Sharp upper bound for the rainbow connection numbers of 2-connected graphs

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Abstract

An edge-colored graph $G$, where adjacent edges may be colored the same, is rainbow connected if any two vertices of $G$ are connected by a path whose edges have distinct colors. The rainbow connection number $rc(G)$ of a connected graph $G$ is the smallest number of colors that are needed in order to make $G$ rainbow connected. In this paper, we give a sharp upper bound that $rc(G) \leq \lceil \frac{n}{2} \rceil$ for any 2-connected graph $G$ of order $n$, which improves the results of Caro et al. to best possible.

Keywords: rainbow connection, noncomplete rainbow path, 2-connected graph

AMS subject classification 2011: 05C15, 05C40

1 Introduction

All graphs considered in this paper are simple, finite and undirected. An edge-coloring of a graph $G$ is a function from the edge set of $G$ to the set of natural numbers. A path in an edge-colored graph $G$ is a rainbow path if no two edges of this path are colored the same. An edge-colored graph is rainbow connected if every pair of vertices is connected by at least one rainbow path. The rainbow connection number of a connected graph $G$, denoted by $rc(G)$, is the smallest number of colors that are needed to rainbow color the graph $G$. We call a rainbow coloring of $G$ with $k$ colors a $k$-rainbow coloring.

*Supported by NSFC No.11071130.
The concept of a rainbow coloring was introduced by Chartrand et al. in [5]. The rainbow connection numbers of several graph classes have been obtained. It is well known that a cycle with \( n \) vertices has a rainbow connection number \( \lceil \frac{n}{2} \rceil \), \( rc(G) = n - 1 \) if and only if \( G \) is a tree of order \( n \) \(( \geq 2)\), and \( rc(G) = 1 \) if and only if \( G \) is a complete graph of order \( n \) \(( \geq 2)\). In [4], Chakraborty et al. gave the following result about the rainbow connection number.

**Theorem 1.1.** [4] Given a graph \( G \), deciding if \( rc(G) = 2 \) is NP-Complete. In particular, computing \( rc(G) \) is NP-Hard.

However, many upper bounds of the rainbow connection number have been given. For a 2-connected graph, Caro et al. proved the following two results.

**Proposition 1.1.** [3] If \( G \) is a 2-connected graph with \( n \) vertices, then \( rc(G) \leq \frac{2n}{3} \).

**Theorem 1.2.** [3] If \( G \) is a 2-connected graph on \( n \) vertices, then \( rc(G) \leq \frac{n}{2} + O(n^{\frac{1}{2}}) \).

One can see that both the above bounds are much greater than \( \lceil \frac{n}{2} \rceil \). However, experience tells us that the best bound should be \( \lceil \frac{n}{2} \rceil \). This paper is to give it a proof. Before proceeding, we need the following notation and terminology.

A *separation* of a connected graph is a decomposition of the graph into two nonempty connected subgraphs which have just one vertex in common. This common vertex is called a *separating vertex* of the graph. A graph is *nonseparable* if it is connected and has no separating vertices; otherwise, it is separable. If a graph \( G \) has at least 3 vertices but no loops, then \( G \) is nonseparable if and only if \( G \) is 2-connected.

Let \( F \) be a subgraph of a graph \( G \). An *ear* of \( F \) in \( G \) is a nontrivial path in \( G \) whose ends are in \( F \) but whose internal vertices are not. A nested sequence of graphs is a sequence \((G_0, G_1, \cdots, G_k)\) of graphs such that \( G_i \subset G_{i+1}, 0 \leq i < k \). An *ear decomposition* of a nonseparable graph \( G \) is a nested sequence \((G_0, G_1, \cdots, G_k)\) of nonseparable subgraphs of \( G \) such that: (1) \( G_0 \) is a cycle; (2) \( G_i = G_{i-1} \cup P_i \), where \( P_i \) is an ear of \( G_{i-1} \) in \( G \), \( 1 \leq i \leq k \); (3) \( G_k = G \). It is clear that every 2-connected graph has an ear decomposition.

