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This is an author produced version of a paper published in IEEE Transactions on Circuits and Systems for Video Technology, 55 (11). pp. 3413-3420. ISSN 1051-8215. This version has been peer-reviewed but does not include the final publisher proof corrections, published layout or pagination.

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Stability of Hybrid Stochastic Retarded Systems
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Abstract—In the past few years, hybrid stochastic retarded systems (also known as stochastic retarded systems with Markovian switching), including hybrid stochastic delay systems, have been intensively studied. Among the key results, Mao et al. proposed the Razumikhin-type theorem on exponential stability of stochastic functional differential equations with Markovian switching and its application to hybrid stochastic delay interval systems. However, the importance of general asymptotic stability has not been considered. This paper is to study Razumikhin-type theorems on general $p$-th moment asymptotic stability of hybrid stochastic retarded systems. The proposed theorems apply to complex systems including some cases when the existing results cannot be used.

Index Terms—Asymptotic stability, Markov chain, Razumikhin-type theorems, retarded systems, stochastic systems.

I. INTRODUCTION

HYBRID systems are employed to model many practical systems where abrupt changes in system structure and parameters may occur (see, e.g., [4] and [8]). An area of particular interest has been the analysis of stability of hybrid systems (see, e.g., [1], [10], [18], and [19]). Recently, hybrid stochastic retarded systems (HSRSs), including hybrid stochastic delay systems (HSDSs), driven by continuous-time Markovian chains have been widely used since stochastic modeling plays an important role in many branches of science and engineering. Consequently, the stability analysis of HSRSs and HSDSs has been studied by many works, see, e.g., [12]–[16], [22]. Mao et al. [13] established a number of exponential stability criteria for stochastic differential delay equations with Markovian switching that apply for systems with constant delay and obtained exponential and asymptotic stability criteria for stochastic differential delay equations with Markovian switching [14], which are useful for systems with sufficient small constant delay. Mao [15] studied the exponential stability of linear stochastic delay interval systems with Markovian switching while Yue et al. [22] considered that of a class of stochastic systems with time delay, nonlinearity, and Markovian switching. These delay-dependent results use linear matrix inequality (LMI) techniques with Lyapunov functionals and require the time delay to be a constant or a differentiable function that varies slowly, or say, the derivative of which is bounded by a constant number less than one. To remove the restriction in [15] and allow the time delay to be a bounded variable only, Mao et al. [12], [16] proposed and employed the Razumikhin-type theorem on exponential stability.

The Razumikhin method is developed to cope with the difficulty arisen from the large, quickly varying, and nondifferentiable time delays. However, the importance of general asymptotic stability has not been considered. In many cases, the exponential stability of the equilibrium of the system is not necessary and to stabilize the system exponentially fast is economically, and sometimes practically, unfeasible. In fact, the criteria for exponential stability of HSRSs implicitly require the diffusion operator associated with the underlying HSRSs of the Lyapunov function along a solution of the system to be negative and have the same order as that of the function itself at some instants, which is not satisfied for many nonlinear systems. In these cases, the existing results (see [12]–[16], and [22]) cannot be applied. For example, consider the following scalar stochastic delay system is driven by a right-continuous Markov chain $r(t)$ that is independent of the one-dimensional (1-D) standard Brownian motion $B(t)$ and takes values in $S = \{1, 2\}$ with generator

$$
\Gamma = \begin{pmatrix}
\gamma_{11} & \gamma_{12} \\
\gamma_{21} & \gamma_{22}
\end{pmatrix}
$$

$$
\gamma_{11} = \gamma_{12} > 0, \quad \gamma_{21} = \gamma_{22} > 0.
$$

This HSDS is described as the following stochastic delay equation with Markovian switching:

$$
dx(t) = -\left[\frac{1}{2}h(t) + \zeta(x(t), r(t))\right] dt + \sigma(x(t), x(t-h(t)), r(t)) d\Gamma B(t) \tag{1}
$$

on $t \geq 0$, where $h: \mathbb{R}_+ \rightarrow [\tau_0, 0)$ is Borel measurable and the nonlinear term $\zeta(x(t), r(t))$ and the diffusion term $\sigma(x(t), x(t-h(t)), i)$ are given as follows:

$$
\zeta(x(t), i) = \begin{cases}
\frac{1}{2}x^2(t), & i = 1 \\
\frac{1}{10}x(t)\sqrt{|x(t)|}, & i = 2
\end{cases}
$$

$$
\sigma(x(t), x(t-h(t)), i) = \begin{cases}
\frac{x^2(t)}{4}, & i = 1 \\
\frac{x(t)}{x(t-h(t))}, & i = 2
\end{cases}
$$

for all $t \geq 0$. We encounter a problem when we attempt to apply the existing results to analyze the stability of the solution to (1). To see this problem, let us set $V(x(t), t, r(t)) = x^2(t)$ and calculate

$$
\Delta V(x_t, t, i) \leq \begin{cases}
-x^2(t) - \frac{1}{10}x^2(t) + x^2(t-h(t)), & i = 1 \\
-x^2(t) - \frac{1}{10}x^2(t) + x^2(t-h(t)), & i = 2
\end{cases}
$$

on $t \geq 0$, where operator $\mathcal{L}$ is defined in (4) or (25) (see, e.g., [12]). The higher order (higher than the order of $V(x)$) of polynomial $-2x(t)\zeta(x(t), r(t))$, time delay, and $\lambda_1 = \lambda_2 = 1$.

Manuscript received July 07, 2007; revised April 29, 2008. First published July 22, 2008; current version published December 12, 2008. The work of L. Huang and X. Mao was supported by the University of Strathclyde, Glasgow, U.K. This paper was recommended by Associate Editor C. Hadjicostis.

