A two-level algorithm for the weak Galerkin discretization of diffusion problems

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Abstract

This paper analyzes a two-level algorithm for the weak Galerkin (WG) finite element methods based on local Raviart-Thomas (RT) and Brezzi-Douglas-Marini (BDM) mixed elements for two- and three-dimensional diffusion problems with Dirichlet condition. We first show the condition numbers of the stiffness matrices arising from the WG methods are of $O(h^{-2})$. We use an extended version of the Xu-Zikatanov (XZ) identity to derive the convergence of the algorithm without any regularity assumption. Finally we provide some numerical results.

Keywords. diffusion problem, weak Galerkin finite element, condition number, two-level algorithm, X-Z identity

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a bounded polyhedral domain. Consider the following diffusion problem:

$$
\begin{aligned}
-\text{div}(a \nabla u) &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{aligned}
$$

where $a \in [L^\infty(\Omega)]^{d \times d}$ is a given symmetric positive-definite permeability tensor, $f \in L^2(\Omega)$.

The weak Galerkin (WG) finite element method was first introduced and analyzed by Wang and Ye [32] for general second order elliptic problems and later developed by their
research group in [37, 38, 34, 39, 36, 33, 35]. It is designed by using a weakly defined
gradient operator over functions with discontinuity. The method, based on local Raviart-
Thomas (RT) elements [40] or Brezzi-Douglas-Marini (BDM) elements [18], allows the
use of totally discontinuous piecewise polynomials in the finite element procedure, as is
common in discontinuous Galerkin methods [3] and hybridized discontinuous Galerkin
methods [22]. As shown in [32, 37, 38, 34, 36], the WG method also enjoys an easy-to-
implement formulation that inherits the physical property of mass conservation locally
on each element. We note that when $a$ in (1.1) is a piecewise-constant matrix, the WG
method, by introducing the discrete weak gradient as an independent variable, is equivalent
to some hybridized version of the corresponding mixed RT or BDM method [2, 18] (cf.
Remark 2.1).

As one knows, multigrid methods are among the most efficient methods for solving
linear algebraic systems arising from the discretization of partial differential equations. By
now, the research of the multigrid methods for second order elliptic problems has reached
a mature stage in some sense (see [5, 6, 7, 8, 9, 10, 11, 12, 29, 41, 42, 43, 44, 45, 46] and
the references therein). Especially, Xu, Chen, and Nochetto [46] presented an overview
of the multigrid methods in an elegant fashion. For the model problem (1.1), Brenner
[14] developed an optimal order multigrid method for the lowest-order Raviart-Thomas
mixed triangular finite element. The algorithm and the convergence analysis are based
on the equivalence between Raviart-Thomas mixed methods and certain nonconforming
methods. In [28] Gopalakrishnan and Tan analyzed the convergence of a variable V-cycle
multigrid algorithm for the hybridized mixed method for Poisson problems. Following the
same idea, Cockburn et al. [23] analyzed the convergence of a non-nested multigrid V-
cycle algorithm, with a single smoothing step per level, for one type of HDG method. One
may refer to [13, 14, 15, 17, 24, 25, 27, 30, 31] for multigrid algorithms for nonconforming
and DG methods.

This paper is to analyze a two-level algorithm for the WG methods. We show the
condition numbers of the WG systems are of $O(h^{-2})$. We follow the basic ideas of [45,
46, 19] to establish an extended version of the Xu-Zikatanov (XZ) identity [45], and then
derive the convergence of the algorithm without any regularity assumption.

The rest of this paper is organized as follows. Section 2 introduces the WG methods.
Section 3 analyzes the conditioning of the WG systems. Section 4 describes the two-level
algorithm, and analyzes its convergence. Section 5 provides some numerical experiments to verify our theoretical results.

2 Weak Galerkin finite element method

2.1 Preliminaries and Notations

Throughout this paper, we shall use the standard definitions of Sobolev spaces and their norms([1]), namely, for an arbitrary open set, $D$, of $\mathbb{R}^d$ and any nonnegative integer $s$,

$$H^s(D) := \{v \in L^2(D) : \partial^\alpha v \in L^2(D), \forall |\alpha| \leq s\},$$

$$\|v\|_{s,D} := \left(\sum_{0 \leq j \leq s} |v|_{j,D}^2 \right)^{\frac{1}{2}},$$

$$|v|_{j,D} := \left(\sum_{|\alpha| = j} \int_D |\partial^\alpha v|^2 \right)^{\frac{1}{2}}.$$

We use $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle_{\partial D}$ to denote the standard $L^2$ inner products on $L^2(D)$ and $L^2(\partial D)$, respectively, and use $\|\cdot\|_D$ and $\|\cdot\|_{\partial D}$ to denote the norms induced by $(\cdot, \cdot)_D$ and $\langle \cdot, \cdot \rangle_{\partial D}$, respectively. In particular, $\|\cdot\|$ abbreviates $\|\cdot\|_\Omega$.

Let $T_h$ be a regular triangulation of $\Omega$. For any $T \in T_h$, we denote by $h_T$ the diameter of $T$ and set $h := \max_{T \in T_h} h_T$. We denote by $F_h$ the set of all faces of $T_h$.

We introduce some mesh-dependent inner products and mesh-dependent norms as follows. We define

$$(\lambda, \mu)_h := \sum_{T \in T_h} h_T \int_{\partial T} \lambda \mu, \forall \lambda, \mu \in L^2(F_h),$$

and

$$(u, \lambda)_h := (u, \lambda)_\Omega + (\lambda, \mu)_h, \forall (u, \lambda), (v, \mu) \in L^2(\Omega) \times L^2(F_h).$$

With a little abuse of notations, we use $\|\cdot\|_h$ to denote the norms induced by the inner products $(\cdot, \cdot)_h$ and $(\cdot, \cdot)_h$, i.e.,

$$\|\mu\|_h := (\mu, \mu)_h^{\frac{1}{2}}, \forall \mu \in L^2(F_h),$$

$$\|(v, \mu)\|_h := ((v, \mu), (v, \mu))_h^{\frac{1}{2}} = \left(\|v\|^2 + \|\mu\|^2_h\right)^{\frac{1}{2}}, \forall (v, \mu) \in L^2(\Omega) \times L^2(F_h).$$

We also need the following elementwise norm and seminorms: for any $\mu \in L^2(F_h)$,

$$\|\mu\|_{h,\partial T} := h_T^{\frac{1}{2}} \|\mu\|_{\partial T},$$

3
\[ |\mu|_{h,\partial T}^2 := h^{-1}\|\mu - m_T(\mu)\|_{\partial T}^2 \quad \text{with} \quad m_T(\mu) := \frac{1}{d+1} \sum_{F \in F_T} |F| \int_F \mu, \]

and

\[ |\mu|_h := \left( \sum_{T \in \mathcal{T}_h} |\mu|_{h,\partial T}^2 \right)^{\frac{1}{2}}, \quad (2.5) \]

where \( \mathcal{F}_T \) denotes the set of all faces of \( T \), and \( |F| \) denotes the \((d-1)\)-dimensional Hausdorff measure of \( F \).

Throughout this paper, \( x \lesssim y \ (x \gtrsim y) \) means \( x \leq C y \ (x \geq C y) \), where \( C \) denotes a positive constant that is independent of the mesh size \( h \). The notation \( x \sim y \) abbreviates \( x \lesssim y \gtrsim x \).

### 2.2 Weak Galerkin formulations

We first introduce two spaces:

\[
egin{align*}
V_h &:= \{ v_h \in L^2(\Omega) : v_h|_T \in V(T), \forall T \in \mathcal{T}_h \}, \\
\mathcal{M}_h^0 &:= \{ \mu_h \in L^2(\mathcal{F}_h) : \mu_h|_F \in M(F), \forall F \in \mathcal{F}_h, \mu_h|_{\partial \Omega} = 0 \},
\end{align*}
\]

where \( V(T) \) and \( M(F) \) denote two local finite dimensional spaces.

