On the union and intersection operations of rough sets based on various approximation spaces

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\begin{abstract}
Algebraic structures and lattice structures of rough sets are basic and important topics in rough sets theory. In this paper we pointed out that a basic problem had been overlooked, that is the closeness of union and intersection operations of rough approximation pairs, i.e. (lower approximation, upper approximation). We present that the union and intersection operations of rough approximation pairs are closed for classical rough sets and two kinds of covering based rough sets, but not for twenty kinds of covering based rough sets and the generalized rough sets based on fuzzy approximation space. Moreover, we proved that the union and intersection operations of rough fuzzy approximation pairs are closed and a bounded distributive lattice can be constructed.
\end{abstract}

1. Introduction

Rough set theory was proposed by Pawlak [22,23]. Rough set is a mathematical tool for dealing with uncertainty and it is widely applied to pattern recognition, image processing, feature selection, rule extraction, decision supporting, granular computing, data mining and knowledge discovery from large data sets [see 24,42].

In the basic theoretical researches on classical rough sets and various generalized rough sets, algebra approach is widely applied (see [1,3–8,10,12,16–19,21,25,27,28,30,32,34,35,38–40,43–45]). For Pawlak’s approximate space \((U, R)\), where \(U\) is the universe and \(R\) is an equivalence relation on \(U\), let \(X \subseteq U\), we consider the approximation pairs \((R(X), \overline{R}(X))\), which is also called rough sets. Denote \(RS(U) = \{(R(X), \overline{R}(X))| X \subseteq U\}\). The union and intersection operations of rough approximation pairs can be defined by:

\[
\begin{align*}
(R(X), \overline{R}(X)) \cup (R(Y), \overline{R}(Y)) &= (R(X) \cup R(Y), \overline{R}(X) \cup \overline{R}(Y)); \\
(R(X), \overline{R}(X)) \cap (R(Y), \overline{R}(Y)) &= (R(X) \cap R(Y), \overline{R}(X) \cap \overline{R}(Y)).
\end{align*}
\]

A natural question is as follows:

Whether the above union and intersection operations are binary operation on \(RS(U)\), that is, whether the union and intersection operations of rough approximation pairs are closed in \(RS(U)\).

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In this paper, we will solve the above problem for various rough set models. The results of this paper show that the union and intersection operations of rough approximation pairs are not closed for some kinds of generalized rough sets, and the closeness problem is often overlooked in some literatures. We will also point out an open problem.

2. On Pawlak's classical rough sets

Let $U$ denote a non-empty set called the universe. Let $R \subseteq U \times U$ be an equivalence relation on $U$. The pair $\text{apr} = (U, R)$ is called an approximation space. The equivalence relation $R$ partitions the set $U$ into disjoint subsets. Let $U/R$ denote the quotient set consisting of equivalence classes of $R$, and $[x]_R$ the equivalence class containing $x$. Given an arbitrary set $A \subseteq U$, in general it may not be possible to describe $A$ precisely in $(U, R)$. One may characterize $A$ by a pair of lower and upper approximations:

$$R(A) = \{x \in U | [x]_R \subseteq A\},$$
$$\bar{R}(A) = \{x \in U | [x]_R \cap A \neq \emptyset\}.$$  

The pair $(R(A), \bar{R}(A))$ is referred to as rough set approximation of $A$.

The union and intersection operations of Pawlak’s classical rough sets (approximation pairs) are closed, which were first obtained by Pomykala and Pomykala in [25] and further discussed in [1,3,9,14].

**Theorem 2.1** [25]. Let $(U, R)$ be a Pawlak’s approximation space. For any $X, Y \subseteq U$, there exists $Z \subseteq U$ such that

$$R(Z) = R(X) \cup R(Y), \quad \overline{R}(Z) = \overline{R}(X) \cup \overline{R}(Y).$$

**Theorem 2.2** [25]. Let $(U, R)$ be a Pawlak’s approximation space. For any $X, Y \subseteq U$, there exists $W \subseteq U$ such that

$$R(W) = R(X) \cap R(Y), \quad \overline{R}(W) = \overline{R}(X) \cap \overline{R}(Y).$$

The above theorems are generalized to strong topological rough algebras (see Theorem 6 and Theorem 7 in [44]). In fact, for Pawlak’s approximation space $(U, R)$, denote the power set of $U$ by $2^U$, then $(2^U, \cap, \cup, \emptyset, U)$ is a Boolean algebra and $(2^U, \cap, \cup, \emptyset, R, \emptyset, U)$ is a strong topological rough algebra (where $\emptyset$ denotes the complement, see Theorem 1 in [44]). Moreover, we can give the formula of the above $Z$ in **Theorem 2.1** and $W$ in **Theorem 2.2**:

$$Z = (\cup\{T \subseteq U | T \subseteq X, T \cap R_1(X, Y) = \emptyset\}) \cup Y,$$

where

$$R_1(X, Y) = \cup\{[s]_R | s \in U, [s]_R \subseteq X \cup Y, [s]_R \subseteq X, [s]_R \subseteq Y\};$$
$$W = \left(\cup\{T \subseteq U | T \subseteq Y', T \cap R_1(X', Y) = \emptyset\}\right)' \cap X'.$$

where

$$R_1(X', Y') = \cup\{[s]_R | s \in U, [s]_R \subseteq (X \cap Y'), [s]_R \subseteq X', [s]_R \subseteq Y'\}.$$

3. On generalized rough sets based on tolerance approximation spaces and determined by preorder relation

Tolerance approximation spaces were introduced by Skowron and Stepaniuk in [29]. Let $U$ be a universe and $\tau \subseteq U \times U$. $\tau$ is called tolerance relation on $U$ if it is reflexive and symmetric. If $\tau$ is tolerance relation on $U$, then $(U, \tau)$ is called tolerance approximation space.

Let $(U, \tau)$ be a tolerance approximation space, denote $\tau(x) = \{y \in U | (x, y) \in \tau\}$, where $x \in U$. Define operations $\tau_+: 2^U \to 2^U$ and $\tau^-: 2^U \to 2^U$ such that, for any $X \subseteq U$:

$$\tau_+(X) = \{x \in U | \tau(x) \subseteq X\},$$
$$\tau^-(X) = \{x \in U | \tau(x) \cap X \neq \emptyset\}.$$  

The sets $\tau_+(X)$ and $\tau^-(X)$ are called $\tau$-lower and $\tau$-upper approximation of $X$ respectively.

By the Example 164(a) and Fig. 25 in [15] we know that, for tolerance approximation spaces, the union and intersection operations of rough approximation pairs $(\tau_+(X), \tau^-(X))$ are not closed.

Based on tolerance approximation spaces, Wasilewski and Slezak introduced the notions of $T$-lower and $T$-upper approximations in [33]. By this generalized rough sets, a double Heyting algebra is established in [33], but we will show that their main results are questionable, because the union and intersection operations of $T$-rough approximation pairs are not proven to be closed.
Definition 3.1 [33]. Let \((U, \tau)\) be a tolerance approximation space. A set \(A \subseteq U\) is a preclass of tolerance \(\tau\), shortly preclass, if for any \(x, y \in U\), if \(x, y \in A\), then \((x, y) \in \tau\). Maximal preclasses with respect to inclusions are called classes. The families of all preclasses and classes of \(\tau\) will be denoted by \(PH_\tau\) and \(H_\tau\) respectively. For \(x \in U\) we put \(H_\tau(x) = \{A \in H_\tau | x \in A\}\).

Definition 3.2 [33]. Let \((U, \tau)\) be a tolerance approximation space. A set \(A_t\) is defined as follows: \(A_t = \{x \in U | \tau(x) \subseteq H_t\}\).