At the IWOCA workshop [7], Hajo Broersma posed a question: what happens with the value \( rc(G) \) for graphs with higher connectivity? Motivated by this question, we study the rainbow connection number of a 2-connected graph and give a sharp upper bound that \( rc(G) \leq \lceil \frac{n}{2} \rceil \) for any 2-connected graph of order \( n \), which improves the results in [3] to best possible.

## 2 Main results

For convenience, we first introduce some new definitions.
Definition 2.1. Let $c$ be a $k$-rainbow coloring of a connected graph $G$. If a rainbow path $P$ in $G$ has length $k$, we call $P$ a complete rainbow path; otherwise, it is a noncomplete rainbow path. A rainbow coloring $c$ of $G$ is noncomplete if for any vertex $u \in V(G)$, $G$ has at most one vertex $v$ such that all the rainbow paths between $u$ and $v$ are complete; otherwise, it is complete.

For a connected graph $G$, if a spanning subgraph has a (noncomplete) $k$-rainbow coloring, then $G$ has a (noncomplete) $k$-rainbow coloring. This simple fact will be used in the following proofs.

Lemma 2.1. Let $G$ be a Hamiltonian graph of order $n$ ($n \geq 3$). Then $G$ has a noncomplete \( \lceil \frac{n}{2} \rceil \)-rainbow coloring, i.e., $rc(G) \leq \lceil \frac{n}{2} \rceil$.

Proof. Since $G$ is a Hamiltonian graph, there is a Hamiltonian cycle $C_n = v_1, v_2, \ldots, v_n, v_{n+1}$ ($= v_1$) in $G$. Define the edge-coloring $c$ of $C_n$ by $c(v_i, v_{i+1}) = i$ for $1 \leq i \leq \lceil \frac{n}{2} \rceil$ and $c(v_i, v_{i+1}) = i - \lceil \frac{n}{2} \rceil$ if $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$. It is clear that $c$ is a $\lceil \frac{n}{2} \rceil$-rainbow coloring of $C_n$, and the shortest path connecting any two vertices in $V(G)$ on $C_n$ is a rainbow path. For any vertex $v_i$ ($1 \leq i \leq n$), only the antipodal vertex of $v_i$ has no noncomplete rainbow path to $v_i$ if $n$ is even. Every pair of vertices in $G$ has a noncomplete rainbow path if $n$ is odd. Hence the rainbow coloring $c$ of $C_n$ is noncomplete. Since $C_n$ is a spanning subgraph of $G$, $G$ has a noncomplete $\lceil \frac{n}{2} \rceil$-rainbow coloring. \( \blacksquare \)

Let $G$ be a 2-connected non-Hamiltonian graph of order $n$ ($n \geq 4$). Then $G$ must have an even cycle. In fact, since $G$ is 2-connected, $G$ must have a cycle $C$. If $C$ is an even cycle, we are done. Otherwise, $C$ is a odd cycle, we then choose an ear $P$ of $C$ such that $V(C) \cap V(P) = \{a, b\}$. Since the lengths of the two segments between $a, b$ on $C$ have different parities, $P$ joining with one of the two segments forms an even cycle. Then, starting from an even cycle $G_0$, there exists a nonincreasing ear decomposition $(G_0, G_1, \ldots, G_t, G_{t+1}, \ldots, G_k)$ of $G$, such that $G_i = G_{i-1} \cup P_i$ ($1 \leq i \leq k$) and $P_i$ is a longest ear of $G_{i-1}$, i.e., $\ell(P_1) \geq \ell(P_2) \geq \cdots \geq \ell(P_k)$. Suppose that $V(P_i) \cap V(G_{i-1}) = \{a_i, b_i\}$ ($1 \leq i \leq k$). We call the distinct vertices $a_i, b_i$ ($1 \leq i \leq k$) the foot vertices of the ear $P_i$. Without loss of generality, suppose that $\ell(P_1) \geq 2$ and $\ell(P_{t+1}) = \cdots = \ell(P_k) = 1$. So $G_t$ is a 2-connected spanning subgraph of $G$. Since $G$ is a non-Hamiltonian graph, we have $t \geq 1$. Denote the order of $G_i$ ($0 \leq i \leq k$) by $n_i$. All these notations will be used in the sequel.

