Codes over rings, complex lattices and Hermitian modular forms

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Received 13 November 2003; received in revised form 10 April 2004; accepted 16 April 2004
Available online 21 July 2004

Abstract

We introduce the finite ring $\mathbb{Z}_{2m} + i\mathbb{Z}_{2m}$. We develop a theory of self-dual codes over this ring and relate self-dual codes over this ring to complex unimodular lattices. We describe a theory of shadows for these codes and lattices. We construct a gray map from this ring to the ring $\mathbb{Z}_{2m}$ and relate codes over these rings, giving special attention to the case when $m = 2$. We construct various Hermitian modular forms from weight enumerators and give the correspondence between the invariant space, where the weight enumerators of codes reside, and the space of Hermitian modular forms.

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Keywords: Self-dual codes; Unimodular lattices; Hermitian modular forms

1. Introduction

Numerous interesting results have arisen by considering a bridge between coding theory and the theory of lattices. More specifically, we refer to that bridge which relates self-dual codes and unimodular lattices. See [10] for a description of these codes and lattices and an extensive bibliography. Initially, the connection was made between binary codes and real unimodular lattices. The difficulty with this connection was that it was unable to produce extremal lattices for all but small lengths. Codes over the rings $\mathbb{Z}_{2m}$ were the natural codes to consider next when building unimodular lattices, since they were canonically connected to extremal unimodular lattices for unbounded lengths by allowing $2m$ to increase. In fact
it was clear that in most ways codes over $\mathbb{Z}_2$ and $\mathbb{Z}_4$ in particular were more naturally connected to unimodular lattices than binary codes.

Another connection was made between codes over ternary and quaternary fields and complex unimodular lattices. This connection was not the canonical connection to complex unimodular lattices. Codes over the ring $\mathbb{F}_2 + u \mathbb{F}_2$ [4] were used to construct complex unimodular lattices and a gray map was used to relate these codes to binary codes. In many ways $\mathbb{F}_2 + u \mathbb{F}_2$ has the same difficulties in constructing complex unimodular lattices that the binary field has in constructing real unimodular lattices. In this work, we show that the canonical bridge is between codes over the rings $\mathbb{S}_2$ and complex unimodular lattices. We shall show that self-dual codes over these rings produce complex unimodular lattices and that the weight enumerators of these codes can produce Hermitian modular forms.

The theory of self-dual codes is intimately related to invariant theory. It has been shown that the complete weight enumerators of codes over certain rings can be considered as an invariant polynomial under a certain finite group. Moreover, this fact has been useful in determining bounds for self-dual codes. Namely, using the theory of invariants one is able to determine the form of a possible weight enumerator and in many instances rule out the existence of codes with certain parameters. The theory of shadows has been extremely useful in this regard. It is also known that one can construct various modular forms from the weight enumerators of codes, by plotting special types of theta-function [2].

Some of the most interesting and important results for codes over rings have been found by examining distance preserving gray maps. In the best known case, the ring $\mathbb{Z}_4$ has a gray map to the binary field. In this work, we construct a gray map from the ring $\mathbb{S}_2$ to $\mathbb{Z}_2$ in much the same way that $\mathbb{Z}_2 + u \mathbb{Z}_2$ is related to $\mathbb{Z}_2$.

2. Codes over the rings $\mathbb{Z}_2 + i \mathbb{Z}_2$

We shall construct a set of rings, an inner product for the space over these rings, and define weights for the rings and we will provide a natural connection to codes over $\mathbb{Z}_2$ via a gray map and to complex unimodular lattices via a construction.

The finite ring $\mathbb{S}_2$ is given by $\mathbb{S}_2 := \mathbb{Z}[i]/2m \mathbb{Z}[i]$. This ring can also be given by $\mathbb{Z}_2 + \omega \mathbb{Z}_2$, where $\omega$ is the element corresponding to $1 + i$. This presentation allows for the ring to be seen as a generalization of $\mathbb{F}_2 + u \mathbb{F}_2$, with $u^2 = 0$. We equip each ring with a corresponding involution corresponding to complex conjugation

$$a + bi = a - bi, \quad a, b \in \mathbb{Z}.$$ 

Note that $|\mathbb{S}_2| = (2m)^2 = 4m^2$.

A code over $\mathbb{S}_2$ is a subset of $\mathbb{S}_2^n$ and a code is said to be linear if it is a submodule of the ambient space. We shall assume all codes are linear unless otherwise specified. Attached to the space $\mathbb{S}_2^n$ is the following natural inner-product corresponding to the Hermitian inner-product:

$$[v, u] = \sum v_i \overline{u_i}. \quad (1)$$
The orthogonal is defined by \( C^\perp = \{ v' \mid [v', v] = 0 \text{ for all } v \in C \} \). It is immediate that the orthogonal of a linear code is linear and that \(|C||C^\perp| = |S_{2m}^n|\). As usual, we say a code is self-orthogonal if \( C \subseteq C^\perp \) and is self-dual if \( C = C^\perp \). The norm of a vector \( v = (v_1, v_2, \ldots, v_n) \) is given by \( \sum N(v_i) = \sum v_i v_i^\perp \).

We define the gray map \( \tilde{\Psi} : S_{2m} \to \mathbb{Z}_{2m}^2 \) by

\[
\tilde{\Psi}(a + bi) = (b, a).
\]

The map \( \tilde{\Psi} \) is obviously linear, since the map \( \tilde{\Psi} \) is extended to \( S_{2m}^n \) by applying it coordinatewise.

**Lemma 2.1.** Let \( C \) be a self-dual code of length \( n \) over \( S_{2m} \). Then \( \tilde{\Psi}(C) \) is a self-dual code of length \( 2n \) over \( \mathbb{Z}_{2m} \).

**Proof.** Let \( C \) be a self-dual code. For vectors \( v \) and \( v' \) in \( C \), we have that \([v, v'] = \sum v_i v'_i = 0 \). Set \( v_j = a_j + b_j i \) and \( v'_j = a'_j + b'_j i \), then

\[
\sum v_i v'_i = \sum (a_j + b_j i)(a'_j + b'_j i) = \sum (a_j + b_j i)(a'_j - b'_j i) = \sum (a_j a'_j + b_j b'_j) + (a'_j b_j - a_j b'_j)i.
\]

Hence \( \sum (a_j a'_j + b_j b'_j) \equiv 0 \pmod{2m} \) and \( \sum (a'_j b_j - a_j b'_j) \equiv 0 \pmod{2m} \).

To show that \( [\tilde{\Psi}(v), \tilde{\Psi}(v')] = 0 \), we see that \( \tilde{\Psi}(v_j) = \tilde{\Psi}(a_j + b_j i) = (b_j, a_j) \) and \( \tilde{\Psi}(v'_j) = \tilde{\Psi}(a'_j + b'_j i) = (b'_j, a'_j) \).

Finally,

\[
[\tilde{\Psi}(v), \tilde{\Psi}(v')] = \sum (b_j b'_j + a_j a'_j) \equiv 0 \pmod{2m}
\]

giving that the codes are self-orthogonal. Then since \( |\tilde{\Psi}(C)| = |C| = (2m)^n \), \( \tilde{\Psi}(C) \) is a self-dual code of length \( 2n \) over \( \mathbb{Z}_{2m} \). \( \square \)

We define the Euclidean weight of a vector \( v \in S_{2m}^n \), denoted \( \text{Euc}(v) \), as the Euclidean weight of its image under \( \tilde{\Psi} \) in \( \mathbb{Z}_{2m}^2 \). The Euclidean weight of a vector \( v = (v_j) \) in \( \mathbb{Z}_{2m}^n \) is \( \sum \min(v_j^2, (2m - v_j)^2) \). The minimum Euclidean weight of a code, \( d_E \), is the minimum of the Euclidean weights of all non-zero vectors in the code.

**Lemma 2.2.** Let \( v = (v_j) \) be a vector in \( S_{2m}^n \), then \( \text{Euc}(v) \equiv N(v) \pmod{2m} \).

**Proof.** For \( a + bi \) in \( S_{2m} \), \( \text{Euc}(a + bi) = \text{Euc}(b, a) \equiv b^2 + a^2 \pmod{2m} \), and \( N(a + bi) = (a + bi)(a + bi) \equiv a^2 + b^2 \pmod{2m} \). The result follows. \( \square \)

Naturally we say that a self-dual code is **Type II** over \( S_{2m} \) if the Euclidean weights of all the vectors are equivalent to \( 0 \pmod{4m} \). A self-dual code that is not Type II is said to be **Type I**.

**Theorem 2.3.** Let \( C \) be a Type II (resp. Type I) code of length \( n \) over \( S_{2m} \). Then \( \tilde{\Psi}(C) \) is a Type II (resp. Type I) code of length \( 2n \) over \( \mathbb{Z}_{2m} \).
Proof. Follows from Lemmas 2.1 and 2.2. □

This puts natural restrictions on the lengths when Type I and Type II codes can exist. If a Type I code exists of length \( n \) then there must exist Type I codes over \( \mathbb{Z}_{2m} \) of length \( 2n \). Moreover, we have the following.