Definition 3.3 [33]. Let \((U, \tau)\) be a tolerance approximation space. For any \(X \subseteq U\) we define operators \(T_\tau : 2^U \rightarrow 2^U\), \(T^\tau : 2^U \rightarrow 2^U\):

\[
T_\tau(X) = \{A \in A_t | A \subseteq X\},
\]

\[
T^\tau(X) = \Sigma^\tau(X) - T_\tau(X).
\]

where \(\Sigma^\tau(X) = \cup\{A \in A_t | A \cap X \neq \emptyset\}\). \(T_\tau(X)\) and \(T^\tau(X)\) are called \(T\)-lower and \(T\)-upper approximations of \(X\).

Denote \(R_t(U) = \{(T_\tau(X), T^\tau(X)) | X \in U\}\). In [33] the authors define operations \(\odot, \odot\) on \(R_t(U)\) in the following way:

\[
(T_\tau(X), T^\tau(X)) \odot (T_\tau(Y), T^\tau(Y)) = (T_\tau(X) \cup T_\tau(Y), T^\tau(X) \cup T^\tau(Y));
\]

\[
(T_\tau(X), T^\tau(X)) \odot (T_\tau(Y), T^\tau(Y)) = (T_\tau(X) \cap T_\tau(Y), T^\tau(X) \cap T^\tau(Y));
\]

Moreover, Theorem 11 in [33] proved that \((R_t(U), \odot, \odot, -, \odot, (0, 0), (U, U))\) is a complete atomic double Heyting algebra (for the definition of \(-, \odot\), please see [33]). However, in [33], the authors have not proved that the union and intersection operations of \(T\)-rough approximation pairs \((T_\tau(X), T^\tau(X))\) are closed, that is, the above operations \(\odot, \odot\) may be not binary operation on \(R_t(U)\), therefore, Theorem 11 in [33] has a fault. So, we point out the following open problem:

**Open Problem** Whether the union and intersection operations of \(T\)-rough approximation pairs \((T_\tau(X), T^\tau(X))\) are closed.

Let \(U\) be a universe. If a binary relation \(R\) on \(U\) is reflexive and transitive, it is called a preorder (or a quasi-order). Then the pair \((U, R)\) is called a preorder approximation space, and the approximation operators are defined by (see [15]):

\[
X^* = \{x \in U | R(x) \subseteq X\},
\]

\[
X^* = \{x \in U | R(x) \cap X \neq \emptyset\}.
\]

for any \(X \subseteq U\), where \(R(x) = \{y \in U | xRy\}\).

In [15], the author point out an open problem (see Fig. 27 in [15]):

Whether \(A\) and \(R\) determined by preorders are lattices?

That is (the definitions of \(A\) and \(R\), please see [15]), whether the operations \(\odot, \odot\) are closed, where \(\odot, \odot\) are defined by:

\[
(X^*, Y^*) \odot (Y^*, Y^*) = (X^* \cup Y^*, X^* \cup Y^*),
\]

\[
(X^*, Y^*) \odot (Y^*, Y^*) = (X^* \cap Y^*, X^* \cap Y^*).
\]

The above problem is solved by Jarvinen et al. in [16], it is proved that for a quasiorder \(R\) on \(U\), the structure \(R\) is always a completely distributive complete lattice. In addition, in [17], it is proved that on \(A\) a Nelson algebra can be defined. Therefore, the union and intersection operations of rough approximation pairs determined by preorder relations are closed.

4. On covering rough sets

Covering rough sets are an important kind of generalized rough set model, into which many authors have investigated (see [11,20,26,31,36,41,46–48]). Particularly, Yao and Yao proposed a general framework for the study of covering based rough sets in [41], and considered twenty-two approximation pairs. In this section, we discuss the union and intersection operations of these rough approximation pairs. We will show that the union and intersection operations are closed for two kinds of covering based rough approximation pairs, but not closed for twenty kinds of covering based rough approximation pairs.

For convenience and consistency, we use the concepts and notations in [41] (although some of these concepts in other literatures have been discussed).

**Definition 4.1** [41]. A mapping \(n : U \rightarrow 2^U\) is called a neighborhood operator. If \(n(x) \neq \emptyset\) for all \(x \in U\), \(n\) is called a serial neighborhood operator. If \(x \in n(x)\) for all \(x \in U\), \(n\) is called a reflexive neighborhood operator.

**Definition 4.2** (Element based definition [41]). Suppose \(n : U \rightarrow 2^U\) is a neighborhood operator. It defines a pair of approximation operators as follows:

\[
\text{ap}_{n}(A) = \{x | x \in U, n(x) \subseteq A\};
\]

\[
\text{ap}^\tau_{n}(A) = \{x | x \in U, n(x) \cap A \neq \emptyset\}.
\]
**Definition 4.3** [13,18,41]. Let $U$ be a universe of discourse and $\mathcal{C}$ be a family of non-empty subsets of $U$. If $\mathcal{C} = U$, $\mathcal{C}$ is called a covering of $U$. The ordered pair $(U, \mathcal{C})$ is called a covering approximation space.

**Definition 4.4** (Granule based definition [41]). For a covering $\mathcal{C}$ of $U$, two pairs of approximation operators $(\text{apr}_c, \overline{\text{apr}}_c)$ and $(\text{apr}_c, \overline{\text{apr}}_c)$ are defined as:

$$\text{apr}_c(A) = \bigcup \{K \in \mathcal{C} \mid K \subseteq A\} = \{x \in U \mid \exists K \in \mathcal{C} \mid x \in K, K \subseteq A\}$$

$$\overline{\text{apr}}_c(A) = \left(\overline{\text{apr}}_c(A)\right)' = \{x \in U \mid \forall K \in \mathcal{C} \mid x \in K \Rightarrow K \cap A = \emptyset\}$$

$$\text{apr}_c(A) = \left(\text{apr}_c(A)\right)' = \{x \in U \mid \forall K \in \mathcal{C} \mid x \in K \Rightarrow K \subseteq A\}$$

$$\overline{\text{apr}}_c(A) = \bigcup \{K \mid K \in \mathcal{C}, K \cap A \neq \emptyset\} = \{x \in U \mid \exists K \in \mathcal{C} \mid x \in K, K \cap A \neq \emptyset\}$$

**Definition 4.5** [41]. Suppose $\mathcal{C}$ is a covering of $U$. A neighborhood system $C(\mathcal{C}, x)$ of $x$ is defined by:

$$C(\mathcal{C}, x) = \{K | K \in \mathcal{C}, x \in K\} \subseteq \mathcal{C}.$$  

**Definition 4.6** [41,47]. Let $(U, \mathcal{C})$ be a covering approximation space, $x \in U$, then set family

$$md(\mathcal{C}, x) = \{K \in C(\mathcal{C}, x) | (\forall S \in C(\mathcal{C}, x), S \subseteq K) \Rightarrow K = S\}.$$  

is called the minimal description of $x$. On the other hand, the set

$$MD(\mathcal{C}, x) = \{K \in C(\mathcal{C}, x) | (\forall S \in C(\mathcal{C}, x), S \supseteq K) \Rightarrow K = S\}.$$  

is called the maximal description of $x$.  