For \( T \in \mathcal{T}_h \), let \( \mathcal{W}(T) \) be a local finite dimensional subspace of \([L^2(T)]^d\). Then, following [32], we introduce the discrete weak gradient \( \nabla_w : L^2(T) \times L^2(\partial T) \to \mathcal{W}(T) \) defined by

\[
\nabla_w(v, \mu) = \nabla^i_w v + \nabla^b_w \mu, \quad \forall (v, \mu) \in L^2(T) \times L^2(\partial T), \quad (2.6)
\]

where \( \nabla^i_w v, \nabla^b_w \mu \in \mathcal{W}(T) \) satisfy, for any \( q \in \mathcal{W}(T) \),

\[
\begin{align*}
(\nabla^i_w v, q)_T &= -(v, \text{div} \ q)_T, \quad (2.7) \\
(\nabla^b_w \mu, q)_T &= \langle \mu, q \cdot n \rangle_{\partial T}. \quad (2.8)
\end{align*}
\]

The WG method for problem (1.1) reads as follows([32]): Seek \((u_h, \lambda_h) \in V_h \times \mathcal{M}_h^0 \) such that

\[
a_h((u_h, \lambda_h), (v_h, \mu_h)) = (f, v_h)_\Omega, \quad \forall (v_h, \mu_h) \in V_h \times \mathcal{M}_h^0, \quad (2.9)
\]

where

\[
a_h((u_h, \lambda_h), (v_h, \mu_h)) := (a \nabla_w(u_h, \lambda_h), \nabla_w(v_h, \mu_h))_\Omega.
\]

For any set \( D \), we denote by \( P_j(D) \) the set of polynomials of degree \( \leq j \) on \( D \). This paper considers two type of WG methods [32] which are based on local RT and BDM mixed elements, respectively:
Type 1. \( V(T) = P_k(T), M(F) = P_k(F), W(T) = [P_k(T)]^d + P_k(T)x. \)

Type 2. \( V(T) = P_{k-1}(T), M(F) = P_k(F), W(T) = [P_k(T)]^d \) \( (k \geq 1). \)

**Remark 2.1.** When \( a \) is a piecewise constant matrix, we can show that the two type of WG methods are equivalent to the hybridized version of the corresponding mixed RT and BDM method ([2, 18]) respectively. In fact, by introducing the vector \( p_h := a \nabla_w(u_h, \lambda_h) \) and the space \( W_h := \{q_h \in [L^2(\Omega)]^d : q_h|_T \in W(T) \} \), it’s straightforward that the WG scheme (2.9) is equivalent to the following problem: Seek \((p_h, u_h, \lambda_h) \in W_h \times V_h \times M_h^0\), such that

\[
(a^{-1}p_h, q_h)_{\Omega} + \sum_{T \in T_h} (u_h, \text{div} q_h)_{T} - \sum_{T \in T_h} \langle \lambda_h, q_h \cdot n \rangle_{\partial T} = 0, \]

\[
- \sum_{T \in T_h} (v_h, \text{div} p_h)_{T} = (f, v_h)_{\Omega}, \]

\[
\sum_{T \in T_h} \langle p_h \cdot n, \mu_h \rangle_{\partial T} = 0
\]

hold for all \((q_h, v_h, \mu_h) \in W_h \times V_h \times M_h^0\). This scheme is no other than the hybridized version of the RT mixed element method (cf. (1.18) in [2]) or the BDM mixed method (cf. (1.13) in [18]).

In the following we give an operator form and a matrix form of the WG discretization (2.9). Let \( \{\phi_i : i = 1, 2, \ldots, M\} \subset V_h \) and \( \{\eta_i : i = 1, 2, \ldots, N\} \subset M_h^0 \) be nodal bases for \( V_h \) and \( M_h^0 \), respectively. Denote by \( \tilde{u}_h, \tilde{v}_h \in \mathbb{R}^M \) the vectors of coefficients of \( u_h, v_h \) in the \( \{\phi_i\} \)-basis, and by \( \tilde{\lambda}_h, \tilde{\mu}_h \in \mathbb{R}^N \) the vectors of coefficients of \( \lambda_h, \mu_h \) in the \( \{\eta_i\} \)-basis, respectively.

Define the operators \( C_h : V_h \to V_h, B_h : V_h \to M_h^0, B_h^T : M_h^0 \to V_h, D_h : M_h^0 \to M_h^0 \), and the matrices \( B_h \in \mathbb{R}^{N \times M}, C_h \in \mathbb{R}^{M \times M}, D_h \in \mathbb{R}^{N \times N} \) respectively by

\[
(C_h u_h, v_h)_{\Omega} := (a \nabla^i_w u_h, \nabla^i_w v_h)_{\Omega} := \tilde{u}_h^T C_h \tilde{v}_h, \quad \forall u_h, v_h \in V_h,
\]

\[
(B_h u_h, \lambda_h)_{\Omega} := (a \nabla^i_w u_h, \nabla^i_w \lambda_h)_{\Omega} := (u_h, B_h^T \lambda_h)_{\Omega} := \tilde{u}_h^T B_h^T \tilde{\lambda}_h, \quad \forall u_h \in V_h, \lambda_h \in M_h^0,
\]

\[
(D_h \lambda_h, \mu_h)_{\Omega} := (a \nabla^b_w \lambda_h, \nabla^b_w \mu_h)_{\Omega} := \tilde{\lambda}_h^T D_h \tilde{\mu}_h, \quad \forall \lambda_h, \mu_h \in M_h^0.
\]

Let \( A_h : V_h \times M_h^0 \to V_h \times M_h^0 \) and \( A_h \in \mathbb{R}^{(M+N) \times (M+N)} \) be defined by

\[
(A_h(u_h, \lambda_h), (v_h, \mu_h)) := a_h((u_h, \lambda_h), (v_h, \mu_h)) := (\tilde{u}_h^T \tilde{\lambda}_h^T) A_h \begin{pmatrix} \tilde{u}_h \\ \tilde{\lambda}_h \end{pmatrix}
\]

(2.10)

for any \((u_h, \lambda_h), (v_h, \mu_h) \in V_h \times M_h^0\). Then we have

\[
A_h = \begin{pmatrix} C_h & B_h^T \\ B_h & D_h \end{pmatrix}, \quad A_h = \begin{pmatrix} C_h & B_h^T \\ B_h & D_h \end{pmatrix}.
\]

(2.11)
and the WG discretization (2.9) is equivalent to the following system: Seek \((u_h, \lambda_h) \in V_h \times M_0^h\) such that
\[
A_h(u_h, \lambda_h) = b_h
\] (2.12)
with \(b_h := (f_h, 0)\) and \(f_h \in V_h\) denoting the standard \(L^2\)-orthogonal projection of \(f\) onto \(V_h\).

### 3 Conditioning of WG methods

In what follows we assume \(\mathcal{T}_h\) to be a quasi-uniform triangulation. We recall that \(|·|_h\), \(\|·\|_h\), \(\|·\|_T\), \(\|·\|_{h,\partial T}\), and \(|·|_{h,\partial T}\) are defined in Subsection 2.1.

We first present a basic estimate as follows.

**Lemma 3.1.** For any \(\mu_h \in M_0^h\), it holds
\[
\|\mu_h\|_h \lesssim |\mu_h|_h.
\] (3.1)

**Proof.** See Appendix A.

For any simplex \(T\), define
\[
M(\partial T) := \{\mu \in L^2(\partial T) : \mu|_F \in M(F), \text{ for each face } F \text{ of } T\}.
\]

The following lemma gives some basic estimates of weak gradients.

**Lemma 3.2.** For any \(T \in \mathcal{T}_h\) and \((v, \mu) \in V(T) \times M(\partial T)\), it holds
\[
\|\nabla_w^i w v\|_T \sim h_T^{-1} \|v\|_T, \quad \|\nabla_w^b w \mu\|_T \sim h_T^{-1} \|\mu\|_{h,\partial T}, \quad \|\nabla_w v(\mu)\|_T \sim h_T^{-1} \|v - m_T(\mu)\|_T + |\mu|_{h,\partial T}.
\] (3.2)

**Proof.** See Appendix B.

In view of Lemmas 3.1-3.2, we have the following conclusion.

