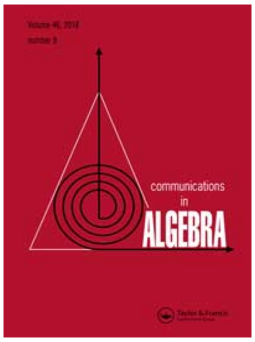

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Foxby equivalences associated to Gorenstein categories #(#,#,#)

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Foxby equivalences associated to Gorenstein categories $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$

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ABSTRACT
In this paper, we establish some new Foxby equivalences between some Gorenstein subcategories in the Auslander class $\mathcal{A}_C(R)$ and that in the Bass class $\mathcal{B}_C(S)$ in a general setting. Our results provide a unification and generalization of some known results and generate some new Foxby equivalences of categories.

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1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary. Let $\mathcal{A}$ be an abelian category and $\mathcal{X},\mathcal{Y},\mathcal{Z}$ additive full subcategories of $\mathcal{A}$. Yang [19] introduced the Gorenstein category $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ as follows

$$\begin{aligned}\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = \{&M \text{ is an object of } \mathcal{A} \mid \text{there exists an exact sequence of objects in } \mathcal{X}\\ &\cdots \to X_1 \to X_0 \to X^0 \to X^1 \to \cdots, \text{which is both } \mathrm{Hom}_{\mathcal{A}}(\mathcal{Y},-)\text{-exact}\\ &\text{and } \mathrm{Hom}_{\mathcal{A}}(-,\mathcal{Z})\text{-exact, such that } M \cong \mathrm{Im}(X_0 \to X^0)\}.\end{aligned}$$

This definition unifies the following notions: Gorenstein projective (injective) modules [7, 8, 11], Ding projective (injective) modules [6, 15, 18], Gorenstein AC-projective (AC-injective) modules [3, 4], $\mathscr{X}$-Gorenstein projective (injective) modules [2, 16], C-Gorenstein projective (injective) modules [10], Ding C-projective (C-injective) modules [20], the Gorenstein category $\mathcal{G}(\mathcal{C})$ [17], and so on.

Let C be a semidualizing module over a commutative noetherian ring R. Enochs et al. [9, Proposition 2.1] established the following Foxby equivalence of categories:

$$\mathcal{A}_C(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{B}_C(R),$$

where $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ denote the Auslander class and the Bass class with respect to C, respectively. Holm and White [12] extended this result to the non-noetherian and non-commutative setting. Later, Foxby equivalences between some special classes of modules in the Auslander class $\mathcal{A}_C(R)$ and that in the Bass class $\mathcal{B}_C(S)$ have been studied by many authors, see [5, 9, 10, 12, 14, 20].

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Let ${}_SC_R$ be a faithfully semidualizing bimodule. It was shown in [10, Theorem 3.11] that there exists the following Foxby equivalence diagram:

$$\mathcal{A}_C(R) \cap \mathcal{G}\text{-}Proj \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{G}_C\text{-}Proj,$$

where $\mathcal{G}$-*Proj* denotes the category of Gorenstein projective left R-modules, and $\mathcal{G}_C$-*Proj* denotes the category of C-Gorenstein projective left S-modules in [10]. Note that $\mathcal{G}_C\text{-}Proj \subseteq \mathcal{B}_C(S)$, thus we have

$$\mathcal{A}_C(R) \cap \mathcal{G}\text{-}Proj \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{B}_C(S) \cap \mathcal{G}_C\text{-}Proj. \qquad (*)$$

Also, Zhang et al. [20] introduced the Ding C-projective modules, and established the similar equivalence of categories.

Inspired by the Foxby equivalence $(*)$ and the similar result in [20], in this paper, we consider the Foxby equivalences between some Gorenstein subcategories in the Auslander class $\mathcal{A}_C(R)$ and that in the Bass class $\mathcal{B}_C(S)$ in a general setting. Let ${}_SC_R$ be a faithfully semidualizing bimodule. We show that if $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \mathcal{A}_C(R)$, then there is an equivalence of categories:

$$\mathcal{A}_C(R) \cap \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{B}_C(S) \cap \mathcal{G}_1(\mathcal{X}, \mathcal{Y}, \mathcal{Z}),$$

where $\mathcal{G}_1(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ denotes the Gorenstein category $\mathcal{G}(C \otimes_R \mathcal{X}, C \otimes_R \mathcal{Y}, C \otimes_R \mathcal{Z})$. Dually, we also show that if $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \mathcal{B}_C(S)$, then there is an equivalence of categories:

$$\mathcal{A}_C(R) \cap \mathcal{G}_2(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{B}_C(S) \cap \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}),$$

where $\mathcal{G}_2(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ denotes the Gorenstein category $\mathcal{G}(\mathrm{Hom}_S(C, \mathcal{X}), \mathrm{Hom}_S(C, \mathcal{Y}), \mathrm{Hom}_S(C, \mathcal{Z}))$. Our results provide a unification and generalization of the known results in [10, Theorem 3.11], [20, Proposition 4.2] and generate some new Foxby equivalences of categories.

