Bounds of complex eigenvalues of structures with interval parameters

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Abstract

Based on the interval finite element method, a new method is presented in this paper to determine the bounds of complex eigenvalues for damping structures with interval parameters. The calculations are done on an element basis, hence, the calculations are greatly simplified. As an application of the method, a numerical example of a spring–mass system with damping is given. In order to illustrate the accuracy of the present method, the results obtained are compared with those obtained by Ref. [14] (Qiu ZP). Interval analysis for static response and eigenvalue problem structures with uncertain parameters. PhD thesis, Jilin University of Technology, 1994. p. 91–100 [in Chinese]). The calculated results show that the proposed method in this paper is effective in evaluating bounds of complex eigenvalues for damping systems with interval parameters. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Interval parameters; Bounds of complex eigenvalues; Interval finite element method

1. Introduction

In many practical engineering problems, the structural parameters are uncertain, for example, there may be inaccuracy of measurements, errors in manufacture, etc. Therefore, the concept of uncertainty plays an important role in the investigation of various engineering problems. The most common approach to uncertain problems is to model the structural parameters as random variables or fields. In this case, all information about the structural parameters is provided by the joint probability density function (or distribution function) of the structural parameters. Unfortunately, probabilistic modeling is not the only way one could describe the uncertainty, and uncertainty does not equal randomness. Indeed, probabilistic approaches are not able to deliver reliable results at the required precision without sufficient experimental data to validate the assumptions made regarding the joint probability densities of the random variables or functions involved.

Since the mid-sixties, a new method called interval analysis has been developed. Moore [1] and his co-workers, Alefeld and Herzberger [2] have carried out the pioneering work. Recently, Chen, Qiu and Elishakoff [3–6] have applied interval set models to the study of the static response and real eigenvalue problems of structures with interval parameters. In their studies, several important results have been obtained. However, these results are based on the assumptions that $\Delta K$, $\Delta f$ are preselected in the equation $K(\alpha)\mathbf{u} = f(\alpha)$ and $\Delta K$, $\Delta M$ are also preselected in the equation $K(\alpha)\mathbf{u} = \mathbf{M}(\alpha)\mathbf{u}$. In general, this assumption is not viable. Chen and Yang presented a new method, the interval finite method, to solve real eigenvalues and static displacements of structures with interval parameters [7], in which several valuable sets of results have been obtained.

The previous discussions are limited to the interval static response and interval real eigenvalue problems. However, in many engineering problems such as the systems with damping, and the analysis of aeroelastics, we have to discuss the complex modal analysis [8–12]. If the parameters of such a system are interval variables, interval complex eigenvalue problems will be encountered. However, except for the small amount of work done in Ref. [14], no other attempts have been made to solve the interval complex eigenproblems. In this study,
by using the interval mathematics and interval finite element method, a new method for estimating the bounds of the complex eigenvalues of structures with interval parameters will be discussed.

In the following, first a brief review of the interval mathematics and the perturbation theory is given for complex eigenvalues. The expressions for the interval element mass, stiffness and damping matrices are established. By using the complex circle plate extension and the upper and the lower bounds of the complex eigenvalues are derived. A numerical example is given to illustrate the application of the present method. The results obtained by the present method are compared with those in Ref. [14].

2. Mathematical backgrounds

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) be a structural parameter vector with bound errors or uncertainties, where 
\[
\alpha_i \in \alpha_i^1 = [\alpha_i^1, \alpha_i^2] = \{a_1 \leq t \leq a_2 | a_1, a_2 \in R\}
\]
then
\[
\alpha \in \alpha^1 = [\alpha^1, \alpha^2] = \{a| a \in \alpha^1, \alpha^2 \}
\]
where 
\[
\alpha^1 = (\alpha_1^1, \alpha_2^1, \ldots, \alpha_n^1)^T
\]
\[
\Delta \alpha = (\Delta \alpha_1, \Delta \alpha_2, \ldots, \Delta \alpha_n)^T
\]
Let \( \alpha \) and \( \bar{\alpha} \) be the lower and upper bound vectors of the structural parameter vector \( \alpha \), respectively. The bound and uncertain parameters \( \alpha_i \) are called interval parameters.