**Theorem 3.1.** For any \((v_h, \mu_h) \in V_h \times M_0^h\), it holds
\[
\|(v_h, \mu_h)\|_h^2 \lesssim a_h((v_h, \mu_h), (v_h, \mu_h)) \lesssim h^{-2} \|(v_h, \mu_h)\|_h^2.
\] (3.3)
Proof. From (3.2c) it follows
\[
\|\nabla w(v_h, \mu_h)\|^2 \sim \sum_{T \in T_h} h_T^{-2} \|v_h - m_T(\mu_h)\|^2_T + |\mu_h|_h^2.
\] (3.4)

Since
\[
|m_T(\mu_h)| \leq \frac{1}{d+1} \sum_{F \in T} |F| \int_F |\mu_h| \lesssim h_T^{-\frac{d+1}{2}} \|\mu_h\|_{\partial T},
\]
we have
\[
\|m_T(\mu_h)\|_T \lesssim h_T^{\frac{d}{2}} \|\mu_h\|_{\partial T} \lesssim \|\mu_h\|_{h, \partial T},
\] (3.5)
which, together with Lemmas 3.1-3.2, implies
\[
\|v_h\|^2 \lesssim \sum_{T \in T_h} \left\{ \|v_h - m_T(\mu_h)\|^2_T + |\mu_h|_h^2 \right\}
\lesssim \sum_{T \in T_h} \|v_h - m_T(\mu_h)\|^2_T + \|\mu_h\|^2_h
\lesssim \sum_{T \in T_h} \|v_h - m_T(\mu_h)\|^2_T + |\mu_h|_h^2.
\] (3.6)

A combination of (3.1), (3.4) and (3.6) yields
\[
\|v_h, \mu_h\|^2_h = \|v_h\|^2 + \|\mu_h\|^2_h \lesssim a_h((v_h, \mu_h), (v_h, \mu_h)).
\] (3.7)

On the other hand, it holds
\[
a_h((v_h, \mu_h), (v_h, \mu_h)) \lesssim \|\nabla w v_h\|^2 + \|\nabla w \mu_h\|^2
\lesssim h^{-2} \|v_h\|^2 + h^{-2} \|\mu_h\|^2_h \text{ by (3.2a) and (3.2b)}
\lesssim h^{-2} \|v_h, \mu_h\|^2_h.
\] (3.8)

The estimates (3.7)-(3.8) lead to the desired result (3.3).

Theorem 3.2. It holds
\[
\sup_{(v_h, \mu_h) \in V_h \times M_h^0} \frac{a_h((v_h, \mu_h), (v_h, \mu_h))}{\|(v_h, \mu_h)\|^2_h} \gtrsim h^{-2}.
\] (3.9)

In addition,
\[
\inf_{(v_h, \mu_h) \in V_h \times M_h^0} \frac{a_h(v_h, \mu_h), (v_h, \mu_h))}{\|(v_h, \mu_h)\|^2_h} \lesssim 1
\] (3.10)
holds if \( h \) is sufficiently small.
Proof. Given \( v_h \in V_h \), from Lemma 3.2 it follows
\[
a_h((v_h,0),(v_h,0)) \sim h^{-2} \|v_h\|^2, \tag{3.11}
\]
which implies (3.9).

Let \( s \) be the smallest eigenvalue of problem (1.1) with \( f = su \) and let \( u_0 \in H^1_0(\Omega) \) be the corresponding eigenvector function. Then it holds
\[
\|\nabla u_0\|^2 \sim s \|u_0\|^2. \tag{3.12}
\]

In the analysis below, we shall denote by \( C \) a positive constant that is independent of the mesh size \( h \) and may take a different value at its each occurrence.

We define \( (v_h, \mu_h) \in V_h \times M^0_h \) by
\[
\begin{align*}
  v_h|_T &= m_T(u_0), \quad \forall T \in T_h, \\
  \mu_h|_F &= \frac{1}{|F|} \int_F u_0, \quad \forall F \in F_h.
\end{align*}
\]

By the definition of \( m_T(\cdot) \) it is easy to see
\[
m_T(\mu_h) = \frac{1}{d+1} \sum_{F \in F_T} \frac{1}{|F|} \int_F \mu_h = m_T(u_0). \tag{3.13}
\]

Standard scaling arguments yield
\[
\begin{align*}
  \|u_0 - m_T(u_0)\|_T &\lesssim h_T|u_0|_{1,T}, \tag{3.14} \\
  \|\mu_h - m_T(\mu_h)\|_{\partial T} &\lesssim h_T^2|u_0|_{1,T}. \tag{3.15}
\end{align*}
\]

Thus, in view of (3.14) and (3.12) we have
\[
\begin{align*}
  \|v_h\|^2 &= \sum_{T \in T_h} \|m_T(u_0)\|_T^2 \geq \sum_{T \in T_h} \left\{ \frac{1}{2} \|u_0\|_T^2 - \|u_0 - m_T(u_0)\|_T^2 \right\} \\
  &\gtrsim \sum_{T \in T_h} \left\{ \|u_0\|_T^2 - Ch_T^2|u_0|_{1,T}^2 \right\} \tag{3.16} \\
  &\gtrsim (1 - sCh^2) \|u_0\|^2,
\end{align*}
\]
which, together with (3.15) and (3.13), further implies
\[
\begin{align*}
  \|\mu_h\|^2 &\geq \sum_{T \in T_h} h_T \left( \frac{1}{2} \|m_T(\mu_h)\|_{\partial T}^2 - \|\mu_h - m_T(\mu_h)\|_{\partial T}^2 \right) \\
  &\gtrsim \sum_{T \in T_h} h_T \|m_T(\mu_h)\|_{\partial T}^2 - Ch_T^2|u_0|_{1,\Omega}^2 \tag{3.17} \\
  &\gtrsim \sum_{T \in T_h} \|m_T(u_0)\|_T^2 - Ch_T^2|u_0|_{1,\Omega}^2 \\
  &\gtrsim (1 - sCh^2) \|u_0\|^2.
\end{align*}
\]
On the other hand, from the definition (2.5) and the estimate (3.15) it follows

\[ |\mu_h|_h \lesssim |u_0|_{1,\Omega}. \quad (3.18) \]

Therefore, it holds

\[
\frac{a_h((v_h, \mu_h), (v_h, \mu_h))}{\| (v_h, \mu_h) \|_h^2} \sim \frac{\| \nabla w(v_h, \mu_h) \|^2}{\| (v_h, \mu_h) \|_h^2} \\
\lesssim \frac{|\mu_h|^2}{\| v_h \|^2 + |\mu_h|^2} \quad (\text{by } (3.2c)) \\
\lesssim \frac{|\mu_h|^2}{(1 - sCh^2) \| u_0 \|^2} \quad (\text{by } (3.16) \text{ and } (3.17)) \\
\lesssim \frac{s \| u_0 \|^2}{(1 - sCh^2) \| u_0 \|^2} \quad (\text{by } (3.18)) \\
\lesssim \frac{s}{1 - sCh^2},
\]

which indicates the inequality (3.10) immediately.

In light of Theorems 3.1-3.2, it’s straightforward to derive the following theorem.

**Theorem 3.3.** Let \( A_h \) be the operator defined by (2.10), then it holds

\[ \kappa(A_h) \lesssim h^{-2}, \quad (3.19) \]

where \( \kappa(A_h) := \frac{\lambda_{\max}(A_h)}{\lambda_{\min}(A_h)} \), with \( \lambda_{\max}(A_h) \), \( \lambda_{\min}(A_h) \) denoting the largest and smallest eigenvalues of \( A_h \) respectively. Furthermore, it holds

\[ \kappa(A_h) = O(h^{-2}) \quad (3.20) \]

if \( h \) is sufficiently small.

**Remark 3.1.** Let \( A_h \) be the stiffness matrix of \( a_h(\cdot, \cdot) \) defined by (2.10), then we easily have \( \kappa(A_h) \sim \kappa(A_h) = O(h^{-2}) \).