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Digital Object Identifier 10.1109/TCSI.2008.2001825.
(cf. Theorem 4.2, [12]) all appear on the right-hand side of inequality (2) and this prevents the exiting results from being used. However, the solution to (1) may be asymptotically stable in mean-square sense though, due to $\lambda_1 = \lambda_2 = 1$, it might be not exponentially stable (see [9] and [20]). This paper is to study the general asymptotic stability of HSRSs with Razumikhin-type arguments, which is a generalization of the result on exponential stability obtained in [12].

**Notation**

Throughout the paper, unless otherwise specified, we will employ the following notation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B(t) = (B_1(t), \ldots, B_m(t))^T$ be an $m$-dimensional Brownian motion defined on the probability space. If $x, y$ are real numbers, then $x \vee y$ denotes the maximum of $x$ and $y$, and $x \wedge y$ stands for the minimum of $x$ and $y$. Let $[\cdot]$ denote the Euclidean norm in $\mathbb{R}^m$. Let $\tau \geq 0$ and $C([-\tau, 0]; \mathbb{R}^m)$ denote the family of all continuous $\mathbb{R}^m$-valued functions $\varphi$ on $[-\tau, 0]$ with the norm $\|\varphi\| = \sup \{|\varphi(\theta)| : -\tau \leq \theta \leq 0\}$. Let $C_{\mathcal{F}_t}^2([-\tau, 0]; \mathbb{R}^m)$ be the family of all $\mathcal{F}_t$-measurable bounded $C([-\tau, 0]; \mathbb{R}^m)$-valued random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$. For $p > 0$ and $\theta \geq 0$, denote by $L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^m)$ the family of all $\mathcal{F}_t$-measurable $C([-\tau, 0]; \mathbb{R}^m)$-valued random processes $\phi = \{\phi(\theta) : -\tau \leq \theta \leq 0\}$ such that $\sup_{-\tau \leq \theta \leq 0} E[|\phi(\theta)|^p] < \infty$.

Let $r(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P\{r(t + \Delta) = j : r(t) = i\} = \begin{cases} \gamma_{ij} \Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii} \Delta + o(\Delta), & \text{if } i = j, \end{cases}$$

where $\Delta > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from $i$ to $j$ if $i \neq j$ while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$.

Assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is known that almost all sample paths of $r(t)$ are right-continuous step functions with a finite number of simple jumps in any finite subinterval of $[0, \infty)$. Let us consider an $n$-dimensional HSRS

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dB(t)$$

on $t \geq 0$ with initial data $x_0 = \{x(\theta) : -r \leq \theta \leq 0\} = \xi \in C_{\mathcal{F}_t}^2([-r, 0]; \mathbb{R}^m)$. Moreover,

$$f : C([-r, 0]; \mathbb{R}^m) \times R_+ \times S \rightarrow \mathbb{R}^m$$

$$g : C([-r, 0]; \mathbb{R}^m) \times R_+ \times S \rightarrow \mathbb{R}^{m \times m}$$

are measurable functions with $f(0, t, i) \equiv 0$ and $g(0, t, i) \equiv 0$ for all $t \geq 0$. Thus, (3) admits a trivial solution $x(t; 0) \equiv 0$. Here, $x(t) = \{x(t + \theta) : -r \leq \theta \leq 0\}$ is regarded as a $C([-r, 0]; \mathbb{R}^m)$-valued stochastic process. We assume that $f$ and $g$ are sufficiently smooth so that (3) only has continuous solutions on $t \geq 0$, any version of which is denoted by $x(t; x_0)$ or $x(t; \xi)$ in this paper. For example, $f$ and $g$ satisfy the local Lipschitz condition and the linear growth condition, see [12]–[16] and references therein.

Let $C^{2,1}(R^m \times R_+ \times S; R_+)$ denote the family of all nonnegative functions $V(x, t, i)$ on $R^m \times R_+ \times S$ that are twice continuously differentiable in $x$ and once in $t$. If $V \in C^{2,1}(R^m \times R_+ \times S; R_+)$, define an operator associated with system (3), $\mathcal{L}$, from $C([-r, 0]; R^m) \times R_+ \times S$ to $R$ by

$$\mathcal{L}(x(t, t, i), V) = V_t(x(t, i)) + V_x(x(t, i))f(x(t, t, i))$$

$$+ \frac{1}{2} \text{trace } \left[ \sum_{j=1}^N \gamma_{ij} V(x(t, j)) \right]$$

$$\text{for } i \in S.$$
2) **Definition 2.2:** The trivial solution of (3) or, simply, (3) is said to be:

1) \(p\)th \((p > 0)\) moment stable if, for every \(\varepsilon > 0\), there exists \(\delta = \delta(\varepsilon) > 0\) such that

\[
E[x(t; \xi)]^p \leq \varepsilon, \quad \forall t \geq 0
\]

whenever \(||\xi||^p < \delta_0\);

2) \(p\)th moment asymptotically stable if it is \(p\)-th moment stable and, moreover, for every \(\varepsilon > 0\), there exist \(\delta_0 = \delta_0(\varepsilon)\) and \(T = T(\varepsilon)\) such that

\[
E[x(t; \xi)]^p \leq \varepsilon, \quad \forall t \geq T
\]

whenever \(||\xi||^p < \delta_0\);

3) globally \(p\)th moment asymptotically stable if it is \(p\)-th moment stable and, moreover, for all \(\xi \in C_{R+0}^p([-r; 0]; R^n)\),

\[
\lim_{t \to \infty} E[x(t; \xi)]^p = 0.
\]

According to the above definitions, it is easy to verify that (global) \(p\)th moment (asymptotic) stability implies (global) stochastic (asymptotic) stability.