From $md(\mathcal{C}, x)$ and $MD(\mathcal{C}, x)$, Yao and Yao [41] defined the following neighborhood operators:

$$N_1(x) = \bigcap \{K \subseteq \mathcal{C} | \mathcal{C} \in md(\mathcal{C}, x)\},$$

$$N_2(x) = \bigcup \{K \subseteq \mathcal{C} | \mathcal{C} \in md(\mathcal{C}, x)\},$$

$$N_3(x) = \bigcap \{K \subseteq \mathcal{C} | \mathcal{C} \in MD(\mathcal{C}, x)\},$$

$$N_4(x) = \bigcup \{K \subseteq \mathcal{C} | \mathcal{C} \in MD(\mathcal{C}, x)\}.$$  

By Definition 4.2, we have the following approximation operators $(i = 1, 2, 3, 4)$:

$$\text{apr}_{N_i}(A) = \{x \mid x \in U, N_i(x) \subseteq A\};$$

$$\overline{\text{apr}}_{N_i}(A) = \{x \mid x \in U, N_i(x) \cap A \neq \emptyset\}.$$  

**Definition 4.7** [41]. A family of subsets of universe $U$ is called a closure system over $U$ if it contains $U$ and is closed under set intersection.

Given a closure system $\mathcal{S}$, one can construct its dual system $\overline{\mathcal{S}}$ as follows: $\overline{\mathcal{S}} = \{X \mid X \in \mathcal{S}\}$. The system $\overline{\mathcal{S}}$ contains $\emptyset$ and is closed under set union.

**Definition 4.8** (Subsystem based definition [41]). Suppose $\mathcal{S} = (\mathcal{S}, \mathcal{S})$ is a pair of subsystems of $2^U$, $\mathcal{S}$ is a closure system and $\overline{\mathcal{S}}$ is the dual system of $\mathcal{S}$. A pair of lower and upper approximation operators $(\text{apr}_\mathcal{S}, \overline{\text{apr}}_\mathcal{S})$ with respect to $\mathcal{S}$ is defined as:

$$\text{apr}_\mathcal{S}(A) = \bigcup \{X \mid X \in \mathcal{S}, X \subseteq A\};$$

$$\overline{\text{apr}}_\mathcal{S}(A) = \bigcap \{X \mid X \in \mathcal{S}, X \supseteq A\}.$$  

4.1. On approximation operator $(\text{apr}_{N_1}, \overline{\text{apr}}_{N_1})$

A neighborhood operator can be defined by using a binary relation. Suppose $R \subseteq U \times U$ is a binary relation. A successor neighborhood operator $R_\mathcal{S} : U \rightarrow 2^U$ can be defined as:

$$R_\mathcal{S}(x) = \{y \mid y \in U, x R y\}.$$  

Conversely, a binary relation can be reconstructed from its successful neighborhood as:

$$x R y \iff y \in R_\mathcal{S}(x).$$
That is, generalized approximations by a neighborhood operator can be equivalently formulated by using a binary relation.

Many authors studied the relationships between relation based rough sets and covering based rough sets [20,36,41,46,49]. By those results and we can also prove that the union and intersection operations of the rough approximation pairs \((\text{apr}_{N_1}, \text{appr}_{N_1})\) are closed.

**Theorem 4.1.** Let \((U, C)\) be a covering approximation space. For any \(X, Y \subseteq U\), there exists \(Z \subseteq U\) such that

\[
\text{apr}_{N_1}(Z) = \text{apr}_{N_1}(X) \cup \text{apr}_{N_1}(Y),
\]

\[
\text{appr}_{N_1}(Z) = \text{appr}_{N_1}(X) \cup \text{appr}_{N_1}(Y).
\]

Moreover, there exists \(W \subseteq U\) such that

\[
\text{apr}_{N_1}(W) = \text{apr}_{N_1}(X) \cap \text{apr}_{N_1}(Y),
\]

\[
\text{appr}_{N_1}(W) = \text{appr}_{N_1}(X) \cap \text{appr}_{N_1}(Y).
\]

**Proof.** By Theorem 3 in [36] (or, Theorem 3.1 in [20], Proposition 1 in [46]), we know that the relation \(R\) induced by \(N_1\) is reflexive and transitive (that is, \(R\) is a preorder relation or quasiorder relation), and

\[
\text{apr}_{N_1}(X) = R(X) = \text{appr}_{N_1}(X) = R(X).
\]

On the other hand, by Theorem 3.3 in [16], the union and intersection operations of rough approximation pairs determined by quasiorder relation (reflexive and transitive relation) are closed. Then the proof is complete. \(\square\)

4.2. On approximation operator \((\text{apr}_{N_1}, \text{appr}_{N_1})\)

The following example demonstrates that the union and intersection operations of covering rough approximation pairs \((\text{apr}_{N_1}, \text{appr}_{N_1})\) are not closed.

**Example 4.1.** Let \(U = \{a, b, c, d\}\), \(K_1 = \{a, b\}\), \(K_2 = \{a, c\}\), \(K_3 = \{c, d\}\), \(C = \{K_1, K_2, K_3\}\). Then \(C\) is a covering of \(U\) and \((U, C)\) is a covering approximation space. By **Definition 4.6** we have

\[
\text{md}(C, a) = \{K_1, K_2\}, \quad \text{md}(C, b) = \{K_1\},
\]

\[
\text{md}(C, c) = \{K_2, K_3\}, \quad \text{md}(c, d) = \{K_3\}.
\]

Then

\[N_2(a) = \{a, b, c\}, \quad N_2(b) = \{a, b\}, \quad N_2(c) = \{a, c, d\}, \quad N_2(d) = \{c\}.
\]

Let \(X = \{a, b\}\), \(Y = \{b, c, d\}\), then

\[
\left(\text{apr}_{N_2}(X), \text{appr}_{N_2}(X)\right) = (\{b\}, \{a, b, c\}),
\]

\[
\left(\text{apr}_{N_2}(Y), \text{appr}_{N_2}(Y)\right) = (\{d\}, \{a, b, c, d\}).
\]

But, there is not any subset \(Z\) of \(U\) such that

\[
\text{apr}_{N_2}(Z) = \text{apr}_{N_2}(X) \cup \text{apr}_{N_2}(Y) = \{b, d\},
\]

\[
\text{appr}_{N_2}(Z) = \text{appr}_{N_2}(X) \cup \text{appr}_{N_2}(Y) = \{a, b, c, d\}.
\]

Similarly, let \(A = \{c, d\}\), \(B = \{a\}\), then

\[
\left(\text{apr}_{N_2}(A), \text{appr}_{N_2}(A)\right) = (\{d\}, \{a, c, d\}), \quad \left(\text{apr}_{N_2}(B), \text{appr}_{N_2}(B)\right) = (\emptyset, \{a, b, c\}).
\]

But, there is not any subset \(W\) of \(U\) such that

\[
\text{apr}_{N_2}(W) = \text{apr}_{N_2}(A) \cap \text{apr}_{N_2}(B) = \emptyset,
\]

\[
\text{appr}_{N_2}(W) = \text{appr}_{N_2}(A) \cap \text{appr}_{N_2}(B) = \{a, c\}.
\]

4.3. On approximation operator \((\text{apr}_{N_1}, \text{appr}_{N_1})\)

It can be proved easily that

\[N_1(x) = \bigcap\{K|K \in \text{md}(C, x)\} = \bigcap\{K|K \in C, x \in K\}.
\]
On the other hand, $N_1(x) = \bigcap \{K | K \in MD(C,x)\}$. This means that $N_1$ is a special $N_1$ for new covering $\{MD(C,x)|x \in U\}$. By Theorem 4.1 we have

**Theorem 4.2.** Let $(U, C)$ be a covering approximation space. For any $X, Y \subseteq U$, there exists $Z \subseteq U$ such that

\[
\begin{align*}
\text{apr}_{N_1}(Z) &= \text{apr}_{N_1}(X) \cup \text{apr}_{N_1}(Y), \\
\text{apr}_{N_1}(Z) &= \text{apr}_{N_1}(X) \cup \text{apr}_{N_1}(Y).
\end{align*}
\]

Moreover, there exists $W \subseteq U$ such that

\[
\begin{align*}
\text{apr}_{N_1}(W) &= \text{apr}_{N_1}(X) \cap \text{apr}_{N_1}(Y), \\
\text{apr}_{N_1}(W) &= \text{apr}_{N_1}(X) \cap \text{apr}_{N_1}(Y).
\end{align*}
\]

4.4. On approximation operator $(\text{apr}_{N_1}, \text{apr}_{N_1})$

The following example demonstrates that the union and intersection operations of covering rough approximation pairs $(\text{apr}_{N_1}, \text{apr}_{N_1})$ are not closed.