### 4 Two-level algorithm

In this section, we analyze a two-level algorithm for the discrete system (2.12). For the sake of clarity, our description is in operator form.
4.1 Algorithm definition

Set
\[ \bar{V}_h := \{ \bar{v}_h \in H^1_0(\Omega) : \bar{v}_h|_T \in P_1(T), \forall T \in \mathcal{T}_h \}. \]  
\(4.1\)

We first define the prolongation operator \( I_h : \bar{V}_h \to V_h \times M_h^0 \) as follows: for any \( \bar{v}_h \in \bar{V}_h \), \( I_h \bar{v}_h := (I_h^l \bar{v}_h, I_h^s \bar{v}_h) \in V_h \times M_h^0 \) satisfies
\[
\begin{align*}
\int_T I_h^l \bar{v}_h v & = \int_T \bar{v}_h v, \quad \forall v \in V(T), \quad \forall T \in \mathcal{T}_h, \\
\int_F I_h^s \bar{v}_h \mu & = \int_F \bar{v}_h \mu, \quad \forall \mu \in M(F), \quad \forall F \in \mathcal{F}_h.
\end{align*}
\]
Then define the adjoint operator, \( I_h^t \), of \( I_h \) by
\[
(I_h^t(v_h, \mu_h), \bar{v}_h)_{\Omega} := ((v_h, \mu_h), I_h \bar{v}_h)_h, \quad \forall (v_h, \mu_h) \in V_h \times M_h^0, \forall \bar{v}_h \in \bar{V}_h.
\]
Define \( \bar{A}_h : \bar{V}_h \to \bar{V}_h \) by
\[
(\bar{A}_h \bar{v}_h, \bar{v}_h)_{\Omega} := (a \nabla \bar{u}_h, \nabla \bar{v}_h)_\Omega, \quad \forall \bar{u}_h, \bar{v}_h \in \bar{V}_h.
\]
\(4.2\)

Remark 4.1. By the definition of \( I_h \), it’s trivial to verify that \( \nabla_v I_h \bar{v}_h = \nabla \bar{v}_h, \quad \forall \bar{v}_h \in \bar{V}_h. \)

Thus we have the following important relationship:
\[
\bar{A}_h = I_h^t A_h I_h.
\]
\(4.3\)

Let \( \bar{R}_h : \bar{V}_h \to \bar{V}_h \) be a good approximation of \( \bar{A}_h^{-1} \) and define \( \bar{R}_h^t \) by
\[
(\bar{R}_h^t \bar{u}_h, \bar{v}_h)_{\Omega} := (\bar{u}_h, \bar{R}_h \bar{v}_h)_{\Omega}, \quad \forall \bar{u}_h, \bar{v}_h \in \bar{V}_h.
\]

Let \( R_h : V_h \times M_h^0 \to V_h \times M_h^0 \) be a good approximation of \( A_h^{-1} \). and let \( R_h^t : V_h \times M_h^0 \to V_h \times M_h^0 \) be defined by
\[
(R_h^t(u_h, \lambda_h), (v_h, \mu_h))_h := ((u_h, \lambda_h), R_h(v_h, \mu_h))_h, \quad \forall (u_h, \lambda_h), (v_h, \mu_h) \in V_h \times M_h^0.
\]

Using the above operators, we define an ingredient operator \( B_h : V_h \times M_h^0 \to V_h \times M_h^0 \) as follows:

Algorithm 1. For any \( b_h \in V_h \times M_h^0 \), define \( B_h b_h = (v^4_h, \mu^4_h) \) by

1. Smooth: \( (v^1_h, \mu^1_h) := R_h b_h \).
2. Correct: \( (v^2_h, \mu^2_h) := (v^1_h, \mu^1_h) + I_h \bar{R}_h^t I_h^t (b_h - A_h (v^1_h, \mu^1_h)) \),
3. Correct: \( (v^3_h, \mu^3_h) := (v^2_h, \mu^2_h) + I_h \bar{R}_h^t I_h^t (b_h - A_h (v^2_h, \mu^2_h)) \),
4. Smooth: \( (v^4_h, \mu^4_h) := (v^3_h, \mu^3_h) + R_h^t (b_h - A_h (v^3_h, \mu^3_h)) \).
We are now in a position to present the two-level algorithm for the system (2.12).

**Algorithm 2.** Set \((u^0_h, \lambda^0_h) = (0, 0)\),

for \(j = 1, 2, \ldots\) till convergence

\[(u^j_h, \lambda^j_h) := (u^{j-1}_h, \lambda^{j-1}_h) + B_h(b_h - A_h(u^{j-1}_h, \lambda^{j-1}_h)).\]

**4.2 Convergence analysis**

At first, we introduce some abstract notations. Let \(X\) be a finite dimensional Hilbert space with inner product \((\cdot, \cdot)\) and its induced norm \(\|\cdot\|\). For any linear SPD operator \(A : X \to X\), the notation \((\cdot, \cdot)_A := (A\cdot, \cdot)\) defines an inner product on \(X\) and we denote by \(\|\cdot\|_A\) the norm induced by \((\cdot, \cdot)_A\). Let \(B : X \to X\) be a linear operator with

\[\|B\|_A := \sup_{0 \neq x \in X} \frac{\|Bx\|_A}{\|x\|_A}.\]

From the definition of \(B_h\) in **Algorithm 1**, we easily obtain the following lemma.

**Lemma 4.1.** It holds

\[I - B_h A_h = (I - R^t_h A_h)(I - I_h \bar{R}_h I^t_h A_h)(I - I_h \bar{R}_h I^t_h A_h)(I - R_h A_h).\] (4.4)

It’s trivial to verify that \(I - B_h A_h\) is symmetric semi-positive definite with respect to the inner product \((\cdot, \cdot)_A\), and thus it follows \(\lambda_{\text{max}}(B_h A_h) \leq 1\) and

\[\|I - B_h A_h\|_{A_h} = 1 - \lambda_{\text{min}}(B_h A_h).\] (4.5)

Now we introduce the symmetrizations of \(R_h\) and \(\bar{R}_h\), i.e.

\[\bar{R}_h := R^t_h + R_h - R^t_h A_h R_h,\] (4.6)
\[\bar{R}_h := \bar{R}^t_h + \bar{R}_h - \bar{R}^t_h \bar{A}_h \bar{R}_h,\] (4.7)

and make the following assumption.

**Assumption I.** The operators \(R_h\) and \(\bar{R}_h\) are such that

\[\|I - \bar{R}_h A_h\|_{A_h} < 1,\] (4.8)
\[\|I - \bar{R}_h \bar{A}_h\|_{\bar{A}_h} < 1.\] (4.9)
Remarök 4.2. It follows from (4.8) that $\overline{R_h}$ is SPD with respect to the inner product $(\cdot, \cdot)_h$.

Then it follows from

$$
\overline{R_h} = R_h^t(R_h^{-t} + R_h^{-1} - A_h)R_h
$$

that $R_h^{-t} + R_h^{-1} - A_h$ is SPD with respect to the inner product $(\cdot, \cdot)_h$. Similarly, $\overline{R_h}$ and $\overline{R_h}^{-t} + \overline{R_h}^{-1} - \overline{A_h}$ are both SPD with respect to the inner product $(\cdot, \cdot)_\Omega$.

Following the basic idea of the X-Z identity ([45,20,19]), we have the following ingredient theorem.

Theorem 4.1. Under Assumption I, $E_h$ is a SPD operator with respect to the inner product $(\cdot, \cdot)_h$, and, for any $(u_h, \lambda_h) \in V_h \times M_h^0$, it holds

$$(B_h^{-1}(u_h, \lambda_h), (u_h, \lambda_h))_h = \inf_{(v_h, \mu_h) \in V_h \times M_h^0} \| (v_h, \mu_h) + R_h^t A_h I_h \tilde{v}_h \|_{\overline{R_h}}^{-1} + \| \tilde{v}_h \|_{\overline{R_h}}^{-1}. \tag{4.10}$$

Further more, it holds the following extended X-Z identity:

$$
\| I - B_h A_h \|_{A_h} = 1 - \frac{1}{K}, \tag{4.11}
$$

where

$$K = \sup_{\| (u_h, \lambda_h) \|_{A_h} = 1} \inf_{(v_h, \mu_h) \in V_h \times M_h^0} \| (v_h, \mu_h) + R_h^t A_h I_h \tilde{v}_h \|_{\overline{R_h}}^{-1} + \| \tilde{v}_h \|_{\overline{R_h}}^{\frac{1}{2}}. \tag{4.12}$$

Proof. The desired results follow from a trivial modification of the proof of the X-Z identity in [19]. For completeness we sketch the proof of this theorem. We note that $\tilde{V}_h \not\subset V_h \times M_h^0$ means the corresponding spaces here are nonnested.

