Joint Power and Admission Control: Non-Convex Approximation and An Efficient Polynomial Time Deflation Approach

Ya-Feng Liu, Yu-Hong Dai, and Shiqian Ma

Abstract

In an interference limited network, joint power and admission control (JPAC) aims at supporting a maximum number of links at their specified signal to interference plus noise ratio (SINR) targets while using a minimum total transmission power. Various convex approximation deflation approaches have been developed for the JPAC problem. In this paper, we propose an efficient polynomial time non-convex approximation deflation approach for solving the problem. The approach is based on the non-convex $\ell_q$-minimization approximation of an equivalent sparse $\ell_0$-minimization reformulation of the JPAC problem where $q \in (0, 1)$. We show that, for any instance of the JPAC problem, there exists a $\bar{q} \in (0, 1)$ such that it can be exactly solved by solving its $\ell_{\bar{q}}$-minimization approximation problem with any $q \in (0, \bar{q}]$. Further, we propose a potential reduction interior-point algorithm, which can return an $\epsilon$-KKT solution of the NP-hard $\ell_q$-minimization approximation problem in polynomial time. The returned solution can be used to check the simultaneous supportability of all links in the network and to guide an iterative link removal procedure, resulting in the polynomial time non-convex approximation deflation approach for the JPAC problem. Numerical simulations show that the proposed approach outperforms the existing convex approximation approaches in terms of the number of supported links and the total transmission power, particularly exhibiting a quite good performance in selecting which subset of links to support.

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Y.-F. Liu and Y.-H. Dai are with the State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China (e-mail: yaflu@lsec.cc.ac.cn; dyh@lsec.cc.ac.cn).

S. Ma is with the Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong (e-mail: sqma@se.cuhk.edu.hk).
Index Terms

Admission control, complexity, non-convex approximation, potential reduction algorithm, power control, sparse optimization.

I. INTRODUCTION

Joint power and admission control (JPAC) has been recognized as an effective tool for interference management in cellular, ad hoc, and cognitive underlay wireless networks for more than two decades [1]–[31]. The goal of JPAC is to support a maximum number of links at their specified signal to interference plus noise ratio (SINR) targets while using a minimum total transmission power when all links in the interference limited network cannot be simultaneously supported. JPAC can not only determine which interfering links must be turned off and rescheduled along orthogonal resource dimensions (such as time, space, or frequency slots), but also alleviate the difficulties of the convergence of stand-alone power control algorithms. For example, a longstanding issue associated with the Foschini-Miljanic algorithm [5] is that, it does not converge when the preselected SINR levels are infeasible. In this case, a JPAC approach must be adopted to determine which links to be removed.

A. Related Work

The JPAC problem can be solved to global optimality by checking the simultaneous supportability of every subset of links. However, the computational complexity of this enumeration approach grows exponentially with the total number of links. Another globally optimal algorithm, which is based on the branch and bound strategy, is given in [10]. Theoretically, the problem is shown to be NP-hard to solve (to global optimality) and to approximate (to constant ratio global optimality) [1], [2], [4]. In recent years, various convex approximation based heuristics algorithms [1]–[11], [16]–[31] have been proposed for the problem, since convex optimization problems, such as linear program (LP), second-order cone program (SOCP), and semidefinite program (SDP), are relatively easy to solve.

Assuming perfect channel state information (CSI), Ref. [1] proposed the so-called linear programming deflation (LPD) algorithm. Instead of solving the original NP-hard problem directly, the LPD algorithm solves an appropriate LP approximation of the original problem at each iteration and uses its solution to

\footnote{For any convex optimization problem and any \( \epsilon > 0 \), the ellipsoid algorithm can find an \( \epsilon \)-optimal solution (i.e., a feasible solution whose objective value is within \( \epsilon \) from being globally optimal) with a complexity that is polynomial in the problem dimension and \( \log(1/\epsilon) \) [32].}
guide the removal of interfering links. The removal procedure is repeated until all the remaining links in the network are simultaneously supportable. In [2], the JPAC problem is shown to be equivalent to a sparse $\ell_0$-minimization problem and then its $\ell_1$-convex relaxation is used to derive an LP, which is different from the one in [1]. Again, the solution to the derived LP is used to guide an iterative link removal procedure (deflation), leading to an efficient new linear programming deflation (NLPD) algorithm. Another convex approximation based heuristics algorithm is proposed in [3]. Assuming the same SINR target for each link, the link that results in the largest increase in the achievable SINR is removed at each iteration until all the remaining links in the network are simultaneously supportable. To determine the removed link, a large number of extreme eigenvalue problems need to be solved at each iteration, making the removal procedure computationally expensive. To reduce the computational complexity, the above idea is approximately implemented in the Algorithm II-B [3]. Similar convex approximation deflation ideas were used in [19], [30] to solve the joint beamforming and admission control problem for the cellular downlink network, where at each iteration an SDP needs to be solved to determine the link to be removed.

Under the imperfect CSI assumption, JPAC has been studied in [1], [28], [29]. In [1], the authors considered the worst-case robust JPAC problem with bounded channel estimation errors. The key there is that the relaxed LP with bounded uncertainty can be equivalently rewritten as an SOCP. The overall approximation algorithm remains similar to LPD for the case of the perfect CSI, except that the SOCP formulation is used to carry out power control and its solution is used to check whether links are simultaneously supportable in the worst case. Ref. [29] studied the JPAC problem under the assumption of the channel distribution information (CDI), and formulated the JPAC problem as a chance (probabilistic) constrained program, where each link’s SINR outage probability is enforced to be less than or equal to a specified tolerance. To circumvent the difficulty of the chance SINR constraint, Ref. [28] employed the sample (scenario) approximation scheme to convert the chance constraints into finitely many simple linear constraints. Then, the sample approximation of the chance SINR constrained JPAC problem is reformulated as a group sparse minimization problem and approximated by an SOCP. The solution of the SOCP approximation problem can be used to check the simultaneous supportability of all links in the network and to guide an iterative link removal procedure.

$^2$Extreme eigenvalue problems are SDP representable.
B. Our Contribution

This paper considers the JPAC problem under the perfect CSI assumption. We remark that similar techniques can be used for the case where the CSI is not perfectly known. As mentioned above, most existing algorithms on JPAC are based on (successive) convex approximations. The main contribution of this paper is to propose an efficient polynomial time non-convex approximation deflation approach for solving the JPAC problem. To our knowledge, this is the first approach that solves the JPAC problem by (successive) non-convex approximations. The basic idea is to approximate the sparse $\ell_0$-minimization reformulation of the JPAC problem by the non-convex $\ell_q$-minimization problem with $q \in (0, 1)$ instead of the convex $\ell_1$-minimization problem as in [1], [2], and to design an efficient polynomial time algorithm for the non-convex $\ell_q$-minimization problem. The main results of this paper are summarized as follows.

• We show that the non-convex $\ell_q$-minimization problem shares the same solution with the $\ell_0$-minimization problem if $q \in (0, \bar{q}]$, where $\bar{q}$ is some value in $(0, 1)$. We also give an example of the JPAC problem, showing that the solution to its non-convex $\ell_q$-minimization approximation problem with any $q \in (0, 1)$ solves the original problem while its convex $\ell_1$-minimization approximation problem fails to do so. We therefore show that the $\ell_q$-minimization problem with $q \in (0, 1)$ approximates the $\ell_0$-minimization JPAC problem better than the $\ell_1$-minimization problem.

• We show that, for any $q \in (0, 1)$, the $\ell_q$-minimization approximation problem is NP-hard. The proof is based on a polynomial time transformation from the partition problem. The complexity result suggests that there is no polynomial time algorithm which can solve the $\ell_q$-minimization approximation problem to global optimality (unless P=NP).

• We reformulate the $\ell_q$-minimization approximation problem and develop a potential reduction interior-point algorithm for solving its equivalent reformulation. We show that, for any given $\epsilon > 0$, the potential reduction algorithm can return an $\epsilon$-KKT solution (its definition will be given later) of the reformulated problem in polynomial time. This is the best we can expect since the $\ell_q$-minimization approximation problem and thus its equivalent reformulation are NP-hard. The obtained $\epsilon$-KKT solution can be used to check the simultaneous supportability of all links in the network and to guide the deflation procedure.