Lemma 2.2. Let $G$ be a 2-connected non-Hamiltonian graph of order $n$ ($n \geq 4$). If $G$ has at most one ear with length 2 in the nonincreasing ear decomposition, then $G$ has a noncomplete $\lceil \frac{n}{2} \rceil$-rainbow coloring, i.e., $rc(G) \leq \lceil \frac{n}{2} \rceil$. 

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Proof. Since $G_t$ ($t \geq 1$) in the nonincreasing ear decomposition is a 2-connected spanning subgraph of $G$, it only needs to show that $G_t$ has a noncomplete $\left\lceil \frac{n}{2} \right\rceil$-rainbow coloring. We will apply induction on $t$.

First, consider the case that $t = 1$. Let $G$ be a 2-connected non-Hamiltonian graph with $t = 1$ in the nonincreasing ear decomposition. The spanning subgraph $G_1 = G_0 \cup P_1$ of $G$ consists of an even cycle $G_0$ and an ear $P_1$ of $G_0$. Without loss of generality, suppose that $G_0 = v_1, v_2, \ldots, v_{2k}, v_{2k+1} (= v_1)$ where $k \geq 2$. We color the edges of $G_0$ with $k$ colors. Define the edge-coloring $c_0$ of $G_0$ by $c_0(v_i, v_{i+1}) = i$ for $1 \leq i \leq k$ and $c_0(v_1, v_{i+1}) = i - k$ if $k + 1 \leq i \leq 2k$. From the proof of Lemma 2.1, the coloring $c_0$ is a noncomplete $k$-rainbow coloring of $G_0$. Now consider $G_1$ according to the parity of $\ell(P_1)$. If $\ell(P_1)$ is even, then $n_1$ is odd and color the edges of $P_1$ with $\ell(P_1)$ new colors. In the first $\ell(P_1)$ edges of $P_1$ the colors are all distinct, and the same ordering of colors is repeated in the last $\ell(P_1)$ edges of $P_1$. It is easy to verify that the obtained coloring $c_1$ of $G_1$ is a noncomplete $\left\lceil \frac{n}{2} \right\rceil$-rainbow coloring and that for any pair of vertices in $G$, there exists a noncomplete rainbow path connecting them. If $\ell(P_1)$ is odd, then $n_1$ is even and color the edges of $P_1$ with $\ell(P_1) - 1$ new colors. The middle edge of $P_1$ receives any color that already appeared in $G_0$. The first $\ell(P_1) - 1$ edges of $P_1$ all receive distinct new colors and in the last $\ell(P_1) - 1$ edges of $P_1$ this coloring is repeated in the same order. It is easy to verify that the obtained coloring $c_1$ of $G_1$ is a noncomplete $\left\lceil \frac{n}{2} \right\rceil$-rainbow coloring.

Let $G$ be a 2-connected non-Hamiltonian graph with $t \geq 2$ in the nonincreasing ear decomposition. Assume that the subgraph $G_i$ ($1 \leq i \leq t - 1$) has a noncomplete $\left\lceil \frac{n}{2} \right\rceil$-rainbow coloring $c_i$ and when $n_i$ is odd, any pair of vertices have a noncomplete rainbow path. We distinguish the following three cases.

**Case 1.** $\ell(P_i) \geq 3$ is odd.

Suppose that $P_i = v_0(= a_i), v_1, \ldots, v_r, v_{r+1}, \ldots, v_{2r}, v_{2r+1}(= b_i)$ where $r \geq 1$. We color the edges of $P_i$ with $r$ new colors to obtain a noncomplete coloring $c_i$ of $G_t$. Define the edge-coloring of $P_i$ by $c(v_{i-1}, v_i) = x_i$ ($1 \leq i \leq r$), $c(v_r, v_{r+1}) = x$ and $c(v_{i-1}, v_i) = x_{i-r-1}$ ($r + 2 \leq i \leq 2r + 1$), where $x_1, x_2, \ldots, x_r$ are new colors and $x$ is a color appeared in $G_{t-1}$. It is easy to check that the obtained coloring $c_i$ of $G_t$ is a $\left\lceil \frac{n}{2} \right\rceil$-rainbow coloring.