Denote $X_h := (V_h \times M_h^0) \times \tilde{V}_h$ and define the inner product $[\cdot, \cdot]$ on $X_h$ by

$$[(a, b), (c, d)] := (a, c)_h + (b, d)_\Omega, \ \forall (a, b), (c, d) \in X_h.$$  

Introduce the operator $\Pi_h : X_h \to V_h \times M_h^0$ and its adjoint operator $\Pi_h^t : V_h \times M_h^0 \to X_h$ with

$$\Pi_h := \begin{pmatrix} I & I_h \end{pmatrix}, \ \text{i.e.} \ \Pi_h(a, b) = a + I_h b \ \text{for any} \ (a, b) \in X_h,$$

$$\Pi_h^t := \begin{pmatrix} I \\ I_h^t \end{pmatrix}, \ \text{i.e.} \ \Pi_h^t a = \begin{pmatrix} a \\ I_h^t a \end{pmatrix} \ \text{for any} \ a \in V_h \times M_h^0.$$  

Obviously, we have $(\Pi_h a, b)_h = [\tilde{a}, \Pi_h^t b], \ \forall \tilde{a} \in X_h, \forall b \in V_h \times M_h^0.$
Now define

\[ \widetilde{A}_h := \begin{pmatrix} A_h & A_h I_h \\ I_h A_h & I_h A_h I_h \end{pmatrix}, \quad \widetilde{B}_h := \begin{pmatrix} \mathcal{R}_h^{-1} & 0 \\ I_h^\top A_h & \mathcal{R}_h^{-1} \end{pmatrix}, \]

and denote by \( \widetilde{D}_h \) the diagonal of \( \widetilde{A}_h \).

For any \( b_h \in V_h \times M_h^0 \), set

\[ w_1 := \widetilde{B}_h \Pi_h b_h, \quad w_2 := w_1 + \widetilde{B}_h (\Pi_h b_h - \widetilde{A}_h w_1). \]

Then it holds \( \Pi_h w_2 = \Pi_h \widetilde{B}_h \Pi_h b_h \), where

\[ \mathcal{R}_h := \begin{pmatrix} \mathcal{R}_h & \widetilde{R}_h \\ \widetilde{R}_h & \mathcal{R}_h \end{pmatrix}, \quad \mathcal{R}_h := \mathcal{R}_h^t \mathcal{R}_h^{-1} \mathcal{R}_h^t \mathcal{R}_h^{-1} \mathcal{R}_h^t. \]

It’s easy to verify that \( \Pi_h w_2 = B_h b_h \), which yields

\[ B_h = \Pi_h \widetilde{B}_h \Pi_h^t. \tag{4.13} \]

Denoting \( \mathcal{R}_h := \text{diag}(\mathcal{R}_h, \widetilde{\mathcal{R}}_h) \), we have

\[ \mathcal{B}_h := \begin{pmatrix} \mathcal{B}_h & \mathcal{B}_h \mathcal{R}_h \mathcal{R}_h \mathcal{B}_h \\ \mathcal{R}_h \mathcal{B}_h \mathcal{R}_h \mathcal{B}_h & \mathcal{R}_h \mathcal{B}_h \mathcal{R}_h \mathcal{B}_h \end{pmatrix}, \]

where \( \mathcal{R}_h := \mathcal{R}_h^t \mathcal{R}_h + \mathcal{R}_h^t \mathcal{R}_h \mathcal{D}_h \mathcal{R}_h \). By (4.3) we also have \( \mathcal{R}_h = \text{diag}(\mathcal{R}_h, \mathcal{R}_h) \). From Remark 4.2, it follows that \( \mathcal{R}_h \) is SPD with respect to \([\cdot, \cdot] \). Thus \( \mathcal{B}_h \) is SPD with respect to \([\cdot, \cdot] \). Then from Theorem 1 in [19] and (4.13) it follows

\[ (B_h^{-1}(u_h, \lambda_h), (u_h, \lambda_h))_h = \inf_{w_h \in X_h} [\mathcal{B}_h^{-1} w_h, w_h]. \tag{4.14} \]

In view of

\[ \mathcal{B}_h^{-1} = \begin{pmatrix} \mathcal{R}_h & \mathcal{R}_h \mathcal{R}_h \mathcal{B}_h \\ \mathcal{R}_h \mathcal{B}_h \mathcal{R}_h \mathcal{B}_h & \mathcal{R}_h \mathcal{B}_h \mathcal{R}_h \mathcal{B}_h \end{pmatrix}, \quad \mathcal{R}_h := \begin{pmatrix} I & 0 \\ I_h^\top A_h \mathcal{R}_h & I \end{pmatrix}, \]

the identity (4.10) follows immediately from (4.14). The extended X-Z identity (4.11) is just a trivial conclusion from (4.10). \[ \square \]
We define the operator $P_h : M^0_h \to \tilde{V}_h$ as follows. For any $\lambda_h \in M^0_h$, $P_h \lambda_h$ satisfies
\[
\begin{align*}
P_h \lambda_h(x) &= \sum_{T \in \omega_x} \frac{\sum_{T \in \omega_x} m_T(\lambda_h)}{\sum_{T \in \omega_x} 1}, \quad \text{for each interior vertex } x \text{ of } T_h, \\
P_h \lambda_h(x) &= 0, \quad \text{for each vertex } x \in \partial \Omega,
\end{align*}
\]
where the set $\omega_x := \{ T \in T_h : x \text{ is a vertex of } T \}$.

As for the operator $P_h$, we have the following important estimates.

**Lemma 4.2.** For any $(u_h, \lambda_h) \in V_h \times M^0_h$, it holds
\[
\begin{align*}
\left\|(I - I^i_h P_h) \lambda_h\right\|_h &\lesssim h \|(u_h, \lambda_h)\|_{A_h}, \quad (4.15) \\
\left\|u_h - I^i_h P_h \lambda_h\right\| &\lesssim h \|(u_h, \lambda_h)\|_{A_h}, \quad (4.16)
\end{align*}
\]
which further indicate
\[
\|(u_h, \lambda_h) - I_h P_h \lambda_h\|_h \lesssim h \|(u_h, \lambda_h)\|_{A_h}. \quad (4.17)
\]

**Proof.** We denote by $\omega_T$ the set $\{ T' \in T_h : T' \text{ and } T \text{ share a vertex} \}$ and by $\mathcal{N}(T)$ the set of all vertexes of $T$. Since
\[
\begin{align*}
h_T \left\|(I - I^i_h P_h) \lambda_h\right\|^2_{\partial T} &\leq h_T \left\|P_h \lambda_h - m_T(\lambda_h)\right\|^2_{\partial T} \\
&\lesssim h_T^d \sum_{x \in \mathcal{N}(T)} |P_h \lambda_h(x) - m_T(\lambda_h)|^2 \\
&\lesssim h_T^d \sum_{x \in \mathcal{N}(T)} \sum_{T_1, T_2 \in \omega_x} |m_{T_1}(\lambda_h) - m_{T_2}(\lambda_h)|^2 \\
&\lesssim h_T^2 \sum_{T' \in \omega_T} |\lambda_h|_{h, \partial T'}^2,
\end{align*}
\]
we have
\[
\begin{align*}
h_T \left\|(I - I^i_h P_h) \lambda_h\right\|_{\partial T} &\lesssim h_T \left\|\lambda_h - m_T(\lambda_h)\right\|_{\partial T}^2 + h_T \left\|I^i_h P_h \lambda_h - m_T(\lambda_h)\right\|_{\partial T}^2 \\
&\lesssim h_T^2 \sum_{T' \in \omega_T} |\lambda_h|_{h, \partial T'}^2.
\end{align*}
\]
Then the estimate (4.15) follows immediately from (3.2c).

On the other hand, since
\[
\begin{align*}
\left\|I^i_h P_h \lambda_h - m_T(\lambda_h)\right\|_{T}^2 &\leq \left\|P_h \lambda_h - m_T(\lambda_h)\right\|_{T}^2 \\
&\lesssim h_T \left\|P_h \lambda_h - m_T(\lambda_h)\right\|_{\partial T}^2 \\
&\lesssim h_T^2 \sum_{T' \in \omega_T} |\lambda_h|_{h, \partial T'}^2, \quad \text{(by (4.18))}
\end{align*}
\]
it holds
\[ \|u_h - I_h^T P_h \lambda_h\|_T^2 \lesssim \|u_h - m_T(\lambda_h)\|_T^2 + \|I_h^T P_h \lambda_h - m_T(\lambda_h)\|_T^2 \]
\[ \lesssim \|u_h - m_T(\lambda_h)\|_T^2 + \sum_{T' \in \omega} h_T^2 |\lambda_h|^2_{h,\partial T'}. \]

Then the estimate (4.16) also follows immediately from (3.2c).