• Based on the above analysis, we propose a non-convex approximation deflation approach for the JPAC problem. The proposed approach enjoys a polynomial time worst-case complexity. Simulation results show that the proposed approach outperforms the existing convex approximation algorithms [1]–[3] in terms of both the number of supported links and the total transmission power. In particular,
the proposed algorithm exhibits a much better performance than the NLPD algorithm \cite{2} in selecting which subset of links to support.

C. Notations

We adopt the following notations in this paper. We denote the index set \( \{1, 2, \ldots, K\} \) by \( K \). Lowercase boldface and uppercase boldface are used for vectors and matrices, respectively. For a given vector \( \mathbf{x} \), the notations \( \max\{\mathbf{x}\} \), \( [\mathbf{x}]_k \) and \( \|\mathbf{x}\|_q^q := \sum_k |[\mathbf{x}]_k|^q \) (\( 0 \leq q \leq 1 \)) stand for its maximum entry, its \( k \)-th entry, and its \( \ell_q \) norm respectively. In particular, when \( q = 0 \), \( \|\mathbf{x}\|_0 \) stands for the number of nonzero entries in \( \mathbf{x} \). For any subset \( \mathcal{I} \subseteq K \), we use \( A_{\mathcal{I}} \) to denote the matrix formed by the rows of \( A \) indexed by \( \mathcal{I} \). For a vector \( \mathbf{x} \), the notation \( \mathbf{x}_{\mathcal{I}} \) is similarly defined. Moreover, for any \( \mathcal{J} \subseteq K \), the notation \( A_{\mathcal{I},\mathcal{J}} \) will denote the submatrix of \( A \) obtained by taking the rows and columns of \( A \) indexed by \( \mathcal{I} \) and \( \mathcal{J} \) respectively. The spectral radius of a matrix \( A \) is denoted by \( \rho(A) \). Finally, we use \( \mathbf{e} \) to represent the vector with all components being one and \( \mathbf{I} \) to represent the identity matrix of an appropriate size, respectively.

II. System Model and Problem Formulation

Consider a \( K \)-link (a link corresponds to a transmitter-receiver pair) interference channel with channel gains \( g_{k,j} \geq 0 \) (from transmitter \( j \) to receiver \( k \)), noise power \( \eta_k > 0 \), SINR target \( \gamma_k > 0 \), and power budget \( \bar{p}_k > 0 \) for \( k, j \in K := \{1, 2, \ldots, K\} \). Denote the power allocation vector by \( \mathbf{p} = (p_1, p_2, \ldots, p_K)^T \) and the power budget vector by \( \mathbf{\bar{p}} = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_K)^T \). Treating interference as noise, we can write the SINR at the \( k \)-th receiver as

\[
\text{SINR}_k = \frac{g_{k,k}p_k}{\eta_k + \sum_{j \neq k} g_{k,j}p_j}, \quad \forall \ k \in K.
\]

To some extent, the JPAC problem can be formulated as a two-stage optimization problem. The first stage maximizes the number of admitted links:

\[
\max_{\mathbf{p}, S} |S|
\]

s.t. \( \text{SINR}_k \geq \gamma_k, \ k \in S \subseteq K, \)

\[
0 \leq \mathbf{p} \leq \mathbf{\bar{p}}.
\]

\(^3\)Strictly speaking, \( \|\mathbf{x}\|_q^q \) with \( 0 \leq q < 1 \) is not a norm, since it does not satisfy the triangle inequality. However, we still call it \( \ell_q \) norm for convenience in this paper.
The optimal solution $S_0$ of problem (1), which may not be unique, is called maximum admissible set. The second stage minimizes the total transmission power required to support the admitted links in $S_0$:

$$\min_{\{p_k\}_{k \in S_0}} \sum_{k \in S_0} p_k$$

s.t. \[ \begin{align*}
\text{SINR}_k & \geq \gamma_k, \quad k \in S_0, \\
0 & \leq p_k \leq \bar{p}_k, \quad k \in S_0.
\end{align*} \]

Due to the special choice of $S_0$, power control problem (2) is feasible and can be efficiently solved by the Foschini-Miljanic algorithm [5].

III. JPAC VIA $\ell_q$-MINIMIZATION DEFLATION

A. Review of the NLPD Algorithm

To develop the non-convex approximation algorithm for the JPAC problem, we briefly review the NLPD algorithm in [2]. The basic idea of the NLPD algorithm is to update the power and check whether all links can be supported. If not, drop one link from the network and update the power again. This process is repeated until all the remaining links are supported.

We begin with the channel normalization. Denote the normalized power allocation vector by $x = (x_1, x_2, \ldots, x_K)^T$ with $x_k = p_k / \bar{p}_k$, and the normalized noise vector by $b = (b_1, b_2, \ldots, b_K)^T$ with $b_k = (\gamma_k \eta_k) / (g_{k,k} \bar{p}_k) > 0$. The normalized channel matrix is denoted by $A \in \mathbb{R}^{K \times K}$ with the $(k,j)$-th entry

$$a_{k,j} = \begin{cases} 
1, & \text{if } k = j; \\
-\gamma_k g_{k,j} \bar{p}_j / g_{k,k} \bar{p}_k, & \text{if } k \neq j.
\end{cases}$$

In fact, $|a_{k,j}|$ is the normalized channel gain. The component $[b - Ax]_k$ measures the excess transmission power that the transmitter of link $k$ needs in the normalized channel in order to be served with its desired SINR target (assuming all other links keep their transmission powers unchanged). It is simple to check that $\text{SINR}_k \geq \gamma_k$ if and only if $[Ax - b]_k \geq 0$.

The two-stage JPAC problem (1) and (2) is reformulated in [2] as a single-stage sparse $\ell_0$-minimization problem

$$\min_x \|b - Ax\|_0 + \alpha \bar{p}^T x$$

s.t. \[ 0 \leq x \leq e, \]

where $\alpha$ is a parameter satisfying

$$0 < \alpha < \alpha_1 := 1 / \bar{p}^T \bar{p}. \quad (4)$$

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Notice that the formulation (3) is capable of finding the maximum admissible set with minimum total transmission power and hence is superior to the two-stage formulation (1) and (2) in case of multiple maximum admissible sets. Since problem (3) is still NP-hard, Ref. [2] further considers its $\ell_1$-convex approximation

$$\min_x \|b - Ax\|_1 + \alpha \bar{p}^T x$$

s.t. $0 \leq x \leq e$, \hspace{1cm} (5)

which is equivalent to the following LP (see Theorem 2 in [2])

$$\min_x e^T (b - Ax) + \alpha \bar{p}^T x$$

s.t. $b - Ax \geq 0$, \hspace{1cm} (6)

$$0 \leq x \leq e.$$

By solving (5), one can check whether all links in the network can be simultaneously supported. If not, the NLPD algorithm drops the link

$$k_0 = \arg \max_{k \in \mathcal{K}} \left\{ \sum_{j \neq k} \left( |a_{k,j}| \| [b - Ax]_j \| + |a_{j,k}| \| [b - Ax]_k \| \right) \right\}.$$ \hspace{1cm} (7)

An easy-to-check necessary condition

$$(\mu^+)^T e - (\mu^- + e)^T b \geq 0$$ \hspace{1cm} (8)

for all links in the network to be simultaneously supported is also derived in [2], where $\mu^+ = \max \{ \mu, 0 \}$, $\mu^- = \max \{ -\mu, 0 \}$, and $\mu = A^T e$. The necessary condition allows to iteratively remove strong interfering links from the network. In particular, the link

$$k_0 = \arg \max_{k \in \mathcal{K}} \left\{ \sum_{j \neq k} |a_{k,j}| + \sum_{j \neq k} |a_{j,k}| + b_k \right\}$$ \hspace{1cm} (9)

is iteratively removed in the NLPD algorithm until (8) becomes true.

The NLPD algorithm can be described as follows.
The NLPD Algorithm

**Step 1.** Initialization: Input data \((A, b, \bar{p})\).

**Step 2.** Preprocessing: Remove link \(k_0\) iteratively according to (9) until condition (8) holds true.

**Step 3.** Power control: Solve problem (5); check whether all links are supported: if yes, go to **Step 5**; else go to **Step 4**.

**Step 4.** Admission control: Remove link \(k_0\) according to (7), set \(K = K \setminus \{k_0\}\), and go to **Step 3**.

**Step 5.** Postprocessing: Check the removed links for possible admission.

B. Non-Convex \(\ell_q\)-Minimization Approximation

The sparse minimization problem (3) is *successively* approximated by the \(\ell_1\)-minimization problem (5) in the NLPD algorithm. Intuitively, the \(\ell_q\)-minimization problem with \(0 < q < 1\),

\[
\min_x \|b - Ax\|_q^q + \alpha \bar{p}^T x \\
\text{s.t.} \quad 0 \leq x \leq e
\]

should approximate (3) “better” than (5). To provide such an evidence, we give the following lemma.