Before one can deal with bounds of complex eigenvalues of structures with interval parameters, one needs to introduce some results from interval analysis [1]. In interval mathematics, a subset of real numbers \( R \) of the form \([a, b]=[a_1 \leq t \leq a_2 | a_1, a_2 \in R]\) is called a closed real interval or an interval, denoted by \( I=[X, X] \), where \( X \) and \( X \) are the lower and upper bounds, respectively. The set of all closed real intervals is denoted by \( I(R) \).

The mid-point and uncertainty (or maximum error) of an interval \( X \) are defined as
\[
X^C = \frac{(X+X)}{2}
\]
and
\[
\Delta X = \frac{(X-X)}{2}
\]
respectively.

An \( n \)-dimensional interval vector can be expressed as
\[
X^1 = (X_1^1, X_2^1, \ldots, X_n^1)^T
\]
The set of all \( n \)-dimensional interval vectors is denoted by \( I(R^n) \).

Similarly, the mid-vector and uncertainty of an interval vector can be defined as
\[
X^C = (X_1^C, X_2^C, \ldots, X_n^C)^T
\]
and
\[
\Delta X = (\Delta X_1, \Delta X_2, \ldots, \Delta X_n)^T
\]
where \( X_i^C \) and \( \Delta X_i \) are given by Eqs. (2) and (3), respectively.

A matrix whose elements are interval parameters is called an interval matrix and denoted by \( A^1=[A, A] \), where \( A \) is a matrix composed of the lower bounds of intervals and \( A \) is a matrix composed of the upper bounds of the intervals. The set of all interval matrices is denoted by \( I(R^{m \times n}) \). The mid-vector and uncertainty of an interval matrix \( A^1 \) are given as
\[
A^C = \frac{(A+A)}{2} \quad \text{or} \quad a^C = \frac{(a+a)}{2}
\]
and
\[
\Delta A = \frac{(A-A)}{2} \quad \text{or} \quad \Delta a = \frac{(a-a)}{2}
\]
where
\[
A^C = (a^C) \quad \text{and} \quad \Delta A = (\Delta a)
\]
An arbitrary interval \( X^1 \in I(R) \) can be written as the sum of its mid-point \( X^C \) and a symmetric interval \( \Delta X^1 = [\Delta X^1, \Delta X^1] = [X^1 - 1, 1] \), i.e.
\[
X^1 = X^C + \Delta X^1
\]
Similar expressions exist for the interval vector and interval matrix. For \( A^1 \in I(R^{m \times n}) \) we have
\[
A^1 = A^C + \Delta A^1
\]
where
\[
\Delta A^1 = [-\Delta A, \Delta A]
\]
Let \( X^1, X^2 \in I(R) \). Then the set
\[
Z^1 = \{a = a_1 + ia_2 | a_1 \in X^1, a_2 \in X^2\}
\]
of complex numbers is called a complex interval.

Let \( z \) be an arbitrary complex and \( r \) be an arbitrary real, with \( r \leq 0 \). Then the bounded close set
\[
Z = \{z = z_0 \in C | |z_0 - z| \leq r\}
\]
is called a complex circle plate.

These basic quantities will play an important role in the following discussions.

Let \( f \) be a complex-valued function of \( n \) complex variables \( z_1, z_2, \ldots, z_n \). A complex circle plate extension of \( f \)
means that a complex circle plate value function $F$ of $n$
complex circle plates $Z_1, Z_2, ..., Z_n$, for all $z_i \in \mathbb{Z}$ ($i=1, 2, ..., n$) possesses the following property
\[ F([z_1, 0], [z_2, 0], ..., [z_n, 0]) = \text{f}(z_1, z_2, ..., z_n) \] (11)

Let $X = [x, y]$ and $Y = [y, x]$ be real interval numbers, respectively, then $X^2 + Y^2, X^3 + Y^3, X^4 + Y^4$ and $[X^3](Y^3)$ are defined by the following formulas:
\[ X^2 + Y^2 = [x, x] + [y, y] = [x + y, x + y] \] (12)
\[ X^3 - Y^3 = [x, x] - [y, y] = [x - y, x - y] \] (13)
\[ X^4 + Y^4 = [x, x] \times [x, x] = [x, x] \times \frac{1}{x, x} \] (15)