**Corollary 2.4.** If there exists a Type II code over \( S_{2m} \) of length \( n \) then \( n \) is a multiple of 4.

**Proof.** If there is a Type II code of length \( n \) then there is a Type II code over \( \mathbb{Z}_{2m} \) of length \( 2n \) which implies \( 2n \equiv 0 \) (mod 8). □

For example, the code generated by
\[
\begin{pmatrix}
1 & 2 + 3i & 3 + i \\
0 & 1 & 3 + 3i & 2 + 3i
\end{pmatrix},
\]
(3)
is a Type II code of length 4 over \( S_4 \). This gives that Type II codes exist for all lengths divisible by 4 over \( S_4 \).

The code
\[
C = \{ 0, 2, 2i, 2 + 2i \}
\]
is a self-dual code of length 1, hence self-dual codes exist over \( S_4 \) for all lengths \( n \).

Over \( S_6 \) there is no self-dual code of odd length \( 2k + 1 \), since if there were there would be a self-dual code of even length \( 4k + 2 \) over \( \mathbb{Z}_6 \) by Lemma 2.1. Then by the inverse of the Chinese remainder theorem (i.e. reading every element (mod 3) there would be a self-dual code over \( F_3 \) of length \( 4k + 2 \equiv 2 \) (mod 4) which is a contradiction. There is a self-dual code of length 2 over \( S_6 \) generated by
\[
(1 \ 2 + i).
\]
(5)

Likewise, similar reasoning gives the following.

**Proposition 2.5.** There are no self-dual codes of odd length over \( S_{2m} \) if there exists a prime congruent to 3 (mod 4) sharply dividing \( 2m \).

Over \( S_8 \) there is a self-dual code of length 1, namely
\[
C = \{ 0, 4, 2 + 2i, 6 + 2i, 4 + 4i, 4i, 6 + 6i, 2 + 6i \}.
\]
(6)

Therefore there are self-dual codes of all lengths over \( S_8 \). Similarly,
\[
\{0, 3 + i, 6 + 2i, 9 + 3i, 2 + 4i, 5 + 5i, 8 + 6i, 1 + 7i, 4 + 8i, 7 + 9i\}
\]
is a self-dual code of length 1 over \( S_{10} \) giving self-dual codes of all lengths for this ring.

Over \( S_4 \) we have additional correspondences. Specifically, denote the usual gray map \( \mathcal{G} : \mathbb{Z}_4 \to \mathbb{Z}_2 \), and \( \mathcal{G}^* := S_4 \to (\mathbb{Z}_2 + u\mathbb{Z}_2)^2 \). We recall that \( \mathcal{G}(0) = 00, \mathcal{G}(1) = 01, \mathcal{G}(2) = 11, \) and \( \mathcal{G}(3) = 10 \). The map \( \mathcal{G} \) is not linear but is distance preserving. The ring \( \mathbb{Z}_2 + u\mathbb{Z}_2 \) has a natural gray map to \( \mathbb{Z}_2^4 \) which connects it to \( S_4 \) as well. The specific correspondences among \( S_4, \mathbb{Z}_4^2, (\mathbb{Z}_2 + u\mathbb{Z}_2)^2 \), and \( \mathbb{F}_4 \) are given in Table 1.


Table 1
Maps

<table>
<thead>
<tr>
<th>Z₄ + iZ₄</th>
<th>Z₄²</th>
<th>Z₄⁺²</th>
<th>(Z₂ + uZ₂)²</th>
<th>Euclidean weight</th>
</tr>
</thead>
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<td>0000</td>
<td>0, 0</td>
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<td>01</td>
<td>0001</td>
<td>0, 1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>02</td>
<td>0011</td>
<td>0, u</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>03</td>
<td>0010</td>
<td>0, 1 + u</td>
<td>1</td>
</tr>
<tr>
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<td>11</td>
<td>0101</td>
<td>1, 1</td>
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</tr>
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<td>12</td>
<td>0111</td>
<td>1, u</td>
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<td>0110</td>
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<tr>
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<td>0100</td>
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<td>1</td>
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<td>1111</td>
<td>u, u</td>
<td>8</td>
</tr>
<tr>
<td>3 + 2i</td>
<td>23</td>
<td>1110</td>
<td>u, 1 + u</td>
<td>5</td>
</tr>
<tr>
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<td>20</td>
<td>1100</td>
<td>u, 0</td>
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<td>32</td>
<td>1011</td>
<td>1 + u, u</td>
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</tbody>
</table>

3. Weight enumerators

We shall now define a series of weight enumerators for codes over \( S_{2m} \).

For a code \( C \) over \( S_{2m} \) define the complete weight enumerator by

\[
\text{cwe}_C(x_0, x_1, \ldots, x_{(2m-1)+(2m-1)i}) = \sum_{v \in C} \prod_{a \in S_{2m}} x_{n_a(v)}^a,
\]  

(8)

where \( n_a(v) = |\{j \mid v_j = a\}| \). Here the ordering is defined as lexicographic order. The relation \( \sim \) on \( S_{2m} \) is defined by \( a \sim b \) if and only if \( a = b\epsilon \) where \( \epsilon \) is a unit in \( S_{2m} \). This relation \( \sim \) forms an equivalence relation and we consider the set of equivalence classes \( S_{2m} = S_{2m}/\sim \). We denote the set of units in \( S_{2m} \) by \( U_{2m} \). For a code \( C \) over \( S_{2m} \) define the symmetrized weight enumerator by

\[
\text{swe}_C(x_{[a]} \mid [a] \in S_{2m}) = \sum_{v \in C} \prod_{[a] \in S_{2m}} x_{sn_{[a]}(v)}^{sn_{[a]}(v)},
\]  

(9)

where \( sn_{[a]}(v) = |\{j \mid v_j \in [a]\}| \). For codes \( C \) and \( D \) over \( S_{2m} \) define the complete joint weight enumerator by

\[
J_{C,D}(X) = \sum \sum \prod_{(a,b) \in S_{2m}^2} x_{n(a,b)(v,v')}^{n(a,b)(v,v')}
\]  

(10)

where \( n(a,b)(v, v') = |\{j \mid v_j = a, v'_j = b\}| \).

The complete weight enumerator of a code is a homogenous polynomial in \((2m)^2\) variables and the complete joint weight enumerator is a homogeneous polynomial in \((2m)^4\) variables.
In order to define the symmetrized joint weight enumerator we need to consider the space \( \Omega = (S^2_{2m})/\sim \), where \((\alpha, \beta) \sim (\alpha', \beta')\) if and only if \(\alpha' = \alpha \epsilon\) and \(\beta' = \beta \epsilon\) for a unit \(\epsilon\). This will be denoted by \((\alpha, \beta) \leftrightarrow (\alpha', \beta') = \epsilon(\alpha, \beta), \) unit \(\epsilon\).

The symmetrized joint weight enumerator for codes \(C\) and \(D\) over \(S^2_{2m}\) is

\[
SJ_{C,D}(X) = \sum_{\alpha \in C} \sum_{\beta \in D} \prod_{(\alpha, \beta) \in (S^2_{2m})/\sim} n_{\epsilon(\alpha, \beta)}(x; y) = S[J_{C,D}(X)]
\]

where \(n_{\epsilon(\alpha, \beta)}(x; y) = \mid\{(j, (v, v')) \in \{(\alpha, \beta)\}\}\mid\). As a specific example, we shall consider codes over \(S_4\). We exhibit the weight enumerators and the gray maps in the following example. Let \(C\) be the code generated by \((1 + i, 1 + i)\) and \((0, 2 + 2i)\). \(|C| = 8^1 2^1 = 16\). The image under \(\bar{\Phi}\) is the Klemm code \(K_4\) and \(G(K_4)\) is the Hamming code. The weight enumerators are given by

\[
W_C(X) = x^2_0 + x^2_1 x^2_i + x^2_2 x^2_{2i} + x^2_3 x^2_{3i} + x^2_{2+2i} + x^2_{1+2i} + x^2_{2+3i} + 2x_1 x_2 + 2x_1 x_3 + 2x_2 x_3 + 2x_0 x_{2+2i}
\]

\[
W_{\bar{\Phi}}(C)(x_0, x_1, x_2, x_3) = x^8 + x^4_1 + x^4_2 + x^6_3 + 6x^6_0 x^2_2 + 6x^6_1 x^2_3
\]

\[
W_G(C)(x, y) = x^8 + 14x^4 y^4 + y^8
\]

\[
W_{\bar{\Phi}}(C)(x_0, x_1, x_u, x_1 + u) = x^8_0 + x^4_1 + x^4_u + x^4_{1+u} + 6x^6_0 x^2_2 + 6x^6_1 x^2_3
\]

where \(G^*\) is the map given in Table 1.