2. Preliminaries

In this section, we will recall some notions and terminologies which we need in the later section.

Semidualizing bimodules. An (S, R)-bimodule $C = {}_SC_R$ is semidualizing if

(1) ${}_SC$ admits a degreewise finite S-projective resolution.
(2) C_R admits a degreewise finite R-projective resolution.
(3) The homothety map ${}_SS_S \xrightarrow{S\gamma} \mathrm{Hom}_R(C, C)$ is an isomorphism.
(4) The homothety map ${}_RR_R \xrightarrow{\gamma_R} \mathrm{Hom}_S(C, C)$ is an isomorphism.
(5) $\mathrm{Ext}_S^{\geqslant 1}(C, C) = 0$.
(6) $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$.

A semidualizing bimodule ${}_SC_R$ is faithfully semidualizing if it satisfies the following conditions for all modules ${}_SN$ and M_R

(a) If $\mathrm{Hom}_S(C, N) = 0$, then $N = 0$.
(b) If $\mathrm{Hom}_R(C, M) = 0$, then $M = 0$.

Auslander class and Bass class with respect to C. The Auslander class $\mathcal{A}_C(R)$ with respect to C consists of all R-modules M satisfying

(1) $\mathrm{Tor}^R_{\geqslant 1}(C, M) = 0$,
(2) $\mathrm{Ext}_S^{\geqslant 1}(C, C \otimes_R M) = 0$,
(3) The natural evaluation homomorphism $\mu_M : M \to \mathrm{Hom}_S(C, C \otimes_R M)$ is an isomorphism.

The Bass class $\mathcal{B}_C(S)$ with respect to C consists of all S-modules N satisfying
(1) $\mathrm{Ext}_S^{\geqslant 1}(C, N) = 0$,
(2) $\mathrm{Tor}^R_{\geqslant 1}(C,\mathrm{Hom}_S(C, N)) = 0$,
(3) The natural evaluation homomorphism $\nu_N : C \otimes_R \mathrm{Hom}_S(C, N) \to N$ is an isomorphism.

***$\mathcal{X}$*-resolution.** Let $\mathcal{X}$ be a class of left (resp., right) R-modules and M a left (resp., right) R-module. A left $\mathcal{X}$-resolution of M is an exact sequence

$$\mathbb{X} = \cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$$

with $X_i \in \mathcal{X}$ for all $i \geq 0$. Dually, one can define the right $\mathcal{X}$-resolution of M.

3. Main results

In the following, we assume that $\mathcal{A}$ is the category of modules, and $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are full subcategories of $\mathcal{A}$. C always stands for a faithfully semidualizing bimodule ${}_SC_R$. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \mathcal{A}_C(R)$, we use $\mathcal{G}_1(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ to denote the Gorenstein category $\mathcal{G}(C \otimes_R \mathcal{X}, C \otimes_R \mathcal{Y}, C \otimes_R \mathcal{Z})$.

Lemma 3.1. *Assume that $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \mathcal{A}_C(R)$ and $M \in \mathcal{B}_C(S)$. Then the following are equivalent:*
(1) *$M \in \mathcal{G}_1(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$.*
(2) *$\mathrm{Hom}_S(C, M) \in \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$.*

Proof.
(1)⇒(2) If $M \in \mathcal{G}_1(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$, then there exists an exact sequence

$$\mathbb{H} : \cdots \to C \otimes_R X_1 \to C \otimes_R X_0 \to C \otimes_R X^0 \to C \otimes_R X^1 \to \cdots$$

with $X^i, X_j \in \mathcal{X}$ and $M \cong \mathrm{Im}(C \otimes_R X_0 \to C \otimes_R X^0)$ such that $\mathrm{Hom}_S(C \otimes_R Y, -)$ and $\mathrm{Hom}_S(-, C \otimes_R Z)$ leave the sequence exact for any $Y \in \mathcal{Y}$ and $Z \in \mathcal{Z}$. Since $M \in \mathcal{B}_C(S)$ and $C \otimes_R \mathcal{X} \subseteq \mathcal{B}_C(S)$, we have that every kernel and cokernel in $\mathbb{H}$ are in $\mathcal{B}_C(S)$ by [12, Corollary 6.3]. Note that $\mathcal{X} \subseteq \mathcal{A}_C(R)$, then $\mathrm{Hom}_S(C, C \otimes_R X) \cong X$ for any $X \in \mathcal{X}$. If we apply the functor $\mathrm{Hom}_S(C, -)$ to the exact sequence $\mathbb{H}$, we obtain an exact sequence