Let $Z_1 = X_1 + iY_1$, $Z_2 = X_2 + iY_2$, where $X_1, Y_1 \in \mathbb{R}$. Then $Z_1 + Z_2, Z_1 - Z_2, Z_1 \times Z_2$ and $[(Z_1)/(Z_2)]$ are defined by the following formulas:
\[ Z_1 + Z_2 = (X_1 + X_2) + i(Y_1 + Y_2) \] (16)
\[ Z_1 - Z_2 = (X_1 - X_2) + i(Y_1 - Y_2) \] (17)
\[ Z_1 \times Z_2 = (X_1 X_2 - Y_1 Y_2) + i(X_1 Y_2 + Y_1 X_2) \] (18)
\[ \frac{Z_1}{Z_2} = \frac{X_1 Y_2 + Y_1 X_2}{(X_2)^2 + (Y_2)^2} + i\frac{X_1 Y_2 - Y_1 X_2}{(X_2)^2 + (Y_2)^2} \] (19)

### 3. Matrix perturbation theory for complex eigenvalues

The vibration equations of the linear damping systems with $n$ degrees of freedom are as follows
\[ M\ddot{q} + C\dot{q} + Kq = Q(t) \] (20)
where $M$, $C$ and $K$ are the mass, damping and stiffness matrices, respectively, and $Q(t)$ is the excitation force vector.

The free vibration equations of the systems are
\[ M\ddot{q} + C\dot{q} + Kq = 0 \] (21)

The corresponding right and left eigenvalue problems are
\[ (MS^2 + CS + K)X = 0 \] (23)
and
\[ YT(\tilde{MS}^2 + \tilde{CS} + K) = 0 \] (24)
Let
\[ M = M_0 + \Delta M \]
\[ C = C_0 + \Delta C \]
\[ K = K_0 + \Delta K \]
where $\Delta M, \Delta C$ and $\Delta K$ are the incremental matrices of $M_0, C_0$ and $K_0$, respectively.

Making use of the matrix perturbation theory, we can obtain the first-order perturbation of the complex eigenvalues [13]
\[ \Delta S = -Y_0^T(\Delta MS^2 + \Delta CS + \Delta K)X_0 \] (25)
where $S_0$, $X_0$ and $Y_0$ are the eigenvalue and corresponding right and left eigenvectors of the system $(M_0, C_0, K_0)$ and satisfy the equations
\[ (MS_0 + CS_0 + K_0)X_0 = 0 \] (26)
and
\[ Y_0^T(M_0S_0^2 + CS_0 + K_0) = 0 \] (27)
and the orthogonal relation of the eigenvectors of the system
\[ (Y_0)^T(M_0(S_0^2 + S_0 + C_0)X_0) = \delta_{ij} \] (28)

### 4. Evaluation of bounds of interval complex eigenvalues

In this section, the bounds of interval complex eigenvalues for damping systems with interval parameters will be discussed.

According to the interval finite method, the stiffness matrix, mass matrix and damping matrix of the element with interval parameters, $\alpha^i$, can be expressed as, respectively
\[ K_i(\alpha^i) = K_i(\alpha^C + \Delta K_i) \] (29)
\[ M_i(\alpha^i) = M_i(\alpha^C + \Delta M_i) \] (30)
\[ C_i(\alpha^i) = C_i(\alpha^C + \Delta C_i) \] (31)
where $K_i(\alpha^C), M_i(\alpha^C)$ and $C_i(\alpha^C)$ are the stiffness, mass and damping matrices of the system with parameter $\alpha^C$, $\Delta K_i, \Delta M_i$ and $\Delta C_i$ are the corresponding uncertain matrices, respectively, and
\[ \Delta K_i = \sum_{j=1}^{m} \frac{\partial K_i}{\partial \alpha_j} \Delta \alpha_j e_j \] (32)
\[ \Delta M_i = \sum_{j=1}^{m} \frac{\partial M_i}{\partial \alpha_j} \Delta \alpha_j e_j \] (33)
\[ \Delta C_i = \sum_{j=1}^{m} \frac{\partial C_i}{\partial \alpha_j} \Delta \alpha_j e_j \] (34)
Let
\[ \alpha^i = [\alpha^C - \Delta \alpha, \alpha^C + \Delta \alpha] \]
\[ \alpha^c=(\alpha_1^c, \alpha_2^c, \ldots, \alpha_m^c)^T \]
\[ \Delta \alpha=(\Delta \alpha_1, \Delta \alpha_2, \ldots, \Delta \alpha_m)^T \]
\[ \epsilon_j=[-1, 1] \]