**Lemma 1.** *For any* \(q \in [0, 1]\), *problem (10) is equivalent to*

\[
\min_x \|b - Ax\|_q^q + \alpha \bar{p}^T x \\
\text{s.t.} \quad Ax \leq b, \quad 0 \leq x \leq e.
\]

The above lemma can be verified in a similar way as the proof of Theorem 2 in [2]. Using Lemma 1 we can show the following result (see Appendix A for its proof).

**Theorem 1.** *For any given instance of problem (3), there exists \(\bar{q} > 0\) (depending on \(A, b, \alpha, \bar{p}\)) such that when \(q \in (0, \bar{q}]\), any global solution to problem (10) is one of the global solutions to problem (3).*

Theorem 1 says that the \(\ell_q\)-minimization problem (10) shares the same solution with the sparse optimization problem (3) if the parameter \(q\) (depending on \(A, b, \alpha, \bar{p}\)) is chosen to be sufficiently small. In general, the \(\ell_1\)-minimization problem (5) does not enjoy this exact recovery property, which is in sharp contrast to the results in [33] and [34].

It is shown in [33] that the problem of minimizing \(\|Ax - b\|_1\) is equivalent to the problem of minimizing \(\|Ax - b\|_0\) with high probability if the vector \(Ax - b\) at the true solution \(x^*\) is sparse, where \(A \in \mathbb{R}^{m \times n}\).
and $m > n$, and if the entries of the matrix $A$ are independent and identically distributed (i.i.d.) Gaussian. The reason why the $\ell_1$-minimization problem (10) fails to recover the solution of problem (3) is that the two assumptions required in [33] do not hold true for problem (3). Specifically, the vector $Ax - b$ may not be sparse even at the optimal power allocation vector $x^*$. This depends on whether the (normalized) channel is strongly interfered or not. In addition, the matrix $A$ in (3) is a square matrix and has a special structure; i.e., all diagonal entries are one and all non-diagonal entries are non-positive.

Now we give an example to illustrate the advantage of the use of $\ell_q$ norm with $0 < q < 1$ over the use of $\ell_1$ norm to approximate problem (3). Suppose $A$, $b$, $\bar{p}$ in (3) are given as follows:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad b = 0.5e, \quad \bar{p} = e.$$ 

It is easy to verify that the optimal solution to the sparse optimization problem (3) is

$$x^* = (0.5, 0.5, 0)^T$$

if the parameter $\alpha$ is chosen satisfying $0 < \alpha < 1/3$ (cf. (4)). We can also obtain the solutions to problems (5) and (10).

- By writing the KKT optimality conditions, we can check that $x = 0$ is the unique global minimizer of problem (5) with any $\alpha \geq 0$.
- Lemma 1 implies that problem (10) is equivalent to

$$\min \begin{array}{l} x_1, x_2, x_3 \end{array} \left\{ (0.5 - x_1 + x_3)^q + (0.5 - x_2 + x_3)^q + (0.5 + x_1 + x_2 - x_3)^q + \alpha(x_1 + x_2 + x_3) \right\}$$

s.t.

$$0.5 - x_1 + x_3 \geq 0,$$

$$0.5 - x_2 + x_3 \geq 0,$$

$$0.5 + x_1 + x_2 - x_3 \geq 0,$$

$$0 \leq x_1, x_2, x_3 \leq 1.$$ 

For any given $q \in (0, 1)$, define

$$\bar{\alpha}_q := \min \{1 + (0.5)^q, 2^q\} - (1.5)^q > 0.$$ 

It can be checked (although tedious) that, as long as $\alpha$ in problem (12) is chosen such that $0 < \alpha \leq \bar{\alpha}_q$, the unique global minimizer of problem (12) is $x^*$.

We remark that, for a given instance of problem (3), it is generally not easy to determine $\bar{q}$ in Theorem 1. In practice, we could set the parameter $q$ in problem (10) to be a constant in $(0, 1)$. Particularly
in this paper, we set $q = 0.5$. Therefore, the solution to problem (10) might not be able to solve the $\ell_0$-minimization problem (3). This is the reason why we do not just use the $\ell_q$-minimization (10) to approximate problem (3), but instead employ a deflation technique to successively approximate problem (3).

C. Complexity Analysis of the $\ell_q$-Minimization Problem (10)

Roughly speaking, convex optimization problems are relatively easy to solve, while non-convex optimization problems are difficult to solve. However, not all non-convex problems are computationally intractable since the lack of convexity may be due to an inappropriate formulation. In fact, many non-convex optimization problems admit a convex reformulation; see [35]–[40] for some examples. Therefore, convexity is useful but unreliable to evaluate the computational intractability of an optimization problem. A more robust tool is the computational complexity theory [41], [42].

In this subsection, we show that problem (10) is NP-hard for any given $q \in (0, 1)$. The NP-hardness proof is based on a polynomial time transformation from the partition problem: given a set of $N$ positive integers $s_1, s_2, \ldots, s_N$, determine whether there exists a subset $S$ of \{1, 2, $\ldots$, $N$\} such that

$$\sum_{n \in S} s_n = \sum_{n \notin S} s_n = \frac{1}{2} N \sum_{n = 1}^{N} s_n.$$ 

The partition problem is known to be NP-complete [41].

**Theorem 2.** For any given $0 < q < 1$, the $\ell_q$-minimization problem (10) is NP-hard.

The proof of Theorem 2 is relegated to Appendix B. Theorem 2 suggests that there is no efficient algorithm which can solve problem (10) to global optimality in polynomial time (unless P=NP), and finding an approximate solution for it is more realistic in practice.

D. A Polynomial Time Potential Reduction Algorithm for Problem (10)

In this subsection, we shall develop a polynomial time potential reduction algorithm for solving problem (10). Based on Lemma 1 by introducing slack variables, we see that problem (10) can be equivalently formulated as

$$\min_{w} f(w) := \tilde{c}^T w_1 + \|w_2\|_q^q$$

subject to

$$\tilde{A}w = \tilde{b},$$

$$w \geq 0,$$

(14)
where

\[ \tilde{A} = \begin{pmatrix} A & I & 0 \\ I & 0 & 1 \end{pmatrix} \in \mathbb{R}^{(2K) \times (3K)} \]

\[ \tilde{b} = \begin{pmatrix} b \\ e \end{pmatrix} \in \mathbb{R}^{2K} \]

\[ \tilde{c} = \alpha \bar{p} \in \mathbb{R}^K \]

\[ w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^{3K} \]

We extend the potential reduction algorithm in [43], [44] to solve problem (14) to obtain one of its \( \epsilon \)-KKT points (the definition of the \( \epsilon \)-KKT point shall be given later). It can be shown that the potential reduction interior-point algorithm returns an \( \epsilon \)-KKT point of problem (14) in no more than

\[ O \left( \left( \frac{K}{\min \{ \epsilon, q \}} \right) \log \left( \frac{1}{\epsilon} \right) \right) \] iterations.

Before going into very details, we first give a high level preview of the proposed algorithm. The two basic ingredients of the potential reduction interior-point algorithm is the potential function (cf. (18)) and the update rule (cf. (25)). The potential function measures the progress of the algorithm, and the update rule guides to compute the next iterate based on the current one. More specifically, the next iterate is chosen as the feasible point that achieves the maximum potential reduction. The algorithm is terminated either when the potential function is below some threshold (cf. (19)) or when the potential reduction (cf. (26)) is smaller than a constant. In the former case, the algorithm returns an \( \epsilon \)-optimal solution of problem (14), and in the latter case an \( \epsilon \)-KKT point or \( \epsilon \)-KKT solution of problem (14). Moreover, the polynomial time convergence of the algorithm can be guaranteed by showing that the value of the potential function is decreased by at least a constant at each iteration and the potential function is bounded below by a threshold.