$$\mathbb{F} : \cdots \to X_1 \to X_0 \to X^0 \to X^1 \to \cdots$$

with $\mathrm{Hom}_S(C, M) \cong \mathrm{Im}(X_0 \to X^0)$. Given any $Y \in \mathcal{Y}$ and $Z \in \mathcal{Z}$, we get that both

$$\mathrm{Hom}_S(C \otimes_R Y, \mathbb{H}) \cong \mathrm{Hom}_R(Y, \mathrm{Hom}_S(C, \mathbb{H})) \cong \mathrm{Hom}_R(Y, \mathbb{F})$$

and

$$\mathrm{Hom}_S(\mathbb{H}, C \otimes_R Z) \cong \mathrm{Hom}_R(\mathbb{F}, \mathrm{Hom}_S(C, C \otimes_R Z)) \cong \mathrm{Hom}_R(\mathbb{F}, Z)$$

are exact. This means that $\mathrm{Hom}_S(C, M) \in \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$.

(2)⇒(1) If $\mathrm{Hom}_S(C, M) \in \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$, then there exists an exact sequence

$$\mathbb{F} : \cdots \to X_1 \to X_0 \to X^0 \to X^1 \to \cdots$$

with $X^i, X_j \in \mathcal{X}$ and $\mathrm{Hom}_S(C, M) \cong \mathrm{Im}(X_0 \to X^0)$ such that $\mathrm{Hom}_R(Y, -)$ and $\mathrm{Hom}_R(-, Z)$ leave the sequence exact for any $Y \in \mathcal{Y}$ and $Z \in \mathcal{Z}$. Given that $M \in \mathcal{B}_C(S)$, then $\mathrm{Hom}_S(C, M) \in \mathcal{A}_C(R)$. Note that $\mathcal{X} \subseteq \mathcal{A}_C(R)$, then every kernel and cokernel in $\mathbb{F}$ are in $\mathcal{A}_C(R)$ by [12, Corollary 6.3]. Applying the functor $C \otimes_R -$ to the exact sequence $\mathbb{F}$, we obtain an exact sequence

$$\mathbb{H} : \cdots \to C \otimes_R X_1 \to C \otimes_R X_0 \to C \otimes_R X^0 \to C \otimes_R X^1 \to \cdots$$

such that $M \cong \mathrm{Im}(C \otimes_R X_0 \to C \otimes_R X^0)$. Given any $Y \in \mathcal{Y}$ and $Z \in \mathcal{Z}$, we have that both $\mathrm{Hom}_S(C \otimes_R Y, \mathbb{H}) \cong \mathrm{Hom}_R(Y, \mathbb{F})$ and $\mathrm{Hom}_S(\mathbb{H}, C \otimes_R Z) \cong \mathrm{Hom}_R(\mathbb{F}, Z)$ are exact. This means that $M \in \mathcal{G}_1(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$. □

Lemma 3.2. *Assume that $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \mathcal{A}_C(R)$ and $M \in \mathcal{A}_C(R)$. Then the following are equivalent:*
(1) $M \in \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$.
(2) $C \otimes_R M \in \mathcal{G}_1(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$.

Proof.
(1)⇒(2) Since $M \in \mathcal{A}_C(R)$, we have $C \otimes_R M \in \mathcal{B}_C(S)$. Note that $\mathrm{Hom}_S(C, C \otimes_R M) \cong M \in \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ by hypothesis, then Lemma 3.1 implies that $C \otimes_R M \in \mathcal{G}_1(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$.

