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On the Lower Semicontinuity of the Solution Mappings to Parametric Weak Generalized Ky Fan Inequality

Z.Y. Peng · X.M. Yang · J.W. Peng

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Abstract In this paper, we obtain some stability results for parametric weak generalized Ky Fan Inequality with set-valued mappings. Under new assumptions, which are weaker than the assumption of C-strict monotonicity, we provide sufficient conditions for the lower semicontinuity of the solution maps to two classes of parametric weak generalized Ky Fan Inequalities in Hausdorff topological vector spaces. These results extend and improve some results in the literature.

Keywords Stability · Lower semicontinuity · f-efficient solutions · Parametric weak generalized Ky Fan inequalities · Scalarization

1 Introduction

It is well known that the Ky Fan Inequality is a very general mathematical model, which embraces the formats of several disciplines, as those for equilibrium problems of Mathematical Physics, those from Game Theory, those from (Vector) Optimization and (Vector) Variational Inequalities, and so on (see [1, 2]).

The stability analysis of solution set map for a parametric Ky Fan Inequality (PKFI, in short) is of considerable interest. Many papers discussed the semicontinuity of solution maps, especially the lower semicontinuity of the solution maps for

Z.Y. Peng (✉)
Department of Mathematics, Inner Mongolia University, Hohhot 010021, P.R. China
e-mail: pengzaiyun@126.com

Z.Y. Peng
College of Science, Chongqing JiaoTong University, Chongqing 400074, China

X.M. Yang (✉) · J.W. Peng
School of Mathematics, Chongqing Normal University, Chongqing 400047, China
e-mail: xmyang@cqnu.edu.cn

PKFI. Cheng and Zhu [3] have obtained a lower semicontinuity result of the solution mapping to a PKFI in finite-dimensional spaces by using a scalarization method. Huang et al. [4] discussed the upper semicontinuity and lower semicontinuity of the solution map for a parametric implicit Ky Fan Inequality. By virtue of a density result and scalarization technique, Gong and Yao [5] first discussed the lower semicontinuity of the set of efficient solutions to a PKFI with vector-valued maps. By using the idea of Cheng and Zhu [3], Gong [6] studied the continuity of the solution maps to a PKFI with vector-valued maps. Recently, Chen et al. [7] studied the stability of solution sets of PKFI without the uniform compactness assumption, which improved the corresponding results of [3] and [5]. Chen and Li [8] established the lower semicontinuity and continuity of the solution mapping to a PKFI with set-valued maps, which extended the corresponding results of [5] and [6]. Subsequently, Li et al. [9] established the lower semicontinuity of the solution maps to a generalized PKFI with set-valued maps, which is different from the model considered in [8].

We observed that the semicontinuity of solution maps of the (generalized) PKFIs has been discussed under assumption of C-strict monotonicity for the vector-valued or set-valued maps, which implies that the f-solution set of the (generalized) PKFIs is a singleton for a linear continuous functional f (see [3, 5–9]). However, it is well known that the f-solution set of the (generalized) PKFIs should be general, but not a singleton. So, in this paper, we aim at studying the lower semicontinuity of the solution maps for two classes of weak generalized PKFIs when the f-solution set is a general set by removing the assumption of C-strict monotonicity.

The rest of the paper is organized as follows. In Sect. 2, we present two classes of weak generalized PKFIs, and recall some concepts and their properties. In Sect. 3, we discuss the lower semicontinuity of the solution mappings for the two classes of PKFIs, and provide some examples to illustrate that our main results extend the corresponding ones in [3, 5–9].

2 Preliminaries

Throughout this paper, if not otherwise specified, let X and Y be two real Hausdorff topological vector spaces, Z be a real topological space and Λ be a nonempty subset of Z. Let Y^* be the topological dual space of Y, and let C be a closed, convex and pointed cone of Y with $\operatorname{int} C \neq \emptyset$, where $\operatorname{int} C$ denotes the interior of C. Let $C^* := \{f \in Y^* : f(y) \geq 0,\ \forall y \in C\}$ be the dual cone of C.