The units in \(S_4\) are \(\{1, 3, 2 + i, 2 + 3i, i, 3i, 3 + 2i, 1 + 2i\}\). The non-units fall into the following equivalence classes under this relation: \(|0| = \{0\}, |2| = \{2, 2i\}, |1 + i| = \{1 + i, 3 + 3i, 3 + i, 1 + 3i\}, \) and \(|2 + 2i| = \{2 + 2i\}\). It follows immediately that if \(a \sim b\), then \(N(a) = N(b)\).

Then we can say that any code over \(\mathbb{Z}_4 + \mathbb{Z}_4i\) is permutation equivalent to a code with generator matrix of the form

\[
\begin{pmatrix}
I_{k_1} & A_{1.2} & A_{1.3} & A_{1.4} & A_{1.5} \\
(1 + i)I_{k_2} & A_{2.3} & A_{2.4} & A_{2.5} \\
0 & 0 & 2I_{k_3} & A_{3.4} & A_{3.5} \\
0 & 0 & 0 & (2 + 2i)I_{k_4} & A_{4.5}
\end{pmatrix}
\]

A code of this form is said to be of type \(\{k_1, k_2, k_3, k_4\}\) and \(|C| = 16^{k_1} 8^{k_2} 4^{k_3} 2^{k_4}\). This follows since \(|S| = 16, |(1 + i)S| = 8, |2S| = 4\) and \(|(2 + 2i)S| = 2\). The rank of a code \(C\) of type \(\{k_1, k_2, k_3, k_4\}\) as a module is \(k_1 + k_2 + k_3 + k_4\).

Next we shall explicitly show the structure of \(S^2_{2m}/\sim\). Each element of \(S^2_4\) is equivalent to \((\alpha, \beta)\), where \(\alpha\) is in \(\{0, 1, 2, 1 + i, \) or \(2 + 2i\}\). So we need to consider how many different classes there can be with an \(\alpha\) in the first coordinate as a representative. Specifically we need to know if there a unit \(m\) such that \(m \alpha = \alpha'\) and \(m \beta = \beta'\).

The first possibility has a 1 in the first coordinate and so there are 16 possibilities for the second coordinate each representing a different class. The second possibility has a 0 in the first coordinate, then the second coordinate is represented by the five equivalence classes of \(S/\sim\). The third possibility has a 2 in the first coordinate. If \(2m = 2\) then \(m = a + bi\) with
$a = 1, 3$ and $b = 0, 2$. There are eight possibilities for the second coordinate corresponding to the following classes: \{1, 3 + 2i, 1 + 2i\}, \{2 + i, 2 + 3i, 3i, i\}, \{0\}, \{2\}, \{2 + 2i\}, \{2i\}, \{3 + i, 1 + 3i\}, and \{1 + i, 3 + 3i\}. The fourth case has $2 + 2i$ as the fourth possibility. All units $m$ have $(2 + 2i)m = 2 + 2i$. So the five equivalence classes of $S/\sim$ are the possibilities for the second coordinate. The fifth possibility has $1 + i$ in the first coordinate. If $m + im = 1 + i$ then $m = 1 + b + bi$ with $b = 0, 2$. The classes for the second coordinate have each non-unit as a class and the classes \{1, 3 + 2i\}, \{3, 1 + 2i\}, \{2 + i, 3i\}, \{2 + 3i, i\}. Hence there are 12 classes. Hence $|\Omega| = |S^2/\sim| = 46$. As a summary we state the following remark.

**Remark 1.**
1. The symmetrized joint weight enumerator over $S_4$ is a homogeneous polynomial in 46 variables.
2. The symmetrized joint weight enumerator of codes $C$ and $D$ is

$$SJ_{C,D}(X | X = (x_{|\alpha|}), [\alpha] \in S^2/\sim) = J_{C,D}(X_{\alpha} | \alpha \in S^2, x_{\alpha} \text{ is identified with } x_{\beta} \text{ if and only if } \alpha \sim \beta).$$

### 3.1. Chinese remainder theorem

Set $I_m = m\mathbb{Z}[i]$ and $S_m = \mathbb{Z}[i]/I_m$. Assume $m_1, m_2, \ldots, m_r$ are pairwise relatively prime. Let $R = \mathbb{Z}[i]/m_1m_2 \ldots m_r\mathbb{Z}[i]$. Define the map

$$\Psi : R \rightarrow (S_{m_1}) \times (S_{m_2}) \times \cdots \times (S_{m_r})$$

by

$$\Psi(\alpha) = (\alpha \text{ (mod } I_{m_1}), \alpha \text{ (mod } I_{m_2}), \ldots, \alpha \text{ (mod } I_{m_r})).$$

The map $\Psi^{-1}$ is a ring isomorphism by the generalized Chinese remainder theorem.

Let $C_1, C_2, \ldots, C_r$ be codes of length $n$ where $C_j$ is a code over $S_{m_j}$ and define the code

$$\text{CRT}(C_1, C_2, \ldots, C_r) = \{ \Psi^{-1}(v_1, v_2, \ldots, v_k) | v_j \in C_j \}.$$  

We say that the code $\text{CRT}(C_1, C_2, \ldots, C_k)$ is the Chinese product of codes $C_1, C_2, \ldots, C_k$. It is clear that $|\text{CRT}(C_1, C_2, \ldots, C_k)| = \prod_{j=1}^k |C_j|$ and that if $C_j$ is self-orthogonal for all $j$ then CRT$(C_1, C_2, \ldots, C_k)$ is self-orthogonal. This gives the following.

**Theorem 3.1.** CRT$(C_1, C_2, \ldots, C_k)$ is a self-dual code over $R$ if and only if it is the Chinese product of self-dual codes $C_1, \ldots, C_k$ over $S_1, \ldots, S_k$, respectively.

### 3.2. MacWilliams relations

Our next task is describe the MacWilliams relations for the various weight enumerators defined above. The matrices which describe these relations are important because the weight enumerators of self-dual codes will be held invariant by these matrices, with a suitable constant multiplier.

Let $T_{2m}$ be a $(2m)^2$ by $(2m)^2$ matrix indexed by the elements of $S_{2m}$, with

$$(T_{2m})_{a+bi,c+di} = \xi_{2m}^{ac+bd}, \quad \xi_{2m} = e^{2\pi i/2m}. \quad (13)$$
Theorem 3.2. If C is a code over $S_{2m}$, then
\[
\text{cwe}_{C \perp}(X) = \frac{1}{|C|} \text{cwe}_{C}(T_{2m} \cdot X).
\] (14)

Proof. The matrix form is given by
\[
\chi_{a+bi}(c + di) = \chi_{1}((a + bi)(c - di)) = \chi_{1}(ac + bd + (bc - ad)i)
\]
and $\chi_{1}(a + bi) = \zeta_{2m}^{a}$ is the character corresponding to 1 in the character group. The MacWilliams relations follows from [3]. □

The results in [3] give the following corollary.

Corollary 3.3. Let C and D be codes over $S_{2m}$, then
\[
\text{J}_{C \perp, D}(X) = \frac{1}{|C|} \text{J}_{C, D}(T_{2m} \otimes T_{2m} \cdot X)
\] (15)
\[
\text{J}_{C \perp, D}(X) = \frac{1}{|D|} \text{J}_{C, D}(T_{2m} \otimes I \cdot X)
\] (16)
\[
\text{J}_{C, D \perp}(X) = \frac{1}{|D|} \text{J}_{C, D}(I \otimes T_{2m} \cdot X).
\] (17)

Let $T'_{2m}$ be a $|S_{2m}|$ by $|S_{2m}|$ matrix indexed by the elements of $S_{2m}$. Set $(T'_{2m})_{c,d} = \sum_{x \sim d}(T_{2m})_{c,x}$. Then we have the following corollary.

Corollary 3.4. Let C be a code over $S_{2m}$, then
\[
\text{swe}_{C \perp}(X) = \frac{1}{|C|} \text{swe}_{C}(T'_{2m} \cdot X).
\] (18)

Proof. Follows from Theorem 3.2 and the explanation of symmetrized weight enumerators given in [11]. □

For $m = 2$ the matrix $T'$ is indexed by [0], [1], [2], [1 + i], [2 + 2i] and is
\[
T' = \begin{bmatrix}
1 & 8 & 2 & 4 & 1 \\
1 & 0 & 0 & 0 & -1 \\
1 & 0 & 2 & -4 & 1 \\
1 & -2 & 0 & 1 \\
1 & -8 & 2 & 4 & 1
\end{bmatrix}
\] (19)

Let $T'_{1}$, $T'_{2}$, and $T'_{3}$ be $|S_{2m}/\sim|$ by $|S_{2m}/\sim|$ matrices, indexed with elements from $S_{2m}^2/\sim$. Set
\[
(T'_{1})_{(a,b),(c,d)} = \sum_{(x,y) \sim (c,d)} (T_{2m} \otimes T_{2m})_{(a,b),(x,y)}
\] (20)
\[
(T'_{2})_{(a,b),(c,d)} = \sum_{(x,y) \sim (c,d)} (T_{2m} \otimes I_{2m})_{(a,b),(x,y)}
\] (21)
and
\[
(T'_3(a,b),(c,d)) = \sum_{(x,y) \sim (c,d)} (I_{2m} \otimes T_{2m})(a,b),(x,y).
\] (22)

**Corollary 3.5.** Let \( C \) and \( D \) be codes over \( S_{2m} \), then the symmetrized joint weight enumerator \( SJ_{C,D} \) satisfies
\[
SJ_{C^+,D^+}(X) = \frac{1}{|C|} \frac{1}{|D|} SJ_{C,D}(T'_1 \cdot X)
\]
\[
SJ_{C^+,D^+}(X) = \frac{1}{|C|} SJ_{C,D}(T'_2 \cdot X)
\]
\[
SJ_{C,D^+}(X) = \frac{1}{|D|} SJ_{C,D}(T'_3 \cdot X).
\] (23-25)

Notice that \( T'_1 \) is not \( T'_{2m} \otimes T'_{2m} \). However, this latter matrix is related to the MacWilliams relations for a different symmetrized joint weight enumerator.