(2)⇒(1) Since $M \in \mathcal{A}_C(R)$, it follows that $M \cong \mathrm{Hom}_S(C, C \otimes_R M)$. Note that $C \otimes_R M \in \mathcal{G}_1(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$, in view of Lemma 3.1, we conclude that $\mathrm{Hom}_S(C, C \otimes_R M) \cong M \in \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$. □

Theorem 3.3. *Assume that $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \mathcal{A}_C(R)$. There is an equivalence of categories:*

$$\mathcal{A}_C(R) \cap \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \underset{Hom_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{B}_C(S) \cap \mathcal{G}_1(\mathcal{X}, \mathcal{Y}, \mathcal{Z}).$$

Proof. By Lemmas 3.1 and 3.2, it is obvious that the functor $C \otimes_R -$ maps $\mathcal{A}_C(R) \cap \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ to $\mathcal{B}_C(S) \cap \mathcal{G}_1(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$, also the functor $\mathrm{Hom}_S(C, -)$ maps $\mathcal{B}_C(S) \cap \mathcal{G}_1(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ to $\mathcal{A}_C(R) \cap \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$. Moreover, note that if $M \in \mathcal{A}_C(R) \cap \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ and $N \in \mathcal{B}_C(S) \cap \mathcal{G}_1(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$, then there exist natural isomorphisms $M \cong \mathrm{Hom}_S(C, C \otimes_R M)$ and $N \cong C \otimes_R \mathrm{Hom}_S(C, N)$. Therefore the desired equivalence of categories follows. □

Let $\mathcal{W}$ be a class of modules, and $0 \to N \to G \to M \to 0$ an arbitrary exact sequence of modules. According to [12], the class $\mathcal{W}$ is closed under extensions provided that both M and N are in $\mathcal{W}$, then $G \in \mathcal{W}$. The class $\mathcal{W}$ is closed under kernels of epimorphisms if whenever G and M are in $\mathcal{W}$, then so is N. Finally, $\mathcal{W}$ is closed under cokernels of monomorphisms if whenever N and G are in $\mathcal{W}$, then so is M. The class $\mathcal{W}$ is projectively resolving if $\mathcal{W}$ contains every projective module, and $\mathcal{W}$ is closed under extensions and kernels of epimorphisms. The notion of injectively resolving is defined dually.

Proposition 3.4. *Let M be a module. If $\mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is closed under extensions, then M has a left $\mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$-resolution if and only if M has a left $\mathcal{X}$-resolution.*

Proof. Note that $\mathcal{X} \subseteq \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$, so if M has a left $\mathcal{X}$-resolution, then M has a left $\mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$-resolution. Conversely, let $0 \to N \to G_0 \to M \to 0$ be an exact sequence with $G_0 \in \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ and N having a left $\mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$-resolution. Since $G_0 \in \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$, there exists an exact sequence $0 \to D \to X_0 \to G_0 \to 0$ with $X_0 \in \mathcal{X}$ and $D \in \mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$. Consider the following pullback diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & D \arrow[r, equal] \arrow[d] & D \arrow[d] & & \\
0 \arrow[r] & H \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since N has a left $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-resolution, there exists an exact sequence $0 \to B \to G_1 \to N \to 0$ with $G_1 \in \mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ and B having a left $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-resolution. Consider the following pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & B \arrow[r, equal] \arrow[d] & B \arrow[d] & \\
0 \arrow[r] & D \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D \arrow[r] & H \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Note that $D, G_1 \in \mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ from the middle horizontal sequence, it follows that $L \in \mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ since $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is closed under extensions by hypothesis. Then H has a left $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-resolution. Note that $0 \to H \to X_0 \to M \to 0$ is exact from the first diagram. By repeating the preceding process, we have that M has a left $\mathcal{X}$-resolution. □

Lemma 3.5. *Assume that $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is projectively resolving. Consider the following exact sequences of modules*

$$0 \to K_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0,$$

$$0 \to L_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to M \to 0,$$

where each G_i and Q_i are in $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$. Then $K_n \in \mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ if and only if $L_n \in \mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$.

Proof. Suppose $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is projectively resolving. It is easy to check that $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ closed under countable direct sums. Moreover, $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is closed under direct summands by [19, Theorem 2.9]. Then the stated result is a direct consequence of [1, Lemma 3.12]. □

Remark 3.6.

(1) Most of the results in [19], the assumptions $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{X} \subseteq \mathcal{Z}$ are used. However, upon reading the proof of [19, Theorem 2.9], it doesn't seem that the containments assumptions are needed. So we don't have to assume that $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{X} \subseteq \mathcal{Z}$ in Lemma 3.5.

(2) In Lemma 3.5, we assume that $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is projectively resolving. On the one hand, this assumption ensures the existence of the left $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-resolution of any module M; on the other hand, it guarantees the rationality of the $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-projective dimension, which is defined in the below. Lemma 3.5 implies that if $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is projectively resolving, then the $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-projective dimension is independent of the choice of the left $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-resolutions.

At this point, we introduce the $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-projective dimension.