**II. ASYMPTOTIC STABILITY OF HRSRS**

As the main results of this paper, we present the Razumikhin-type theorems on general stability of HRSRSs (3) as follows. ■

1) **Theorem 3.1:** Let \(p > 0\), \(v \in V_{K_{\infty}}, v \in C_{K_{\infty}}, \) and \(w : R^n \times R_+ \times S \to R_+\) be a nonnegative continuous function with \(w(x, t, i) > 0\) if \(E[x(t)]^p > 0\). Assume that there exists a function \(V \in C^2_{R+0}(R^n \times R_+ \times S; R_+)\) such that

\[
u(x^p) \leq V(x, t, i) \leq v(x^p), \quad \forall (x, t, i) \in R^n \times [-\tau, \infty)
\]

and, moreover, for all \(1 \leq i \leq N\)

\[
EVL(\phi, i, \phi) \leq -w(\phi(0), t, i)
\]

for all \(t \geq 0\) and those \(\phi \in L^p_{R+0}([-\tau, 0]; R^n)\) satisfying

\[
\min_{1 \leq k \leq N} E[V(\phi(\theta), t + \theta, k)] < q\left(\max_{1 \leq k \leq N} E[V(\phi(0), t, k)], i\right)
\]

on \(-\tau \leq \theta \leq 0\), where \(q : R \times S \to R\) is a continuous nondecreasing function with respect to \(s \in R\) for all \(s \geq 0\) and \(1 \leq i \leq N\). Moreover, \(q(s, i) > s\) for all \(s > 0\) and \(1 \leq i \leq N\). Then, the trivial solution of HRSRS (3) is globally \(p\)th moment asymptotically stable.

**Proof:** Fix the initial data \(\xi \in C_{R+0}^p([-\tau, 0]; R^n)\) and extend \(r(t)\) to \([-\tau, 0]\) by setting \(r(t) = r(0)\) for all \(t \in [-\tau, 0]\). Noting that \(x(t; \xi)\) is continuous and \(r(t)\) is right continuous for all \(t \geq 0\), we see that \(EV(x(t), t, r(t))\) is right continuous on \(t \geq -\tau\). Define

\[
U(t) = \sup_{-\tau \leq \theta \leq 0} EV(x(t + \theta), t + \theta, r(t + \theta)) \quad \forall t \geq 0.
\]

We claim that

\[
U(t) = \sup_{-\tau \leq \theta \leq 0} EV(x(t + \theta), t + \theta, r(t + \theta)) \quad \forall t \geq 0.
\]

To show inequality (9), for each \(t \geq 0\) (fix \(t\) for the moment), we define

\[
\overline{\theta} = \max\{\theta \in [-\tau, 0] | EV(x(t + \theta), t + \theta, r(t + \theta)) = U(t)\}
\]

(10)

Obviously, \(\overline{\theta}\) is either less than 0 or equal to 0.

If \(\overline{\theta} < 0\), then

\[
EV(x(t + \theta), t + \theta, r(t + \theta)) < EV(x(t + \overline{\theta}, t + \overline{\theta}, r(t + \overline{\theta})) = U(t) \quad \forall \theta \in (\overline{\theta}, 0).
\]

(11)

It follows from the right continuity of \(EV(x(t), t, r(t))\) that for every sufficiently small \(h > 0\)

\[
EV(x(t + h), t + h, r(t + h)) \leq U(t)
\]

hence

\[
U(t + h) \leq U(t) \text{ and } D_h U(t) \leq 0.
\]

If \(\overline{\theta} = 0\), then

\[
EV(x(t + \theta), t + \theta, r(t + \theta)) \leq EV(x(t), t, r(t)) = U(t) \quad \forall \theta \in [-\tau, 0].
\]

(12)

Note that either \(EV(x(t), t, r(t)) = 0\) or \(EV(x(t), t, r(t)) > 0\). In the former case, i.e., \(EV(x(t), t, r(t)) = 0\), inequalities (12) and (5) yield that \(x(t + \theta) = 0\) a.s. for all \(-\tau \leq \theta \leq 0\). Recalling that \(f(0, t, i) = 0\) and \(g(0, t, i) = 0\), we see \(x(t + h) = 0\) for all \(h > 0\), hence \(U(t + h) = 0\) and \(D_h U(t) = 0\). In the other case when \(EV(x(t), t, r(t)) > 0\), the above inequality (12) implies

\[
EV(x(t + \theta), t + \theta, r(t + \theta)) \leq EV(x(t), t, r(t)) < q\left(\frac{EV(x(t), t, r(t))}{r(t)}\right), \quad \forall \theta \in [-\tau, 0].
\]

(13)

Consequently, inequality (7) holds, that is,

\[
\min_{1 \leq k \leq N} \left(\frac{EV(x(t + \theta), t + \theta, k)}{r(t)}\right) < q\left(\max_{1 \leq k \leq N} \left(\frac{EV(x(t), t, k)}{r(t)}\right)\right)
\]

on all \(-\tau \leq \theta \leq 0\). Moreover, by condition (5) and Jensen’s inequality, \(EV(x(t), t, r(t)) > 0\) yields \(E[x(t)]^p > 0\). Thus, by condition (6), we have

\[
EVL(x(t), t, i) < 0
\]