Now we show that $c_i$ is noncomplete. For any pair of vertices $u, v \in V(G_{t-1}) \times V(G_{t-1})$, the rainbow path $P$ from $u$ to $v$ in $G_{t-1}$ is noncomplete in $G_t$, because the new colors $x_{1}, x_{2}, \ldots, x_r$ ($r \geq 1$) do not appear in $P$. For any pair of vertices $u, v \in V(P_i) \times V(P_i)$, if there exists a rainbow path $P$ from $u$ to $v$ on $P_i$, then $P$ is noncomplete in $G_t$, since some color in $G_{t-1}$ does not appear in $P$; if not, there exists a noncomplete rainbow path $P$ from $u$ to $v$ through some vertices of $G_{t-1}$ such that at least one new color does not appear in $P$. For any pair of vertices $umv \in V(G_{t-1}) \times (V(P_i) \setminus \{v_r, v_{r+1}\})$, there exists a noncomplete rainbow path from $u$ to $v$ in which at least one new color does not appear.
If there exists a vertex all of whose rainbow paths to $a_t$ (resp. $b_t$) in $G_{t-1}$ are complete, we denote the vertex by $a'_t$ (resp. $b'_t$). For vertex $v_r$ (resp. $v_{r+1}$), only the vertex $a'_t$ (resp. $b'_t$) possibly has no noncomplete rainbow path to $v_r$ (resp. $v_{r+1}$) in $G_t$. So there possibly exist two pairs of vertices $a'_t, v_r$ and $b'_t, v_r + 1$ which have no noncomplete rainbow path. Since $a'_t, b'_t$ are distinct in $G_{t-1}$, the rainbow coloring $c_t$ is noncomplete. If $n_t$ is odd, then $n_{t-1}$ is odd. By induction, $a'_t, b'_t$ do not exist when $n_{t-1}$ is odd. Hence every pair of vertices have a noncomplete rainbow path.

**Case 2.** $\ell(P_t) \geq 2$ is even and $n_{t-1}$ is even.

In this case, $n_t$ is odd. Suppose that $P_t = v_0(= a_t), v_1, \ldots, v_r, v_{r+1}, \ldots, v_{2r-1}, v_{2r}(= b_t)$ where $r \geq 1$. Define the edge-coloring of $P_t$ by $c(v_i v_i) = x_i$ for $1 \leq i \leq r$ and $c(v_i v_i) = x_i$ for $r + 1 \leq i \leq 2r$. It is clear that the obtained coloring $c_t$ of $G_t$ is a $\lceil \frac{n_t}{2} \rceil$-rainbow coloring.

Now we prove that $c_t$ is noncomplete. For any pair of vertices in $V(G_{t-1}) \times V(G_{t-1})$ or $V(P_t) \times V(P_t)$, there is a noncomplete rainbow path connecting them in $G_t$, similar to the Case 1. For any pair of vertices $u \in V(G_{t-1}), v \in V(P_t)$ ($v \neq v_r$), there is a noncomplete rainbow path $P$ from $u$ to $v$ such that at least one new color does not appear in $P$. For any vertex $u \in V(G_{t-1})$, since the coloring $c_{t-1}$ is noncomplete, $u$ has a noncomplete rainbow path $P'$ in $G_{t-1}$ to one of $a_t, b_t$ (say $a_t$). Then $P'$ joining with $a_t P_t v_r$ is a noncomplete rainbow path from $u$ to $v_r$ in $G_t$. Therefore, the rainbow coloring $c_t$ of $G_t$ is noncomplete such that any pair of vertices has a noncomplete rainbow path.

**Case 3.** $\ell(P_t) \geq 2$ is even and $n_{t-1}$ is odd.

In this case, $n_t$ is even. We consider the following three subcases.