Definition 3.7. Suppose $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is projectively resolving. The $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-projective dimension, $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-pd$(M)$, of a module M is defined by declaring that $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-pd$(M) \leq n$ if and only if M has a left $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-resolution

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0.$$

If no such finite sequence exists, define $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-pd$(M)=\infty$; otherwise, if n is the least such integer, define $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-pd$(M)=n$.

Proposition 3.8. *Assume that $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is projectively resolving. Let M be a left S-module with finite $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$-projective dimension and n a nonnegative integer. Then the following are equivalent:*

(1) *$\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$-pd$(M)\leq n$.*

(2) *There exists an exact sequence $0\to C\otimes_R X_n\to\cdots\to C\otimes_R X_2\to C\otimes_R X_1\to G\to M\to 0$ with $G\in\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$ and $X_i\in\mathcal{X}$ for all $1\leq i\leq n$.*

(3) *For any exact sequence of left S-modules $0\to K_n\to G_{n-1}\to\cdots\to G_1\to G_0\to M\to 0$ with $G_i\in\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$ for all $0\leq i\leq n-1$, then $K_n\in\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$.*

Proof.

(2)⇒(1) is obvious. (3)⇔(1) is clear by Lemma 3.5.

(1)⇒(2) Assume that $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$-pd$(M)\leq n$. Then there exists an exact sequence

$$0\to G_n\xrightarrow{f}G_{n-1}\to\cdots\to G_0\to M\to 0$$

with $G_i\in\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$ for any $0\leq i\leq n$. Since $G_n\in\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$, there exists an exact sequence $0\to G_n\xrightarrow{g}C\otimes_R X_n\to N\to 0$ with $X_n\in\mathcal{X}$ and $N\in\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$. Consider the pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & G_n \arrow[r, "f"] \arrow[d, "g"] & G_{n-1} \arrow[r] \arrow[d] & G_{n-2} \arrow[r] & \cdots \\
0 \arrow[r] & C\otimes_R X_n \arrow[r, "h"] \arrow[d] & D_{n-1} \arrow[d] & & \\
 & N \arrow[r, equal] \arrow[d] & N \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Then $D_{n-1}\in\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$ since $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is closed under extensions by hypothesis. Clearly, f and h have isomorphic cokernels, so we have the following exact sequence

$$0\to C\otimes_R X_n\xrightarrow{h}D_{n-1}\to G_{n-2}\to\cdots\to G_0\to M\to 0.$$

Continuing this process yields the sequence. □

Let n be a nonnegative integer. If $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is projectively resolving, we use $\mathcal{G}_P(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}$ to denote the class of modules with $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-projective dimension at most n. In particular, if $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is projectively resolving, $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}$ denotes the class of left S-modules with $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$-projective dimension at most n. The symbol $\mathcal{P}_C(S)$ stands for the class $C\otimes_R P$ with P a projective left R-module.

Theorem 3.9. *Let n be a nonnegative integer and $\mathcal{X},\mathcal{Y},\mathcal{Z}\subseteq\mathcal{A}_C(R)$. If both $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ and $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$ are closed under extensions and kernels of epimorphisms, and $\mathcal{X}$ contains every projective left R-module, then there is an equivalence of categories:*

$$\mathcal{A}_C(R)\cap\mathcal{G}_P(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}\ \underset{Hom_S(C,-)}{\overset{C\otimes_R-}{\rightleftarrows}}\ \mathcal{B}_C(S)\cap\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}\,.$$

Proof. Since $\mathcal{X}$ contains every projective left R-module, we have that the left $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-resolution of any module M exists by Proposition 3.4. Note that ${}_SC_R$ is a faithfully semidualizing bimodule, then the class $\mathcal{P}_C(S)$ is projectively resolving by [12, Corollary 6.4]. Thus for any projective left S-module N, we have $N \in \mathcal{P}_C(S) \subseteq C \otimes \mathcal{X} \subseteq \mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$, and so the left $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$-resolution of any module M always exists. Moreover, both $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ and $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$ are projectively resolving.

If $n=0$, the result is true by Theorem 3.3. In the following, we assume that $n \geq 1$.

Let $M \in \mathcal{A}_C(R) \cap \mathcal{G}_P(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}$. Then by Proposition 3.8 with $C = {}_RR_R$, there exists an exact sequence of R-modules

$$0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to G \to M \to 0$$

with $G \in \mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ and $X_i \in \mathcal{X}$ for any $1 \leq i \leq n$. By [12, Corollary 6.3], we have that G and every cokernel in the above sequence are in $\mathcal{A}_C(R)$. Applying the functor $C \otimes_R -$ to the exact sequence, we get the following exact sequence

$$0 \to C \otimes_R X_n \to C \otimes_R X_{n-1} \to \cdots \to C \otimes_R X_1 \to C \otimes_R G \to C \otimes_R M \to 0$$

with $C \otimes_R G$ and all $C \otimes_R X_i$ are in $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z}) \cap \mathcal{B}_C(S)$ by Lemma 3.2, which means that $C \otimes_R M \in \mathcal{B}_C(S) \cap \mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}$.