Let A be a nonempty subset of X, and $F : A \times A \rightrightarrows Y \setminus \{\emptyset\}$ be a set-valued mapping. When the set A and the function F are perturbed by a parameter λ which varies over a set Λ of Z, we consider the following two classes of weak generalized PKFIs of finding $x \in A(\lambda)$ such that

$$F(x, y, \lambda) \cap (Y \setminus -\operatorname{int} C) \neq \emptyset, \quad \forall y \in A(\lambda); \tag{1}$$

and of finding $x \in A(\lambda)$ such that

$$F(x, y, \lambda) \subseteq Y \setminus -\operatorname{int} C, \quad \forall y \in A(\lambda). \tag{2}$$

where $A : \Lambda \rightrightarrows X\backslash\{\emptyset\}$ is a set-valued mapping and $F : B \times B \times \Lambda \subset X \times X \times Z \rightrightarrows Y\backslash\{\emptyset\}$ is a set-valued mapping with $A(\Lambda) = \bigcup_{\lambda\in\Lambda} A(\lambda) \subset B$.

Special case

(i) When F is a vector-valued mapping, the models (1) and (2) simultaneously reduce to the parametric generalized Ky Fan inequality in [10].

(ii) If for any $\lambda \in \Lambda, x, y \in A(\lambda)$, $F(x, y, \lambda) := \varphi(x, y, \lambda) + \psi(y, \lambda) - \psi(x, \lambda)$, $\varphi : A(\mu) \times A(\mu) \times \Lambda \to Y$ and $\psi : A(\mu) \times \Lambda \to Y$ be three maps, then (1) and (2) reduce to the parametric weak vector equilibrium problem in [5–7, 11].

For each $\lambda \in \Lambda$, the solution set of (1) is defined by

$$V^I(F, \lambda) := \{x \in A(\lambda) | F(x, y, \lambda) \nsubseteq -\mathrm{int}\, C, \forall y \in A(\lambda)\};$$

and for each $\lambda \in \Lambda$, the solution set of (2) is defined by

$$V^{II}(F, \lambda) := \{x \in A(\lambda) | F(x, y, \lambda) \subseteq Y \setminus -\mathrm{int}\, C, \forall y \in A(\lambda)\}.$$

For each $f \in C^*\backslash\{0\}$ and for each $\lambda \in \Lambda$, the f-efficient solutions set of (1) is defined by

$$V^I_f(F, \lambda) := \{x \in A(\lambda) | \forall y \in A(\lambda), \exists z \in F(x, y, \lambda), \text{ such that } f(z) \geq 0\};$$

and the f-efficient solution set of (2) is defined by

$$V^{II}_f(F, \lambda) = \Big\{x \in A(\lambda) | \inf_{z\in F(x,y,\lambda)} f(z) \geq 0, \forall y \in A(\lambda)\Big\}.$$

Throughout this paper, we always assume $V^I(F, \lambda) \neq \emptyset$ and $V^{II}(F, \lambda) \neq \emptyset$ for all $\lambda \in \Lambda$. This paper aims at investigating the lower semicontinuity of the solution mappings $V^I(F, \lambda)$ and $V^{II}(F, \lambda)$ as two set-valued maps from the set Λ into X. Now we recall some basic definitions and their properties.

Definition 2.1 Let $F : X \times X \times \Lambda \rightrightarrows Y\backslash\{\emptyset\}$ is a trifunction.

(i) $F(x, \cdot, \lambda)$ is called C-function on $A(\lambda)$ (i.e.,$F(x, A(\lambda), \lambda) + C$ is covex), iff for every $x_1, x_2 \in A(\lambda)$ and $t \in [0, 1]$, $tF(x, x_1, \lambda) + (1 - t)F(x, x_2, \lambda) \subset F(x, tx_1 + (1 - t)x_2, \lambda) + C$.

(ii) $F(x, \cdot, \lambda)$ is called C-like-function on $A(\lambda)$, iff for any $x_1, x_2 \in A(\lambda)$ and any $t \in [0, 1]$, there exists $x_3 \in A(\lambda)$ such that $tF(x, x_1, \lambda) + (1 - t)F(x, x_2, \lambda) \subset F(x, x_3, \lambda) + C$.