**Definition of \( \epsilon \)-KKT point.** The \( \epsilon \)-KKT point of problem (14) can be defined in a similar way as in [43]–[45]. Suppose that \( w^* \) is a local minimizer of problem (14), and define \( S = \{ j \mid |w^*_j| > 0 \} \). Then \( w^*_S = (w^*_1; |w^*_2|_S; w^*_3) \) should be a local minimizer of problem

\[
\min_{w_S} \quad \tilde{c}^T w_1 + \| [w_2]_S \|_q^q \\
\text{s.t.} \quad \tilde{A}_S w_S = \tilde{b}, \\
w_S \geq 0,
\]

where \( \tilde{A}_S = \begin{pmatrix} A & I^T_S & 0 \\ I & 0 & 1 \end{pmatrix} \in \mathbb{R}^{(2K) \times (2K + |S|)} \). The first-order KKT condition of problem (15) is:

there exists a Lagrange multiplier vector \( \lambda^* \in \mathbb{R}^{2K} \) such that

\[
\nabla_{w_S} f(w^*) - \tilde{A}_S^T \lambda^* \geq 0
\]

and

\[ (\nabla_{w_S} f(w^*) - \tilde{A}_S^T \lambda^*)^T w_S^* = 0. \]
Notice that given $\lambda^*$, if $[w^*]_n = 0$ for $n - K \notin S$, we still have

$$\left[ \nabla f(w^*) - \tilde{A}^T \lambda^* \right]_n [w^*]_n = q [w^*]^q_n - \left[ \tilde{A}^T \lambda^* \right]_n [w^*]_n = 0.$$ 

Therefore, the $\epsilon$-KKT point of problem (14) can be defined as follows.

**Definition 1.** $w^*$ is called an $\epsilon$-KKT point of problem (14) if

(a) it is feasible;
(b) there exists $\lambda^*$ such that (16) holds true; and
(c) the complementarity gap

$$\frac{(\nabla f(w^*) - \tilde{A}^T \lambda^*)^T w^*}{\hat{f} - \underline{f}} \leq \epsilon,$$

where $\hat{f}$ and $\underline{f}$ are upper and lower bounds on the objective value of problem (14), respectively.

In addition, $w^*$ is called an $\epsilon$-optimal solution to problem (14) if $f(w^*) \leq \epsilon$.

It is worthwhile remarking that if $\epsilon = 0$ in (17), the above definition reduces to the definition of the KKT point. For simplicity, we set $\underline{f} = 0$ in (17) in this paper, since the objective function of problem (14) is always nonnegative.

**Potential Function.** For any given strictly feasible $w$, define the following potential function

$$\phi(w) = \rho \log(f(w)) - \sum_{k=1}^{3K} \log([w]_k),$$

where $\rho$ is a parameter to be specified later.

**Lemma 2.** Let $\epsilon > 0$ and $\rho > K^q$ be fixed. Suppose that $w$ is strictly feasible and satisfies

$$\phi(w) \leq \left( \rho - \frac{K}{q} \right) \log(\epsilon) + \frac{K}{q} \log(K) + K \log(4).$$

Then $w$ is an $\epsilon$-optimal solution to problem (14).

**Proof:** Since $w$ is feasible, it follows that

$$\sum_{k=1}^{K} \log([w_1)_k[w_3]_k) = \sum_{k=1}^{K} \log([w_1]_k(1 - [w_1]_k)) \leq -K \log(4),$$

and

$$\frac{K}{q} \log(||w_2||_q^q) - \sum_{k=1}^{K} \log([w_2]_k) \geq \frac{K}{q} \log K,$$

where (21) comes from

$$\frac{||w_2||_q^q}{K} = \frac{\sum_{k=1}^{K} [w_2]_k^q}{K} \geq \left( \prod_{k=1}^{K} [w_2]_k^q \right)^{1/K}.$$
By the definition of $\phi(w)$ (cf. (18)), we obtain, for any strictly feasible $w$,

$$
\phi(w) = \rho \log(f(w)) - \sum_{k=1}^{K} \log([w_1]_k) - \sum_{k=1}^{K} \log([w_2]_k) - \sum_{k=1}^{K} \log([w_3]_k)
\geq \left( \rho - \frac{K}{q} \right) \log(f(w)) + \left( \frac{K}{q} \log(\|w_2\|_q^q) - \sum_{k=1}^{K} \log([w_2]_k) \right) - \sum_{k=1}^{K} \log([w_1][w_3]_k)
\geq \left( \rho - \frac{K}{q} \right) \log(f(w)) + \frac{K}{q} \log(K) + K \log(4),
$$

(22)

where the first inequality is because $\tilde{c}^T w_1 \geq 0$ for any feasible $w$, and the second is by (20) and (21). Therefore, if (19) holds, we must have $f(w) \leq \epsilon$, which shows that $w$ is an $\epsilon$-optimal solution.

**Update Rule.** Consider one iteration update from $w$ to $w^+$ by minimizing the potential reduction $\phi(w^+) - \phi(w)$. Suppose that $w^+ = w + d > 0$, where $d$ satisfies $\tilde{A}d = 0$. From the concavity of $\log(f(w))$, we have

$$
\log(f(w^+)) - \log(f(w)) \leq \frac{1}{f(w)} \nabla f(w)^T d.
$$

(23)

On the other hand, we have the following standard lemma [46, Theorem 9.5].

**Lemma 3.** Let $W = \text{Diag}(w)$. Suppose that $\|W^{-1}d\| \leq \beta < 1$. Then, we have

$$
-\sum_{k=1}^{3K} \log([w^+]_k) + \sum_{k=1}^{3K} \log([w]_k) \leq -e^T W^{-1} d + \frac{\beta^2}{2(1-\beta)}.
$$

It is worthwhile remarking that if $\|W^{-1}d\| \leq \beta < 1$, then $w^+ = w + d > 0$. By combining (23) and Lemma 3, we have

$$
\phi(w^+) - \phi(w) \leq \left( \frac{\rho}{f(w)} \nabla f(w)^T W - e^T \right) W^{-1} d + \frac{\beta^2}{2(1-\beta)}.
$$

(24)

Let $\tilde{d} = W^{-1}d$. To achieve the maximum potential reduction, one can solve the following problem

$$
\min_{\tilde{d}} \quad v^T \tilde{d}
\text{s.t.} \quad \tilde{A} W \tilde{d} = 0,
\|\tilde{d}\|^2 \leq \beta^2,
$$

(25)

where

$$
v = \frac{\rho}{f(w)} W \nabla f(w) - e.
$$

Problem (25) is simply a projection problem. The minimal value of problem (25) is

$$
- \beta \|g(w)\|,
$$

(26)
and the solution to problem (25) is \( \tilde{d} = \frac{\beta}{\|g(w)\|^2} g(w) \), where

\[
g(w) = e - \frac{\rho}{f(w)} W \left( \nabla f(w) - \tilde{A}^T \lambda \right),
\]

and

\[
\lambda = \left( \tilde{A} W^2 \tilde{A}^T \right)^{-1} \tilde{A} W \left( W \nabla f(w) - \frac{f(w)}{\rho} e \right).
\]

**Analysis of Polynomial Time Complexity.** Consider the following two cases.

- If \( \|g(w)\| > 1 \), then we know
  \[
  \phi(w^+) - \phi(w) < -\beta + \frac{\beta^2}{2(1 - \beta)}.
  \]
  The potential function value is reduced by a constant \(2 - \sqrt{3}\) if we set \( \beta = 1 - \frac{\sqrt{3}}{3} < 1 \). If this case would hold for \( O \left( \left( \rho - \frac{K}{q} \right) \log \left( \frac{1}{\epsilon} \right) \right) \) iterations (cf. (19)), we would obtain an \( \epsilon \)-optimal solution of (14).

- If \( \|g(w)\| \leq 1 \), then, from the definition of \( g(w) \), we must have
  \[
  0 \leq \frac{\rho}{f(w)} W (\nabla f(w) - \tilde{A}^T \lambda) \leq 2e.
  \]
  In other words,
  \[
  \left[ \nabla f(w) - \tilde{A}^T \lambda \right] \geq 0, \quad \frac{w_k \left[ \nabla f(w) - \tilde{A}^T \lambda \right]}{f(w)} \leq \frac{2}{\rho}, \quad \forall k.
  \]
  By choosing \( \rho \geq \frac{6K}{\epsilon} \), we have
  \[
  \frac{w^T(\nabla f(w) - \tilde{A}^T \lambda)}{f(w)} \leq \epsilon.
  \]
  Therefore,
  \[
  \frac{w^T(\nabla f(w) - \tilde{A}^T \lambda)}{f(w)} \leq \frac{w^T(\nabla f(w) - \tilde{A}^T \lambda)}{f(w)} \leq \epsilon.
  \]
  Recalling Definition 1, we know that \( w \) is an \( \epsilon \)-KKT point of problem (14).