(14)

for all \(1 \leq i \leq N\). By the right continuity of the processes concerned, we see that, for all \(h > 0\) sufficiently small, we have

\[
EVL(x, s, i) \leq 0 \quad \forall t \leq s \leq t + h, 1 \leq i \leq N.
\]
By a formula derived from generalized Itô’s lemma (see [17] and [12]) and Fubini’s theorem, we observe

\[ EV(x(t+h), t+h, r(t+h)) = EV(x(t), t, r(t)) + \int_0^h EV(x(s), s, r(s))ds \leq EV(x(t), t, r(t)). \]  

Hence, we have

\[ U(t+h) = U(t) = EV(x(t), t, r(t)) D_\gamma U(t) = 0. \]

Inequality (9) has been proved. It follows immediately that

\[ U(t) \leq U(0), \quad \forall \ t \geq 0. \]  

Together with the definition of \( U(t) \), condition (5) and Jensen’s inequality, the above inequality (16) yields

\[ E|x(t)|^p \leq \nu^{-1}(u(\epsilon)), \quad \forall \ t \geq 0. \]  

So, for any \( \epsilon > 0 \), we can find \( \delta(\epsilon) = \nu^{-1}(u(\epsilon)) \) such that

\[ E|x(t)|^p \leq \epsilon, \quad \forall \ t \geq 0 \]

whenever \( |\xi| < \delta(\epsilon) \). The \( p \)th moment stability is proved. Now, we proceed to show the convergence of \( E|x(t)|^p \) to 0 as \( t \to \infty \). Fix any initial data \( \xi \in C_{\mathcal{F}_0}([\tau_0, \tau]; \mathbb{R}^n) \). Let \( \delta > 0 \) and \( \epsilon > 0 \) be such that \( |\xi| < \delta \) and \( U(0) < \nu(\delta) = u(\epsilon) \). So, by inequalities (16) and (17), we have \( EV(x(t), t, r(t)) < \nu(\delta) \) and \( E|x(t)|^p < \epsilon \) for all \( t \geq 0 \). Suppose 0 < \( \beta < \epsilon \) is arbitrary. We need to show there is a number \( T = T(\beta, \delta) \) such that \( E|x(t)|^p \leq \beta \) for all \( t \geq T \). This will be true by condition (5) and Jensen’s inequality if we show that \( EV(x(t), t, r(t)) \leq u(\beta) \) for all \( t \geq T \).

From the property of function \( q(s, i) \), there is a positive real number \( a > 0 \) such that \( q(s, i) = s > a \) for all \( u(\beta) \leq s \leq \nu(\delta) \) and \( 1 \leq i \leq N \). Let \( J \) be the minimal nonnegative integer such that \( u(\beta) + Ja > \nu(\delta) \), and \( \gamma = \inf \{ w(x(t), t, i) : \beta \leq E|x(t)|^p \leq \epsilon, \gamma \geq 0, 1 \leq i \leq N \} \). So \( \gamma > 0 \), since \( w(x(t), t, i) \geq 0 \) with \( w(x(t), t, i) > 0 \) if \( E|x(t)|^p > 0 \). Let \( \gamma = \tau \vee \gamma \vee \gamma \vee \cdots \)

We claim that \( EV(x(t), t, r(t)) \leq u(\beta) \) for all \( t \geq T \). First we show that \( EV(x(t), t, r(t)) \leq u(\beta) + (J-1)a \) for all \( t \geq T \). Let \( t_1 = \inf \{ t \geq T_0 : EV(x(t), t, r(t)) \leq u(\beta) + (J-1)a \} \) for all \( t \geq T_1 \). If \( t_1 > T_1 \), then, \( \forall T_0 \leq t \leq T_1 \), we have

\[ q( \max_{1 \leq k \leq N} EV(x(t), t, k), r(t)) \geq q(EV(x(t), t, r(t)), r(t)) > EV(x(t), t, r(t)) + a \]

Thus, by condition (6), implies

\[ EV(x(t), t, r(t)) \leq -u(\epsilon(t)), \quad \forall T_0 \leq t \leq T_1. \]

Consequently, by formula (15), we see

\[ EV(x(t_1), t_1, r(t_1)) \leq EV(x(t_0), t_0, r(t_0)) - \gamma(\tilde{T}_1 - \tilde{T}_0) \leq 0 \]

which contradicts the positive property of \( EV(x(t), t, r(t)) \). So, \( t_1 \leq T_1 \) and \( EV(x(t_1), t_1, r(t_1)) \leq -\gamma \). In fact, \( \forall \tilde{T}_1 \in [t \geq T_0 : EV(x(t), t, r(t)) = u(\beta) + (J-1)a] \), we have \( EV(x(t_1), t_1, r(t_1)) \leq -\gamma \) because

\[ q( \max_{1 \leq k \leq N} EV(x(t_1), t_1, k), r(t)) \geq q(EV(x(t_1), t_1, r(t_1)), r(t)) > u(\beta) + Ja \]

\[ > \nu(\delta) \]

\[ > EV(x(t_1 + \theta), t_1 + \theta, r(t_1 + \theta)) \]

\[ \geq \min_{1 \leq k \leq N} EV(x(t_1 + \theta), t_1 + \theta, k) \quad \forall \theta \in [-\gamma, 0]. \]

Thus, we have \( EV(x(t), t, r(t)) \leq u(\beta) + (J-1)a \) for all \( t \geq T_1 \). Define \( t_j = \inf \{ t \geq T_{j-1} : EV(x(t), t, r(t)) \leq u(\beta) + (J-j)a \} \) for \( j = 2, 3, \ldots, J \). By the same type of reasoning as above, we have

\[ EV(x(t), t, r(t)) \leq u(\beta) + (J-j)a \]

for all \( t \geq T_j \) and \( j = 2, 3, \ldots, J \).