\[ \{[\partial K_j]/(\partial \alpha_j)\}, \{[\partial M_j]/(\partial \alpha_j)\} \text{ and } \{[\partial C_j]/(\partial \alpha_j)\} \] are the derivatives of \( K, \ M \) and \( C \) with respect to \( \alpha \), respectively.

Making use of the assembled procedure of the usual finite element method and the natural interval extension given by Section 2, Eqs. (29)–(31) become
\[ K(\alpha^b)=K(\alpha^c)+\Delta K^b \]  \hspace{1cm} (35)
\[ M(\alpha^b)=M(\alpha^c)+\Delta M^b \]  \hspace{1cm} (36)
\[ C(\alpha^b)=C(\alpha^c)+\Delta C^b \]  \hspace{1cm} (37)
where
\[ \Delta K^b=\sum_{i=1}^{n} \Delta K_i=\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial K_i}{\partial \alpha_j} \Delta \alpha_j \epsilon_j \]  \hspace{1cm} (38)
\[ \Delta M^b=\sum_{i=1}^{n} \Delta M_i=\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial M_i}{\partial \alpha_j} \Delta \alpha_j \epsilon_j \]  \hspace{1cm} (39)
\[ \Delta C^b=\sum_{i=1}^{n} \Delta C_i=\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial C_i}{\partial \alpha_j} \Delta \alpha_j \epsilon_j \]  \hspace{1cm} (40)
in which \( n \) denotes the total element number of the structure, and \( m \) denotes the number of the interval parameters.

If changes to the structural parameters are introduced, the eigenvalues can be expressed by the perturbation method
\[ S_i^k=S_i^c+\Delta S_i^k \]  \hspace{1cm} (41)
where \( S_i^c \) is the \( k \)th middle-complex eigenvalue of the system with parameter \( \alpha^c \), and \( \Delta S_i^k \) is the first-order perturbation of the \( k \)th complex eigenvalue.

By circle plate extension, expression (41) becomes
\[ S_i^k=S_i^c+\Delta S_i^k \]  \hspace{1cm} (42)
where
\[ \Delta S_i^k=-(Y_i^c)^T(\Delta M_i(S_i^c)^2+\Delta C_i S_i^c+\Delta K_i)X_i^c \]  \hspace{1cm} (43)
Substituting Eqs. (38)–(40) into Eq. (43), one has
\[ \Delta S_i^k=-(Y_i^c)^T \left[ \sum_{j=1}^{m} \sum_{j=1}^{m} \frac{\partial M_j}{\partial \alpha_j} \Delta \alpha_j \epsilon_j (S_i^c)^2 \right. \]
\[ + \left. \sum_{j=1}^{m} \sum_{j=1}^{m} \frac{\partial C_j}{\partial \alpha_j} \Delta \alpha_j \epsilon_j S_i^c + \sum_{j=1}^{m} \sum_{j=1}^{m} \frac{\partial K_j}{\partial \alpha_j} \Delta \alpha_j \epsilon_j \right] X_i^c \]
\[ =-[\sum_{j=1}^{m} \sum_{j=1}^{m} (Y_i^c)^T \frac{\partial M_j}{\partial \alpha_j} X_i^c (S_i^c)^2 \Delta \alpha_j \epsilon_j \]
\[ + \sum_{j=1}^{m} \sum_{j=1}^{m} (Y_i^c)^T \frac{\partial C_j}{\partial \alpha_j} X_i^c S_i^c \Delta \alpha_j \epsilon_j \]
\[ + \sum_{j=1}^{m} \sum_{j=1}^{m} (Y_i^c)^T \frac{\partial K_j}{\partial \alpha_j} X_i^c S_i^c \Delta \alpha_j \epsilon_j \]
\[ =\Delta S_i^k \]  \hspace{1cm} (44)