Therefore, we have the following theorem.

**Theorem 3.** The potential reduction interior-point algorithm returns an \( \epsilon \)-KKT point or \( \epsilon \)-optimal solution of problem (14) (equivalent to problem (10)) in no more than \( O \left( \left( \frac{K}{\min \{\epsilon, q\}} \right) \log \left( \frac{1}{\epsilon} \right) \right) \) iterations.

One may ask why we restrict ourselves to use interior-point algorithms for solving problem (10). The reasons are the following. First, the objective function of problem (10) is differentiable in the interior
feasible region. Moreover, we are actually interested in finding an feasible $x$ such that $Ax - b$ is as sparse as possible; if we start from an $x$, some entries of $Ax - b$ are already zero, then it is very hard to make it nonzero. In contrast, if we start from an interior point, the interior-point algorithm may generate a sequence of interior points that bypasses solutions with the wrong zero supporting set and converges to the true one. This is exactly the idea of the interior-point algorithm developed in [44] for the non-convex quadratic programming.

E. A Polynomial Time Non-Convex Approximation Deflation Approach

The proposed $\ell_q$-minimization deflation (LQMD) algorithm is given as follows. The framework of the LQMD algorithm is the same as the one of the NLPD algorithm [2]. A few remarks on the proposed LQMD algorithm are in order. First, power control is carried out in the NLPD algorithm by solving LP (6); while in the proposed algorithm, power is updated by solving the non-convex $\ell_q$-minimization approximation problem (10). Second, in the NLPD algorithm, the parameter $\alpha$ is given by

$$\alpha = \begin{cases} 
c_1 \alpha_1, & \text{if } \rho(I - A) \geq 1; \\
c_2 \min \{\alpha_1, \alpha_2\}, & \text{if } \rho(I - A) < 1, 
\end{cases}$$

(27)

where $0 < c_1 \leq c_2 < 1$ are two constants, and $\alpha_1$ is determined by the equivalence between problem (3) and the joint problem (1) and (2) (cf. (4)), and $\alpha_2$ is determined by the so-called “Never-Over-Removal” property [2]. Since the $\ell_q$-minimization problem (10) is “closer” to the sparse $\ell_0$-minimization problem (3), we relax the parameter $\alpha$ in (27) to

$$\alpha = \begin{cases} 
c_1 \alpha_1, & \text{if } \rho(I - A) \geq 1; \\
\min \{c_2 \alpha_1, c_3 \alpha_2\}, & \text{if } \rho(I - A) < 1, 
\end{cases}$$

(28)

where $c_3 > c_2$, $0 < c_1, c_2 < 1$ are three constants. Finally, we compare the proposed LQMD algorithm and the existing algorithms in [1]–[3] in terms of the computational complexity needed to drop one link from the network. Since both the LPD algorithm [1] and the NLPD algorithm [2] require solving an LP, their complexity are $O(|K|^3)$ [32]; while the complexity of the Algorithm II-B [3] is $O(|K|^4)$, as it needs to solve $O(|K|)$ eigenvalue problems [47]. By comparison, the proposed LQMD algorithm has a complexity of $O(|K|^4)$, since it needs to solve $O(|K|)$ projection problems in the form of (25) (cf. Theorem 3) and the complexity of solving problem (25) is $O(|K|^3)$. Notice that the LQMD algorithm will drop at most $|K|$ links. Thus, its worst-case computational complexity is $O(|K|^5)$. We remark that all constants $\epsilon$ and $q$ are neglected in the above complexity analysis.
The LQMD Algorithm

**Step 1.** Initialization: Input data \((A, b, p)\).

**Step 2.** Preprocessing: Remove link \(k_0\) iteratively according to (9) until condition (8) holds true.

**Step 3.** Power control: Compute the parameter \(\alpha\) by (28) and solve problem (10); check whether all links are supported: if yes, go to **Step 5**; else go to **Step 4**.

**Step 4.** Admission control: Remove link \(k_0\) according to (7), set \(K = K \setminus \{k_0\}\), and go to **Step 3**.

**Step 5.** Postprocessing: Check the removed links for possible admission.

### IV. Numerical Simulations

In this section, we present some numerical simulation results to illustrate the effectiveness of the proposed LQMD algorithm.

**Simulation Setup.** We generate the same channel parameters as in [1] in our simulations; i.e., each transmitter’s location obeys the uniform distribution over a 2 Km \(\times\) 2 Km square and the location of its corresponding receiver is uniformly generated in a disc with radius 400 m; channel gains are given by \(g_{k,j} = 1/d_{k,j}^4\) (\(\forall k, j \in K\)), where \(d_{k,j}\) is the Euclidean distance from the link of transmitter \(j\) to the link of receiver \(k\). Each link’s SINR target is set to be \(\gamma_k = 2\) dB (\(\forall k \in K\)) and the noise power is set to be \(\eta_k = -90\) dBm (\(\forall k \in K\)). The power budget of the link of transmitter \(k\) is \(\tilde{p}_k = 2p_{k}^{\text{min}}\) (\(\forall k \in K\)), where \(p_{k}^{\text{min}}\) is the minimum power needed for link \(k\) to meet its SINR requirement in the absence of any interference from other links.

**Algorithm Parameters and Comparison Metrics.** The parameter \(q\) in problem (10) is set to be 0.5 and the ones in (28) are set to be \(c_1 = c_2 = 0.2\) and \(c_3 = 4\). We remark that the numerical performance of the LQMD algorithm is not sensitive to the choice of \(c_1\), \(c_2\), and \(c_3\). The number of supported links, the total transmission power, and the CPU time are the metrics we employ to compare the performance of the proposed LQMD algorithm with that of the LPD algorithm in [1], the NLPD algorithm in [2], and the Algorithm II-B in [3], since all of them have been reported to have close-to-optimal performance in terms of the number of supported links. All figures are obtained by averaging over 200 Monte-Carlo runs.

**Simulation Results and Analysis.** Figs. [1] to [3] plot the performance comparison of aforementioned various admission and power control algorithms. Fig. [1] shows that the proposed LQMD algorithm and
Fig. 1. Average number of supported links versus the number of total links.

Fig. 2. Average transmission power versus the number of total links.

the NLPD algorithm can support more links than the other two algorithms (the LPD algorithm and the Algorithm II-B) over the whole range of the tested number of total links. Figs. 1 and 2 show that, compared to the LPD algorithm and the NLPD algorithm, the proposed LQMD algorithm can support
more links with much less total transmission power. One can also see from Fig. 2 that when the number of total links is greater than or equal to 35, the Algorithm II-B transmits the least power among all the tested algorithms. This is because the Algorithm II-B supports the least number of links in these cases; see Fig. 1. In fact, as discussed in [2], a small difference of the number of supported links may lead to a large difference of the total transmission power. The significant reduction of the CPU time of the LQMD algorithm and the NLPD algorithm over the ones of the LPD algorithm and the Algorithm II-B shown in Fig. 3 is due to the Preprocessing step; i.e., the use of the necessary condition (8) to accelerate the deflation.\footnote{We remark that the necessary condition (8) can also be used to accelerate other deflation algorithms like the LPD algorithm and the Algorithm II-B. However, in this paper we do not use the necessary condition to accelerate these two algorithms.} One may say that it is not intuitive that the NLPD algorithm is slightly slower than the LQMD algorithm as shown in Fig. 3 since it takes the NLPD algorithm $O(|\mathcal{K}|^{3.5})$ arithmetic operations to drop one link from the network, while it takes the LQMD algorithm $O(|\mathcal{K}|^{4})$ operations. One possible reason for this is because a general-purpose subroutine \texttt{linprog} in MATLAB is used to solve problem (10), while a customized program is developed for solving problem (10).

**LQMD vs NLPD.** Next, we focus on the performance comparison of the LQMD algorithm and the NLPD algorithm, since these two algorithms outperform the other two in terms of the number of supported links. The comparison results are presented in Figs. 4 and 5. The vertical axis “Win Ratio” in
Fig. 4. Win ratio comparison of LQMD and NLPD versus the number of total links.

Fig. 5. Average transmission power comparison of NLPD and LQMD when the two algorithms find the admissible set with same cardinality versus the number of total links.