(iii) $F(\cdot, \cdot, \cdot)$ is called C-strictly monotone on $A(\Lambda) \times A(\Lambda) \times \Lambda$, iff for any given $\lambda \in \Lambda$, for all $x, y \in A(\lambda)$ and $x \neq y$, $F(x, y, \lambda) + F(y, x, \lambda) \subset -\mathrm{int}\, C$.

Definition 2.2 [12, 13] Let X and Y be topological spaces, $T : X \rightrightarrows Y$ be a set-valued mapping.

(i) T is said to be upper semicontinuous (u.s.c., for short) at $x_0 \in X$ iff for any open set V containing $T(x_0)$, there exists an open set U containing x_0 such that $T(x) \subseteq V$ for all $x \in U$.

(ii) T is said to be lower semicontinuous (l.s.c., for short) at $x_0 \in X$ iff for any open set V with $T(x_0) \cap V \neq \emptyset$, there exists an open set U containing x_0 such that $T(x) \cap V \neq \emptyset$ for all $x \in U$.

(iii) T is said to be continuous at $x_0 \in X$, if it is both l.s.c. and u.s.c. at $x_0 \in X$. T is said to be l.s.c. (resp. u.s.c.) on X, iff it is l.s.c. (resp. u.s.c.) at each $x \in X$.

Proposition 2.1 [13, 14] *Let X and Y be topological spaces, $T : X \rightrightarrows Y$ be a set-valued mapping.*

(i) *T is l.s.c. at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and any $y_0 \in T(x_0)$, there exists $y_\alpha \in T(x_\alpha)$ such that $y_\alpha \to y_0$.*

(ii) *If T has compact values (i.e., $T(x)$ is a compact set for each $x \in X$), then T is u.s.c. at x_0 if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and for any $y_\alpha \in T(x_\alpha)$, there exist $y_0 \in T(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$, such that $y_\beta \to y_0$.*

3 Lower Semicontinuity of the Solution Map to Weak Generalized PKFIs

In this section, we discuss the lower semicontinuity of the solutions to two classes of weak generalized PKFIs.

Using a similar method of Lemma 3.3 in [9], we can get the following results.

Lemma 3.1 *For each $\lambda \in \Lambda$, and for each $x \in A(\lambda)$,*

(i) *Suppose that for all $y \in A(\lambda)$, there exists a selection $z(y)$ of $F(x, y, \lambda) \setminus -\operatorname{int} C$ (that is, $z(y) \in F(x, y, \lambda) \setminus -\operatorname{int} C$), such that $\bigcup_{y \in A(\lambda)} z(y) + C$ be a convex set, then*

$$V^I(F, \lambda) = \bigcup_{f \in C^* \setminus \{0\}} V_f^I(F, \lambda).$$

(ii) *Suppose that $F(x, A(\lambda), \lambda) + C$ be a convex set, then*

$$V^{II}(F, \lambda) = \bigcup_{f \in C^* \setminus \{0\}} V_f^{II}(F, \lambda).$$

Theorem 3.1 *Let $f \in C^* \setminus \{0\}$. Suppose that the following conditions be satisfied:*

(i) *$A(\cdot)$ is continuous with nonempty compact value on Λ;*

(ii) *$F(\cdot, \cdot, \cdot)$ is u.s.c. with nonempty compact values on $B \times B \times \Lambda$;*

(iii) *For each $\lambda \in \Lambda$, $x \in A(\lambda) \setminus V_f^I(F, \lambda)$, there exists $y \in V_f^I(F, \lambda)$, such that*

$$F(x, y, \lambda) + F(y, x, \lambda) + B(0, d(x, y)) \subset -C.$$

Then, $V_f^I(F, \cdot)$ is l.s.c. on Λ.

Proof By the contrary, assume that there exists $\lambda_0 \in \Lambda$, such that $V_f^I(F, \cdot)$ is not l.s.c. at λ_0. Then there exist λ_α with $\lambda_\alpha \to \lambda_0$ and $x_0 \in V_f^I(F, \lambda_0)$, such that for any $x_\alpha \in V_f^I(F, \lambda_\alpha)$ with $x_\alpha \nrightarrow x_0$.