We shall examine the group of invariants for Type I and Type II codes. The complete weight enumerator of a self-dual code over \( S_{2m} \) is held invariant by the action of the matrix \((1/2m)T_{2m}\) since it is self-dual.

In addition since every monomial represents self-orthogonal vectors the complete weight enumerator of a self-dual code is held invariant by the matrix \( P_S \)
\[
P_{\alpha,\beta} = \begin{cases} 
\zeta_{2m}^{N(\alpha)} & \text{if } \alpha = \beta \\
0 & \text{if } \alpha \neq \beta
\end{cases}
\] (26)
where \( \alpha, \beta \) run over the elements of \( S_{2m} \) and \( N(\alpha) \) is the norm of \( \alpha \).

Moreover the code is linear, so multiplication of a vector by a unit permutes the vectors of the codes (meaning it sends each vector to a uniquely defined vector). Hence the complete weight enumerator is held invariant by the matrix \( U \) for \( \alpha \) a unit, where \( U \) is the matrix associated with the permutation on the \( x_j \) given by \( \sigma(x_j) = x_{\alpha j} \). Let \( \mathcal{U} = \{U_{\alpha} \mid \alpha \text{ a unit}\} \).

Let \( G_S^I = \{(1/2m)T_{2m}, P_S, U\} \). Then \( G_S^I \) is the group of invariants for the complete weight enumerator of a self-dual code over \( S \).

If a code is Type II then the complete weight enumerator is also held invariant by the matrix \( iI_{2m} \) since the length must be a multiple of 4.

Additionally, since each monomial represents a doubly even vector the complete weight enumerator is held invariant by the following matrix \( D_S \):
\[
(D_S)_{\alpha,\beta} = \begin{cases} 
\eta_{4m}^{N(\alpha)} & \text{if } \alpha = \beta \\
0 & \text{if } \alpha \neq \beta
\end{cases}
\] (27)
where \( \alpha, \beta \) run over the elements of \( S \) and \( \eta_{4m} \) is a primitive \( 4m \)th root of unity.

Let \( G_S^{II} = \{(1/2m)T_{2m}, D_S, iI_{2m}, \mathcal{U}\} \). Then \( G_S^{II} \) is the group of invariants for the complete weight enumerator of a Type II code over \( S_{2m} \).

### 4. Complex unimodular lattices

In this section we shall describe a bridge between self-dual codes over \( S_{2m} \) and complex unimodular lattices. The Gaussian integers \( \mathbb{Z}[i] \) are denoted by \( \mathcal{O} \). A lattice in \( \mathbb{C}^n \) is a free
\(\mathcal{O}\)-module. The standard inner product is attached to \(\mathbb{C}^n\):

\[
v \cdot u = \sum v_i u_i
\]

where \(a + bi = a - bi\).

We define \(L^* = \{u \in \mathbb{C}^n \mid u \cdot v \in \mathcal{O} \text{ for all } v \in L\}\). If a lattice has \(L \subseteq L^*\) then we say it is integral and if a lattice has \(L = L^*\) then we say it is unimodular. The norm of a vector \(v\) is given by \(N(v) = v \cdot v\). If the norm of every vector in a unimodular lattice is even then we say it is an even lattice.

We denote the reduction map modulo \(2m\) by

\[
\tilde{h} : \mathcal{O}^n \to S_{2m}^n.
\]

It is a group homomorphism and it can be seen that \(\tilde{h}^{-1}(C)\), the preimage of a code \(C\) defined over \(S_{2m}\), is a free \(\mathcal{O}\)-module. The lattice induced from a code \(C\) is defined as follows:

\[
\Lambda(C) := \left\{ \frac{1}{\sqrt{2m}} \tilde{h}^{-1}(C) = \left\{ \frac{1}{\sqrt{2m}} v \in \mathcal{O}^n \mid v \pmod{2m\mathcal{O}} \in C \right\} \right\}.
\]

**Theorem 4.1.** If \(C\) is a self-dual code over \(S_{2m}\), then \(\Lambda(C)\) is a complex unimodular lattice. Moreover, if \(C\) is Type II, then \(\Lambda(C)\) is an even lattice. The minimum norm of the lattice is \(\min \{ \frac{1}{\sqrt{2m}}, 2m \} \).

**Proof.** Assume \(v = (v_j) = (a_j + b_j i)\) and \(v' = (v'_j) = (c_j + d_j i)\) are vectors in \(C\), then \(\sum v_j v'_j = 0\). Consider a single coordinate:

\[
(a_j + b_j i)(c_j + d_j i) = (a_j + b_j i)(c_j - d_j i) = (a_j c_j + b_j d_j) + (b_j c_j - a_j d_j)i.
\]

Now \((1/\sqrt{2m})\tilde{h}^{-1}(v) = (1/\sqrt{2m})(a_j + b_j i + 2m\mathcal{O})\) and \((1/\sqrt{2m})\tilde{h}^{-1}(v') = (1/\sqrt{2m})(c_j + d_j i + 2m\mathcal{O})\). We shall show that a vector \(z\) in \((1/\sqrt{2m})\tilde{h}^{-1}(v)\) has integral inner product with any vector \(z'\) in \((1/\sqrt{2m})\tilde{h}^{-1}(v')\), i.e. \(\sum z_j \cdot z'_j \in \mathbb{Z}\). Consider a single coordinate:

\[
(a_j + b_j i) \cdot (c_j + d_j i) = (a_j c_j + b_j d_j) + (b_j c_j - a_j d_j)i.
\]

From the above computation we have that \(\sum (a_j c_j + b_j d_j) \equiv 0 \pmod{2m}\) and that \(\sum (b_j c_j - a_j d_j) \equiv 0 \pmod{2m}\), so we have that \(\Lambda(C) = (1/\sqrt{2m})\tilde{h}^{-1}(C)\) is integral. The norm of \(a + bi\) is congruent to its Euclidean weight \((\text{mod } 4m)\). This together with the above computation shows that the image of a Type II code is an even lattice. □

It is clear from this theorem why it is important to study codes over \(S_{2m}\) for all \(m\), since for a particular \(m\) the highest attainable minimum norm is \(2m\).

We shall describe the situation completely for the ring \(S_4\).

To complete the definitions of the maps we define \(\mathcal{G}^* : S_4 \to (\mathbb{Z}_2 + u\mathbb{Z}_2)^2\) by the relation given in Table 1 and \(\mathcal{G}^{**} : \mathcal{O}^n \to \mathcal{O}^{2n}\) so that Diagram 1 commutes.
Example. Take the $S_4$ code $C = \{0, 2, 2 + 2i\}$. Then $\tilde{\Psi}(C) = \{00, 02, 20, 22\}$, $h^{-1}(C) = 2O$, $G(h\tilde{\Psi}(C)) = \{0000, 0011, 1100, 1111\}$, $G^*(C) = \{00, 0u, u0, uu\}$, and $h^{-1}(G^*(C)) = (\sqrt{2}O)^2$.

5. Shadows

In this section we shall develop a theory of shadows for complex unimodular lattices and for codes over $S_{2m}$. We describe the orthogonality relations between the various cosets and use them to produce shadow sum constructions.

5.1. Lattices

Let $\Lambda$ be a unimodular lattice, and let

$$A_0 = \{ v \mid v \in \Lambda, N(v) \in 2\mathbb{Z} \}.$$ 

An even vector $v$ is a vector such that $v \cdot v \in 2\mathbb{Z}$. If $v$ and $w$ are both even vectors in $\Lambda$ then $(v + w) \cdot (v + w) = v \cdot v + w \cdot w + 2(v \cdot w)$ and hence even. Also if $v \in \mathcal{O}$ then $N(\alpha v) = \alpha^2 N(v)$ which is even. Hence $A_0$ is linear. If $\Lambda$ is even then $\Lambda = A_0$. If $\Lambda$ is odd, then $A_0$ is index 2 in $\Lambda$ and let $A^* = A_0 \cup A_1 \cup A_2 \cup A_3$ with $A_2 = \Lambda - A_0$. Set $\Sigma = A^*_0 - A$ with $\Sigma = A_1 \cup A_3$.