Fig. 4 shows the ratio of the number that the LQMD algorithm (the NLPD algorithm) wins the NLPD (the LQMD algorithm) to the total run number 200. Given an instance of the JPAC problem,
the LQMD algorithm is said to win the NLPD algorithm if the former can support strictly more links than the latter for this instance. In a similar fashion, we can define that the NLPD algorithm wins the LQMD algorithm. It can be observed from Fig. 4 that the proposed LQMD algorithm wins the NLPD algorithm with a higher ratio over the whole tested range, and the gap of the win ratios between the two algorithms becomes larger as the number of total links in the network increases. As depicted in Fig. 4 when there are \( K = 50 \) links in the network, the LQMD algorithm wins the NLPD algorithm 53 times, while the NLPD algorithm wins the LQMD algorithm 10 times (among the total 200 runs). The two algorithms find the admissible set with same cardinality for the remaining 137 times. However, this does not mean the two algorithms find the same admissible set in these cases. Fig. 5 plots the average total transmission power when the two algorithms can support the same number of links, which demonstrates that the LQMD algorithm is able to select a “better” subset of links to support, and can use much less total transmission power to support the same number of links (compared to the NLPD algorithm). As the number of total links in the network increases, the LQMD algorithm saves more power.

In a nutshell, the proposed LQMD algorithm outperforms the NLPD algorithm in terms of the number of supported links and the total transmission power. As can be seen from Figs. 1 and 4 the LQMD algorithm can support slightly more links than the NLPD algorithm. In fact, it is impossible for the LQMD algorithm to achieve a large margin of the number of supported links over the NLPD algorithm, since it has been shown in [2] that the NLPD algorithm can achieve more than 98% of global optimality (by “brute force” enumeration) in terms of the number of supported links when \( K \leq 18 \). However, the LQMD algorithm exhibits a substantially better performance (than the NLPD algorithm) in selecting which subset of links to support, and thus yields much better total transmission power performance. The performance improvement of the proposed LQMD algorithm over the NLPD algorithm is mainly attributed to the \( \ell_q \)-minimization approximation problem (10). All of the above simulation results are consistent with our intuition and analysis (cf. Theorem 1) that the \( \ell_q \)-minimization problem (10) with \( q \in (0, 1) \) is capable of approximating the \( \ell_0 \)-minimization problem (3) better than the \( \ell_1 \)-minimization problem (5).

V. Conclusions

In this paper, we have proposed an efficient polynomial time non-convex approximation deflation approach for the NP-hard joint power and admission control (JPAC) problem. Different from the existing convex approximation approaches, the proposed one solves the JPAC problem by successive non-convex \( \ell_q \)-minimization approximations with \( q \in (0, 1) \). We have shown exact recovery of the \( \ell_q \)-minimization
problem, i.e., any global solution to the $\ell_q$-minimization problem is one of the global solutions to the JPAC problem as long as the parameter $q$ is chosen to be sufficiently small. We have also developed a polynomial time potential reduction interior-point algorithm for solving the $\ell_q$-minimization problem, which makes the proposed deflation approach enjoy a polynomial time worst-case complexity. Numerical simulations demonstrate that the proposed approach is both effective and efficient, exhibiting a much better performance in selecting which subset of links to support compared to the existing convex approximation approaches.

**APPENDIX A**

**PROOF OF THEOREM 1**

To prove Theorem 1, we first introduce the following lemma.

**Lemma 4** (2). Suppose $\mathcal{S}$ is an admissible set of problem (3) and $\mathcal{S}^c = \mathcal{K} \setminus \mathcal{S}$ is its complement. Then the following statements hold.

1. For any admissible set $\mathcal{S}$, $A_{\mathcal{S},\mathcal{S}}$ is invertible and $(A_{\mathcal{S},\mathcal{S}})^{-1} \succeq 0$.
2. For any admissible set $\mathcal{S}$ and any feasible $x$ satisfying $[Ax - b]_\mathcal{S} \succeq 0$, $x_\mathcal{S} \succeq (A_{\mathcal{S},\mathcal{S}})^{-1}b_\mathcal{S} > 0$. \hfill (29)
3. Let $\bar{\mathcal{S}}$ be the optimal maximum admissible set of problem (3). Then $x^*$ with $x^*_\mathcal{S} = (A_{\mathcal{S},\mathcal{S}})^{-1}b_\mathcal{S}$, $x^*_\mathcal{S}^c = 0$ \hfill (30)

is the solution to problem (3).

We are now ready to prove Theorem 1. We first prove that the theorem is true under the assumptions that the solution of problem (3) is unique and the maximum admissible set of problem (3) is also unique. Then, we remove these two assumptions and prove that the theorem remains true.

Case I: Assume $x^*$ in (30) is the unique global minimizer of problem (3) and $\bar{\mathcal{S}}$ is the corresponding maximum admissible set, which is also unique. Next, we show $x^*$ is the unique global minimizer of problem (10). By Lemma 1, it is equivalent to show $x^*$ is the unique global minimizer of problem (11). We divide the proof into two parts.

5The subset $\mathcal{S}$ is called admissible if there exists a feasible $x$ such that $[Ax - b]_\mathcal{S} \succeq 0$. 

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Part A: In this part, we show that when $q$ is sufficiently small, $x^*$ is the unique global minimizer of problem

$$
\min_x \| [b - Ax]_{S_c} \|_q^q + \alpha \bar{p}^T x \\
\text{s.t. } [b - Ax]_{S} = 0, \\
[b - Ax]_{S_c} > 0, \\
0 \leq x \leq e.
$$

Consider the following problem

$$
\min_x \min \{ [b - Ax]_{S_c} \} \\
\text{s.t. } [b - Ax]_{S} = 0, \\
[b - Ax]_{S_c} \geq 0, \\
0 \leq x \leq e.
$$

We claim that the optimal value of problem (32) is greater than or equal to $\delta := \delta(A, b) > 0$. Otherwise, there must exist a feasible point $x$ of problem (32) such that $[b - Ax]_j = 0, \forall j \in \bar{S} \cup k$, where $k \in \bar{S}^c$. This contradicts the fact that $\bar{S}$ is the maximum admissible set.

Suppose $g(x)$ is the gradient of the objective function in (31). Then

$$
g(x) = \alpha \bar{p} - q (A_{\bar{S}^c, \bar{S}}, A_{\bar{S}^c, \bar{S}^c})^T [b - Ax]_{\bar{S}^c} \geq \alpha \bar{p} - q \bar{q}_{\bar{S}^c} \geq (|A_{\bar{S}^c, \bar{S}}|, |A_{\bar{S}^c, \bar{S}^c}|)^T e,
$$

where $|A|$ denotes the entry-wise absolute value of the matrix $A$. If $q$ is sufficiently small (say, $q \leq \bar{q}_1(A, b, \alpha, \bar{p})$, where $\bar{q}_1(A, b, \alpha, \bar{p})$ is a positive number such that for any $q \in (0, \bar{q}_1(A, b, \alpha, \bar{p}))$, we have $\alpha \bar{p} - q \bar{q}_{\bar{S}^c} \geq (|A_{\bar{S}^c, \bar{S}}|, |A_{\bar{S}^c, \bar{S}^c}|)^T e > 0$), then the gradient $g(x)$ of the objective function in (31) is component-wise positive at any feasible point. In addition, to guarantee $[b - Ax]_{\bar{S}} = 0$, we need $x_{\bar{S}} \geq x_{\bar{S}} = A_{\bar{S}, \bar{S}}^{-1} b_{\bar{S}}$ (cf. (29) and (30)). Therefore, $x^*$ is the unique global minimizer of problem (31), since for any feasible $x$, we have $x_{\bar{S}} \geq x_{\bar{S}}$ and $x_{\bar{S}^c} \geq 0$.

Part B: In this part, we show that when $q$ is sufficiently small, $x^*$ is the unique global minimizer of problem (11). To show this, it suffices to show that, for any given admissible set $S \subset K$ with $|S| < |\bar{S}|$, the minimum value of problem

$$
\min_x \| [b - Ax]_S \|_q^q + \alpha \bar{p}^T x \\
\text{s.t. } [b - Ax]_S = 0, \\
[b - Ax]_{S^c} > 0, \\
0 \leq x \leq e,
$$

is greater than or equal to $\delta := \delta(A, b) > 0$. Otherwise, there must exist a feasible point $x$ of problem (33) such that $[b - Ax]_j = 0, \forall j \in \bar{S} \cup k$, where $k \in \bar{S}^c$. This contradicts the fact that $\bar{S}$ is the maximum admissible set.
is greater than the one of problem (31). Without loss of generality, suppose the solution $x^*(S)$ of problem (33) is attainable. Otherwise, there must exist $k \in S^c$ such that $[b - Ax]_k \to 0$ at the optimal point. In this case, we consider problem (33) with $S$ replaced by $S \cup k$. According to the assumption that the maximum admissible set of problem (3) is unique, we still have $|S \cup k| < |\bar{S}|$ unless $S \cup k = \bar{S}$.