Finally, the result (4.17) is a trivial conclusion from (4.15) and (4.16).

Lemma 4.3. For any \((u_h, \lambda_h) \in V_h \times M_h^0\), it holds
\[ \|I_h P_h \lambda_h\|_{A_h} \leq \|P_h \lambda_h\|_{\tilde{A}_h} \lesssim \|(u_h, \lambda_h)\|_{A_h}. \] (4.19)

Proof. The relation \(\|I_h P_h \lambda_h\|_{A_h} = \|P_h \lambda_h\|_{\tilde{A}_h}\) follows from (4.3). It suffices to prove the inequality of (4.19). Since
\[ |P_h \lambda_h|_{1,T}^2 = |P_h \lambda_h - m_T(\lambda_h)|_{1,T}^2 \]
\[ \lesssim h_T^{-2} \|P_h \lambda_h - m_T(\lambda_h)\|_T^2 \quad \text{(by inverse estimate)} \]
\[ \lesssim h_T^{-1} \|P_h \lambda_h - m_T(\lambda_h)\|_T^2 \]
\[ \lesssim \sum_{T' \in \omega} h_T^2 |\lambda_h|^2_{h,\partial T'}, \quad \text{(by (4.18))} \]
we have
\[ \|P_h \lambda_h\|_{\tilde{A}_h} \sim |P_h \lambda_h|_{1,\Omega} \lesssim |\lambda_h|_h, \]
which, together with (3.2c), implies the desired conclusion.

Assumption II. The smoother \(\mathcal{R}_h : V_h \times M_h^0 \to V_h \times M_h^0\) is SPD with respect to \((\cdot, \cdot)_h\) and satisfies
\[ \sigma(\mathcal{R}_h A_h) \subset (0, 1], \] (4.20)
where \(\sigma(\mathcal{R}_h A_h)\) denotes the set of all eigenvalues of \(\mathcal{R}_h A_h\). What’s more, for any \((u_h, \lambda_h) \in V_h \times M_h^0\), it holds
\[ \|(u_h, \lambda_h)\|_{\overline{A}_h}^2 \leq C_R \lambda_{\max}(A_h) \|(u_h, \lambda_h)\|_h^2, \] (4.21)
where \(\overline{A}_h\) is the symmetrization of \(\mathcal{R}_h\), and \(C_R\) denotes a positive constant.

Remark 4.3. If we take \(\mathcal{R}_h = \frac{1}{\lambda_{\max}(A_h)} I\), then it holds \(\overline{R}_h^{-1} = \lambda_{\max}^2(A_h)(2 \lambda_{\max}(A_h) I - A_h)^{-1}\). In this case it is obvious that \(C_R = 1\). If we take \(\mathcal{R}_h\) to be the symmetric Gauss-Seidel smoother, then \(C_R\) is a bounded positive constant independent of the mesh size \(h\).
Remark 4.4. Suppose Assumption II holds, then the relation \( I - \mathcal{R}_h A_h = (I - \mathcal{R}_h A_h)^2 \) leads to \( \sigma(I - \mathcal{R}_h A_h) \subset [0, 1) \) and it follows \( \| I - \mathcal{R}_h A_h \|_{A_h} < 1 \).

Lemma 4.4. Under Assumption II, for any \((u_h, \lambda_h) \in V_h \times M_h^0\), it holds

\[
\| R_h A_h(u_h, \lambda_h) \|_{\mathcal{R}_h^{-1}} \leq \|(u_h, \lambda_h)\|_{A_h}. \tag{4.22}
\]

Proof. Denoting \( S_h := \mathcal{R}_h A_h \) and thanks to \( \mathcal{R}_h = 2R_h - R_h A_h R_h = (2S_h - S_h^2)A_h^{-1}, \)
we have

\[
\| R_h A_h(u_h, \lambda_h) \|_{\mathcal{R}_h^{-1}} = \frac{1}{\mathcal{R}_h} R_h A_h(u_h, \lambda_h), \quad R_h A_h(u_h, \lambda_h) \|
\]

\[
= (A_h(2S_h - S_h^2)^{-1}S_h(u_h, \lambda_h), R_h A_h(u_h, \lambda_h))_h \quad \tag{4.23}
\]

\[
= (S_h(2S_h - S_h^2)^{-1}S_h(u_h, \lambda_h), (u_h, \lambda_h))_{A_h},
\]

which, together with the fact that \( S_h \) is SPD with respect to \((\cdot, \cdot)_{A_h}\) and the inequality

\[
t(2t - t^2)^{-1}t \leq 1, \quad t \in (0, 1],
\]

yields the desired estimate (4.22).

Finally, we state the following convergence theorem.

Theorem 4.2. Under Assumptions I-II, it holds

\[
\|(I - B_h A_h)\|_{A_h} \leq 1 - \frac{1}{K}, \tag{4.24}
\]

where

\[
K \lesssim \left( 1 + C_R + \frac{1}{1 - \| I - \mathcal{R}_h A_h \|_{A_h}} \right). \quad \tag{4.25}
\]

Proof. For any \((u_h, \lambda_h) \in V_h \times M_h^0\), set

\[
\tilde{v}_h := P_h \lambda_h, \quad (v_h, \mu_h) := (u_h, \lambda_h) - I_h \tilde{v}_h,
\]

we then obtain

\[
\|(v_h, \mu_h) + \mathcal{R}_h A_h I_h \tilde{v}_h \|_{\mathcal{R}_h^{-1}} \lesssim \|(v_h, \mu_h)\|_{\mathcal{R}_h^{-1}} + \| \mathcal{R}_h A_h I_h \tilde{v}_h \|_{\mathcal{R}_h^{-1}} \leq (1 + C_R) \|(u_h, \lambda_h)\|_{A_h}^2 \tag{by (4.19)}
\]

We have

\[
\mathcal{R}_h = 2R_h - R_h A_h R_h = (2S_h - S_h^2)A_h^{-1},
\]

therefore

\[
\| \mathcal{R}_h A_h I_h \tilde{v}_h \|_{\mathcal{R}_h^{-1}} \leq \| \mathcal{R}_h A_h \|_{A_h} \| I_h \tilde{v}_h \|_{A_h} \leq C_R \| \mathcal{R}_h A_h \|_{A_h} \| I_h \tilde{v}_h \|_{A_h} \leq C_R \| \mathcal{R}_h A_h \|_{A_h} \| I_h \tilde{v}_h \|_{A_h} \leq C_R \lambda_{max}(A_h) \| (v_h, \mu_h) \|_{A_h} + \| (u_h, \lambda_h) \|_{A_h} \| (v_h, \mu_h) \|_{A_h} + \| (u_h, \lambda_h) \|_{A_h} \| (v_h, \mu_h) \|_{A_h} \leq C_R \lambda_{max}(A_h) \| (v_h, \mu_h) \|_{A_h} + \| (u_h, \lambda_h) \|_{A_h} \| (v_h, \mu_h) \|_{A_h} \leq (1 + C_R) \|(u_h, \lambda_h)\|_{A_h}^2 \tag{by Assumption II}
\]

Finally, we state the following convergence theorem.
where, in the last inequality, we have used the estimate (4.17) and the fact \( \lambda_{\text{max}}(A_h) \sim h^{-2} \) derived from Theorems 3.1-3.2. Similar to (4.5), we have

\[
\| I - \overline{\mathcal{R}}_h A_h \|_{A_h} = 1 - \lambda_{\min}(\overline{\mathcal{R}}_h A_h),
\]

and it follows

\[
\frac{1}{\lambda_{\min}(\overline{\mathcal{R}}_h A_h)} \| \tilde{\mathbf{v}}_h \|_{A_h}^2 = \frac{1}{1 - \| I - \overline{\mathcal{R}}_h A_h \|_{A_h}} \| \tilde{\mathbf{v}}_h \|_{A_h}^2 \leq (1 + C_R + \frac{1}{1 - \| I - \overline{\mathcal{R}}_h A_h \|_{A_h}}) \| (u_h, \lambda_h) \|_{A_h}^2.
\]

Therefore, we have

\[
\| (v_h, \mu_h) + \mathcal{R}_h A_h I_h \tilde{\mathbf{v}}_h \|_{A_h}^2 + \| \tilde{\mathbf{v}}_h \|_{\overline{\mathcal{R}}_h}^2 \leq (1 + C_R + \frac{1}{1 - \| I - \overline{\mathcal{R}}_h A_h \|_{A_h}}) \| (u_h, \lambda_h) \|_{A_h}^2,
\]

which implies

\[
\sup_{\| (u_h, \lambda_h) \|_{A_h} = 1} \inf_{(v_h, \mu_h) + \mathcal{R}_h A_h I_h \tilde{\mathbf{v}}_h, (v_h, \mu_h) \in V_h \times M_h, \tilde{\mathbf{v}}_h \in \tilde{V}_h} \| (v_h, \mu_h) + \mathcal{R}_h A_h I_h \tilde{\mathbf{v}}_h \|_{A_h}^2 + \| \tilde{\mathbf{v}}_h \|_{\overline{\mathcal{R}}_h}^2 \leq (1 + C_R + \frac{1}{1 - \| I - \overline{\mathcal{R}}_h A_h \|_{A_h}}).
\]

As a result, the desired estimate (4.24) follows from the extended X-Z identity (4.11) in Theorem 4.1.