In particular, \( EV(x(t), t, r(t)) \leq u(\beta) \) for all \( t \geq T_j \). This completes the proof.

2) Theorem 3.2: Let \( p \geq 0 \), \( u \in V_{K_\infty} \), \( v \in C_{K_\infty} \), and \( w : R^n \times R_+ \times S \to R_+ \) be a nonnegative continuous function with \( w(x, t, i) > 0 \) if \( E|x(t)|^p > 0 \). Assume that there exists a function \( V \in C^{2,1}(R^n \times R_+ \times S; R_+) \) such that

\[ u(|x|^p) \leq V(x, t, i) \leq v(|x|^p), \quad \forall (x, t) \in R^n \times [-\gamma, \infty) \]

and, moreover, for all \( 1 \leq i \leq N \)

\[ EV(\phi, t, i) \leq -w(\phi, t, i) \]

for all \( t \geq 0 \) and those \( \phi \in L^p_{\mathcal{F}_t}([-\gamma, 0]; R^n) \) satisfying

\[ \min_{1 \leq k \leq N} EV(\phi, t, \theta + \theta, k) < \max_{1 \leq k \leq N} EV(\phi, \theta, t, k), \theta \]

on \( -\gamma \leq \theta \leq 0 \), where \( \bar{q} : R \times S \to R \) is a continuous nondecreasing function with respect to \( s \in R \) for all \( s \geq 0 \) and \( 1 \leq i \leq N \). Moreover, for all \( 1 \leq i \leq N \), \( \bar{q}(s, i) > s \) for all \( s > 0 \) and \( \bar{q}(s, i) > s \) as \( s \to \infty \). Then, the trivial solution of HSRS (3) is globally \( p \)th moment asymptotically stable.
Proof: As above, the proof is composed of tow parts. The first part to show the $p$th moment stability of (3) is similar to that for Theorem 3.1. One only needs to note that from the property of $q(s,i)$ holds

\[
E\tilde{q}(V(x(t),t,r(t)),r(t)) \\
\geq \int_{0<V<\infty} \tilde{q}(V(x(t),t,r(t)),r(t))d\mathbb{P} \\
+ \int_{V<\infty} \tilde{q}(V(x(t),t,r(t)),r(t))d\mathbb{P} \\
> \int_{0<V<\infty} V(x(t),t,r(t))d\mathbb{P} \\
+ \int_{V<\infty} V(x(t),t,r(t))d\mathbb{P} \\
= EV(x(t),t,r(t)) \forall t \geq 0. 
\]

(21)

Inequalities (12) and (21) imply that condition (20) is satisfied. Moreover, $EV(x(t),t,r(t)) > 0$ implies $E|x(t)|^p > 0$. Thus, by condition (19) and the property of $u(x,t,i)$, we are led to (14) in the case when $EV(x(t),t,r(t)) > 0$.

The other part to show the convergence of $E|x(t)|^p \to 0$ as $t \to \infty$ is slightly different and given as follows.

Numbers $\delta$, $\varepsilon$, $\gamma$, and $\hat{\tau}$ are defined as above while the positive real number $\hat{a} = a_1 \wedge a_2$, where $a_1 > 0$ and $a_2 > 0$ are such that, for all $1 \leq i \leq N$, we have

\[
\tilde{q}(s,i) - s > a_1 \forall u(\beta) \leq s < \infty \\
\tilde{q}(s,i) - s > a_2 \text{ as } s \to \infty.
\]

Let us now consider the expectation of function $V(x(t),t,r(t))$ for any $t \geq 0$

\[
EV(x(t),t,r(t)) = \int_{V<\infty} V(x(t),t,r(t))d\mathbb{P} \\
+ \int_{u(\beta)\leq V<\infty} V(x(t),t,r(t))d\mathbb{P} \\
+ \int_{V=\infty} V(x(t),t,r(t))d\mathbb{P}.
\]

Obviously, there is a positive number $0 < \tilde{p} < 1$ such that

\[
\alpha_1 \vee \alpha_2 \geq \tilde{p} 
\]

for any $t \geq 0$ whenever $EV(x(t),t,r(t)) \geq u(\beta)$, where

\[
\alpha_1 = P\{u(\beta) \leq V(x(t),t,r(t)) < \infty\} \\
\alpha_2 = \int_{V=\infty} V(x(t),t,r(t))d\mathbb{P}.
\]

Let $J$ be the minimal nonnegative integer such that $u(\beta) + (J-1)\tilde{p} \geq \nu(\delta)$, and $T_j = \hat{\tau} + j\tilde{p}$ with $j = 0, 1, \ldots, J$.

To prove that $EV(x(t),t,r(t)) \leq u(\beta)$ for all $t \geq T_j$, we first show that $EV(x(t),t,r(t)) \leq u(\beta) + (J-1)\tilde{p}$ for all $t \geq T_j$. Let $T_j = \inf\{t \geq T_0 : EV(x(t),t,r(t)) \leq u(\beta) + (J-1)\tilde{p}\}$. If $T_j > T_1$, then $\forall T_0 \leq t \leq T_j$, we have