**Example 4.2.** Let $U = \{a, b, c, d, e\}$, $K_1 = \{a\}$, $K_2 = \{a, b, d\}$, $K_3 = \{a, c, d\}$, $K_4 = \{c, e\}$, $C = \{K_1, K_2, K_3, K_4\}$. Then $C$ is a covering of $U$ and $(U, C)$ is a covering approximation space. By Definition 4.6 we have

\[
\begin{align*}
MD(C,a) &= \{K_2, K_3\}, \quad MD(C,b) = \{K_2\}, \\
MD(C,c) &= \{K_3, K_4\}, \quad MD(C,d) = \{K_2, K_3\}, \quad MD(C,e) = \{K_4\}.
\end{align*}
\]

Then

\[
\begin{align*}
N_1(a) &= \{a, b, c, d\}, \quad N_1(b) = \{a, b, d\}, \quad N_1(c) = \{a, c, d, e\}, \\
N_1(d) &= \{a, b, c, d\}, \quad N_1(e) = \{c, e\}.
\end{align*}
\]

Let $A = \{a\}$, $B = \{c\}$, then

\[
\begin{align*}
\left(\text{apr}_{N_1}(A), \text{apr}_{N_1}(A)\right) &= (\emptyset, \{a, b, c, d\}), \\
\left(\text{apr}_{N_1}(B), \text{apr}_{N_1}(B)\right) &= (\emptyset, \{a, c, d, e\}).
\end{align*}
\]

But, there is not any subset $W$ of $U$ such that

\[
\begin{align*}
\text{apr}_{N_1}(W) &= \text{apr}_{N_1}(A) \cap \text{apr}_{N_1}(B) = \emptyset, \\
\text{apr}_{N_1}(W) &= \text{apr}_{N_1}(A) \cap \text{apr}_{N_1}(B) = \{a, c, d\}.
\end{align*}
\]

This means that the intersection operation of covering rough approximation pairs $(\text{apr}_{N_1}, \text{apr}_{N_1})$ is not closed. By Theorem 1 in [41], $\text{apr}_{N_1}$ and $\text{apr}_{N_1}$ are dual approximation operations, therefore, the union operation of approximation pairs $(\text{apr}_{N_1}, \text{apr}_{N_1})$ is not closed.

4.5. On approximation operator $(\text{apr}_{c}, \text{apr}_{c})$

The following example demonstrates that the union and intersection operations of covering rough approximation pairs $(\text{apr}_{c}, \text{apr}_{c})$ are not closed.

**Example 4.3.** Let $U = \{a, b, c, d\}$, $K_1 = \{a\}$, $K_2 = \{a, b, d\}$, $K_3 = \{b, c\}$, $C = \{K_1, K_2, K_3\}$. Then $C$ is a covering of $U$ and $(U, C)$ is a covering approximation space. Let $X = \{a\}$, $Y = \{c\}$, by Definition 4.4 we have

\[
\begin{align*}
\left(\text{apr}_{c}(X), \text{apr}_{c}(X)\right) &= (\{a\}, \{a\}), \quad \left(\text{apr}_{c}(Y), \text{apr}_{c}(Y)\right) = (\emptyset, \{c\}).
\end{align*}
\]

But, there is not any subset $Z$ of $U$ such that

\[
\begin{align*}
\text{apr}_{c}(Z) &= \text{apr}_{c}(X) \cup \text{apr}_{c}(Y) = \{a\}, \\
\text{apr}_{c}(Z) &= \text{apr}_{c}(X) \cup \text{apr}_{c}(Y) = \{a, c, d\}.
\end{align*}
\]
This means that the union operation of covering rough approximation pairs \( \text{apr}_c, \overline{\text{apr}}_c \) is not closed. By Definition 4.4 in [41], \( \text{apr}_c \) and \( \overline{\text{apr}}_c \) are dual approximation operations, therefore, the intersection operation of approximation pairs \( \text{apr}_c, \overline{\text{apr}}_c \) is not closed.

4.6. On approximation operator \( \text{apr}_c, \overline{\text{apr}}_c \)

The following example demonstrates that the union and intersection operations of covering rough approximation pairs \( \text{apr}_c, \overline{\text{apr}}_c \) are not closed.

**Example 4.4.** Let \( U = \{a, b, c, d\} \), \( K_1 = \{a\}, K_2 = \{a, b, d\}, K_3 = \{b, c\} \), \( C = \{K_1, K_2, K_3\} \). Then \( C \) is a covering of \( U \) and \((U, C)\) is a covering approximation space. Let \( X = \{b, c\}, Y = \{a, b, d\} \), by Definition 4.4 we have

\[
\begin{align*}
\text{apr}_c(X, \overline{\text{apr}}_c(X)) &= (\{c\}, \{a, b, c, d\}), \\
\text{apr}_c(Y, \overline{\text{apr}}_c(Y)) &= (\{a, d\}, \{a, b, c, d\}).
\end{align*}
\]

But, there is not any subset \( Z \) of \( U \) such that

\[
\begin{align*}
\text{apr}_c(Z) &= \text{apr}_c(Y) \cup \overline{\text{apr}}_c(Y) = \{a, b, d\}, \\
\overline{\text{apr}}_c(Z) &= \overline{\text{apr}}_c(Y) \cup \overline{\text{apr}}_c(Y) = \{a, b, c, d\}.
\end{align*}
\]

This means that the union operation of covering rough approximation pairs \( \text{apr}_c, \overline{\text{apr}}_c \) is not closed. By Theorem 2 in [41], \( \text{apr}_c \) and \( \overline{\text{apr}}_c \) are dual approximation operations, therefore, the intersection operation of approximation pairs \( \text{apr}_c, \overline{\text{apr}}_c \) is not closed.

4.7. On approximation operators \( \text{apr}_{c_1}, \overline{\text{apr}}_{c_1} \) and \( \text{apr}_{c_2}, \overline{\text{apr}}_{c_2} \)

In [41], \( C_1 \) is defined as the following:

\[
C_1 = \bigcup \{md(C, x)|x \in U\}.
\]

Then \( C_1 \) may be equal to \( C \), for example, for the covering \( C \) in Examples 4.3 and 4.4, we have \( C_1 = C \). Therefore, by Examples 4.3 and 4.4, the union and intersection operations of covering rough approximation pairs \( \text{apr}_{c_1}, \overline{\text{apr}}_{c_1} \) and \( \text{apr}_{c_2}, \overline{\text{apr}}_{c_2} \) are not closed.

4.8. On approximation operators \( \text{apr}_{c_1}, \overline{\text{apr}}_{c_1} \) and \( \text{apr}_{c_2}, \overline{\text{apr}}_{c_2} \)

In [41], \( C_2 \) is defined as following:

\[
C_2 = \bigcup \{MD(C, x)|x \in U\}.
\]

The following example demonstrates that the union and intersection operations of covering rough approximation pairs \( \text{apr}_{c_1}, \overline{\text{apr}}_{c_1} \) and \( \text{apr}_{c_2}, \overline{\text{apr}}_{c_2} \) are not closed.