Let
\[ M_k+iM_i=(Y_i^c)^T \frac{\partial M_i}{\partial \alpha_j} X_i^c (S_i^c)^2 \Delta \alpha_j \]  \hspace{1cm} (45)
\[ C_k+iC_i=(Y_i^c)^T \frac{\partial C_i}{\partial \alpha_j} X_i^c S_i^c \Delta \alpha_j \]  \hspace{1cm} (46)
\[ K_k+iK_i=(Y_i^c)^T \frac{\partial K_i}{\partial \alpha_j} X_i^c S_i^c \Delta \alpha_j \]  \hspace{1cm} (47)
Then one arrives at
\[ \Delta S_i^k=-\sum_{j=1}^{m} \sum_{j=1}^{m} (M_k+iM_i) \epsilon_j + (C_k+iC_i) \epsilon_j + (K_k+iK_i) \epsilon_j \]  \hspace{1cm} (48)
\[ +C_k+iC_i \epsilon_j \]  \hspace{1cm} (49)
\[ +iK_i \epsilon_j \]  \hspace{1cm} (50)
\[ +iC_k+iC_i \epsilon_j \]  \hspace{1cm} (51)
where \( \epsilon_j=[-1, 1] \).

Let
\[ \Delta S_i^{(k)}=\sum_{i=1}^{m} \sum_{i=1}^{m} |M_k+C_k+K_k| \]  \hspace{1cm} (49)
\[ \Delta S_i^{(k)}=\sum_{i=1}^{m} \sum_{i=1}^{m} |M_k+C_k+K_k| \]  \hspace{1cm} (50)
Then
\[ S_i^k=S_i^c+\Delta S_i^k=S_i^c+\Delta S_i^{(k)} e_\Delta+i\Delta S_i^{(k)} e_\Delta \]  \hspace{1cm} (51)
If \( S_i=[S_i^{kr}+iS_i^{ki}, S_i^{kr}+iS_i^{ki}] \), the lower and the upper bounds of the real parts and complex parts of the complex eigenvalues \( S_i^{kr}, S_i^{ki}, S_i^{kl} \) and \( S_i^{kl} \) can be obtained as follows
\[ S_i^{kr}=S_i^{ck}-\Delta S_i^{(k)} \]  \hspace{1cm} (52)
\[ S_i^{ki}=S_i^{ck}+\Delta S_i^{(k)} \]  \hspace{1cm} (53)
\[ S_i^{kl}=S_i^{kl}+\Delta S_i^{(k)} \]  \hspace{1cm} (54)
\[ S_i^{kl}=S_i^{kl}+\Delta S_i^{(k)} \]  \hspace{1cm} (55)
where \( S_i^{kr} \) and \( S_i^{ci} \) are the real parts and the complex parts of \( S_i^{kr} \), respectively.
The lower and upper bounds of complex eigenvalues (β = 0.005)

<table>
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<tr>
<th>$k$</th>
<th>$S_{LR}$</th>
<th>$S_{UI}$</th>
<th>$S_{LR}^{C}$</th>
<th>$S_{UI}^{C}$</th>
<th>$S_{LR}^{I}$</th>
<th>$S_{UI}^{I}$</th>
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<tr>
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<td>-0.134721E-01</td>
<td>1.08134</td>
</tr>
<tr>
<td>4</td>
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<td>-0.204338E-01</td>
<td>1.31652</td>
<td>-0.199554E-01</td>
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<tr>
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<td>1.55371</td>
<td>-0.133440E-01</td>
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(b) Obtained by Ref. [14]

<table>
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<th>$S_{UI}$</th>
<th>$S_{LR}^{C}$</th>
<th>$S_{UI}^{C}$</th>
<th>$S_{LR}^{I}$</th>
<th>$S_{UI}^{I}$</th>
</tr>
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<td>-0.142447E-02</td>
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<td>-0.971112E-02</td>
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5. Numerical example