\[
\max_{1 \leq k \leq N} E\tilde{q}(V(x(t),t,k),r(t)) \\
\geq E\tilde{q}(V(x(t),t,r(t)),r(t)) \\
= \int_{V<u(\beta)} \tilde{q}(V(x(t),t,r(t)),r(t))d\mathbb{P} \\
+ \int_{u(\beta)\leq V<\infty} \tilde{q}(V(x(t),t,r(t)),r(t))d\mathbb{P} \\
+ \int_{V=\infty} \tilde{q}(V(x(t),t,r(t)),r(t))d\mathbb{P} \\
> \int_{V<u(\beta)} V(x(t),t,r(t))d\mathbb{P} \\
+ \int_{u(\beta)\leq V<\infty} [V(x(t),t,r(t)) + \tilde{p}]d\mathbb{P} \\
+ (1 + \hat{a}) \int_{V=\infty} V(x(t),t,r(t))d\mathbb{P} \\
\geq EV(x(t),t,r(t)) + \tilde{p} \tilde{a} \\
\geq u(\beta) + J\tilde{p} \tilde{a} \\
\geq \nu(\delta) \\
\geq EV(x(t+\theta),t+\theta,r(t+\theta)) \\
\geq \min_{1 \leq k \leq N} EV(x(t+\theta),t+\theta,k) 
\]

for all $\theta \in [-\tau, 0]$. This, by condition (19), implies

\[
EV(x(t),t,r(t)) \leq -u(x(t),t,r(t)) \leq -\gamma, \\
\forall T_0 \leq t \leq T_j.
\]

Consequently, we see

\[
EV(x(T_0),T_0,r(T_0)) \\
\leq EV(x(T_1),T_1,r(T_1)) - \gamma(T_1 - T_0) \\
< \nu(\delta) - \gamma\hat{\tau} \\
\leq 0
\]

which contradicts the positive property of $EV(x(t),t,r(t))$. Thus, $T_j \leq T_1$ and $EV(x(T_j),T_j,r(T_j)) \leq -\gamma$. Moreover, $\forall T_{11} \in \{t \geq T_0 : EV(x(t),t,r(t)) = u(\beta) + (J-1)\tilde{p}\}$, we have $EV(x(T_{11}),T_{11},r(T_{11})) \leq -\gamma$ because inequality (20), or say, (23) holds on $t = T_{11}$. Thus, we have $EV(x(t),t) \leq u(\beta) + (J-1)\tilde{p} \tilde{a}$ for all $t \geq T_1$.

Define $T_j = \inf\{t \geq T_{j-1} : EV(x(t),t,r(t)) \leq u(\beta) + (J-j)\tilde{p}\}$ for $j = 2, 3, \ldots, J$. By the same type of reasoning, we have

\[
EV(x(t),t,r(t)) \leq u(\beta) + (J-j)\tilde{p} \tilde{a}
\]

for all $t \geq T_j$ and $j = 2, 3, \ldots, J$.

Therefore, $EV(x(t),t,r(t)) \leq u(\beta)$ for all $t \geq T_j$. The proof is complete.

III. APPLICATION

Hybrid stochastic delay systems (HSDSs) described with stochastic differential delay equations with Markovian switching
are an important class of HSRs that are frequently used in engineering. As an illustrative example of applications of our new results, we consider the following HSDEs.

Let us consider the HSDEs of the form

\[ \text{dx}(t) = F(x(t), x(t-h(t)), t, r(t))dt + G(x(t), x(t-h(t)), t, r(t))dB(t) \quad (24) \]

on \( t \geq 0 \) with initial data \( x_0 = \xi \in C_{R+}^0([\tau, 0]; \mathbb{R}^m) \), where \( h : R_+ \rightarrow [0, \tau] \) is Borel measurable while

\[ F : \mathbb{R}^n \times \mathbb{R}^m \times R_+ \times S \rightarrow \mathbb{R}^n \]

and

\[ G : \mathbb{R}^n \times \mathbb{R}^m \times R_+ \times S \rightarrow \mathbb{R}^{n \times m} \]

are measurable functions with \( F(0,0,t,i) \equiv 0 \) and \( g(0,0,t,i) \equiv 0 \) for all \( t \geq 0 \). Assume that (24) only has continuous solutions. This is a special case of (3) with

\[ f(\phi, t, i) = F(\phi(0), \phi(-h(t)), t, i) \]

\[ g(\phi, t, i) = G(\phi(0), \phi(-h(t)), t, i) \]

for \( (\phi, t, i) \in C([\tau, 0]; \mathbb{R}^m) \times R_+ \times S \). If \( V \in C^{2,1}(R^n \times R_+ \times S; R^n) \), for the special case of (24) the operator \( \mathcal{L} \) defined in (4) becomes from \( R^n \times R^m \times R_+ \times S \) to \( R \) as

\[ \mathcal{L}V(x, y, t, i) = V_t(x, t, i) + V_x(x, t, i)F(x, y, t, i) \]

\[ + \frac{1}{2}\text{trace} \]

\[ \times \left[ G^T(x, y, t, i)V_{xx}(x, t, i)G(x, y, t, i) \right] \]

\[ + \sum_{j=1}^{N} \gamma_j V(x, t, j), \quad (25) \]

To give our new result for the HSDEs (24), let us introduce one more notation that \( L_{f_{R+}}^p(\Omega; \mathbb{R}^m) \) are the collection of all \( f_{R+} \) measurable \( C([\tau, 0]; \mathbb{R}^m) \)-valued random variables \( X \) such that \( E[|X|^p] < \infty \) and state the corresponding version of Theorem 3.2 for (24) as follows.