**Example 4.5.** Let \( U = \{a, b, c, d\} \), \( K_1 = \{a\}, K_2 = \{b, c\}, K_3 = \{b, c, d\} \), \( K_4 = \{a, b, d\} \), \( C = \{K_1, K_2, K_3, K_4\} \). Then

\[
\begin{align*}
MD(C, a) &= \{K_4\}, \\
MD(C, b) &= \{K_3, K_4\}, \\
MD(C, c) &= \{K_3, K_4\}, \\
MD(C, d) &= \{K_3, K_4\}.
\end{align*}
\]

By the definition of \( C_1 \) we have \( C_1 = \{\{a, b, d\}, \{b, c, d\}\} \).

Let \( X = \{a\}, Y = \{c\} \), then

\[
\begin{align*}
\text{apr}_{c_1}(X, \overline{\text{apr}}_{c_1}(X)) &= (\emptyset, \{a\}), \\
\text{apr}_{c_1}(Y, \overline{\text{apr}}_{c_1}(Y)) &= (\emptyset, \{c\}).
\end{align*}
\]

But, there is not any subset \( W \) of \( U \) such that

\[
\begin{align*}
\text{apr}_{c_1}(W) &= \text{apr}_{c_1}(X) \cup \text{apr}_{c_1}(Y) = \emptyset, \\
\overline{\text{apr}}_{c_1}(W) &= \overline{\text{apr}}_{c_1}(X) \cup \overline{\text{apr}}_{c_1}(Y) = \{a, c\}.
\end{align*}
\]
This means that the union operation of covering rough approximation pairs \( (\text{apr}^c_{C_3}, \overline{\text{apr}}^c_{C_3}) \) is not closed. By Theorem 2 in [41], \( \text{apr}^c_{C_3} \) and \( \overline{\text{apr}}^c_{C_3} \) are dual approximation operations, therefore, the intersection operation of approximation pairs \( (\text{apr}^c_{C_3}, \overline{\text{apr}}^c_{C_3}) \) is not closed.

Let \( A = \{a\}, B = \{c\} \), then
\[
\left( \text{apr}^c_{C_3}(A), \overline{\text{apr}}^c_{C_3}(A) \right) = (\emptyset, \{a, b, d\}), \quad \left( \text{apr}^c_{C_3}(B), \overline{\text{apr}}^c_{C_3}(B) \right) = (\emptyset, \{b, c, d\}).
\]
But, there is not any subset \( W \) of \( U \) such that
\[
\text{apr}^c_{C_3}(W) = \text{apr}^c_{C_3}(A) \cap \text{apr}^c_{C_3}(B) = \emptyset,
\]
\[
\overline{\text{apr}}^c_{C_3}(W) = \overline{\text{apr}}^c_{C_3}(A) \cap \overline{\text{apr}}^c_{C_3}(B) = \{b, d\}.
\]
This means that the union operation of covering rough approximation pairs \( (\text{apr}^c_{C_3}, \overline{\text{apr}}^c_{C_3}) \) is not closed. By dual properties (Theorem 2 in [41]), the intersection operation of approximation pairs \( (\text{apr}^c_{C_3}, \overline{\text{apr}}^c_{C_3}) \) is not closed.

4.9. On approximation operators \( (\text{apr}^c_{C_3}, \overline{\text{apr}}^c_{C_3}) \) and \( (\text{apr}^c_{C_3}, \overline{\text{apr}}^c_{C_3}) \)

By Definition 13, Theorem 5 and Table 1 in [37], we have
\[
C_3 = \{N_3(x) | x \in U\}.
\]
The following examples demonstrate that the union and intersection operations of rough approximation pairs \( (\text{apr}^c_{C_3}, \overline{\text{apr}}^c_{C_3}) \) and \( (\text{apr}^c_{C_3}, \overline{\text{apr}}^c_{C_3}) \) are not closed.

**Example 4.6.** Let \( U = \{a, b, c, d, e\} \), \( K_1 = \{a, b\} \), \( K_2 = \{b, c, d\} \), \( K_3 = \{a, d, e\} \), \( C = \{K_1, K_2, K_3\} \). Then
\[
C_3 = \{\{a, b\}, \{b\}, \{b, c, d\}, \{d\}, \{a, d, e\}\}.
\]
Let \( X = \{b\}, Y = \{e\} \), then
\[
\left( \text{apr}^c_{C_3}(X), \overline{\text{apr}}^c_{C_3}(X) \right) = (\{b\}, \{b, c\}), \quad \left( \text{apr}^c_{C_3}(Y), \overline{\text{apr}}^c_{C_3}(Y) \right) = (\emptyset, \{e\}).
\]
But, there is not any subset \( W \) of \( U \) such that
\[
\text{apr}^c_{C_3}(W) = \text{apr}^c_{C_3}(X) \cup \text{apr}^c_{C_3}(Y) = \{b\},
\]
\[
\overline{\text{apr}}^c_{C_3}(W) = \overline{\text{apr}}^c_{C_3}(X) \cup \overline{\text{apr}}^c_{C_3}(Y) = \{b, c, e\}.
\]
This means that the union operation of covering rough approximation pairs \( (\text{apr}^c_{C_3}, \overline{\text{apr}}^c_{C_3}) \) is not closed. By dual properties (Theorem 2 in [41]), the intersection operation of approximation pairs \( (\text{apr}^c_{C_3}, \overline{\text{apr}}^c_{C_3}) \) is not closed.

**Example 4.7.** Let \( U = \{a, b, c, d\} \), \( K_1 = \{a\} \), \( K_2 = \{b, c\} \), \( K_3 = \{c, d\} \), \( C = \{K_1, K_2, K_3\} \). Then
\[
C_3 = \{\{a\}, \{b, c\}, \{c\}, \{c, d\}\}.
\]
Let \( X = \{b, c\}, Y = \{c, d\} \), then
\[
\left( \text{apr}^c_{C_3}(X), \overline{\text{apr}}^c_{C_3}(X) \right) = (\{b\}, \{b, c\}), \\ 
\left( \text{apr}^c_{C_3}(Y), \overline{\text{apr}}^c_{C_3}(Y) \right) = (\{d\}, \{b, c\}).
\]
But, there is not any subset \( W \) of \( U \) such that
\[
\text{apr}^c_{C_3}(W) = \text{apr}^c_{C_3}(X) \cup \text{apr}^c_{C_3}(Y) = \{b, d\},
\]
\[
\overline{\text{apr}}^c_{C_3}(W) = \overline{\text{apr}}^c_{C_3}(X) \cup \overline{\text{apr}}^c_{C_3}(Y) = \{b, c, d\}.
\]
This means that the union operation of covering rough approximation pairs \( (\text{apr}^c_{C_3}, \overline{\text{apr}}^c_{C_3}) \) is not closed. By dual properties (Theorem 2 in [41]), the intersection operation of approximation pairs \( (\text{apr}^c_{C_3}, \overline{\text{apr}}^c_{C_3}) \) is not closed.

4.10. On approximation operators \( (\text{apr}^c_{C_3}, \overline{\text{apr}}^c_{C_3}) \) and \( (\text{apr}^c_{C_3}, \overline{\text{apr}}^c_{C_3}) \)

By Definition 13, Theorem 5 and Table 1 in [41], we have
\[
C_4 = \{N_4(x) | x \in U\}.
\]
The following examples demonstrate that the union and intersection operations of covering rough approximation pairs \( \text{apr}_{c_{16}} \) and \( \text{apr}_{c_{17}} \) are not closed.