Let $\Psi : \mathbb{C}^n \rightarrow \mathbb{R}^{2n}$ by

$$\Psi(v_1, v_2, \ldots, v_n) = (\text{Re}(v_1), \text{Im}(v_1), \text{Re}(v_2), \text{Im}(v_2), \ldots, \text{Re}(v_n), \text{Im}(v_n)).$$

If $\Lambda$ is unimodular then $\Psi(\Lambda)$ is a real unimodular lattice. Moreover, if $N(v) \in 2\mathbb{Z}$ then $N(\Psi(v)) \in 2\mathbb{Z}$.

**Theorem 5.1.** Let $\Lambda$ be a complex unimodular lattice, then $\Psi(A_j) = (\Psi(A_j))$ for $j = 0, 2$ and $\Psi(A_j) = (\Psi(A_j))$ up to labeling for $j = 1, 3$.

**Corollary 5.2.** If $\Lambda$ is a complex unimodular lattice then the norms of the vectors in $\Sigma$ are $\frac{n}{2}$ (mod 2).

**Proof.** The image of the shadow of a complex unimodular lattice is the shadow of the image. The vectors in the image of the shadow are of length $2n$ and therefore have norm $(2n/4)$ (see [5]). Since $N(v) = N(\tilde{\Psi}(v))$, we have the result. □

**Corollary 5.3.** The glue group of $A^*_0/\Lambda_0$ is isomorphic to the Klein 4 group for all $n$. 
Table 2
Lattice orthogonal relations for odd \( n \)

<table>
<thead>
<tr>
<th></th>
<th>( \Lambda_0 )</th>
<th>( \Lambda_1 )</th>
<th>( \Lambda_2 )</th>
<th>( \Lambda_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Lambda_0 )</td>
<td>( \perp )</td>
<td>( \perp )</td>
<td>( \perp )</td>
<td>( \perp )</td>
</tr>
<tr>
<td>( \Lambda_1 )</td>
<td>( \perp )</td>
<td>( \not\perp )</td>
<td>( \not\perp )</td>
<td>( \perp )</td>
</tr>
<tr>
<td>( \Lambda_2 )</td>
<td>( \perp )</td>
<td>( \not\perp )</td>
<td>( \not\perp )</td>
<td>( \perp )</td>
</tr>
<tr>
<td>( \Lambda_3 )</td>
<td>( \perp )</td>
<td>( \not\perp )</td>
<td>( \not\perp )</td>
<td>( \perp )</td>
</tr>
</tbody>
</table>

Table 3
Lattice orthogonal relations for even \( n \)

<table>
<thead>
<tr>
<th></th>
<th>( \Lambda_0 )</th>
<th>( \Lambda_1 )</th>
<th>( \Lambda_2 )</th>
<th>( \Lambda_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Lambda_0 )</td>
<td>( \perp )</td>
<td>( \perp )</td>
<td>( \perp )</td>
<td>( \perp )</td>
</tr>
<tr>
<td>( \Lambda_1 )</td>
<td>( \perp )</td>
<td>( \not\perp )</td>
<td>( \not\perp )</td>
<td>( \perp )</td>
</tr>
<tr>
<td>( \Lambda_2 )</td>
<td>( \perp )</td>
<td>( \not\perp )</td>
<td>( \not\perp )</td>
<td>( \perp )</td>
</tr>
<tr>
<td>( \Lambda_3 )</td>
<td>( \perp )</td>
<td>( \not\perp )</td>
<td>( \not\perp )</td>
<td>( \perp )</td>
</tr>
</tbody>
</table>

**Proof.** For any vector \( v \) in \( \Sigma \), \( N(v + v) = N(2v) = 4N(v) \equiv 2n \pmod{2} \) and so \( v + v \in \Lambda_0 \). □

There exist vectors \( t \) and \( s \) such that \( \Lambda = \langle \Lambda_0, t \rangle \), \( \Lambda_1 = \langle \Lambda_0, s \rangle \), \( \Lambda_3 = \langle \Lambda_0, s + t \rangle \), and \( \Sigma = \langle \Lambda + s \rangle \). Hence the orthogonal relations are easily determined (see Tables 2 and 3), since \( s \) and \( t \) are not orthogonal (by design) and \( s \cdot t = \frac{n}{4} \pmod{2} \).

5.2. Codes

Let \( C \) be a Type I code over \( S_{2m} \) of length \( n \). Let

\[
C_0 = \{ v \mid v \in C, \text{Euc}(v) \equiv 0 \pmod{4m} \}. \tag{32}
\]

Note that \( \tilde{\Psi}(C_0) = \tilde{\Psi}(C)_0 \) by definition, giving that \( C_0 \) is index 2 in \( C \), and we define \( S := C_0^\perp - C \), with \( S = C_1 \cup C_3 \) and \( C = C_0 \cup C_2 \). It is clear that \( \tilde{\Psi}(C_j) = \tilde{\Psi}(C) \) for \( j = 0, 2 \) and for \( j = 1, 3 \) up to labeling.

**Lemma 5.4.** Let \( C \) be a Type I code over \( S_{2m} \) and \( C_0 \) described as above, then

\[
cw_{\text{e}}(X) = \frac{1}{2}(\text{cw}_{\text{e}}(X) + \text{cw}_{\text{e}}(X')) \tag{33}
\]

where \( X = \langle x_0, x_1, \ldots, x_{2(2m-1)+(2m-1)i} \rangle \) and \( X' \) is formed by replacing \( x_{a+bi} \) with \( \zeta_{4m}^{\text{Euc}(a+bi)} x_{a+bi} \), where \( \zeta_{4m} \) is the complex 4\(^m\)th root of unity.

**Proof.** Vectors that have Euclidean weight congruent to 0 \( \pmod{4m} \) are counted positively in \( \text{cw}_{\text{e}}(X) \) and \( \text{cw}_{\text{e}}(X') \). Vectors that have Euclidean weight congruent to \( 2m \pmod{4m} \) are counted positively in \( \text{cw}_{\text{e}}(X) \) and negatively in \( \text{cw}_{\text{e}}(X') \). □

**Theorem 5.5.** Let \( C \) be a Type I code over \( S_{2m} \) and \( S \) its shadow. Then

\[
cw_{\text{e}}(X) = \frac{1}{|C|} \text{cw}(T \cdot X') \tag{34}
\]
Table 4
Orthogonality relations for \( n \) even

<table>
<thead>
<tr>
<th></th>
<th>( C_0 )</th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_0 )</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
</tr>
<tr>
<td>( C_1 )</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
</tr>
</tbody>
</table>

Table 5
Orthogonality relations for \( n \) odd

<table>
<thead>
<tr>
<th></th>
<th>( C_0 )</th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_0 )</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
</tr>
<tr>
<td>( C_1 )</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
<td>⊥</td>
</tr>
</tbody>
</table>

Proof. Let \( C_0 \) be the subcode described above, then we have

\[
\text{cwe}_S(X) = \text{cwe}_{C_0}(X) - \text{cwe}_C(X) = \frac{1}{|C_0|}\left(\frac{1}{2}(\text{cwe}_C(T \cdot X) + \text{cwe}_C(T \cdot X'))\right) - \text{cwe}_C(X) = \frac{1}{|C|}\text{cwe}_C(T \cdot X) - \text{cwe}_C(X) + \frac{1}{|C|}\text{cwe}_C(T \cdot X') = \frac{1}{|C|}\text{cwe}_C(T \cdot X').\]

As an example consider the code given in Eq. (4) then \( C_0 = \{0, 2 + 2i\}, C_2 = \{2, 2i\}, C_1 = C_0 + (1 + i) = \{1 + i, 3 + 3i\} \) and \( C_3 = C_2 + (1 + i) = \{3 + i, 1 + 3i\}. \)

Theorem 5.6. Let \( C \) be a Type I code over \( S_{2m} \). Then \( \tilde{h}(C_j) = \tilde{h}(C)_j \) for \( j = 0, 2 \) and for \( j = 1, 3 \) up to labeling.

Proof. Follows from a straightforward computation noticing that the norms match.

Theorem 5.7. Let \( C \) be a self-dual code over \( S_{2m} \) of length \( n \).

1. If \( n \) is even then Table 4 holds, where the symbol \( \perp \) in position \( (i, j) \) means that \( [x, y] \equiv 0 \pmod{2m} \) for any vector \( x \in C_i \) and any vector \( y \in C_j \), and the symbol \( \not\perp \) means that \( x \cdot y \equiv \not0 \pmod{2m} \) for any vector \( x \in C_i \) and any vector \( y \in C_j \).

2. If \( n \) is odd then Table 5 holds where the symbol \( \perp \) in position \( (i, j) \) means that \( x \cdot y \equiv 0 \pmod{2m} \) for any vector \( x \in C_i \) and any vector \( y \in C_j \), and the symbol \( \not\perp \) means that \( x \cdot y \equiv \not0 \pmod{2m} \) for any vector \( x \in C_i \) and any vector \( y \in C_j \).

