ax^2 + \frac{1}{2} a^2 x^2 + ..., \) then
\[ 6a = -1 + 0(x), \]
\[ a = \frac{-1}{6}, \]
\[ y = \frac{-1}{6} x^2. \]  
(22)

From (22) the solution of (20) can be supposed as
\[ y = \frac{-1}{6} x^2 + ax^3, \]  
(23)

substitute (23) into (20) and neglect higher order terms, we have
\[ -1 + 12ax = -e^{(-\frac{1}{6} x^2 + ax^3)}, \]

by Taylor series
\[ e^{(-\frac{1}{6} x^2 + ax^3)} = 1 - \frac{1}{6} x^2 + ax^3 + ..., \]  
then
\[ -1 + 12ax = 1 - \frac{1}{6} x^2 + ax^3 + ..., \]
\[ -1 + 12ax = 1 - 0(x^2), \]
\[ 12a = 0, \]

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From (24) the solution can be substitute as
\[ y = -\frac{1}{6}x^2 + ax^4, \] (25)
substitute (25) into (20) and neglect higher order terms, we have
\[ -1 + 20ax^2 = e^{(-\frac{1}{6}x^2 + ax^4)}, \]
\[ e^{(-\frac{1}{6}x^2 + ax^4)} = 1 - \frac{1}{6}x^2 + ..., \]
then
\[ -1 + 20ax^2 = -1 + \frac{1}{6}x^2 + 0(x^3), \]
\[ 20a = \frac{1}{6}, \]
\[ a = \frac{1}{120}, \]
\[ y = -\frac{1}{6}x^2 + \frac{1}{120}x^4. \]
So that other components can be evaluated in alike manner, the solution in a series form is given by
\[ y(x) = -\frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{8}{21 \times 6!}x^6 - \frac{122}{81 \times 8!}x^8 + ... \] (26)

3.4. Richardson’s theory of thermionic currents

Richardson [4] introduced a counterpart of Eq. (20) in which \( e^y(x) \) is replaced by \( e^{-y(x)} \). The model is controlled by the nonlinear differential equation
\[ y'' + \frac{2}{x}y' = -e^{-y}, \] (27)
subject to the boundary conditions
\[ y(0) = y'(0) = 0. \] (28)
According to [2] this model has been used in Richardson’s theory of thermionic currents. For a thorough discussion of the formulation of (27) and the physical behavior of the emission of electricity from hot bodies, see [2, 4].
Following the discussion of the previous application, this model can be handled by applying the method presented above. The solution of (27) can be supposed as
\[ y = y(0) + y'(0)x + ax^2, \] (29)
substitute (29) into (27) and neglect higher order terms, we have
\[ 6a = -e^{-ax^2}, \]
by Taylor series
\[ e^{-ax^2} = 1 - ax^2 + ..., \]
\[ 6a = -1 + 0(x). \]
\[ a = \frac{-1}{6}, \]
then
\[ y = -\frac{1}{6}x^2. \] (30)
From (30) the solution can be supposed as
\[ y = -\frac{1}{6}x^2 + ax^3, \] (31)
substitute (31) into (27) and neglect higher order terms, we have
\[ -1 + 12ax = e^{-x^2 + ax^3}, \]
\[ -1 + 12ax = -1 - \frac{1}{6}x^2 + ax^3 + ..., \]
\[ -1 + 12ax = -1 + 0(x^2), \]
\[ 12a = 0, \]
\[ a = 0, \]
\[ y = -\frac{1}{6}x^2, \]
the solution can be supposed as
\[ y = -\frac{1}{6}x^2 + ax^4, \] (32)
substitute (32) into (27) and neglect higher order terms, we have
\[ -1 + 20ax^2 = e^{-x^2 + ax^4}, \]
\[ -1 + 20ax^2 = -1 - \frac{1}{6}x^2 - (-a + \frac{1}{72})x^4, \]
\[ -1 + 20ax^2 = -1 - \frac{1}{6}x^2 + 0(x^3), \]
Where other components can be evaluated in a like manner. The solution in a series form is given by

\[ y(x) = \frac{-1}{6}x^2 - \frac{1}{120}x^4. \]  

It is interesting to point out that eight possible cases are discussed in [2] in which \(f(y)\) is specialized by

\[ f(y) = \pm \sin y, \pm \cos y, \pm \sinh y, \pm \cosh y. \]

In the following, the method will be applied to one trigonometric function. Other cases can be handled in alike manner.