Since $x_0 \in A(\lambda_0)$ and $A(\cdot)$ is l.s.c. at λ_0, there exists a net $\hat{x}_\alpha \in A(\lambda_\alpha)$ such that $\hat{x}_\alpha \to x_0$. Obviously, $\hat{x}_\alpha \in A(\lambda) \setminus V_f^I(F,\lambda_\alpha)$. By (iii), there exists $y_\alpha \in V_f^I(F,\lambda_\alpha)$, such that

$$F(\hat{x}_\alpha, y_\alpha, \lambda_\alpha) + F(y_\alpha, \hat{x}_\alpha, \lambda_\alpha) + B\big(0, d(\hat{x}_\alpha, y_\alpha)\big) \subset -C. \tag{3}$$

For $y_\alpha \in V_f^I(F,\lambda_\alpha) \subset A(\lambda_\alpha)$, because $A(\cdot)$ is u.s.c. at λ_0 with compact values, there exist $y_0 \in A(\lambda_0)$ and a subnet $\{y_{\alpha_k}\}$ of $\{y_\alpha\}$ such that $y_{\alpha_k} \to y_0$. In particular, for (3), we have

$$F(\hat{x}_{\alpha_k}, y_{\alpha_k}, \lambda_{\alpha_k}) + F(y_{\alpha_k}, \hat{x}_{\alpha_k}, \lambda_{\alpha_k}) + B\big(0, d(\hat{x}_{\alpha_k}, y_{\alpha_k})\big) \subset -C.$$

Taking the limit as $\alpha_k \to +\infty$, we have

$$F(\hat{x}_0, y_0, \lambda_0) + F(y_0, \hat{x}_0, \lambda_0) + B\big(0, d(\hat{x}_0, y_0)\big) \subset -C. \tag{4}$$

It follows from $x_0 \in V_f^I(F,\lambda_0)$ and $y_0 \in A(\lambda_0)$ that there exists $z_{0_1} \in F(x_0, y_0, \lambda_0)$ such that

$$f(z_{0_1}) \geq 0. \tag{5}$$

On the other hand, since $y_{\alpha_k} \in V_f^I(F,\lambda_{\alpha_k})$ and $\hat{x}_{\alpha_k} \in A(\lambda_{\alpha_k})$, there exists $z_{\alpha_k} \in F(y_{\alpha_k}, \hat{x}_{\alpha_k}, \lambda_{\alpha_k})$ such that

$$f(z_{\alpha_k}) \geq 0. \tag{6}$$

Since $F(\cdot,\cdot,\cdot)$ is u.s.c. at (y_0, x_0, λ_0) with compact values, there exists $z_{0_2} \in F(y_0, x_0, \lambda_0)$ such that $z_{\alpha_k} \to z_{0_2}$ (taking a subnet if necessary). It follows from the continuity of f and (6) that

$$f(z_{0_2}) \geq 0. \tag{7}$$

By (5), (7) and the linearity of f, we get

$$f(z_{0_1} + z_{0_2}) = f(z_{0_1}) + f(z_{0_2}) \geq 0. \tag{8}$$

Assume that $x_0 \neq y_0$, by (4), we can obtain that

$$F(x_0, y_0, \lambda_0) + F(y_0, x_0, \lambda_0) \subset -\operatorname{int} C.$$

Then it follows from $f \in C^* \setminus \{0\}$ and $z_{0_1} + z_{0_2} \in -\operatorname{int} C$ that

$$f(z_{0_1} + z_{0_2}) < 0,$$

which is a contradiction to (8). Therefore, $x_0 = y_0$. This is impossible by the contradiction assumption. Therefore, $V_f^I(F,\cdot)$ is l.s.c. on Λ. The proof is complete. □

Remark 3.1 In [3, 5–9], under the condition of C-strict monotonicity, the continuity of the f-efficient solutions to the (GKFI) or (PKFI) is obtained. However, this condition is so strict that the f-efficient solution set for various PKFIs is confined to be a singleton. In our paper, we use assumption (iii) of Theorem 3.1 to weaken this condition. Furthermore, the f-efficient solution set may be a general set, but not a singleton. The following example is given to illustrate the case.