**Example 4.8.** Let \( U = \{ a, b, c, d \} \), \( K_1 = \{ a \} \), \( K_2 = \{ b, c \} \), \( K_3 = \{ a, c, d \} \), \( C = \{ K_1, K_2, K_3 \} \). Then

\[
C_4 = \{ \{ a, c, d \}, \{ b, c \}, \{ a, b, c, d \}, \{ a, c, d \} \}
\]

Let \( X = \{ a \} \), \( Y = \{ b \} \), then

\[
(\text{apr}_{c_4}(X), \overline{\text{apr}}_{c_4}(X)) = (\emptyset, \{ a, d \}) \quad \text{and} \quad (\text{apr}_{c_4}(Y), \overline{\text{apr}}_{c_4}(Y)) = (\emptyset, \{ b \})
\]

But, there is no any subset \( W \) of \( U \) such that

\[
\text{apr}_{c_4}(W) = \text{apr}_{c_4}(X) \bigcup \text{apr}_{c_4}(Y) = \emptyset
\]

This means that the union operation of covering rough approximation pairs \( \text{apr}_{c_{16}} \) is not closed. By dual properties (Theorem 2 in [41]), the intersection operation of approximation pairs \( \text{apr}_{c_{17}} \) is not closed.

**Example 4.9.** Let \( U = \{ a, b, c, d \} \), \( K_1 = \{ a \} \), \( K_2 = \{ b, c \} \), \( K_3 = \{ c, d \} \), \( C = \{ K_1, K_2, K_3 \} \). Then

\[
C_4 = \{ \{ a \}, \{ b, c \}, \{ b, c, d \}, \{ c, d \} \}
\]

Let \( X = \{ a \} \), \( Y = \{ b \} \), then

\[
(\text{apr}_{c_4}(X), \overline{\text{apr}}_{c_4}(X)) = (\{ a \}, \{ a \}) \quad \text{and} \quad (\text{apr}_{c_4}(Y), \overline{\text{apr}}_{c_4}(Y)) = (\emptyset, \{ b, c \})
\]

But, there is no any subset \( W \) of \( U \) such that

\[
\text{apr}_{c_4}(W) = \text{apr}_{c_4}(X) \bigcup \text{apr}_{c_4}(Y) = \{ a \}
\]

This means that the union operation of covering rough approximation pairs \( \text{apr}_{c_{17}} \) is not closed. By dual properties (Theorem 2 in [41]), the intersection operation of approximation pairs \( \text{apr}_{c_{16}} \) is not closed.

4.11. On approximation operators \( \text{apr}_{c_{16}} \) and \( \text{apr}_{c_{17}} \)

By Definition 15 in [41], the intersection reduct of \( C \) is defined as

\[
\cap - \text{reduct}(C) = C - \{ K | K \in C, \exists C_1 \subseteq (C - K) | K = \bigcap C_1 \}
\]

Let \( U = \{ a, b, c, d \} \), \( K_1 = \{ a \} \), \( K_2 = \{ a, b \} \), \( K_3 = \{ b, c \} \), \( K_4 = \{ b \} \), \( C = \{ K_1, K_2, K_3, K_4 \} \). Then

\[
\cap - \text{reduct}(C) = \{ K_1, K_2, K_3 \}
\]

This means that the \( \cap - \text{reduct}(C) \) is equal to the covering \( C \) in Examples 4.3 and 4.4. Therefore, by Definition 4.4 and the results of Examples 4.3 and 4.4, we know that the union and intersection operations of covering rough approximation pairs \( \text{apr}_{c_{16}} \) and \( \text{apr}_{c_{17}} \) are not closed.

4.12. On approximation operators \( \text{apr}_{c_{16}} \) and \( \text{apr}_{c_{17}} \)

By Definition 15 in [41], the union reduct of \( C \) is defined as

\[
\cup - \text{reduct}(C) = C - \{ K | K \in C, \exists C_1 \subseteq (C - K) | K = \bigcup C_1 \}
\]

Let \( U = \{ a, b, c, d \} \), \( K_1 = \{ a \} \), \( K_2 = \{ a, b \} \), \( K_3 = \{ b, c \} \), \( K_4 = \{ a, b, c \} \), \( C = \{ K_1, K_2, K_3, K_4 \} \). Then

\[
\cup - \text{reduct}(C) = \{ K_1, K_2, K_3 \}
\]

This means that the \( \cup - \text{reduct}(C) \) is equal to the covering \( C \) in Examples 4.3 and 4.4. Therefore, by Definition 4.4 and the results of Examples 4.3 and 4.4, we know that the union and intersection operations of covering rough approximation pairs \( \text{apr}_{c_{16}} \) and \( \text{apr}_{c_{17}} \) are not closed.
4.13. On approximation operators \((\text{apr}_S, \overline{\text{apr}}_S)\) and \((\overline{\text{apr}}_S, \text{apr}_S)\)

**Definition 4.9** [41]. Let \(\mathcal{C}\) be a covering of a universe \(U\). The intersection closure of \(\mathcal{C}\), denoted by \(\cap - \text{closure}(\mathcal{C})\), is the minimum subset of \(2^U\) that contains \(\mathcal{C}, \emptyset\) and \(U\), and is closed under set intersection. The union closure, \(\cup - \text{closure}(\mathcal{C})\), is the minimum subset of \(2^U\) that contains \(\mathcal{C}, \emptyset\) and \(U\), and is closed under set union.

**Definition 4.10** [41]. Let \(\mathcal{C}\) be a covering of a universe \(U\). Two pairs of subsystems \(S_1\) and \(S_2\) induced by \(\mathcal{C}\) are defined as:

\[
S_1 = ((\cap - \text{closure}(\mathcal{C})), \cap - \text{closure}(\mathcal{C})),
\]

\[
S_2 = (\cup - \text{closure}(\mathcal{C}), (\cup - \text{closure}(\mathcal{C}))').
\]

The following examples demonstrate that the union and intersection operations of covering rough approximation pairs \((\text{apr}_S, \overline{\text{apr}}_S)\) and \((\overline{\text{apr}}_S, \text{apr}_S)\) are not closed.

**Example 4.10.** Let \(U = \{a, b, c, d\}\), \(K_1 = \{a\}\), \(K_2 = \{a, b, d\}\), \(K_3 = \{b, c\}\), \(C = \{K_1, K_2, K_3\}\). Then

\[
\cap - \text{closure}(\mathcal{C}) = \{U, \emptyset, \{a\}, \{a, b, d\}, \{b, c\}, \{b\}\},
\]

\[
(\cap - \text{closure}(\mathcal{C}))' = \{\emptyset, U, \{b, c\}, \{a, d\}, \{a, c, d\}\}.
\]

Let \(X = \{a\}\), \(Y = \{c\}\), then

\[
(\text{apr}_S(X), \overline{\text{apr}}_S(X)) = (\emptyset, \{a\}), \quad (\text{apr}_S(Y), \overline{\text{apr}}_S(Y)) = (\{c\}, \{b, c\}).
\]

But, there is not any subset \(W\) of \(U\) such that

\[
\text{apr}_S(W) = \text{apr}_S(X) \cup \text{apr}_S(Y) = \{c\},
\]

\[
\overline{\text{apr}}_S(W) = \overline{\text{apr}}_S(X) \cup \overline{\text{apr}}_S(Y) = \{a, b, c\}.
\]

This means that the union operation of covering rough approximation pairs \((\text{apr}_S, \overline{\text{apr}}_S)\) is not closed. By dual properties (Table 2 in [41]), the intersection operation of approximation pairs \((\overline{\text{apr}}_S, \text{apr}_S)\) is not closed.