1) Theorem 4.1: Let \( p > 0 \), \( c_2 \geq c_1 > 0 \) and \( w : R^m \times R_+ \times S \rightarrow R_+ \) be a nonnegative continuous function with \( w(X, t, i) > 0 \) for \( E[|X|^p] > 0 \). Assume that there exists a function \( V \in C^{2,1}(R^n \times R_+ \times S; R^n) \) such that

\[ c_1 |X|^p \leq V(X, t, i) \leq c_2 |X|^p, \quad \forall (x, t) \in R^n \times [-\tau, \infty) \quad (26) \]

and, moreover, for all \( 1 \leq i \leq N \), let

\[ E\mathcal{L}V(X, Y, t, i) \leq -w(X, t, i) \quad (27) \]

for all \( t \geq 0 \) and those \( X, Y \in L_{f_{R+}}^p(\Omega; \mathbb{R}^n) \) satisfying

\[ \min_{1 \leq k \leq N} E\mathcal{L}V(Y, t-h(t), i) \leq \min_{1 \leq k \leq N} E\mathcal{L}V(X, t, k, i) \quad (28) \]

with \( \mathcal{L} \rightarrow R \times S \rightarrow R \) is a continuous nondecreasing function with respect to \( s \in R \) for all \( s \geq 0 \) and \( 1 \leq i \leq N \). Moreover, for all \( 1 \leq i \leq N \), \( \overline{q}(s, i) > s \) for all \( s > 0 \) and \( \overline{q}(s, i)/s > 1 \) as \( s \to \infty \). Then, the trivial solution of HSDS (24) is globally \( p \)th moment asymptotically stable.

This is a corollary from Theorem 3.2 and will be used to establish the following useful result.

2) Theorem 4.2: Let \( p > 0 \), \( c_2 \geq c_1 > 0 \), \( \lambda_0 \geq \lambda_1 \geq 0 \) and \( \lambda : R \times S \rightarrow R \) be a continuous nondecreasing function with respect to \( s \in R \) for all \( s \geq 0 \) and \( 1 \leq i \leq N \). Moreover \( \lambda(s, i)/s \geq 0 \) for all \( s > 0 \) and \( 1 \leq i \leq N \). Assume that there exists a function \( V \in C^{2,1}(R^n \times R_+ \times S; R^n) \) such that inequality (26) is satisfied and, moreover, for all \( X, Y \in R^n \), \( t \geq 0 \), and \( 1 \leq i \leq N \), assume

\[ \mathcal{L}V(X, Y, t, i) \leq -\lambda_0 \max_{1 \leq k \leq N} V(X, t, k) \]

\[ + \lambda_1 \min_{1 \leq k \leq N} V(Y, t-h(t), k) \]

\[ - \lambda_i \max_{1 \leq k \leq N} V(X, t, k, i). \quad (29) \]

Then, the trivial solution of HSDS (24) is globally \( p \)th moment asymptotically stable.

Proof: In condition (28), let

\[ \beta(s, i) = s + \frac{1}{2(1 + \lambda_i)} \lambda(s, i), \quad (30) \]

For all \( t \geq 0 \) and \( X, Y \in L_{f_{R+}}^p(\Omega; \mathbb{R}^n) \) satisfying condition (28) with function (30), i.e.,

\[ \min_{1 \leq k \leq N} E\mathcal{L}V(Y, t-h(t), i) \]

\[ < \min_{1 \leq k \leq N} E\mathcal{L}V(X, t, i) + \frac{1}{2(1 + \lambda_i)} E\mathcal{L}(\max_{1 \leq k \leq N} V(X, t, i)) \]

from inequality (29), Fatou’s lemma, and condition (26), we have

\[ E\mathcal{L}V(X, Y, t, i) \leq -\lambda_0 \max_{1 \leq k \leq N} E\mathcal{L}V(X, t, k) \]

\[ + \lambda_1 \min_{1 \leq k \leq N} E\mathcal{L}V(Y, t-h(t), k) \]

\[ - \lambda_i \max_{1 \leq k \leq N} E\mathcal{L}V(X, t, k, i) \]

\[ \leq -\lambda_i E\mathcal{L}(\max_{1 \leq k \leq N} V(X, t, k, i)) \]

\[ \leq -\frac{1}{2} E\mathcal{L}(c_1 |X|^p, i) \]

for all \( 1 \leq i \leq N \). Since \( \lambda(s, i)/s \geq 0 \) for all \( s > 0 \) and \( 1 \leq i \leq N \), it is easy to verify that \( E\mathcal{L}(c_1 |X|^p, i) > 0 \) if \( E[|X|^p] > 0 \). Let \( u(X, t, i) = -1/2E\mathcal{L}(c_1 |X|^p, i) \) in condition (27), then the conclusion follows from Theorem 4.1.

3) Remark 4.1: In many cases, this useful criterion may be applied with \( \lambda(s, i) = \lambda_i s^{k_i} \), \( k_i \geq 1 \), and \( \lambda_i \geq 0 \) for \( 1 \leq i \leq N \). In a special case when \( \lambda(s, i) = \lambda_0 s \), \( \lambda_0 > 0 \) for all \( 1 \leq i \leq N \), the above result is exactly [12, Theorem 4.2]. However, our result works for the particular cases when \( \lambda_0 = \lambda_1 = 0 \) for some \( 1 \leq i \leq N \), to which the existing results (see [12–16] and [22]) do not apply.

Using the above skills, Theorem 4.2 can be developed to cope with systems with multiple delays of the form

\[ dx(t) = F(x(t), x(t-h_1(t)), \ldots, x(t-h_L(t)), t, r(t))dt + G(x(t), x(t-h_1(t)), \ldots, x(t-h_L(t)), t, r(t))dB(t) \quad (31) \]
on \( t \geq 0 \), where \( h_t : R_+ \to [0, \tau] \) is Borel measurable, \( l = 1, 2, \ldots, L \).

Let us state the following generalized result, which can be proven in the same way as in the proof of Theorem 4.2.