Example 3.1 Let $X = Z = Y = \mathbb{R}$, $C = \mathbb{R}_+$, $\Lambda = [0, 1]$ be a subset of Z. Let $F : X \times X \times \Lambda \rightrightarrows Y$ be a mapping defined by $F(x, y, \lambda) = [-10+3\lambda, (y^2+1)(x-\lambda)]$ and let $A : \Lambda \rightrightarrows X$ defined by $A(\lambda) = [\lambda^2, 1]$. For any given $\lambda \in \Lambda$, let $f(F(x, y, \lambda)) = \frac{3}{2}z$, $\forall z \in F(x, y, \lambda)$. Clearly, conditions (i)–(ii) of Theorem 3.1 are satisfied. It follows from direct computations that $A(\Lambda) = [0, 1]$, $V_f^I(F, \lambda) = [\lambda, 1]$, $\forall \lambda \in \Lambda$.

Obviously, the f-solution set to the (GKFI1) is not a singleton. The assumption (iii) in Theorem 3.1 can be checked as follows: For any $x \in A(\lambda) \setminus V_f^I(F, \lambda) = [0, \lambda)$, there exists $y = \lambda \in V_f^I(F, \lambda) = [\lambda, 1]$, such that

$$F(x, y, \lambda) + F(y, x, \lambda) + B(0, d(x, y)) \subset -C.$$

However, for $\lambda \in [0, \frac{1}{3}]$, there exists $x = 0, y = 1$ with $y \neq x$, such that $F(x, y, \lambda) + F(y, x, \lambda) \not\subseteq -\operatorname{int} C$, which implies that $F(\cdot, \cdot, \cdot)$ is not C-strictly monotone on $A(\Lambda) \times A(\Lambda) \times \Lambda$.

Theorem 3.2 *For each $f \in C^* \setminus \{0\}$. Suppose that the following conditions be satisfied*:

(i) *$A(\cdot)$ is continuous with nonempty compact value on Λ*;
(ii) *$F(\cdot, \cdot, \cdot)$ is u.s.c. with nonempty compact values on $B \times B \times \Lambda$*;
(iii) *For each $\lambda \in \Lambda$, $x \in A(\lambda) \setminus V_f^I(F, \lambda)$, there exists $y \in V_f^I(F, \lambda)$, such that*

$$F(x, y, \lambda) + F(y, x, \lambda) + B(0, d(x, y)) \subset -C.$$

(iv) *For each $\lambda \in \Lambda$, $x \in A(\lambda)$ and for all $y \in A(\lambda)$, there exists a selection $z(y)$ of $F(x, y, \lambda) \setminus -\operatorname{int} C$, such that $\bigcup_{y \in A(\lambda)} z(y) + C$ be a convex set.*

Then, $V^I(F, \cdot)$ is l.s.c. on Λ.

Proof It follows from assumption (iv) and Lemma 3.1 that for each $\lambda \in \Lambda$,

$$V^I(F, \lambda) = \bigcup_{f \in C^* \setminus \{0\}} V_f^I(F, \lambda).$$

By Theorem 3.1, for each $f \in C^* \setminus \{0\}$, $V_f^I(F, \cdot)$ is l.s.c. on Λ. Therefore, in view of Theorem 2 (p. 114 in [14]), we have $V^I(F, \cdot)$ is l.s.c. on Λ. This completes the proof. □

Now, we give an example to illustrate that our result extends that of [9].

Example 3.2 Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $\Lambda = [-1, 1]$ be a subset of Z. Let $F : X \times X \times \Lambda \rightrightarrows Y$ be a mapping defined by $F(x, y, \lambda) = [-10 + |x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} - \lambda, (\lambda^2 + 2)x]$ and let $A : \Lambda \rightrightarrows X$ defined by $A(\lambda) = [-\lambda^2, 1]$. Then, it can be shown that $A(\lambda) = [-1, 1]$ and $V^I(F, \lambda) = [0, 1]$. For each $f \in C^* \setminus \{0\}$, it follows from a direct computation that if $V_f^I(F, \lambda) \neq \emptyset$, then $0 \in V_f^I(F, \lambda)$. Clearly, conditions (i), (ii), (iv) of Theorem 3.2 are satisfied. For any $x \in A(\lambda) \setminus V_f^I(F, \lambda)$, there exists

$y=0\in V_f^I(F,\lambda)$, such that

$$\begin{aligned}&F(x,y,\lambda)+F(y,x,\lambda)+B(0,d(x,y))\\&\quad=\left[-20+2|x|^{\frac{1}{2}}-2\lambda,(\lambda^2+2)x\right]+B(0,d(x,0))\subset -C.\end{aligned}$$

Thus, the condition (iii) of Theorem 3.2 is also satisfied. By Theorem 3.2, $V^I(F,\cdot)$ is l.s.c. on Λ.