Conversely, let $N \in \mathcal{B}_C(S) \cap \mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}$. Then by Proposition 3.8, there exists an exact sequence of S-modules

$$0 \to C \otimes_R X_n \to C \otimes_R X_{n-1} \to \cdots \to C \otimes_R X_1 \to G \to N \to 0$$

with $G \in \mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$ and $X_i \in \mathcal{X}$ for any $1 \leq i \leq n$. By [12, Corollary 6.3], we have that G and every kernel in the above sequence are in $\mathcal{B}_C(S)$. Applying the functor $\mathrm{Hom}_S(C,-)$ to it, we have the following exact sequence of R-modules

$$0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to \mathrm{Hom}_S(C,G) \to \mathrm{Hom}_S(C,N) \to 0.$$

Since $G \in \mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z}) \cap \mathcal{B}_C(S)$, we have that $\mathrm{Hom}_S(C,G) \in \mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ by Lemma 3.1. Thus $\mathrm{Hom}_S(C,N) \in \mathcal{A}_C(R) \cap \mathcal{G}_P(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}$.

Moreover, take any $M \in \mathcal{A}_C(R) \cap \mathcal{G}_P(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}$ and $N \in \mathcal{B}_C(S) \cap \mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}$, there exist natural isomorphisms $M \cong \mathrm{Hom}_S(C, C \otimes_R M)$ and $N \cong C \otimes_R \mathrm{Hom}_S(C,N)$. Therefore the desired equivalence of categories follows. □

We use $\mathcal{P}(R)$, $\mathcal{F}(R)$, $\mathcal{I}(S)$ and $\mathcal{FI}(S)$ to denote the classes of projective left R-modules, flat left R-modules, injective left S-modules and FP-injective left S-modules, respectively. Then $\mathcal{P}(R) \subseteq \mathcal{F}(R) \subseteq \mathcal{A}_C(R)$ and $\mathcal{I}(S) \subseteq \mathcal{B}_C(S)$ by [12, Lemma 4.1].

Corollary 3.10. *Let n be a nonnegative integer. If $\mathcal{X} = \mathcal{Y} = \mathcal{P}(R) \subseteq \mathcal{Z} \subseteq \mathcal{A}_C(R)$, then there is an equivalence of categories:*

$$\mathcal{A}_C(R) \cap \mathcal{G}_P(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n} \underset{Hom_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}.$$

Proof. Clearly, $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z}) \subseteq \mathcal{B}_C(S)$ by [10, Proposition 3.5]. Then $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n} \subseteq \mathcal{B}_C(S)$ by [12, Corollary 6.3]. Moreover, $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is closed under extensions and kernels of epimorphisms by [19, Corollary 2.5]. Take any $Z \in \mathcal{Z}$ and for any pair of projective left R-modules P_1, P_2, we have

$$\mathrm{Ext}^i_S(C \otimes_R P_1, C \otimes_R P_2) \cong \mathrm{Ext}^i_R(P_1, P_2) = 0$$

and

$$\mathrm{Ext}^i_S(C \otimes_R P_2, C \otimes_R Z) \cong \mathrm{Ext}^i_R(P_2, Z) = 0,$$

for any $i \geq 1$, by [12, Theorem 6.4]. According to [19, Corollary 2.5], we obtain that $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is closed under extensions. Furthermore, given any exact sequence $\mathbb{H}: 0 \to M \to Q \to N \to 0$ with $Q, N \in \mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$, we have $M \in \mathcal{B}_C(S)$ by [12, Corollary 6.3], then $\mathrm{Hom}_S(C,\mathbb{H})$ is exact. In view of [12, Lemma 6.1], we conclude that $\mathrm{Hom}_S(C \otimes_R P,\mathbb{H})$ is exact for any projective left R-module P. Thus $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is closed under kernels of epimorphisms by [19, Corollary 2.5]. Then the result follows from Theorem 3.9. □

Suppose $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is injectively resolving. Dually, we can define $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-injective dimension, $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-id$(M)$, of M.