**Remark 4.5.** In our analysis, we do not use any regularity assumption of the model problem (1.1). Thus our theory applies to the case that (1.1) doesn’t have full elliptic regularity. However, if \( \overline{\mathcal{R}}_h \) is construted by standard multigrid methods, as shown in [8]-[10], the lack of full regularity may affect the convergence rate \( \| I - \overline{\mathcal{R}}_h A_h \|_{A_h} \).

5 Numerical experiments

This section reports some numerical results in two space dimensions to verify our theoretical results. For the model problem (1.1), we set \( \mathbf{a} \in \mathbb{R}^{2 \times 2} \) to be the identity matrix, \( \Omega = (0, 1) \times (0, 1) \) and we shall use the **Type 2** WG method (\( k = 1 \)). When given a coarse triangulation \( \mathcal{T}_0 \), we produce a sequence of uniformly refined triangulations
\( \{ T_i : i = 0, 1, \ldots, 5 \} \) (cf. Figure 1 for \( T_0 \) and \( T_1 \)) by a simple procedure: \( T_{j+1} \) is obtained by connecting the midpoints of all edges of \( T_j \) for \( j = 0, 1, 2, 3, 4 \).

In our first experiment, we compute the smallest eigenvalue \( \lambda_{\text{min}}(A_h) \), the largest eigenvalue \( \lambda_{\text{max}}(A_h) \) and the condition number \( \kappa(A_h) \) of the stiffness matrix \( A_h \) on each triangulation \( T_i \) and list them in Table 1. The results imply \( \kappa(A_h) \sim \kappa(A_h) = O(h^{-2}) \), which is conformable to Theorem 3.3.

<table>
<thead>
<tr>
<th>( T_i )</th>
<th>( T_0 )</th>
<th>( T_1 )</th>
<th>( T_2 )</th>
<th>( T_3 )</th>
<th>( T_4 )</th>
<th>( T_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_{\text{min}}(A_h) )</td>
<td>0.55</td>
<td>0.28</td>
<td>0.075</td>
<td>0.019</td>
<td>0.0048</td>
<td>0.0012</td>
</tr>
<tr>
<td>( \lambda_{\text{max}}(A_h) )</td>
<td>27.63</td>
<td>33.31</td>
<td>33.46</td>
<td>33.47</td>
<td>33.47</td>
<td>33.47</td>
</tr>
<tr>
<td>( \kappa(A_h) )</td>
<td>50.1</td>
<td>121.1</td>
<td>444.8</td>
<td>1746.7</td>
<td>6954.6</td>
<td>27792</td>
</tr>
</tbody>
</table>

In our second experiment, for each triangulation \( T_j \), we set \( T_h = T_j \) and take \( R_h \) to be the \( m \)-times symmetric Gauss-Seidel iteration with \( \tilde{R}_h = \tilde{A}_h^{-1} \). We are to solve the problem \( A_h x = b \), where \( b \) is a zero vector, In order to verify the convergence, in Algorithm 2, we take \( x_0 = (1, 1, \ldots, 1)^t \) as the initial value, rather than the zero vector. We stop the two-level algorithm when the initial error, i.e. \( \sqrt{x_0^t A_h x_0} \), is reduced by a factor of \( 10^{-8} \). The corresponding results listed in Table 2 show that the two-level algorithm is efficient.

Our third experiment is a modification of the second one. In this experiment, the operator \( \tilde{R}_h \) is constructed by using the standard \( V \)-cycle multigrid method based on the nested triangulations \( T_0, T_1, \ldots, T_j \), rather than by simply setting \( \tilde{R}_h = \tilde{A}_h^{-1} \). Here we set all smoothers encountered to be the \( m \)-times symmetric Gauss-Seidel iterations.
is a practical multi-level algorithm. The numerical results listed in Tables 3-4 show that the multi-level algorithm is efficient.

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A Appendix: Proof of Lemma 3.1

For any simplex \( T \) with vertexes \( a_1, a_2, \ldots, a_{d+1} \), let \( \lambda_i \) be the barycentric coordinate function associated with the vertex \( a_i \) for \( i = 1, 2, \ldots, d + 1 \). We first introduce

\[
\Lambda(T) := Q_1(T) + Q_2(T) + \ldots Q_{d+1}(T),
\]
where
\[
Q_i(T) = \left( \prod_{j \neq i} \lambda_j \right) \text{span} \left\{ \prod_j \lambda_j^{\alpha_j} : \sum_j \alpha_j = k, \alpha_i = 0 \right\}, i = 1, 2, \ldots, d + 1.
\]

Then we define the operator \( S : L^2(\partial T) \rightarrow \Lambda(T) \) as follows: For any \( \mu \in L^2(\partial T) \), \( S\mu \) satisfies
\[
\int_F S\mu q = \int_F \mu q, \quad \forall q \in P_k(F), \text{ for each face } F \text{ of } T.
\]

Finally, we define \( R : L^2(\partial T) \rightarrow P_1(T) + \Lambda(T) \) by
\[
R\mu := \Pi^{CR}\mu + S(\mu - \Pi^{CR}\mu).
\]

where \( \Pi^{CR}\mu \in P_1(T) \) satisfies
\[
\int_F \Pi^{CR}\mu := \int_F \mu, \text{ for each face } F \text{ of } T.
\]

By recalling \( M(\partial T) := \{ \mu \in L^2(\partial T) : \mu|_F \in M(F), \text{ for each face } F \text{ of } T \} \) and using standard scaling arguments, it is easy to derive the following lemma.

**Lemma A.1.** For any \( \mu \in M(\partial T) \), it holds
\[
\|\mu\|_{h,\partial T} \sim \|R\mu\|_{T}, \quad (A.1)
\]
\[
|\mu|_{h,\partial T} \sim |R\mu|_{1,T}. \quad (A.2)
\]

For any \( \mu_h \in M^0_h \), it is obvious that \( R\mu_h \) satisfies the 0-th order weak continuity, i.e., \( R\mu_h \) is continuous at the gravity point of each interior face of \( T_h \). In addition, it holds \( R\mu_h|_{\partial \Omega} = 0 \). Therefore, from discrete Poincaré-Friedrichs inequalities ([16]) we have
\[
\|R\mu_h\| \lesssim \left( \sum_{T \in T_h} |R\mu_h|^2_{1,T} \right)^{1/2}.
\]

Then it follows
\[
\|\mu_h\|^2_h = \sum_{T \in T_h} \|\mu_h\|^2_{h,\partial T}
\]
\[
\sim \sum_{T \in T_h} \|R\mu_h\|^2_T \quad \text{(by (A.1))}
\]
\[
\lesssim \sum_{T \in T_h} |R\mu_h|^2_{1,T}
\]
\[
\lesssim |\mu_h|^2_h. \quad \text{(by (A.2))}
\]
B Appendix: Proof of Lemma 3.2

Denote by $\hat{T}$ the referential unit simplex. For any simplex $T$, there exists an invertible affine map $F : \hat{T} \rightarrow T$ with $F(\hat{x}) = A\hat{x} + b$ for $\hat{x} \in \hat{T}$, $A \in \mathbb{R}^{d \times d}$ a nonsingular matrix and $b \in \mathbb{R}^d$. For any $p \in [L^2(T)]^s (s = 1, 2, 3)$ and $\mu \in L^2(\partial T)$, we understand $\hat{p}$ and $\hat{\mu}$ by

$$\hat{p}(\hat{x}) = p(x), \quad \hat{\mu}(\hat{x}) = \mu(x), \quad (B.1)$$

where $x = F(\hat{x})$ for $\hat{x} \in \hat{T}$.