In order to demonstrate the procedure presented by this study, a spring–mass system with five degrees of freedom, shown in Fig. 1, is considered. The physical parameters of this system are as follows: $m_i^n=m_i^s=m_i^c=2$, $m_1=1$, $k_1=k_6=k_3=k_5=k_4=k_2=1$; $c_1=c_5=c_3=2c$, $c_2=c_4=c_6=0.01$. The uncertainties of parameters are given by $m_i^n=m_i^s+\beta m_i^c$, $c_i=c_i^c+\beta c_i^e$, $k_i=k_i^c+\beta k_i^e$, where $\beta$ is a small parameter. The results obtained by the present method and Ref. [14] are listed in Tables 1–4. In the tables, $k$ is the number of modes; $S_{LR}$ is the lower bound of the real part of the $k$th complex eigenvalue; $S_{UI}$ is the upper bound of the real part of the $k$th complex eigenvalue; $S_{LR}^C$ is the real part of the $k$th complex eigenvalue; $S_{UI}^C$ is the complex part of the $k$th complex eigenvalue; $S_{LR}^I$ is the lower bound of the complex part of the $k$th eigenvalue; $S_{UI}^I$ is the upper bound of the complex part of the $k$th eigenvalue.

From the results listed in the tables, it can be seen that the present method is more accurate than that of Ref. [14] for estimating the lower and upper bounds of complex eigenvalues for a damping system with interval parameters. The results indicate that if the interval range of the structural parameters is large, the method pro-

Table 2

The lower and upper bounds of complex eigenvalues (β = 0.01)

<table>
<thead>
<tr>
<th>$k$</th>
<th>$S_{LR}$</th>
<th>$S_{UI}$</th>
<th>$S_{LR}^{C}$</th>
<th>$S_{UI}^{C}$</th>
<th>$S_{LR}^{I}$</th>
<th>$S_{UI}^{I}$</th>
</tr>
</thead>
<tbody>
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<td>-0.142447E-02</td>
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(b) Obtained by Ref. [14]

<table>
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<tr>
<th>$k$</th>
<th>$S_{LR}$</th>
<th>$S_{UI}$</th>
<th>$S_{LR}^{C}$</th>
<th>$S_{UI}^{C}$</th>
<th>$S_{LR}^{I}$</th>
<th>$S_{UI}^{I}$</th>
</tr>
</thead>
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<td>0.345352</td>
<td>-0.142447E-02</td>
<td>0.373219</td>
<td>0.126383E-01</td>
<td>0.401086</td>
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<tr>
<td>2</td>
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<td>3</td>
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<td>4</td>
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<td>5</td>
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<td>-0.135411E-01</td>
<td>1.55371</td>
<td>-0.577048E-02</td>
<td>1.56938</td>
</tr>
</tbody>
</table>
posed by Ref. [14] cannot be used, however, the present method can still be used to estimate the lower and upper bounds of complex eigenvalues for damping systems with interval parameters. For example, if $\beta=0.1$, the lower and upper bounds of the first eigenvalues are $S_{LR}=-0.178420E-02$, $S_{IR}=-0.106474E-02$, $S_{LM}=0.355897$, $S_{IM}=0.410541$ from the present method, and the corresponding results obtained by Ref. [14] are $S_{LR}=-0.142052$, $S_{IR}=0.139203$, $S_{LM}=0.945478E-01$, $S_{IM}=0.651890$, respectively. The reason for this is that $\Delta K$, $\Delta M$ and $\Delta C$ are the increments in the global stiffness matrix, mass matrix and damping matrix in the approach presented by Ref. [14], and the corresponding calculations of the proposed method in this paper are done on an element basis, thus simplifying the complex interval algorithm and ensuring the reliability of the algorithm.

6. Conclusions

With the interval finite element method and the complex circle plate extension, a new method is presented in this paper to evaluate bounds of complex eigenvalues for damping systems with interval parameters. The calculations are based on the element and can be simplified. From the numerical example, it can be seen that the method is effective in evaluating bounds of complex eigenvalues for damping systems with interval parameters. The results obtained by this study are very useful for the robust analysis of the control problem and the dynamic stability analysis of nonlinear systems. It should be noted that although the numerical example presented was a spring–mass damping system in one dimension, the present method can be used to solve two-dimensional or even three-dimensional structures.

Acknowledgements

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References