5.3. Shadow sums

We shall show how the theory of shadow sums applies to complex lattices and self-dual codes over the rings \( S_{2m} \).
Theorem 5.8. Let $\Lambda$ and $\Lambda'$ be Type I lattices in dimensions $n$ and $k$, respectively with the four cosets $\Lambda_j$ and $\Lambda'_j$ ($j = 0, 1, 2, 3$). Let

\begin{align*}
M &= (\Lambda_0, \Lambda'_0) \cup (\Lambda_1, \Lambda'_1) \cup (\Lambda_2, \Lambda'_2) \cup (\Lambda_3, \Lambda'_3), \\
M' &= (\Lambda_0, \Lambda'_0) \cup (\Lambda_1, \Lambda'_3) \cup (\Lambda_2, \Lambda'_2) \cup (\Lambda_3, \Lambda'_1),
\end{align*}

where $(\Lambda_j, \Lambda'_j) = \{(l, l') \mid l \in \Lambda_j, l' \in \Lambda'_j\}$.

(1) Suppose that $n \equiv k \equiv 0 \pmod{2}$. Then $M$ and $M'$ are unimodular lattices in dimensions $n + k$. Moreover $M$ and $M'$ are Type II if and only if $n + k \equiv 0 \pmod{4}$.

(2) Suppose that $n \equiv k \equiv 1 \pmod{2}$. Then $M$ and $M'$ are unimodular lattices in dimension $n + k$. Moreover $M$ and $M'$ are Type II if and only if $n + k \equiv 0 \pmod{8}$.

Proof. Similar to the proofs in [6]. □

Similarly we have the following.

Theorem 5.9. Let $C$ and $D$ be Type I, $\mathbb{Z}_{2m}$-codes of lengths $n$ and $k$, respectively. Let

\begin{align*}
E &= (C_0, D_0) \cup (C_1, D_1) \cup (C_2, D_2) \cup (C_3, D_3), \\
F &= (C_0, D_0) \cup (C_1, D_3) \cup (C_2, D_2) \cup (C_3, D_1),
\end{align*}

where $(C_j, D_j) = \{(c, d) \mid c \in C_j, d \in D_j\}$.

(1) Suppose that $n \equiv k \equiv 0 \pmod{2}$. Then $E$ and $F$ are self-dual codes of length $n + k$. Moreover $E$ and $F$ are Type II if and only if $n + k \equiv 0 \pmod{4}$. $G$ is a Type II code if and only if $n \equiv k \equiv 2 \pmod{4}$.

(2) Suppose that $n \equiv k \equiv 1 \pmod{2}$. Then $E$ and $F$ are self-dual codes of length $n + k$. Moreover $E$ and $F$ are Type II if and only if $n + k \equiv 0 \pmod{8}$.

6. Construction of Hermitian Jacobi forms

There has been extensive research connecting invariant theory and coding theory. Specifically, the complete weight enumerator of codes, seen as an invariant polynomial under a certain finite group, is used to construct various modular forms using special types of theta-function [2]. In this section we extend this idea by studying the connection between weight enumerators of codes over $S_{2m}$ and Hermitian Jacobi forms. More precisely, the Jacobi Hermitian theta-series formed from the complete weight enumerators of the codes over $S_{2m}$ is a Hermitian Jacobi form over the Gaussian ring $O$.

Also Hermitian modular forms of higher genus are derived from the joint weight enumerators of codes over $S_{2m}$.

6.1. Hermitian Jacobi form

We recall the definition of Hermitian Jacobi forms and theta-functions. We follow the definition given in [7].
Let
\[ H_g := \left\{ \tau \in M_{2g \times 2g}(\mathbb{C}) \left| \frac{\tau - \bar{\tau}}{2i} > 0 \right. \right\}. \]
Here \( \bar{\tau} = t^T \bar{\tau} \). The Hermitian symplectic group
\[ SP_g(\mathbb{C}) := \{ M \in M_{2g \times 2g}(\mathbb{C}) \mid JM = J \}, \]
acts on \( H_g \) in the usual way. The Hermitian modular group of genus \( g \) associated with \( O \) is defined by
\[ \Gamma_g(O) := SP_g(\mathbb{C}) \cap M_{2g \times 2g}(O). \]

**Definition 1.** A holomorphic function \( f : H_1 \times \mathbb{C}^2 \to \mathbb{C} \) is said to be a Hermitian Jacobi form of weight \( k \) and index \( m \) with respect to \( O \) if it satisfies
1. \( (f |_{k,m} M)(\tau, z_1, z_2) := (c\tau + d)^{-k} e^{-2\pi im(z_1 + z_2)} f \left( M\tau, \frac{z_1}{c\tau + d}, \frac{z_2}{c\tau + d} \right) = f(\tau, z_1, z_2), \forall M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_1(O), \)
2. \( (f |_{m} [\lambda, \mu])(\tau, z_1, z_2) := e^{2\pi im(\lambda\tau + \bar{\lambda}z_1 + \lambda z_2)} f(\tau, z_1 + \lambda\tau + \mu, z_2 + \bar{\lambda}\tau + \bar{\mu}). \)

It has the following Fourier expansion:
3. \( f(\tau, z_1, z_2) = \sum_{n=0}^{\infty} \sum_{t \in O, N(t) \leq 4mn} c(n, t) e^{2\pi i(nt + iz_1 + iz_2)} \).

**Remark 2.**
1. It is known (see page 55 in [9]) that \( \Gamma_g(O) \) is generated by the matrices
\[ \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}, \begin{pmatrix} 1_g & \alpha \\ 0 & 1_g \end{pmatrix}, \forall \alpha = \overline{\alpha} \in \text{Sym}(g, O), \]
\[ \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}, \forall \beta \in \text{GL}(g, O). \]
2. To check modularity it is enough to check the transformation laws of \( f \) for only the above generators.

The \( \mathbb{C} \)-vector space of Jacobi forms of weight \( k \) and index \( m \) is denoted by \( J_{k,m}(\Gamma_1(O)). \)

6.2. **Theta-series**

The following theta-function was first introduced and studied in [7, 8] to show the correspondence between the space of Hermitian Jacobi forms and that of the vector valued...
Hermitian modular forms. For each $\mu \in \mathbb{S}_{2m}$, let

$$
\theta_{m, \mu}(\tau, z_1, z_2) := \sum_{r \in \mathcal{O}, r \equiv \mu \pmod{2m\mathcal{O}}} q^{|\frac{N(r)}{4m}| \xi_1^2 \xi_2^2},
$$

$q = e^{2\pi i \tau}, \xi_1 = e^{2\pi i z_1}, \xi_2 = e^{2\pi i z_2}.

(35)$

Then, by the Poisson summation formula, the theta-series satisfies the following transformation formula [8].

**Lemma 6.1.**

1. $(\theta_{m, \mu} \mid_{1,m} \begin{pmatrix} a & b \\ c & d \end{pmatrix})(\tau, z_1, z_2) = e^{2\pi i \frac{N(b\mu)}{4m}} \theta_{m, \mu}(\tau, z_1, z_2), \forall \alpha = \frac{a}{c} \in \mathcal{O}$

2. $(\theta_{m, \mu} \mid_{1,m} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})(\tau, z_1, z_2) = e^{\frac{\pi i}{2m}} \sum_{v \in \mathcal{O}/2m\mathcal{O}} e^{2\pi i \frac{\text{Re}(\mu)v}{2m}} \theta_{m, v}(\tau, z_1, z_2)$.

**Proof.** The standard tool using the Poisson summation formula gives the result which was stated in [7, 8].

**6.3. Complete weight enumerators of the codes and Hermitian Jacobi forms**

In this section, we show that a certain theta-series defined over the lattices induced from codes over $\mathbb{F}_{2m}$ is a Hermitian Jacobi form.

For each $Y$ in the lattice, consider the theta-series $\Theta_{A,Y} : \mathcal{H}_1 \times \mathbb{C}^2 \to \mathbb{C}$ associated with a lattice $A$:

$$
\Theta_{A,Y}(\tau, z_1, z_2) := \sum_{x \in A} e^{2\pi i \frac{\text{Re}(x) \tau + \text{Im}(x) (z_1 + i z_2)}{2m}}.
$$

(36)

The following theorem gives a connection between a theta-series defined over the lattices induced from codes and their complete weight enumerators.

**Theorem 6.2.** Let $C$ be a code over $\mathbb{F}_{2m}$. Let $\Lambda(C)$ be a lattice induced from $C$ over $\mathbb{F}_{2m}$, i.e. $\Lambda(C) = (1/\sqrt{2m})\tilde{h}^{-1}(C)$. From the complete weight enumerator $\text{cw}_{\mathbb{C}}(x_0, \ldots, x_{(2m-1)+i(2m-1)})$, one constructs the following theta-series associated with $\Lambda(C)$:

$$
\Theta_{\Lambda(C)}(\tau, z_1, z_2) = \text{cw}_{\mathbb{C}}(\theta_{m, \mu}(\tau, z_1, z_2) \mid \mu \in \mathbb{S}_{2m}),
$$

(37)

where $\{\theta_{m, \mu}\}$ is given in Lemma 6.1.