### Subcase 3.1 $[V(P_t) \cap V(P_{t-1})] \setminus V(G_{t-2}) = \emptyset$.

If $\ell(P_{t-1})$ is odd, let $G'_{t-1} = G_{t-2} \cup P_t$ and $G_t = G'_{t-1} \cup P_{t-1}$. By induction, $G'_{t-1}$ has a noncomplete $\lceil \frac{n_{t-1}}{2} \rceil$-rainbow coloring ($n_{t-1}$ is the order of $G'_{t-1}$). Similar to Case 1, we can obtain a noncomplete $\lceil \frac{n_t}{2} \rceil$-rainbow coloring of $G_t$ from $G'_{t-1}$.

Suppose that $\ell(P_{t-1})$ is even. By induction, $G_{t-2}$ has a noncomplete $\lceil \frac{n_{t-2}}{2} \rceil$-rainbow coloring $c_{t-2}$. Suppose that $P_{t-1} = v_0(= a_{t-1}), v_1, \ldots, v_r, v_{r+1}, \ldots, v_{2r-1}, v_{2r}(= b_{t-1})$ and $P_t = v'_0(= a_t), v'_1, \ldots, v'_{s}, v'_{s+1}, \ldots, v'_{2s-1}, v'_{2s}(= b_t)$, where $r \geq 2, s \geq 1$. Since $c_{t-2}$ is noncomplete and $a_t, b_t$ ($1 \leq i \leq k$) are two distinct vertices, then $a_{t-1}$ has a noncomplete rainbow path $P'$ to one of $a_t, b_t$ (say $a_t$) and $b_{t-1}$ has a noncomplete rainbow path $P''$ to the other vertex. Suppose that $x$ is the color in $G_{t-2}$ that does not appear in $P'$. Now color the edges of $P_{t-1}, P_t$ with $r + s - 1$ new colors and the color $x$. Define an edge-coloring of $P_{t-1}$ by $c(v_i v_i) = x_i$ ($1 \leq i \leq r$) and $c(v_i v_i) = x_{i-r}$ ($r + 1 \leq i \leq 2r$), where $x_1, x_2, \ldots, x_r$ are new colors. And define an edge-coloring of $P_t$ by $c(v'_i v'_i) = y_i$ ($1 \leq i \leq s - 1$), $c(v'_{s-1} v'_s) = x, c(v'_{s-1} v'_{s+1}) = x_1$ and $c(v'_{s-1} v'_i) = y_{i-s-1}$ ($1 \leq i \leq 2s$), where
$y_1, y_2, \ldots, y_{s-1}$ are new colors.

Similar to Case 2, the obtained coloring $c_{t-1}$ of $G_{t-1}$ is a noncomplete $\lceil \frac{n_t - 1}{2} \rceil$-rainbow coloring such that every pair of vertices have a noncomplete rainbow path. It is obvious that $G_t$ is rainbow connected. The path $(v'_t P(a_t) P'(a_{t-1} P_{t-1} v_r))$ is a rainbow path from $v'_t$ to $v_r$ which is possibly complete. For any other pair of vertices in $G_t$, there is a noncomplete rainbow path connecting them. Hence the rainbow coloring $c_t$ of $G_t$ is noncomplete.