Let n be a nonnegative integer. If $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is injectively resolving, we use $\mathcal{G}_I(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}$ to denote the class of modules with $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$-injective dimension at most n. Let $\mathcal{X},\mathcal{Y},\mathcal{Z} \subseteq \mathcal{B}_C(S)$. Denote the Gorenstein category $\mathcal{G}(\mathrm{Hom}_S(C,\mathcal{X}),\mathrm{Hom}_S(C,\mathcal{Y}),\mathrm{Hom}_S(C,\mathcal{Z}))$ by $\mathcal{G}_2(\mathcal{X},\mathcal{Y},\mathcal{Z})$. In particular, if $\mathcal{G}_2(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is injectively resolving, $\mathcal{G}_2(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}$ stands for the class of left R-modules with $\mathcal{G}_2(\mathcal{X},\mathcal{Y},\mathcal{Z})$-injective dimension at most n.

We note that all the foregoing results have the dual version. So we have the following results.

Theorem 3.11. *Assume that $\mathcal{X},\mathcal{Y},\mathcal{Z} \subseteq \mathcal{B}_C(S)$, then there is an equivalence of categories:*

$$\mathcal{A}_C(R) \cap \mathcal{G}_2(\mathcal{X},\mathcal{Y},\mathcal{Z}) \underset{Hom_S(C,-)}{\overset{C\otimes_R-}{\rightleftarrows}} \mathcal{B}_C(S) \cap \mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z}).$$

Theorem 3.12. *Let n be a nonnegative integer and $\mathcal{X},\mathcal{Y},\mathcal{Z} \subseteq \mathcal{B}_C(S)$. If both $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ and $\mathcal{G}_2(\mathcal{X},\mathcal{Y},\mathcal{Z})$ are closed under extensions and cokernels of monomorphisms, and $\mathcal{X}$ contains every injective left S-module, then there is an equivalence of categories:*

$$\mathcal{A}_C(R) \cap \mathcal{G}_2(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n} \underset{Hom_S(C,-)}{\overset{C\otimes_R-}{\rightleftarrows}} \mathcal{B}_C(S) \cap \mathcal{G}_I(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}.$$

Corollary 3.13. *Let n be a nonnegative integer. If $\mathcal{X} = \mathcal{Z} = \mathcal{I}(S) \subseteq \mathcal{Y} \subseteq \mathcal{B}_C(S)$, then there is an equivalence of categories:*

$$\mathcal{G}_2(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n} \underset{Hom_S(C,-)}{\overset{C\otimes_R-}{\rightleftarrows}} \mathcal{B}_C(S) \cap \mathcal{G}_I(\mathcal{X},\mathcal{Y},\mathcal{Z})_{\leq n}.$$

4. Applications and examples

As the applications of our results, in this section, we give the following examples.

Example 4.1. If $\mathcal{X} = \mathcal{Y} = \mathcal{Z} = \mathcal{P}(R)$, then $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = \mathcal{G}$-*Proj*, the subcategory of Gorenstein projective left R-modules, and $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z}) = \mathcal{G}_C$-*Proj*, the subcategory of C-Gorenstein projective left S-modules in [10]. Also, if $\mathcal{X} = \mathcal{Y} = \mathcal{Z} = \mathcal{I}(S)$, then $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = \mathcal{G}$-*Inj*, the subcategory of Gorenstein injective left S-modules, and $\mathcal{G}_2(\mathcal{X},\mathcal{Y},\mathcal{Z}) = \mathcal{G}_C$-*Inj*, the subcategory of C-Gorenstein injective left R-modules in [10]. According to Corollary 3.10 and Corollary 3.13, we obtain the following equivalences of categories:

$$\mathcal{A}_C(R) \cap \mathcal{G}\text{-}Proj_{\leq n} \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R-}{\rightleftarrows}} \mathcal{G}_C\text{-}Proj_{\leq n},$$

$$\mathcal{G}_C\text{-}Inj_{\leq n} \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R-}{\rightleftarrows}} \mathcal{B}_C(S) \cap \mathcal{G}\text{-}Inj_{\leq n}.$$

This is the result in [10, Corollary 3.12]. In particular, if $n = 0$, we obtain the result in [10, Theorem 3.11] as follows

$$\mathcal{A}_C(R) \cap \mathcal{G}\text{-}Proj \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{G}_C\text{-}Proj,$$

$$\mathcal{G}_C\text{-}Inj \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{B}_C(S) \cap \mathcal{G}\text{-}Inj.$$

Recall from [4] that an R-module M over a ring R is said to be of type FP_∞ if M has a projective resolution by finitely generated projective modules, and a left R-module N is called level if $\mathrm{Tor}_1^R(M,N) = 0$ for all right R-modules M of type FP_∞. Denote by $\mathcal{L}(R)$ the class of level left R-modules. A left R-module M is called Gorenstein AC-projective [3, 4] if $M \in \mathcal{G}(\mathcal{P}(R),\mathcal{P}(R),\mathcal{L}(R))$. A left S-module N is called absolutely clean [4] if $\mathrm{Ext}_S^1(M,N) = 0$ for all left S-modules M of type FP_∞. Denote by $\mathcal{AC}(S)$ the class of absolutely clean left S-modules. A left S-module M is called Gorenstein AC-injective [4] if $M \in \mathcal{G}(\mathcal{I}(S),\mathcal{AC}(S),\mathcal{I}(S))$.