**Example 4.11.** Let \(U = \{a, b, c, d\}\), \(K_1 = \{a\}\), \(K_2 = \{a, b, d\}\), \(K_3 = \{b, c\}\), \(C = \{K_1, K_2, K_3\}\). Then

\[
\cup - \text{closure}(\mathcal{C}) = \{U, \emptyset, \{a\}, \{a, b, d\}, \{b, c\}, \{a, b, c\}\},
\]

\[
(\cup - \text{closure}(\mathcal{C}))' = \{\emptyset, U, \{b, c\}, \{a, d\}, \{a, c, d\}\}.
\]

Let \(X = \{a\}\), \(Y = \{c\}\), then

\[
(\text{apr}_S(X), \overline{\text{apr}}_S(X)) = (\{a\}, \{a, d\}), \quad (\text{apr}_S(Y), \overline{\text{apr}}_S(Y)) = (\emptyset, \{c\}).
\]

But, there is not any subset \(W\) of \(U\) such that

\[
\text{apr}_S(W) = \text{apr}_S(X) \cup \text{apr}_S(Y) = \{a\},
\]

\[
\overline{\text{apr}}_S(W) = \overline{\text{apr}}_S(X) \cup \overline{\text{apr}}_S(Y) = \{a, c, d\}.
\]

This means that the union operation of covering rough approximation pairs \((\text{apr}_S, \overline{\text{apr}}_S)\) is not closed. By dual properties (Table 2 in [41]), the intersection operation of approximation pairs \((\overline{\text{apr}}_S, \text{apr}_S)\) is not closed.

4.14. On non-dual approximation operators \((\text{FL}_C, FH_C)\) and \((\text{SL}_C, SH_C)\)

**Definition 4.11** [46–48]. Let \((U, \mathcal{C})\) be a covering approximation space, \(X \subseteq U\). The set

\[
\text{FL}_C(X) = \cup \{K \in \mathcal{C} | K \subseteq X\}
\]

is called the first type of covering lower approximation of \(X\). The set family

\[
\text{Bn}(X) = \{\text{md}(x) | x \in X - \text{FL}_C(X)\}
\]

is called the covering boundary approximation set family of \(X\). The set

\[
FH_C(X) = \cup \{\{K \in \mathcal{C} | K \subseteq X\} \cup \text{Bn}(X)\}
\]

is called the first type of covering upper approximation of \(X\).
**Example 4.12.** Let $U = \{a, b, c, d\}$, $K_1 = \{a, b\}$, $K_2 = \{a, c\}$, $K_3 = \{c, d\}$, $C = \{K_1, K_2, K_3\}$. Then $C$ is a covering of $U$ and $(U, C)$ is a covering approximation space. By Definition 4.6 we have

\[
\begin{align*}
md(C, a) &= \{K_1, K_2\}, \quad md(C, b) = \{K_1\}, \\
md(C, c) &= \{K_2, K_3\}, \quad md(C, d) = \{K_3\}.
\end{align*}
\]

Let $X = \{a, c\}$, $Y = \{b, c, d\}$, then

\[
\begin{align*}
(FL_c(X), FH_c(X)) &= (\{a, c\}, \{a, b, c, d\}), \\
(FL_c(Y), FH_c(Y)) &= (\{c, d\}, \{a, b, c, d\}).
\end{align*}
\]

But, there is not any subset $Z$ of $U$ such that

\[
(FL_c(Z), FH_c(Z)) = (FL_c(X) \cup FL_c(Y), FH_c(X) \cup FH_c(Y)) = (\{a, c, d\}, \{a, b, c, d\}).
\]

Similarly, let $A = \{a\}$, $B = \{c\}$, then

\[
(FL_c(A), FH_c(A)) = (\emptyset, \{a, b, c\}), \quad (FL_c(B), FH_c(B)) = (\emptyset, \{b, c, d\}).
\]

But, there is not any subset $W$ of $U$ such that

\[
(FL_c(W), FH_c(W)) = (FL_c(A) \cap FL_c(B), FH_c(A) \cap FH_c(B)) = (\emptyset, \{b, c\}).
\]

For the second type of covering-based rough sets (see Definition 9 in [48]), we have the similar result, that is, the union and intersection operations of the second type of covering rough approximation pairs are not closed.

**Definition 4.12** [48]. Let $(U, C)$ be a covering approximation space, $X \subseteq U$. The set

\[
SL(X) = \cup \{K \in C | K \subseteq X\}
\]

is called the second type of covering lower approximation of $X$. The set

\[
SH(X) = \cup \{K \in C | K \cap X \neq \emptyset\}
\]

is called the second type of covering upper approximation of $X$.

**Example 4.13.** Let $U = \{a, b, c, d, e\}$, $K_1 = \{a, b\}$, $K_2 = \{b, c\}$, $K_3 = \{c\}$, $K_4 = \{a, d\}$, $K_5 = \{d\}$, $K_6 = \{d, e\}$, $C = \{K_1, K_2, K_3, K_4, K_5, K_6\}$. Then $C$ is a covering of $U$ and $(U, C)$ is a covering approximation space. Let $X = \{a\}$, $Y = \{d\}$, and by Definition 4.5 we have

\[
\begin{align*}
( SL(X), SH(X)) &= (\emptyset, \{a, b, d\}), \\
( SL(Y), SH(Y)) &= (\{d\}, \{a, d, e\}).
\end{align*}
\]

But, there is not any subset $Z$ of $U$ such that

\[
( SL(Z), SH(Z)) = (SL(X) \cup SL(Y), SH(X) \cup SH(Y)) = (\{d\}, \{a, b, d, e\}).
\]

Moreover, there is not any subset $W$ of $U$ such that

\[
( SL(W), SH(W)) = (SL(X) \cap SL(Y), SH(X) \cap SH(Y)) = (\emptyset, \{a, d\}).
\]

5. On fuzzy rough sets

Dubois and Prade firstly introduced the notion of fuzzy rough sets combining fuzzy set and rough set theory in [13]. Let $U$ be a non empty set, a fuzzy set $R$ on $U \times U$ is called a fuzzy similarity relation (see Page 200 in [13]), if it satisfies:

1. $\mu_R(x, x) = 1$ (reflexivity);
2. $\mu_R(x, y) = \mu_R(y, x)$ (symmetry);
3. $\mu_R(x, z) \geq \min\{\mu_R(x, y), \mu_R(y, z)\}$ (min-transitivity).

Then the pair $(U, R)$ is called a fuzzy approximation space, and the approximation operators are defined by (for any $A \in F(U)$, where $F(U)$ denote the fuzzy power set): for all $x \in U$,

\[
\begin{align*}
\mu_{apr_L}(A)(x) &= \inf \{\max\{\mu_R(y, x)\} | y \in U\}, \\
\mu_{apr_R}(A)(x) &= \sup \{\min\{\mu_R(x, y)\} | y \in U\}.
\end{align*}
\]

Fuzzy set $\text{apr}_L(A)$ is called a lower approximation of $A$, and $\text{apr}_R(A)$ is called an upper approximation of $A$. The pair $(\text{apr}_L(A), \text{apr}_R(A))$ is called a fuzzy rough approximation of $A$, and it is also called a fuzzy rough set.
We give the following example to show that the union and intersection operations of fuzzy rough approximation pairs are not closed.