4) **Theorem 4.3:** Let \( p > 0, c_2 \geq c_1 > 0, \) and \( \lambda_{0i} \geq 0, \lambda_{1i} \geq 0 \) such that \( \lambda_{0i} \geq \sum_{i=1}^{N} \lambda_{0i} \) for all \( 1 \leq i \leq N \). Let \( \lambda : R \times S \to R_+ \) be a continuous nondecreasing function with respect to \( s \in R \) for all \( s \geq 0 \) and \( 1 \leq i \leq N \). Moreover \( \lambda(s, i)/s > 0 \) for all \( s > 0 \) and \( 1 \leq i \leq N \). Assume that there exists a function \( V \in C^{2,1}(R^n \times R_+ \times S; R_+) \) such that inequality (26) is satisfied and, moreover, for all \( X, Y_1, \ldots, Y_L \in R^n, t \geq 0 \) and \( 1 \leq i \leq N \),

\[
\begin{align*}
LV(X, Y_1, \ldots, Y_L, t, i) & \leq -\lambda_{0i}\max_{1 \leq k \leq N} V(X, t, k) + \lambda_{1i}\min_{1 \leq k \leq N} V(Y_1, t - h_1(t), k) + \cdots + \lambda_{Li}\min_{1 \leq k \leq N} V(Y_L, t - h_L(t), k) \\
& \quad - \lambda_{i} \max_{1 \leq k \leq N} V(X, t, k), \tag{32}
\end{align*}
\]

Then, the trivial solution of HSDS (31) is globally \( p \)th moment asymptotically stable.

**IV. examples**

1) **Example 5.1:** Let us now return to the scalar HSDS (1). For the previous calculation (2), let

\[
\begin{align*}
\lambda_{01} &= \lambda_{11} = 1, \quad \lambda_{12} = 0, \\
\lambda_{02} &= \lambda_{12} = 1, \quad \lambda_{22} = 0
\end{align*}
\]

in condition (29). It immediately follows from Theorem 4.2 that the trivial solution of system (1) is mean-square asymptotically stable. Clearly, this is in fact an application of Theorem 3.2. Alternatively, we can use Theorem 3.1 and have the same conclusion. Let

\[
g(s, i) = \begin{cases} 
  s + \frac{1}{4} s^2, & i = 1 \\
  s + \frac{1}{10} s^{5/4}, & i = 2
\end{cases}
\]

in condition (7), then the previous calculation (2) yields

\[
E(LV(x, t, i) \leq \begin{cases} 
  -\frac{1}{2} E [x^2(t)], & i = 1 \\
  -\frac{1}{10} E [x^2(t) \sqrt{x(t)}], & i = 2
\end{cases}
\]

when condition (7) is satisfied. Let

\[
w(x, t, i) = \begin{cases} 
  \frac{1}{2} (E x^2(t)), & i = 1 \\
  \frac{1}{10} (E x^2(t))^{5/4}, & i = 2
\end{cases}
\]

in inequality (6), then the inequality holds. According to Theorem 3.1, this implies that the trivial solution of system (1) is mean-square asymptotically stable.

2) **Example 5.2:** Let \( \tau(t) \) be a right-continuous Markov chain taking values in \( S = 1, 2 \) with generator

\[
\Gamma = (\gamma_{ij})_{2 \times 2} = \begin{pmatrix}
  0 & 1 \\
  1 & 0
\end{pmatrix}
\]

and independent of the scalar Brownian motion \( B(t) \). Let \( a_i, b_i, c_i, d_i \) be positive numbers with \( b_i > c_i^2 \) for \( i = 1, 2 \) and \( h : R_+ \to [0, \tau] \) be Borel measurable. Consider the following HSDS:

\[
dx(t) = [\alpha(x(t), r(t)) + \beta(x(t), r(t))] dt + \sigma(x(t), x(t - h(t)), r(t)) dB(t) \tag{33}
\]

on \( t \geq 0 \), where

\[
\begin{align*}
\alpha(1) &= a_1, \quad \beta(x, 1) = -b_1 x \\
\alpha(2) &= -a_2, \quad \beta(x, 2) = -b_2 x^3
\end{align*}
\]

in condition (7), then the previous calculation (2) yields

\[
(1 - 2a_1) \gamma_1 - \gamma_2 \geq 2b_1^2 \gamma_1 \\
(1 + 2a_2) \gamma_2 - \gamma_1 \geq 2b_2^2 \gamma_2
\]

when the following inequalities are satisfied:

\[
\begin{align*}
0 + d_1^2 + d_1^2 + d_2^2 & \leq 1 + 2(a_2 - d_2^2) \\
1 - 2(a_1 + d_1^2) & \leq 1 + 2(a_2 - d_2^2)
\end{align*}
\]

Since \( b_i > c_i^2 \) for \( i = 1, 2 \), by Theorem 4.2, we can conclude that system (33) is mean-square asymptotically stable if the above inequalities hold.

**V. Conclusion**

In this paper, the general \( p \)th moment asymptotic stability of HSRSSs (3) is studied with Razumikhin-type arguments. Theorems on asymptotic stability are established. Their applications to HSDSs (24) and (31) are also proposed. The Razumikhin-type theorems work for many HSRSSs including some complicated cases to which the existing results do not apply. In a special case of the above results when \( w(x(t), t, r(t)) = \alpha(t) E V(x(t), t, r(t)) \) for all \( t \geq 0 \) with \( \alpha(t) > 0 \), using the techniques similar to [16], Razumikhin-type theorems on generalized exponential stability of HSRSSs (3) may be obtained. By Fatou’s lemma, we note that conditions (7) and (20) are less conservative than that in the existing results (see [12] and [16]) and are convenient in application.
ACKNOWLEDGMENT

The authors would like to thank the reviewers, Associate Editor, and Editor-in-Chief for their helpful comments.

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