**Proof.** Note that $\sqrt{2m} := (1/\sqrt{2m})(2m, \ldots, 2m, 2m) \in \Lambda(C)$. Let $v = (v_1, \ldots, v_n)$ be any given codeword in $C$ and, for each $\mu \in \mathbb{S}_{2m}$, $n_\mu(v) = |\{j \mid v_j = \mu\}|$. If we let $\tilde{h}(\tilde{v}) = v$, then the image can be arranged in the form $\{\tilde{h}^{-1}(v)\} = \{\tilde{h}^{-1}(\tilde{v}) + \tilde{v} \mid \tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_n)\} = (\tilde{v}_j = a_j + b_j \bar{w}, 0 \leq a_j, b_j < 2m)$ and the number of $\mu, (\mu \in \mathbb{S}_{2m}), \text{of } \tilde{v}_1, \ldots, \tilde{v}_n$ is exactly $n_\mu(v)$. Thus, for each $v \in C$,

$$
\sum_{x \in \sqrt{2m} \tilde{h}^{-1}(v)} e^{2\pi i \frac{\text{Re}(x) \tau + \text{Im}(x) (z_1 + i z_2)}{2m}} = \sum_{x \in \tilde{h}^{-1}(0)} e^{2\pi i \frac{\text{Re}(x) \tau + \text{Im}(x) (z_1 + i z_2)}{2m}}.
$$

(38)
Second, let 

\[
\begin{align*}
\sum_{x_1 \in 2mO+\bar{v}_1} \cdots \sum_{x_n \in 2mO+\bar{v}_n} e^{2\pi i \left( \frac{N(x_1)+N(x_2)+\cdots+N(x_n)}{2m} \right)} e^{2\pi i \left( \frac{N(x_1)+N(x_2)+\cdots+N(x_n)}{2m} \right)}
\quad = \prod_{\mu \in S_{2m}} \theta_{m,\mu}(\tau, z_1, z_2)^{\nu(\mu)}.
\end{align*}
\]

Therefore, we have \( \Theta_{\Lambda(C),2(1,\ldots,1)}(\tau, z_1, z_2) = \text{cwe}_C(\theta_{m,\mu}(\tau, z) \mid \mu \in S_{2m}) \) This finishes the proof. \( \square \)

**Theorem 6.3.** Let \( C \) be a Type II code of length \( n \) over \( S_{2m} \). Then

\[
\text{cwe}_C(\theta_{m,\mu}(\tau, z_1, z_2) \mid \mu \in S_{2m})
\]

is a Hermitian Jacobi form of weight \( n \) and index \( mn \).

**Proof.** For convenience, let \( g(\tau, z_1, z_2) := \text{cwe}_C(\theta_{m,\mu}(\tau, z_1, z_2) \mid \mu \in S_{2m}) \). To check the modularity of \( g(\tau, z_1, z_2) \) it is enough to check the transformation formula under two types of generator (see Remark 2) \( \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \) and \( \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \) of \( \Gamma_1(C), \alpha = \bar{\alpha} \in O \). Since \( \alpha \) is an integer in this case, it is enough to consider the matrix of the form \( \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \). First,

\[
\begin{align*}
g \mid_{n,mn} \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) (\tau, z_1, z_2) &= \text{cwe}_C(\theta_{m,\mu}(\tau + 1, z_1, z_2) \mid \mu \in S_{2m}) \\
&= \text{cwe}_C(\theta_{m,\mu}(\tau, z_1, z_2) \mid \mu \in S_{2m}) \\
&= g(\tau, z_1, z_2) \quad (\text{since all weights of codeword in } C \text{ are divisible by } 4m).
\end{align*}
\]

Second,

\[
\begin{align*}
\left( g \mid_{n,mn} \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \right) (\tau, z_1, z_2) &= \tau^{-n} e^{2\pi i \frac{mn+1}{2} \frac{\bar{z}_2}{\bar{z}}} \text{cwe}_C \left( \theta_{m,\mu} \left( -\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau} \right) \mid \mu \in S_{2m} \right) \\
&= \tau^{-n} e^{2\pi i \frac{mn+1}{2} \frac{\bar{z}_2}{\bar{z}}} \text{cwe}_C \left( \frac{i\tau}{2m} e^{2\pi i \frac{mn+1}{2} \frac{z_2}{\bar{z}}} T_{2m} \cdot (\theta_{m,\mu}(\tau, z_1, z_2) \mid \mu \in S_{2m}) \right) \\
&= \frac{1}{(2m)^n} \text{cwe}_C(T_{2m} \cdot (\theta_{m,\mu}(\tau, z_1, z_2) \mid \mu \in S_{2m})) \\
&= \text{cwe}_C(\theta_{m,\mu}(\tau, z_1, z_2) \mid \mu \in S_{2m}) = g(\tau, z_1, z_2) \quad (\text{from Theorem 3.2}).
\end{align*}
\]
Next, one needs to check the elliptic property, specifically

\[ e^{2\pi i mn(\lambda \tau + \bar{\lambda} + \lambda z_1 + \bar{\lambda} z_2)} g(\tau, z_1 + \lambda \tau + \mu, z_2 + \bar{\lambda} \tau + \bar{\mu}) = e^{2\pi i \sqrt{2m} \cdot (\lambda z_1 + \bar{\lambda} z_2)} e^{2\pi i \sum_{\lambda \in \Lambda(C)} (\lambda \tau + \mu)(\bar{\lambda} \tau + \bar{\mu})} \]

\[ = \sum_{\lambda \in \Lambda(C)} e^{2\pi i \sqrt{2m} \cdot (\lambda z_1 + \bar{\lambda} z_2)} e^{2\pi i \sum_{\lambda \in \Lambda(C)} (\lambda \tau + \mu)(\bar{\lambda} \tau + \bar{\mu})} = g(\tau, z_1, z_2) \quad \text{(by replacing } x \rightarrow (x + \sqrt{2m})). \]

The proper Fourier expansion can be checked easily and we omit the detailed proof. □

7. Construction of Hermitian modular form of genus \( g \)

In this section, we consider a higher genus Hermitian modular form and derive a connection between the joint weight enumerators of codes over \( S_{2m} \) and the symmetrized weight enumerators of codes as well.

7.1. Hermitian modular form of genus \( g \)

We define a Hermitian modular form of higher genus.

**Definition 2.** A holomorphic function \( F : \mathcal{H}_g \rightarrow \mathbb{C} \) is called a Hermitian modular form of weight \( k \) of genus \( g \) if

\[ F(M \tau) = \text{Det}(C \tau + C)^k F(\tau), \forall M = \left( \begin{array}{cc} * & * \\ C & D \end{array} \right) \in \Gamma_g, \]

with a proper holomorphic condition at each cusp in the case of \( g = 1 \).

Consider, for each \( \mu \in S_{2m}^g \), the theta-function \( \theta_{m,\mu}^{(g)} : \mathcal{H}_g \times \mathbb{C}^g \times \mathbb{C}^g \rightarrow \mathbb{C} \);

\[ \theta_{m,\mu}^{(g)}(\tau, z_1, z_2) = \sum_{r \in \mathbb{C}^g, r \equiv \mu \text{ (mod } (2m\mathbb{C})^g)} e^{2\pi i \frac{\tau \cdot r}{2} + \sum_{v \in S_{2m}^g} \text{Re} \left( \frac{\tau \cdot r}{2} \right)} \theta_{m,\mu}(\tau, z_1, z_2). \] (38)

Then the following can be derived.