3.5. Lane-Emden type where \(f(y) = \sin y\).

Consider the differential equation

\[ y'' + \frac{2}{x}y' = -\sin y, \tag{34} \]

subject to the boundary conditions

\[ y(0) = 1, y'(0) = 0. \]

From boundary conditions, the solutions of (34) can be supposed as

\[ y = y(0) + y'(0)x + ax^2 = 1 + ax^2, \tag{35} \]

substitute (35) into (34) and neglect higher order terms, we have

\[ 6a = -\sin(1 + ax^2), \]

by Taylor series

\[ \sin(1 + ax^2) = \sin 1 + (a \cos 1)x^2 + ..., \]

then

\[ 6a = -\sin 1 - (a \cos 1)x^2 + ..., \]

\[ 6a = -\sin 1 + 0(x), \]

\[ a = -\sin 1, \]

\[ y = 1 - \frac{\sin 1}{6}x^2. \tag{36} \]

From (36) the solution can be supposed as

\[ y = 1 - \frac{\sin 1}{6}x^2 + ax^3, \tag{37} \]

substitute (37) into (34) and neglect higher order terms, we have

\[ -\sin 1 + 12ax = -\sin(1 - \frac{\sin 1}{6}x^2 + ax^3), \]

by Taylor series method

\[ \sin(1 - \frac{\sin 1}{6}x^2 + ax^3) = \sin 1 - \frac{1}{6}(\sin 1 \cos 1)x^2 + ...; \]

\[ -\sin 1 + 12ax = -\sin 1 + \frac{1}{6}(\sin 1 \cos 1)x^2 + ..., \]

\[ \sin 1 + 12ax = -\sin 1 + 0(x^2), \]

\[ 12a = 0, \]

\[ a = 0, \]

then

\[ y = 1 - \frac{\sin 1}{6}x^2. \]

The solution can be supposed as

\[ y = 1 - \frac{\sin 1}{6}x^2 + ax^4. \tag{38} \]

In this manner, substitute (38) into (34) and neglect higher order terms, we have

\[ -\sin 1 + 20ax^2 = -\sin(1 - \frac{\sin 1}{6}x^2 + ax^4), \]

by Taylor series

\[ -\sin 1 + 20ax^2 = -\sin 1 + \frac{1}{6}(\sin 1 \cos 1)x^2 + 0(x^3), \]

\[ 20ax^2 = \frac{1}{6}(\sin 1 \cos 1)x^2, \]

\[ 20a = \frac{1}{6}(\sin 1 \cos 1), \]

\[ a = \frac{1}{5!}(\sin 1 \cos 1), \]

then

\[ y = 1 - \frac{\sin 1}{6}x^2 + \frac{1}{5!}(\sin 1 \cos 1)x^4. \]
Repeating above procedure we have
\[
y = 1 - \sin \frac{1}{6} x^2 + \frac{1}{5!} (\sin 1 \cos 1) x^4 \\
+ \sin 1 \frac{1}{7!} \cos 2 1 - \frac{1}{5040} \sin^2 1 x^6 + \ldots \tag{39}
\]
In alike manner we can derive the series solution for the \(-\sin y, \pm \cos y, \pm \sinh y, \pm \cosh y\) models.