However, for $\forall x\in A(\lambda)$, exists $y=-x\in A(\lambda)$ with $y\neq x$, such that $F(x,y,\lambda)+F(y,x,\lambda)\nsubseteq -\operatorname{int}C$, which implies that $F(\cdot,\cdot,\cdot)$ is not C-strictly monotone on $A(\Lambda)\times A(\Lambda)\times\Lambda$. Thus, Theorem 3.7 in [9] is not applicable.

Now, we discuss the lower semicontinuity of the solutions to weak generalized PKFI (2).

Theorem 3.3 *Let $f\in C^*\setminus\{0\}$. For the problem (2), suppose that the following conditions be satisfied:*

(i) *$A(\cdot)$ is continuous with nonempty compact value on Λ;*
(ii) *$F(\cdot,\cdot,\cdot)$ is u.s.c. with nonempty compact values on $B\times B\times\Lambda$;*
(iii) *For each $\lambda\in\Lambda$, $x\in A(\lambda)\setminus V_f^{II}(F,\lambda)$, there exists $y\in V_f^{II}(F,\lambda)$, such that*

$$F(x,y,\lambda)+F(y,x,\lambda)+B(0,d(x,y))\subset -C.$$

Then, $V_f^{II}(F,\cdot)$ is l.s.c. on Λ.

Proof In a similar way to the proof of Theorem 3.1, with suitable modifications, we can obtain the conclusion. □

The following example illustrates that the assumption (iii) in Theorem 3.3 is essential.

Example 3.3 Let $X=Y=\mathbb{R}$, $C=\mathbb{R}_+$. Let $\Lambda=[1,2]$ be a subset of Z. For each $\lambda\in\Lambda$, $x,y\in X$, let $A(\lambda)=[\lambda-1,1]$ and $F(x,y,\lambda)=[(10-\lambda^2)\lambda x(x-y),+\infty)$. Obviously, assumptions (i) and (ii) of Theorem 3.3 are satisfied, and $A(\lambda)=[0,1]$, $\forall\lambda\in\Lambda$. For any given $\lambda\in\Lambda$, let $f(F(x,y,\lambda))=2z$, $\forall z\in F(x,y,\lambda)$. Then, it follows from a direct computation that

$$V_f^{II}(F,1)=\{0,1\}\quad\text{and}\quad V_f^{II}(F,\lambda)=1,\quad\forall\lambda\in(1,2].$$

Hence $V_f^{II}(F,\lambda)$ is even not l.c.s at $\lambda=1$. The reason is that the assumption (iii) is violated. Indeed, if $x'=0\in V_f^{II}(F,\lambda)$, for $\lambda=1$, there exist $y=\frac{1}{4}\in A(\lambda)\setminus V_f^{II}(F,\lambda)=(0,1)$, such that

$$\begin{aligned}&F(x,y,\lambda)+F(y,x,\lambda)+B(0,d(x,y))\\&\quad=\left[(10-\lambda^2)\lambda x(x-y),+\infty\right)+\left[(10-\lambda^2)\lambda y(y-x),+\infty\right)+B(0,d(x,y))\\&\quad=\left[(10-\lambda^2)\lambda(x-y)^2,+\infty\right)+B(0,d(x,y))\end{aligned}$$

$$= \left[\frac{9}{16}, +\infty\right) + B\left(0, \frac{1}{4}\right) \nsubseteq -C;$$

if $x' = 1 \in V_f^{II}(F,\lambda)$, for $\lambda = 1$, there exist $y = \frac{1}{4} \in A(\lambda) \setminus V_f^{II}(F,\lambda)$, using a similar method, we have $F(x,y,\lambda) + F(y,x,\lambda) + B(0,d(x,y)) = [\frac{81}{16}, +\infty) + B(0,\frac{3}{4}) \nsubseteq -C$. Therefore, (iii) is violated.