Suppose $x^*(S)$ is achievable, there must exist $\delta_S > 0$ such that
\[
[b - Ax^*(S)]_j \geq \delta_S > 0, \quad \forall \ j \in S^c,
\] (34)
where $\delta_S$ depends on $A, b, \alpha, \vec{p}, S$. Since the number of admissible sets $S$, with which the solution $x^*(S)$ of problem (33) is achievable, is finite, then
\[
\delta := \min \left\{ \min_{S \subset K} \{ \delta_S \}, \min \{ b / 2 \} \right\} > 0.
\] (35)
Here, $\delta$ only depends on $A, b, \alpha$ and $\vec{p}$. Define
\[
\Delta := \max_{0 \leq x \leq e, b - Ax \geq 0} \| b - Ax \|_{\infty}.
\] (36)
Let $\bar{q}_2(A, b, \alpha, \vec{p})$ be a positive number such that for any $q \in (0, \bar{q}_2(A, b, \alpha, \vec{p})$, we have
\[
(K - |\bar{S}| + 1) \delta^q \geq (K - |\bar{S}|) \Delta^q + \alpha \vec{p}^T x^*.
\] (37)
Therefore, if
\[
0 < q \leq \bar{q} := \min \{ \bar{q}_1(A, b, \alpha, \vec{p}), \bar{q}_2(A, b, \alpha, \vec{p}) \},
\] (38)
for any admissible $S$ with $|S| < |\bar{S}|$ :

- if $x^*(S) \neq 0$, there holds
  \[
  \| [b - Ax^*(S)]_q + \alpha \vec{p}^T x^*(S) \|_q \geq (K - |S|) \delta^q \geq (K - |\bar{S}| + 1) \delta^q \geq (K - |\bar{S}|) \Delta^q + \alpha \vec{p}^T x^* \geq \| b - Ax^* \|_q + \alpha \vec{p}^T x^*,
  \] (39)
  where (39) is due to (34), (35), and $x^*(S) \neq 0$, (40) is due to the fact $|S| < |\bar{S}|$, (41) is due to (37) and (38), and (42) is by the definition of $\Delta$ (cf. (36)).

- if $x^*(S) = 0$, then
  \[
  \| b - Ax^*(S) \|_q + \alpha \vec{p}^T x^*(S) = \| b \|_q > K \delta^q \geq \| b - Ax^* \|_q + \alpha \vec{p}^T x^*,
  \]
where the first strict inequality is due to (35), and the second inequality can be obtained in a similar fashion as in the case of $x^*(S) \neq 0$. 

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Case II: Consider the case when problem (3) has multiple maximum admissible sets, but its solution \( x^* \) remains unique. Then, for any feasible set \( S \neq \bar{S} \) such that \( |S| = |\bar{S}| \), we have

\[
\|Ax^*(S) - b\|_0 + \alpha p^T x^*(S) > \|Ax^* - b\|_0 + \alpha p^T x^*,
\]

where \( x^*(S) \) is the solution to problem (3). The above strict inequality is because \( x^* \) is the unique solution to problem (3). Therefore, there exists \( q(A, b, \alpha, p, S) > 0 \) such that for all \( q \in (0, q(A, b, \alpha, p, S)] \), there holds

\[
\|Ax^*(S) - b\|_q + \alpha p^T x^*(S) > \|Ax^* - b\|_q + \alpha p^T x^*.
\]

Since problem (3) has at most \( \binom{K}{|\bar{S}|} \) maximum admissible sets, we can take the minimum among \( \{q(A, b, \alpha, p, S)\} \), and obtain a \( q(A, b, \alpha, p) > 0 \) such that when \( q \in (0, q(A, b, \alpha, p)] \), it has

\[
\|Ax^*(S) - b\|_q + \alpha p^T x^*(S) > \|Ax^* - b\|_q + \alpha p^T x^*, \quad \forall \ S \neq S^*, \quad |S| = |S^*|.
\]

This, together with Case I, implies that when \( q \) is sufficiently small, \( x^* \) is the unique global minimizer of problem (11) under the assumption that the solution \( x^* \) of problem (3) is unique.

Case III: The remaining case is that when problem (3) has multiple solutions. Without loss of generality, we assume that there are two different solutions \( x_1^* \) and \( x_2^* \). Then, the choice of \( \alpha \) (cf. (4)) immediately implies

\[
\|Ax_1^* - b\|_0 = \|Ax_2^* - b\|_0, \quad p^T x_1^* = p^T x_2^*.
\]

If there exists an bijective mapping \( \pi \) from \( \{1, 2, \ldots, K\} \) to \( \{1, 2, \ldots, K\} \) such that \( [Ax_1^* - b]_j = [Ax_2^* - b]_{\pi(j)} \) for all \( j \in K \), then both \( x_1^* \) and \( x_2^* \) are global minimizers of problem (11). Otherwise, we can find \( q(A, b, \alpha, p) > 0 \) such that when \( q \in (0, q(A, b, \alpha, p)] \), we have either

\[
\|Ax_1^* - b\|_q + \alpha p^T x_1^* < \|Ax_2^* - b\|_q + \alpha p^T x_2^*.
\]

or

\[
\|Ax_1^* - b\|_q + \alpha p^T x_1^* > \|Ax_2^* - b\|_q + \alpha p^T x_2^*.
\]

Combining the above with Cases I and II, we know that \( x_1^* \) (or \( x_2^* \)) is the global minimizer of problem (11). This completes the proof of Theorem 1.

**APPENDIX B**

**PROOF OF THEOREM 2**

Given an instance of the partition problem with \( s_1, s_2, \ldots, s_N \), define \( S = \sum_{n=1}^N s_n \). Next, we construct an instance of problem (10), where
- $K = 2N + 2$;
- all entries of $\mathbf{p} \in \mathbb{R}^{2N+2}$ are set to be 1;
- the first $2N$ entries of $\mathbf{b} \in \mathbb{R}^{2N+2}$ are set to be 1, and the last two entries of $\mathbf{b}$ are set to be 0.5;
- all diagonal entries of $\mathbf{A} \in \mathbb{R}^{(2N+2) \times (2N+2)}$ are one, and non-diagonal entries of $\mathbf{A}$ are
  - for $k = 2n-1$, $n = 1, 2, \ldots, N$, set $a_{k,j} = 0$ except $a_{k,k+1} = -1$;
  - for $k = 2n$, $n = 1, 2, \ldots, N$, set $a_{k,j} = 0$ except $a_{k,k-1} = -1$;
  - for $k = 2N + 1$, set $a_{k,j} = 0$ except $a_{k,2n-1} = -\frac{s_n}{S}$ for $n = 1, 2, \ldots, N$;
  - for $k = 2N + 2$, set $a_{k,j} = 0$ except $a_{k,2n} = -\frac{s_n}{S}$ for $n = 1, 2, \ldots, N$; and
- the parameter $\alpha$ satisfies
  \[ 0 < \alpha \leq \frac{2 - 2^q}{2}. \] (43)

Then the constructed instance of problem (10) is

\[
\begin{align*}
\min_{x_1, \ldots, x_{2N+2}} & \quad \sum_{n=1}^{N} F(x_{2n-1}, x_{2n}) + \sum_{i=0}^{1} H(x_{1+i}, x_{3+i}, \ldots, x_{2N+1+i}) \\
\text{s.t.} & \quad 0 \leq x_n \leq 1, \; n = 1, 2, \ldots, 2N+2,
\end{align*}
\] (44)

where

\[
F(x_{2n-1}, x_{2n}) = |1 + x_{2n} - x_{2n-1}|^q + |1 + x_{2n-1} - x_{2n}|^q + \alpha(x_{2n-1} + x_{2n}), \; n = 1, 2, \ldots, N,
\]

\[
H(x_{1+i}, x_{3+i}, \ldots, x_{2N+1+i}) = \left| 0.5 + \frac{1}{S} \sum_{n=1}^{N} s_n x_{2n-1+i} - x_{2N+1+i} \right|^q + \alpha x_{2N+1+i}, \; i = 0, 1.
\]

Notice that $0 \leq x_{2n-1}, x_{2n} \leq 1$ for all $n = 1, 2, \ldots, N$, it follows that

\[
F(x_{2n-1}, x_{2n}) = (1 + x_{2n} - x_{2n-1})^q + (1 + x_{2n-1} - x_{2n})^q + \alpha(x_{2n-1} + x_{2n}), \; n = 1, 2, \ldots, N. \tag{45}
\]

Next, we claim that the partition problem has a “yes” answer if and only if the optimal value of problem (44) is less than or equal to $2^q N + \alpha(N + 2)$. We prove the “if” and “only if” directions separately.