4 Generalization

A generalization of the lane-Emden-like equation (1) has been studied by wazwaz [10]. In a parallel manner. The standard coefficient of \(y'\) in Lane-Emden equation is \(\frac{2}{x}\). However, if we replace \(\frac{2}{x}\) by \(\frac{n}{x}\), for real \(n\), it is \(negative integral\), then we write down Lane-Emden like equation a general fashion as
\[
y'' + \frac{n}{x} y + y^m = 0
\]
we can rewrite it in the form
\[
y'' + \frac{n}{x} y = -y^m, \tag{40}
\]
where the boundary condition, which are of most interest are the following:
\[
y(0) = 1, \quad y'(0) = 0.
\]
From boundary condition, the solution of (40) can be supposed as
\[
y = y(0) + y'(0)x + ax^2 = 1 + ax^2, \tag{41}
\]
substitute (41) into (40) and neglect higher order terms, we have
\[
2a + 2na = -(1 + ax^2)^m,
\]
by Taylor series
\[
(1 + ax^2)^m = 1 + amx^2 + \ldots,
\]
\[
2a(1 + n) = -1 - amx^2 + \ldots,
\]
\[
2a(1 + n) = -1 + 0(x^2),
\]
\[
2a(1 + n) = -1,
\]
\[
a = -\frac{1}{2(n + 1)},
\]
then
\[
y = 1 - \frac{1}{2(n + 1)} x^2. \tag{42}
\]
From (42) the solutions of (40) can be supposed as
\[
y = 1 - \frac{1}{2(n + 1)} x^2 + ax^3, \tag{43}
\]
substitute (43) into (40) and neglect higher order terms, we have
\[
-1 + 12ax = -(1 - \frac{1}{2(n + 1)} x^2 + ax^3)^m,
\]
\[
-1 + 12ax = -1 + \frac{m}{2(2(1+n))} x^2 + \ldots,
\]
\[
-1 + 12ax = -1 + 0(x^2),
\]
\[
a = 0,
\]
\[
y = 1 - \frac{1}{2(n + 1)} x^2,
\]
the solution can be supposed as
\[
y = 1 - \frac{1}{2(n + 1)} x^2 + ax^4, \tag{44}
\]
substitute (44) into (40) and neglect higher order terms, we have
\[
-1 + 12ax^2 + 4nx^2 = -(1 - \frac{1}{2(n + 1)} x^2 + ax^4)^m,
\]
\[
-1 + 12ax^2 + 4nx^2 = -1 + \frac{m}{2(1+n)} x^2 + \ldots,
\]
\[
-1 + 12ax^2 + 4nx^2 = -1 + \frac{m}{2(1+n)} x^2 + 0(x^3),
\]
\[
4a(3 + n)x^2 = \frac{m}{2(1+n)} x^2,
\]
\[
a = \frac{m}{8(n + 1)(n + 3)},
\]
\[
y = 1 - \frac{1}{2(n + 1)} x^2 + \frac{m}{8(n + 1)(n + 3)} x^4. \tag{45}
\]
Repeating above procedure we have
\[
y = 1 - \frac{1}{2(n + 1)} x^2 + \frac{m}{8(n + 1)(n + 3)} x^4 - \frac{(2n + 4)m^2 - (n + 3)m}{48(n + 1)^2(n + 3)(n + 5)} x^6 + 0(x^7). \tag{46}
\]
As indicated before, the general solution obtained in (46) works for all real values of \(n\) and \(m\). For example, for fixed \(n = 0\), the following solutions
\[
y(x) = 1 - \frac{1}{2} x^2,
\]
y(x) = \cos x,
y(x) = 1 - \frac{1}{2} x^2 + \frac{1}{12} x^4 - \frac{1}{72} x^6 + \frac{1}{504} x^8 + \ldots,
\]
can be obtained for \(m = 0\), \(1\) and \(2\) respectively upon using (46), where \(x = 0\) is an ordinary point, in addition, for fixed \(n = 0, 5\), we find
\[
y(x) = 1 - \frac{1}{3} x^2,
y(x) = 1 - \frac{1}{3} x^2 + \frac{1}{42} x^4 + \frac{1}{83160} x^8 + \ldots,
y(x) = 1 - \frac{1}{3} x^2 + \frac{1}{21} x^4 - \frac{1}{2079} x^6 + \frac{23}{31185} x^8 + \ldots,
\]
for \(m = 0, 1, 2\) respectively, where \(x = 0\) is a singular point. Moreover, the solutions
\[
y(x) = 1 - \frac{1}{4} x^2,
y(x) = 1 - \frac{1}{4} x^2 + \frac{1}{64} x^4 - \frac{1}{1304} x^6 + \frac{1}{147456} x^8 + \ldots,
y(x) = 1 - \frac{1}{4} x^2 + \frac{1}{32} x^4 - \frac{1}{289} x^6 + \frac{13}{36864} x^8 + \ldots,
\]
obtained for \(n = 1\) and \(m = 0, 1, 2\) respectively, where \(x = 0\) is a singular point, for fixed point \(m = 0\), we find
\[
y(x) = 1 - x^2,
y(x) = 1 - \frac{2}{3} x^2,
y(x) = 1 + x^2,
\]
obtained for \(n = -\frac{1}{2}, -\frac{1}{4}, \) and \(-\frac{3}{2}\) respectively.

## 5 Conclusion

In the above discussion it was shown that, with the proper investment of the power series method, it is possible to attain an analytic solution to Lane-Emden type of equations. The difficulty in using Adomian decomposition method to this type of equations, due to the existence of singular point at \(x = 0\), is overcome here.

Our goal has been achieved by formally deriving analytical approximations with a high degree of accuracy.

Lane-Emden like equation was generalized, by changing the coefficient of \(y'\), and the proposed technique was presented in a general way so that it can be used in applied sciences for similar cases.

In closing, we point out that other types of ODEs with singular coefficients, such as Legendre’s equation, Bessel’s equation, were handled differently, but successively, by [11-13].

### References


**Vitae**

Yahya Qaid Hasan, male, was born in the city of Taiz, Taiz Province, Yemen. He got bachelor degrees on Mathematics from Sana’a University in 1993, he got Master degree on Applied Mathematics from Anhui Normal University (China) in 2005 and he got PhD degree from Harbin Institute of Technology (China) in 2009. He works in Thamar University. His research interest includes Differential Equations and its applications, numerical solution of Differential Equations.