**Subcase 3.2** $[V(P_t) \cap V(P_{t-1})] \setminus V(G_{t-2}) = \{ b_t \}$.

If $\ell(P_{t-1})$ is odd, suppose that $P_{t-1} = v_0(= a_{t-1}), v_1, \ldots, v_r, v_{r+1}, \ldots, v_{2r}, v_{2r+1}(= b_{t-1})$. Since $P_{t-1}$ is a longest ear of $G_{t-2}$ and $b_t \in V(P_{t-1}) \setminus V(G_{t-2})$, we have $r \geq 2$. Define an edge-coloring of $P_{t-1}$ by $c(v_{i-1} v_i) = x_i$ ($1 \leq i \leq r$), $c(v_r v_{r+1}) = x$ and $c(v_{i-1} v_i) = x_{i-r} (r+2 \leq i \leq 2r+1)$, where $x_1, x_2, \ldots, x_r$ are new colors and $x$ is a color appeared in $G_{t-2}$. Similar to Case 1, the obtained coloring $c_{t-1}$ of $G_{t-1}$ is a noncomplete $\lceil \frac{n_t - 1}{2} \rceil$-rainbow coloring such that every pair of vertices have a noncomplete rainbow path. If $\ell(P_{t-1})$ is even, suppose that $P_{t-1} = v_0(= a_{t-1}), v_1, \ldots, v_r, v_{r+1}, \ldots, v_{2r-1}, v_{2r}(= b_{t-1})$, where $r \geq 2$. Define an edge-coloring of $P_{t-1}$ by $c(v_{i-1} v_i) = x_i$ ($1 \leq i \leq r$), and $c(v_{i-1} v_i) = x_{i-r} (r+1 \leq i \leq 2r)$, where $x_1, x_2, \ldots, x_r$ are new colors. Similar to Case 2, the obtained coloring $c_{t-1}$ of $G_{t-1}$ is a noncomplete $\lceil \frac{n_t - 1}{2} \rceil$-rainbow coloring such that every pair of vertices have a noncomplete rainbow path.

Without loss of generality, assume that $b_t$ belongs to the first half of $P_{t-1}$ and that $P_t = v'_0(= a_t), v'_1, \ldots, v'_s, v'_{s+1}, \ldots, v'_{2s-1}, v'_{2s}(= b_t)$, where $s \geq 1$. We color the edges of $P_t$ with $s-1$ new colors. Define an edge-coloring of $P_t$ by $c(v'_{i-1} v'_i) = y_i$ ($1 \leq i \leq s-1$), $c(v'_{s-1} v'_s) = x_1, c(v'_s v'_{s+1}) = y$ and $c(v'_{i-1} v'_i) = y_{i-s} (s+2 \leq i \leq 2s)$, where $y_1, y_2, \ldots, y_{s-1}$ are new colors and the color $y$ is different from color $x$ in $G_{t-2}$. It is easy to verify that the obtained coloring $c_t$ of $G_t$ is a $\lceil \frac{n_t}{2} \rceil$-rainbow coloring.

For any pair of vertices $v' \in V(P_t)$ and $v \in V(G_{t-1})$, there exists a noncomplete rainbow path $P$ connecting them since the path from $v'$ to one foot vertex of $P_t$ colored by new colors joining with the noncomplete rainbow path from the foot vertex to $v$ in $V(G_{t-1})$ is a noncomplete rainbow path from $v'$ to $v$ in $G_t$. For $v'_s$, there is a noncomplete rainbow path from $v'_s$ to any vertex in $V(G_{t-2}) \cup V(b_t P_{t-1} v_{r+2})$ through edge $e = v'_{s-1} v'_s$; and a noncomplete rainbow path from $v'_s$ to any vertex in $V(a_{t-1} P_{t-1} v_{r+1})$ through $e = v'_{s+1} v'_{s+2}$. For any pair of vertices in $V(P_t) \times V(P_t)$, there is a noncomplete rainbow path connecting them obviously. Hence the rainbow coloring $c_t$ is noncomplete.

**Subcase 3.3** $[V(P_t) \cap V(P_{t-1})] \setminus V(G_{t-2}) = \{ a_t, b_t \}$.

We can prove this subcase in a way similar to Subcase 3.2. Without loss of generality, we can assume that $a_t = v_p (1 \leq p \leq r-1)$ and $b_t = v_q (q \geq p + 2)$. Color all the edges
of \( P_{t-1} \) and \( P_t \) as in Subcase 3.2 but only the edge \( e = v'_{t-1}v'_t \) which is colored by \( x_{j+1} \) instead. The obtained coloring \( c_t \) of \( G_t \) is a noncomplete \( \lceil \frac{n}{2} \rceil \)-rainbow coloring. ■

Lemma 2.3. Let \( G \) be a 2-connected non-Hamiltonian graph of order \( n \) \((n \geq 4)\). If \( G \) has at least 2 ears of length 2 in the nonincreasing ear decomposition, then \( \text{rc}(G) \leq \lceil \frac{n}{2} \rceil \).