Theorem 3.4 *For the problem* (2), *suppose that the following conditions be satisfied*:

(i) $A(\cdot)$ *is continuous with nonempty compact value on* Λ;
(ii) $F(\cdot,\cdot,\cdot)$ *is u.s.c. with nonempty compact values on* $B \times B \times \Lambda$;
(iii) *For each* $f \in C^* \setminus \{0\}$, $\lambda \in \Lambda$, *and* $x \in A(\lambda) \setminus V_f^{II}(F,\lambda)$, *there exists* $y \in V_f^{II}(F,\lambda)$, *such that*

$$F(x,y,\lambda) + F(y,x,\lambda) + B(0,d(x,y)) \subset -C.$$

(iv) *For each* $\lambda \in \Lambda$ *and for each* $x \in A(\lambda)$, $F(x,\cdot,\lambda)$ *is* C*-like-function on* $A(\lambda)$.

Then, $V^{II}(F,\cdot)$ *is l.s.c. on* Λ.

Proof For each $\lambda \in \Lambda$ and for each $x \in A(\lambda)$, since $F(x,\cdot,\lambda)$ is C-like-function on $A(\lambda)$, $F(x, A(\lambda), \lambda) + C$ is convex. Thus, by virtue of Lemma 3.1(ii), for each $\lambda \in \Lambda$, it holds

$$V^{II}(F,\lambda) = \bigcup_{f \in C^* \setminus \{0\}} V_f^{II}(F,\lambda).$$

By Theorem 3.3, for each $f \in C^* \setminus \{0\}$, $V_f^{II}(F,\cdot)$ is *l.s.c.* on Λ. Therefore, in view of Theorem 2 in [14, p. 114], we have $V^{II}(F,\cdot)$ is l.s.c. on Λ. The proof is complete. □

Remark 3.2 Theorem 3.4 generalizes Theorem 3.1 in [10] for the mapping F from vector-valued version to set-valued version.

Now, we give an example to illustrate that our result extends those of [8] and [5–7].

Example 3.4 Let $X = Z = Y = \mathbb{R}$, $C = \mathbb{R}_+$, $\Lambda = [0, 2^{\frac{1}{2}}]$. And let $A : \Lambda \rightrightarrows X$ defined by $A(\lambda) = [\lambda^2, 2]$, $F(x,y,\lambda) = (-\infty, (y+1)(\lambda^2+1)(x-\lambda)]$. Then, $V^{II}(F,\lambda) = [\lambda, 2]$, $\forall \lambda \in \Lambda = [0, 2^{\frac{1}{2}}]$. It is easy to verify that all assumptions of Theorem 3.4 are satisfied. By Theorem 3.4, $V^{II}(F,\cdot)$ is l.s.c. on Λ.

However, for any $\lambda \in [0,1] \subset \Lambda = [0, 2^{\frac{1}{2}}]$, there exists $x = \lambda \in A(\lambda)$, $y = 2 \in A(\lambda)$ with $y \neq x$, such that

$$F(x,y,\lambda) + F(y,x,\lambda) = \big(-\infty, (\lambda+1)\big(\lambda^2+1\big)(2-\lambda)\big] \nsubseteq -\mathrm{int}\, C,$$

which implies that $F(\cdot,\cdot,\cdot)$ is not $\mathbb{R}_+$-strictly monotone on $A(\Lambda) \times A(\Lambda) \times \Lambda$. Then, Theorems 3.1–3.2 in [8] are not applicable, and the corresponding results in references (e.g. [5–7]) are also not applicable.

4 Conclusion

In this paper, under new assumptions, which are weaker than C-strict monotonicity, we establish sufficient conditions for the lower semicontinuity of the solutions to two class of weak generalized PKFIs with set-valued maps in the case where f-solution set is a general set-valued one. These results extend and improve the corresponding ones obtained in [3, 5–10]. We have also showed some examples to illustrate the case.

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