Lemma 4.2 **([13, Proposition 3.3]).** *$\mathcal{L}(R) \subseteq \mathcal{A}_C(R)$ and $\mathcal{AC}(S) \subseteq \mathcal{B}_C(S)$.*

Example 4.3. If $\mathcal{X} = \mathcal{Y} = \mathcal{P}(R)$ and $\mathcal{Z} = \mathcal{L}(R)$, then $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = \mathcal{GP}_{\mathcal{AC}}(R)$, the subcategory of Gorenstein AC-projective left R-modules in [3, 4]. Denote by $\mathcal{GP}^C_{\mathcal{AC}}(S)$ the subcategory $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z})$. Also, if $\mathcal{X} = \mathcal{Z} = \mathcal{I}(S)$ and $\mathcal{Y} = \mathcal{AC}(S)$, then $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = \mathcal{GI}_{\mathcal{AC}}(S)$, the subcategory of Gorenstein AC-injective left S-modules in [3, 4]. Denote by $\mathcal{GI}^C_{\mathcal{AC}}(R)$ the category $\mathcal{G}_2(\mathcal{X},\mathcal{Y},\mathcal{Z})$. According to Corollaries 3.10 and 3.13, we obtain the following equivalences of categories:

$$\mathcal{A}_C(R) \cap \mathcal{GP}_{\mathcal{AC}}(R)_{\leq n} \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{GP}^C_{\mathcal{AC}}(S)_{\leq n},$$

$$\mathcal{GI}^C_{\mathcal{AC}}(R)_{\leq n} \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{B}_C(S) \cap \mathcal{GI}_{\mathcal{AC}}(S)_{\leq n}.$$

In particular, if $n = 0$, we obtain the following equivalences of categories:

$$\mathcal{A}_C(R) \cap \mathcal{GP}_{\mathcal{AC}}(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{GP}^C_{\mathcal{AC}}(S),$$

$$\mathcal{GI}^C_{\mathcal{AC}}(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{B}_C(S) \cap \mathcal{GI}_{\mathcal{AC}}(S).$$

Example 4.4. If $\mathcal{X} = \mathcal{Y} = \mathcal{P}(R)$ and $\mathcal{Z} = \mathcal{F}(R)$, then $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = \mathcal{DP}(R)$, the subcategory of Ding projective left R-modules in [6, 18], and $\mathcal{G}_1(\mathcal{X},\mathcal{Y},\mathcal{Z}) = \mathcal{D}(\mathcal{P}_C(S))$, the subcategory of Ding C-projective left S-modules in [20]. Also, if $\mathcal{X} = \mathcal{Z} = \mathcal{I}(S)$ and $\mathcal{Y} = \mathcal{FI}(S) \subseteq \mathcal{AC}(S) \subseteq \mathcal{B}_C(S)$ by Lemma 4.2, then $\mathcal{G}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = \mathcal{DI}(S)$, the subcategory of Ding injective left S-modules in [15, 18], and $\mathcal{G}_2(\mathcal{X},\mathcal{Y},\mathcal{Z}) = \mathcal{D}(\mathcal{I}_C(R))$, the subcategory of Ding C-injective left R-modules in [20]. According to Corollaries 3.10 and 3.13, we obtain the following equivalences of categories:

$$\mathcal{A}_C(R) \cap \mathcal{DP}(R)_{\leq n} \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{D}(\mathcal{P}_C(S))_{\leq n},$$

$$\mathcal{D}(\mathcal{I}_C(R))_{\leq n} \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{B}_C(S) \cap \mathcal{DI}(S)_{\leq n}.$$

This is the result in [20, Theorem 4.4]. In particular, if $n = 0$, we obtain the following equivalences of categories:

$$\mathcal{A}_C(R) \cap \mathcal{DP}(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{D}(\mathcal{P}_C(S)),$$

$$\mathcal{D}(\mathcal{I}_C(R)) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{B}_C(S) \cap \mathcal{DI}(S),$$

which appeared in [20, Proposition 4.2].

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