**Lemma 7.1.** Let \( \theta_{m,\mu}^{(g)}(\tau, z_1, z_2) \) be the function defined in (38). Then it satisfies the following transformation formula for \( \xi_{2m} = e^{2\pi i} \):

\[ \theta_{m,\mu}^{(g)}(-\tau^{-1}, \bar{\tau}^{-1} z_1, \tau^{-1} z_2) = \left( \frac{i}{2m} \right)^g \text{Det}(\tau) e^{2m\pi i \xi_{2m} \cdot \tau^{-1} z_2} \times \sum_{\nu \in S_{2m}^g} \xi_{2m}^\nu \theta_{m,\nu}(\tau, z_1, z_2). \]
7.2. Joint weight enumerators and Hermitian modular of genus g

Proof. This can be derived using the Poisson summation formula and we omit the detailed proof (see also [8]). □

Theorem 7.2. Let $C_j, 1 \leq j \leq g$, be a code of length $n$ over $S_{2m}$ and $A_j$ be an induced lattice from the code $C_j$, i.e., $A_j = (1/\sqrt{2m}) \widehat{h}^{-1}(C_j)$. Let $J_{C_1, C_2, \ldots, C_g}(X)$ be the complete joint weight enumerator of the codes $C_j$, $1 \leq j \leq g$. Then the following holds for $Y = \sqrt{2m} : (1/\sqrt{2m})(2m, 2m, \ldots, 2m)$:

$$
\Theta_{A_1, A_2, \ldots, A_g}(\tau, z_1, z_2) = J_{C_1, C_2, \ldots, C_g}(\theta^{(e)}_{m, \mu}(\tau, z_1, z_2) \mid \mu \in S^g_{2m}).
$$

Proof. First note that $Y = (1/\sqrt{2m})(2m, 2m, \ldots, 2m)^t \in A_1 \cap A_2 \cap \cdots \cap A_g$. Let $\tilde{h} : \mathcal{O}^n \times \cdots \times \mathcal{O}^n \to \mathcal{S}_{2m}^n \times \cdots \times \mathcal{S}_{2m}^n$ be a homomorphism induced from $h$ in (29). For each $v \in C_1 \times C_2 \times \cdots \times C_g$, let $\tilde{h}^{-1}(v) = \tilde{h}^{-1}(0) + (\tilde{v}_i)$ be a preimage of $v$, all of whose entries $((\tilde{v}_i)) = (a_{ij} + b_{ij} w)$ are the forms such that $0 \leq a_{ij}, b_{ij} < 2m$. Then

$$
\sum_{x \in \tilde{h}^{-1}(v)} \exp \left( 2\pi i \left( \frac{\text{Tr}(\tilde{x} \cdot \tau \cdot x)}{4m} + \frac{(x \sqrt{2m}) \cdot z_1 + z_2 \cdot (x \sqrt{2m})}{2} \right) \right)
$$

$$
= \sum_{x \in \tilde{h}^{-1}(0)} \exp \left( 2\pi i \text{Tr} \left( \frac{(x_1 + \tilde{v}_1, \ldots, x_n + \tilde{v}_n) \cdot \tau \cdot (x_1 + \tilde{v}_1, \ldots, x_n + \tilde{v}_n)}{4m} \right) \right.
$$

$$
+ \frac{(x_1 + \tilde{v}_1, \ldots, x_n + \tilde{v}_n) \sqrt{2m}) \cdot z_1 + z_2 \cdot ((x_1 + \tilde{v}_1, \ldots, x_n + \tilde{v}_n) \sqrt{2m})}{2}
$$

$$
= \left( \sum_{x_1 \in (2m\mathcal{O})^g} \exp \left( 2\pi i \text{Tr} \left( \frac{(x_1 + \tilde{v}_1) \tau (x_1 + \tilde{v}_1)}{4m} \right) \right) \right)
$$

$$
\times \exp \left( 2\pi i \left( \frac{\tilde{x}_1 \cdot z_1 + z_2 \cdot x_1}{2} \right) \right) \ldots
$$

$$
\left( \sum_{x_n \in (2m\mathcal{O})^g} \exp \left( 2\pi i \text{Tr} \left( \frac{(x_n + \tilde{v}_n) \tau (x_n + \tilde{v}_n)}{4m} \right) \right) \right).
$$
\[
\times \exp \left( 2\pi i \left( \frac{x_n \sqrt{2m} \cdot z_1 + z_2 \cdot x_n \sqrt{2m}}{2} \right) \right)
\]
\[
= \prod_{a \in S_{2m}^g} \theta^{(g)}_{m,a}(\tau, z_1, z_2)^{n_a(v_1, \ldots, v_n)}
\]

from the fact that the number of \( a \) in \( S_{2m}^g \) which are equal to \( v_1, \ldots, v_n \) is exactly \( n_a(v_1, \ldots, v_n) \). □

**Theorem 7.3.** Let \( C, 1 \leq j \leq g \), be a Type II code of length \( n \) over \( S_{2m} \). Let \( J_{C_1, C_2, \ldots, C_g}(X) \) be the complete joint weight enumerator of the codes \( C_j, 1 \leq j \leq g \). Then

\[
J_{C_1, C_2, \ldots, C_g}(\theta^{(g)}_{m,\mu}(\tau, 0, 0) \mid \mu \in S_{2m}^g)
\]

is a Hermitian modular form of weight \( n \) and genus \( g \).

**Proof.** For simplicity let \( H(\tau) := J_{C_1, C_2, \ldots, C_g}(\theta^{(g)}_{m,\mu}(\tau, 0, 0) \mid \mu \in S_{2m}^g) \). It is enough to check the transformation law of \( H(\tau) \) under the three types of generator of \( \Gamma(\mathcal{O}) \) (see Remark 2):

\[
\begin{pmatrix}
0 & -1_g \\
1_g & 0 \\
\end{pmatrix}, \begin{pmatrix}
\bar{u} & 0 \\
0 & u^{-1} \\
\end{pmatrix}, \forall u \in \text{GL}(g; \mathcal{O}), \begin{pmatrix}
1 & \alpha \\
0 & 1 \\
\end{pmatrix}, \forall \alpha = \tilde{\alpha} \in \text{Sym}(g; \mathcal{O}).
\]

Then,

\[
\left( \begin{array}{c}
H \\
\cdot \\
& \begin{pmatrix}
0 & -1_g \\
1_g & 0 \\
\end{pmatrix}
\end{array} \right)(\tau) = \text{Det}(\tau)^{-n} H(-\tau^{-1})
\]

\[
= J_{C_1, C_2, \ldots, C_g}(\theta^{(g)}_{m,\mu}(-\tau^{-1}, 0, 0) \mid \mu \in S_{2m}^g)
\]

\[
= J_{C_1, C_2, \ldots, C_g}(\left( \frac{i}{2m} \right)^g \text{Det}(\tau) T \otimes T \otimes \cdots \otimes T \cdot \theta^{(g)}_{m,\mu}(\tau, 0, 0) \mid \mu \in S_{2m}^g)
\]

\[
= \left( \frac{i}{2m} \right)^g J_{C_1, C_2, \ldots, C_g}(\theta^{(g)}_{m,\mu}(\tau, 0, 0)) = H(\tau)
\]

(since \( n \equiv 0 \pmod{4m} \) and from the MacWilliams identity given in Corollary 3.3). □

We now state the following corollary.

**Corollary 7.4.** Let \( C, 1 \leq j \leq g \), be a Type II code of length \( n \) over \( S_{2m} \). Then

\[
SJ_{C_1, C_2, \ldots, C_g}(\theta^{(g)}_{m,\mu}(\tau, 0, 0) \mid \mu \in S_{2m}^g/\sim)
\]

is a Hermitian modular form of weight \( n \) and genus \( g \).

**Proof.** From the definition, for each \( \mu \in S_{2m}^g \), \( \sim \mu' \Leftrightarrow \mu' = \epsilon \mu \), for a unit \( \epsilon \) in \( S_{2m} \). So, note that \( \theta^{(g)}_{m,\mu}(\tau, z_1, z_2) = \theta^{(g)}_{m,\mu}(\tau, \bar{\epsilon} z_1, \bar{\epsilon} z_2) \). In particular, \( \theta_{m,\mu}(\tau, 0, 0) = \theta_{m,\epsilon \mu}(\tau, 0, 0) \). Secondly, from the definition

\[
SJ_{C_1, C_2, \ldots, C_g}(X \mid X = (x_{[\alpha]}), [\alpha] \in S_{2m}^g/\sim)
\]

\[
= J_{C_1, C_2, \ldots, C_g}(X \mid X = (x_{[\beta]}), x_{[\beta]} \text{ is identified with } x_{[\beta]} \text{ if } \beta = \epsilon \alpha)
\]
Therefore, $\text{SJ}_{C_1,\ldots,C_g}(\theta^{(g)}(m,\mu(\tau,0,0)|\mu \in S^g_{2m}/\sim)$ is a Hermitian Jacobi form. So, Theorem 7.3 implies the result. □

Remark 3. The symmetrized biweight enumerators of the code $C$ over $S_2 = \mathbb{Z}_2 + u\mathbb{Z}_2$ have been studied including a connection with Hermitian modular forms of genus 2 in [1].

8. Conclusion

The interaction between coding theory, the theory of lattices, and modular forms has been a source of many interesting results. Beginning with codes over fields and real lattices, it progressed to codes over $\mathbb{Z}_4$ and then to codes over $\mathbb{Z}_{2m}$. Next the relationship with lattices and modular forms was developed. Codes over $\mathbb{Z}_2 + u\mathbb{Z}_2$ were connected to complex lattices and their associated forms. In this paper we generalized this relationship constructing the ring $\mathbb{Z}_{2m} + u\mathbb{Z}_{2m}$. We developed the necessary coding theory over this ring including the MacWilliams relations and investigated self-dual codes. We then used these codes to build unimodular complex lattices. We used a set of weight enumerators over these codes to construct Hermitian Jacobi forms and Hermitian forms.

Acknowledgements

This work was partially supported by KRF 2003-070-C00001 and KOSEF R01-2003-000011596-0.

References