Let us first prove the “only if” direction. Suppose the partition problem has a “yes” answer and let $S$ be the subset of $\{1, 2, \ldots, N\}$ such that

\[
\sum_{n \in S} s_n = S/2. \tag{46}
\]

We show that there exists a feasible power allocation vector $\{x_n\}_{n=1}^{2N+2}$ such that the optimal value of problem (44) is less than or equal to $2^q N + \alpha(N + 2)$. In particular, let

\[
\hat{x}_{2n-1} = \begin{cases} 
1, & \text{if } n \in S; \\
0, & \text{if } n \notin S,
\end{cases} \quad \hat{x}_{2n} = 1 - \hat{x}_{2n-1}, \; n = 1, 2, \ldots, N, \tag{47}
\]
and

\[ \hat{x}_{2N+1} = \hat{x}_{2N+2} = 1. \]

It is simple to check

\[
F(\hat{x}_{2n-1}, \hat{x}_{2n}) = 2^q + \alpha, \quad n = 1, 2, \ldots, N,
\]

\[
H(\hat{x}_1, \hat{x}_3, \ldots, \hat{x}_{2N+1}) = 0.5 + \frac{1}{S} \sum_{n=1}^N s_n \hat{x}_{2n-1} - \hat{x}_{2N+1} \right| \right| q + \alpha \hat{x}_{2N+1}
\]

\[
= 0.5 + \frac{1}{S} \sum_{n \in S} s_n - 1 + \alpha \quad \text{(from (47))}
\]

\[
= \alpha, \quad \text{(from (46))}
\]

\[
H(\hat{x}_2, \hat{x}_4, \ldots, \hat{x}_{2N+2}) = \alpha.
\]

Thus, we have

\[
\sum_{n=1}^N F(\hat{x}_{2n-1}, \hat{x}_{2n}) + \sum_{i=0}^1 H(\hat{x}_{1+i}, \hat{x}_{3+i}, \ldots, \hat{x}_{2N+1+i}) = N2^q + (N + 2)\alpha,
\]

which implies that the optimal value of problem (44) is less than or equal to \( N2^q + (N + 2)\alpha \).

To show the “if” direction, suppose that the optimal solution of problem (44) is less than or equal to \( N2^q + (N + 2)\alpha \). Consider a relaxation of problem (44) by dropping the constraints \( x_{2N+1} \leq 1 \) and \( x_{2N+2} \leq 1 \):

\[
\min_{x_1, \ldots, x_{2N+2}} \sum_{n=1}^N F(\hat{x}_{2n-1}, \hat{x}_{2n}) + \sum_{i=0}^1 H(\hat{x}_{1+i}, \hat{x}_{3+i}, \ldots, \hat{x}_{2N+1+i})
\]

\[
\text{s.t.} \quad 0 \leq x_n \leq 1, \quad n = 1, 2, \ldots, 2N,
\]

\[
x_{2N+1} \geq 0, \quad x_{2N+2} \geq 0.
\]

(48)

Clearly, the optimal value of problem (48) is less than or equal to the optimal value of problem (44).

The relaxed problem (48) can be equivalently rewritten as

\[
\min_{x_1, \ldots, x_{2N+2}} \sum_{n=1}^N F(\hat{x}_{2n-1}, \hat{x}_{2n}) + \sum_{i=0}^1 \hat{H}(x_{1+i}, x_{3+i}, \ldots, x_{2N-1+i})
\]

\[
\text{s.t.} \quad 0 \leq x_n \leq 1, \quad n = 1, 2, \ldots, 2N,
\]

where for \( i = 0, 1, \)

\[
\hat{H}(x_{1+i}, x_{3+i}, \ldots, x_{2N-1+i}) := \min_{x_{2N+1+i}} H(x_{1+i}, x_{3+i}, \ldots, x_{2N-1+i}, x_{2N+1+i})
\]

\[
\text{s.t.} \quad x_{2N+1+i} \geq 0.
\]

(49)
Since problem (49) is an univariate optimization problem and \( \alpha \) satisfies (43), we can verify that, for any \( x_{1+i}, x_{3+i}, \ldots, x_{2N+1+i} \geq 0 \), there holds

\[
H(x_{1+i}, x_{3+i}, \ldots, x_{2N+1+i}) \geq H(x_{1+i}, x_{3+i}, \ldots, \hat{x}_{2N+1+i}) = \frac{\alpha}{2} + \frac{\alpha}{S} \sum_{n=1}^{N} s_n x_{2n-1+i},
\]

where

\[
\hat{x}_{2N+1+i} = 0.5 + \frac{1}{S} \sum_{n=1}^{N} s_n x_{2n-1+i}, \quad i = 0, 1.
\]

By definition (49), we have

\[
\hat{H}(x_{1+i}, x_{3+i}, \ldots, x_{2N+1+i}) = \frac{\alpha}{2} + \frac{\alpha}{S} \sum_{n=1}^{N} s_n x_{2n-1+i}, \quad i = 0, 1.
\]

As a result, problem (48) can be decomposed into \( N \) subproblems

\[
\begin{align*}
\min_{x_{2n-1}, x_{2n}} & \quad \hat{F}(x_{2n-1}, x_{2n}) := F(x_{2n-1}, x_{2n}) + \frac{\alpha s_n}{S} (x_{2n-1} + x_{2n}) \\
\text{s.t.} & \quad 0 \leq x_{2n-1}, x_{2n} \leq 1.
\end{align*}
\]

We know from (45) that \( \hat{F}(x_{2n-1}, x_{2n}) \) in (52) is strictly concave with respect to \( x_{2n-1} \) and \( x_{2n} \) in \( [0, 1] \times [0, 1] \). Since the minimum of a strictly concave function is always attained at a vertex (48), we immediately obtain that the optimal solution of (52) must be \((0, 0), (0, 1), (1, 0), \) or \((1, 1)\). It is easy to see that

\[
\hat{F}(0, 0) = 2, \quad \hat{F}(0, 1) = \hat{F}(1, 0) = 2^q + \alpha \left(1 + \frac{s_n}{S}\right), \quad \hat{F}(1, 1) = 2 + 2\alpha \left(1 + \frac{s_n}{S}\right).
\]

This, together with the facts \( 0 < s_n/S < 1 \) and \( \alpha \leq \frac{2-2^q}{2} \) (cf. (43)), shows the optimal solution of (52) is

\[
(\hat{x}_{2n-1}, \hat{x}_{2n}) = (0, 1) \text{ or } (1, 0).
\]

Now, we can use (51) and (53) to conclude that the optimal value of problem (48) is

\[
\sum_{n=1}^{N} \left(2^q + \alpha + \frac{\alpha s_n}{S}\right) = N2^q + (N+2)\alpha.
\]

Since the optimal value of problem (44) is less than or equal to \( N2^q + (N+2)\alpha \) (the assumption of the “if” direction), it follows from (51) that

\[
\hat{x}_{2N+1} = 0.5 + \frac{1}{S} \sum_{n=1}^{N} s_n \hat{x}_{2n-1} \leq 1, \quad \hat{x}_{2N+2} = 0.5 + \frac{1}{S} \sum_{n=1}^{N} s_n \hat{x}_{2n} \leq 1.
\]

Combining this with (53) yields

\[
\sum_{n=1}^{N} s_n \hat{x}_{2n-1} = \sum_{n \in S} s_n = \frac{S}{2}, \quad \sum_{n=1}^{N} s_n \hat{x}_{2n} = \sum_{n \notin S} s_n = \frac{S}{2}.
\]
where $S = \{ n | \hat{x}_{2n-1} = 1 \}$. Therefore, there exists a subset $S$ such that \( (46) \) holds true, which shows that the partition problem has a “yes” answer.

Finally, this transformation can be finished in polynomial time. Since the partition problem is NP-complete, we conclude that problem \( (10) \) is NP-hard.

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**REFERENCES**