We state two well-known results as follows [21]:

$$\|A\| \sim h_T, \quad (B.3)$$
$$\|A^{-1}\| \sim h_T^{-1}, \quad (B.4)$$

where the matrix norm $\|\cdot\| : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}$ is defined by

$$\|A\| = \max_{0 \neq x \in \mathbb{R}^d} \frac{\|Ax\|}{\|x\|}, \forall A \in \mathbb{R}^{d \times d}. \quad (B.5)$$

Based on the above two results, it’s straightforward to obtain

$$\|A^{-1}x\| \sim h_T^{-1} \|x\|, \forall x \in \mathbb{R}^d. \quad (B.6)$$

Using the same techniques as in the proof the properties of the famous Piola transformation ([4]), we easily obtain the lemma below.

**Lemma B.1.** For any $(v, \mu) \in L^2(T) \times L^2(\partial T)$, it holds

$$\hat{\nabla}^b_w \hat{\mu} = A^T \nabla^b_w \mu, \quad (B.7)$$
$$\hat{\nabla}^i_w \hat{\nu} = A^T \nabla^i_w v, \quad (B.8)$$
$$\hat{\nabla}_w(\hat{\nu}, \hat{\mu}) = A^T \nabla_w (v, \mu). \quad (B.9)$$

**Lemma B.2.** For any simplex $T$, there exist two positive constants $c_T$ and $C_T$, which only depend on $T$ and $k$, such that

$$c_T \|\mu\|_{\partial T} \leq \left\| \nabla^b_w \mu \right\|_T \leq C_T \|\mu\|_{\partial T}, \forall \mu \in M(\partial T). \quad (B.10)$$

**Proof.** Assuming $\nabla^b_w \mu = 0$, by the definition of $\nabla^b_w$, i.e. (2.8) we have

$$\langle \mu, q \cdot n \rangle_{\partial T} = 0, \forall q \in \mathbf{W}(T),$$

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which implies \( \mu = 0 \). This means the semi-norm \( \| \nabla_w^b \|_T \) is a norm on \( M(\partial T) \). Since different norms on a finite dimensional space are equivalent, this lemma follows immediately. \( \square \)

**Theorem B.1.** For any simplex \( T \), it holds

\[
\left\| \nabla_w^b \mu \right\|_T \sim h_T^{-1} \| \mu \|_{h, \partial T}, \quad \forall \mu \in M(\partial T).
\]

(B.11)

**Proof.** In view of \( T = A \hat{T} + b \), we have

\[
\left\| \nabla_w^b \mu \right\|_T \sim h_T^{\frac{d}{2}} \left\| \nabla_w^b \mu \right\|_{\hat{T}}
\]

\[
\sim h_T^{\frac{d}{2}} \left\| A^{-T} \nabla_w^b \mu \right\|_{\hat{T}}
\]

(by Lemma B.1)

\[
\sim h_T^{\frac{d}{2} - 1} \left\| \nabla_w^b \mu \right\|_{\hat{T}}
\]

(by (B.6))

\[
\sim h_T^{\frac{d}{2} - 1} \| \hat{\mu} \|_{\partial \hat{T}}
\]

(by Lemma B.2)

\[
\sim h_T^{-\frac{1}{2}} \| \mu \|_{\partial T}
\]

\[
\sim h_T^{-1} \| \mu \|_{h, \partial T}.
\]

\( \square \)

Similarly, we can easily prove the following theorem.

**Theorem B.2.** For any simplex \( T \), it holds

\[
\left\| \nabla^i w v \right\|_T \sim h_T^{-1} \| v \|_T, \quad \forall v \in V(T).
\]

(B.12)

**Lemma B.3.** For any simplex \( T \), there exist two positive constants \( c_T \) and \( C_T \) that only depend on \( T \) and \( k \), such that

\[
c_T(\| v \|_T + \| \mu \|_{\partial T}) \leq \| \nabla_w(v, \mu) \|_T \leq C_T(\| v \| + \| \mu \|_{\partial T}), \quad \forall (v, \mu) \in \Sigma(T),
\]

where \( \Sigma(T) := \{(v, \mu) \in V(T) \times M(\partial T) : m_T(\mu) = 0\} \).

**Proof.** It’s easy to know

\[
(v, \mu) \mapsto \| v \|_T + \| \mu \|_{\partial T}, \quad \forall (v, \mu) \in \Sigma(T)
\]

defines a norm on \( \Sigma(T) \).

Next we show

\[
(v, \mu) \rightarrow \| \nabla_w(v, \mu) \|_T, \quad \forall (v, \mu) \in \Sigma(T)
\]
also defines a norm on $\Sigma(T)$. In fact, if $\|\nabla_w(v,\mu)\|_T = 0$, then by the definition of $\nabla_w$, i.e. (2.6) we have

$$\langle \nabla v, q \rangle_T + \langle \mu - v, q \cdot n \rangle_{\partial T} = 0, \ \forall q \in W(T).$$

(B.14)

This relation, together with the properties of the BDM elements ([18]) and the RT elements [40], shows $v = \mu = \text{constant}$. Thus the relation $m_T(\mu) = 0$ leads to $(v, \mu) = 0$.

Finally, the desired conclusion follows from the equivalence of the above two norms. □

**Lemma B.4.** For any simplex $T$, it holds

$$\|\nabla_w(v,\mu)\|_T \sim h_T^{-\frac{1}{2}}\|v\|_T + h_T^{-\frac{1}{2}}\|\mu\|_{\partial T}, \ \forall (v, \mu) \in \Sigma(T).$$

(B.15)

**Proof.** In light of $T = A\hat{T} + b$ and $m_T(\mu) = m_{\hat{T}}(\hat{\mu})$ for all $\mu \in L^2(\partial T)$, we obtain

$$\|\nabla_w(v,\mu)\|_T \sim h_T^{-\frac{1}{2}}\|\nabla_w(\hat{v},\hat{\mu})\|_{\hat{T}}$$

(by Lemma B.1)

$$\sim h_T^{-\frac{3}{2}}\|A^{-T}\hat{\nabla}_w(\hat{v},\hat{\mu})\|_{\hat{T}}$$

(by (B.6))

$$\sim h_T^{-\frac{3}{2}}\|\hat{\nabla}_w(\hat{v},\hat{\mu})\|_{\hat{T}}$$

(by Lemma B.3)

$$\sim h_T^{-1}\|v\|_T + h_T^{-\frac{1}{2}}\|\mu\|_{\partial T}.$$ 

□

**Theorem B.3.** For any simplex $T$, it holds

$$\|\nabla_w(v,\mu)\|_T \sim h_T^{-\frac{1}{2}}\|v - m_T(\mu)\|_T + |\mu|_{h,\partial T}, \ \forall (v, \mu) \in V(T) \times M(\partial T).$$

(B.16)

**Proof.** By (2.6) we have

$$\langle \nabla_w(v,\mu), q \rangle_T = -(v, \text{div} q)_T + \langle \mu, q \cdot n \rangle_{\partial T}$$

$$= -(v - m_T(\mu), \text{div} q)_T + \langle \mu - m_T(\mu), q \cdot n \rangle_{\partial T}$$

$$= (\nabla_w(v - m_T(\mu), \mu - m_T(\mu)), q)_T, \ \forall q \in W(T),$$

which implies

$$\nabla_w(v,\mu) = \nabla_w(v - m_T(\mu), \mu - m_T(\mu)).$$

Thus it follows

$$\|\nabla_w(v,\mu)\|_T = \|\nabla_w(v - m_T(\mu), \mu - m_T(\mu))\|_T$$

$$\sim h_T^{-\frac{1}{2}}\|v - m_T(\mu)\|_T + h_T^{-\frac{1}{2}}\|\mu - m_T(\mu)\|_{\partial T}$$

(by Lemma B.4)

$$\sim h_T^{-1}\|v - m_T(\mu)\|_T + |\mu|_{h,\partial T}.$$ 

□
A combination of Theorems B.1-B.3 proves Lemma 3.2.

References