Proof. We only need to prove that there exists a rainbow coloring \( c_t \) of the spanning subgraph \( G_t \) in the nonincreasing ear decomposition that uses at most \( \lceil \frac{n}{2} \rceil \) colors. If \( G \) has 2 or 3 ears of length 2 in the nonincreasing ear decomposition, then \( G_{t-2} \) has at most one ear of length 2 and \( \ell(P_{t-1}) = \ell(P_t) = 2 \). From Lemmas 2.1 and 2.2, \( G_{t-2} \) has a noncomplete \( \lceil \frac{n-2}{2} \rceil \)-rainbow coloring \( c_{t-2} \). Assume that \( P_{t-1} = a_{t-1}, v, b_{t-1} \) and \( P_t = a_t, v', b_t \). Since \( P_{t-1} \) is a longest ear of \( G_{t-2} \), we have \( a_t, b_t \in V(G_{t-2}) \). Since the coloring \( c_{t-2} \) is noncomplete, \( a_{t-1} \) has a noncomplete rainbow path \( P \) to one of \( a_t, b_t \) (say \( a_t \)) such that the color \( x \) in \( G_{t-2} \) does not appear in \( P \). Define an edge-coloring of \( P_{t-1} \) and \( P_t \) by \( c(a_{t-1}v) = c(b_{t-1}v) = c(b_tv') = x_1 \) and \( c(a_tv') = x \), where \( x_1 \) is a new color. It is clear that \( va_{t-1}Pa_tv' \) is a rainbow path from \( v \) to \( v' \), and the obtained coloring of \( G_t \) is a \( \lceil \frac{n}{2} \rceil \)-rainbow coloring.

Now consider the case that \( G \) has at least 4 ears of length 2 in the nonincreasing ear decomposition. Suppose that \( \ell(P_{t-1}) \geq 3 \) and \( \ell(P_t) = \ell(P_{t+1}) = \cdots = \ell(P_t) = 2 \). Since \( P_t(1 \leq i \leq k) \) is a longest ear, we have that \( a_{t'}, b_{t'}, \cdots, a_t, b_t \in V(G_{t-1}) \). From Lemmas 2.1 and 2.2, there exists a \( \lceil \frac{n'-1}{2} \rceil \)-rainbow coloring \( c_{t-1} \) of \( G_{t-1} \). Color one edge of \( P_t(t' \leq i \leq t) \) with \( x_1 \) and the other with \( x_2 \), where \( x_1, x_2 \) are two new colors. It is obvious that \( G_t \) is rainbow connected. Since \( G \) has at least 4 ears of length 2, the rainbow coloring of \( G_t \) uses at most \( \lceil \frac{n}{2} \rceil \) colors. ■

From the above three lemmas and the fact that \( \text{rc}(C_n) = \lceil \frac{n}{2} \rceil \) \((n \geq 4)\), we can derive our following main result.

Theorem 2.1. Let \( G \) be a 2-connected graph of order \( n \) \((n \geq 3)\). Then \( \text{rc}(G) \leq \lceil \frac{n}{2} \rceil \), and the upper bound is sharp for \( n \geq 4 \).

Since for any two distinct vertices in a \( k \)-connected graph \( G \) of order \( n \), there exist at least \( k \) internal disjoint paths connecting them, the diameter of \( G \) is no more than \( \lceil \frac{n}{k} \rceil \). One could think of generalizing Theorem 2.1 to the case of higher connectivity in the obvious way, and pose the following conjecture.

Conjecture 2.1. Let \( G \) be a \( k \)-connected graph \( G \) of order \( n \). Then \( \text{rc}(G) \leq \lceil \frac{n}{k} \rceil \).
References


[6] M. Krivelevich, R. Yuster, The rainbow connection of a graph is (at most) reciprocal to its minimum degree, J. Graph Theory 63(2010), 185-191.