**Example 5.1.** Let \( U = \{a, b, c\} \), define fuzzy relation \( R \) on \( U \) as follows (by a fuzzy matrix):
\[
\begin{bmatrix}
1 & 0.4 & 0.6 \\
0.4 & 1 & 0.4 \\
0.6 & 0.4 & 1
\end{bmatrix}.
\]
That is,
\[
\mu_R(a, a) = \mu_R(b, b) = \mu_R(c, c) = 1, \quad \mu_R(a, b) = \mu_R(b, a) = 0.4,
\mu_R(a, c) = \mu_R(c, a) = 0.6, \quad \mu_R(b, c) = \mu_R(c, b) = 0.4.
\]
Then \( R \) is a fuzzy similarity relation on \( U \) and \((U, R)\) is a fuzzy approximation space. Let \( A, B \) are fuzzy sets in \( U \) defined by:
\[
\mu_A(a) = 0.3, \quad \mu_A(b) = 0.5, \quad \mu_A(c) = 0.7;
\mu_B(a) = 0.8, \quad \mu_B(b) = 0.2, \quad \mu_B(c) = 0.35.
\]
By the definition of fuzzy rough sets, we have
\[
\text{apr}_R(A)(a) = 0.3, \quad \text{apr}_R(A)(b) = 0.5, \quad \text{apr}_R(A)(c) = 0.4,
\text{apr}_R(A)(a) = 0.6, \quad \text{apr}_R(A)(b) = 0.5, \quad \text{apr}_R(A)(c) = 0.7;
\text{apr}_R(B)(a) = 0.4, \quad \text{apr}_R(B)(b) = 0.2, \quad \text{apr}_R(B)(c) = 0.35,
\text{apr}_R(B)(a) = 0.8, \quad \text{apr}_R(B)(b) = 0.4, \quad \text{apr}_R(B)(c) = 0.6.
\]
Thus (now, we use a vector to denote a fuzzy set),
\[
\text{apr}_R(A) \cap \text{apr}_R(B) = (0.3, 0.2, 0.35);
\text{apr}_R(A) \cap \overline{\text{apr}_R(B)} = (0.6, 0.4, 0.6).
\]
We prove that there is not any fuzzy set \( W \) in \( U \) such that \( \text{apr}_R(W) = (0.3, 0.2, 0.35), \overline{\text{apr}_R(W)} = (0.6, 0.4, 0.6) \).

In fact, if \( W = (k_1, k_2, k_3) \in F(U) \) such that \( \text{apr}_R(W) = (0.3, 0.2, 0.35), \overline{\text{apr}_R(W)} = (0.6, 0.4, 0.6) \), then (by the definition of fuzzy rough set).

\[
\begin{align*}
(1) \min\{k_1, \max\{0.6, k_2\}, \max\{0.4, k_3\}\} &= 0.3, \\
(2) \min\{\max\{0.6, k_1\}, k_2, \max\{0.6, k_3\}\} &= 0.2, \\
(3) \min\{\max\{0.4, k_1\}, \max\{0.6, k_2\}, k_3\} &= 0.35; \\
(4) \max\{k_1, \min\{0.4, k_2\}, \min\{0.6, k_3\}\} &= 0.6, \\
(5) \max\{\min\{0.4, k_1\}, k_2, \min\{0.4, k_3\}\} &= 0.4, \\
(6) \max\{\min\{0.6, k_1\}, \min\{0.4, k_2\}, k_3\} &= 0.6.
\end{align*}
\]

By (2) we get \( k_2 = 0.2 \). From this and (1), we get \( \min\{k_1, 0.6, \max\{0.4, k_3\}\} = 0.3 \), it follows that \( k_1 = 0.3 \). Moreover, by (3), we have \( \min\{0.4, 0.6, k_3\} = 0.35 \), that is, \( k_3 = 0.35 \). Therefore, by (4) we get
\[
\max\{0.3, 0.2, 0.35\} = 0.6.
\]
This is a contradiction. Hence, there is not any fuzzy set \( W \) in \( U \) such that
\[
\text{apr}_R(W) = \text{apr}_R(A) \cap \text{apr}_R(B); \\
\overline{\text{apr}_R(W)} = \overline{\text{apr}_R(A)} \cap \overline{\text{apr}_R(B)}.
\]
Similarly, we can construct counterexamples to show that the union operation of fuzzy rough approximation pairs is not closed in general.

**Remark 5.1.** For the notion of fuzzy similarity relation in this paper, we use the definition introduced by Dubois and Prade in [13], in which the three conditions (reflexivity, symmetry and min-transitivity) are required (see Page 200 in [13]). But, some scholars used a general definition, that is, two conditions (reflexivity and symmetry) are required. We call the second case “generalized fuzzy similarity relation”. Then, the pair \((U, R)\) is called a generalized fuzzy approximation space, where \( R \) is a generalized fuzzy similarity relation on \( U \). The following example shows that, for generalized fuzzy approximation space, the union and intersection of fuzzy rough approximation pairs are not closed in general (we are grateful to one reviewer for this suggestion).
Example 5.2. Let \( U = \{a, b, c, d, e\} \), define fuzzy relation \( R \) on \( U \) as follows:
\[
\begin{align*}
\mu_R(a, a) &= \mu_R(b, b) = \mu_R(c, c) = \mu_R(d, d) = \mu_R(e, e) = 1, \\
\mu_R(a, b) &= \mu_R(b, a) = 1, \quad \mu_R(b, c) = \mu_R(c, b) = 1, \\
\mu_R(c, d) &= \mu_R(d, c) = 1, \quad \mu_R(d, e) = \mu_R(e, d) = 1, \\
\text{otherwise,} \quad \mu_R(x, y) &= 0.
\end{align*}
\]
That is, \( R \) is the tolerance relation in Example 164 (a) in [15]. Obviously, \( R \) is a generalized fuzzy similarity relation on \( U \), but it is not a fuzzy similarity relation, since
\[
\mu_R(a, c) = 0 \neq \min(\mu_R(a, b), \mu_R(b, c)) = 1.
\]
That is, the min-transitivity is not satisfied. For generalized fuzzy approximation space \( (U, R) \), applying the result of Example 164 (a) in [15], we can easily verify that: (1) there is not any fuzzy set \( Z \) in \( U \) such that
\[
\mu_{\text{appr}}(Z) = \mu_{\text{appr}}(X) \cup \mu_{\text{appr}}(Y), \quad \mu_{\text{appr}}(Z) = \mu_{\text{appr}}(X) \cup \mu_{\text{appr}}(Y),
\]
where \( \mu_X, \mu_Y, \mu_{\text{appr}}(X), \mu_{\text{appr}}(Y), \mu_{\text{appr}}(X), \mu_{\text{appr}}(Y) \) are the characterization functions of the sets \( \{a, b\}, \{b, c\}, \{a\}, \{a, b, c\}, \emptyset, \{a, b, c, d\} \), respectively. (2) there is not any fuzzy set \( W \) in \( U \) such that
\[
\mu_{\text{appr}}(W) = \mu_{\text{appr}}(X) \cap \mu_{\text{appr}}(Y), \quad \mu_{\text{appr}}(W) = \mu_{\text{appr}}(X) \cap \mu_{\text{appr}}(Y),
\]
where \( \mu_X, \mu_Y, \mu_{\text{appr}}(X), \mu_{\text{appr}}(X), \mu_{\text{appr}}(X), \mu_{\text{appr}}(X) \) are the characterization functions of the sets \( \{a, b, c\}, \{a, b, d\}, \{a, b\}, \{a, b, c, d\}, \{a\}, \{a, b, c, d, e\} \), respectively.

6. On rough fuzzy sets based on classical approximation spaces

Liu [18] investigated the lattice structures of rough fuzzy sets. We will show that the basic results in [18] are tenable, but there are two problems:

(1) the closeness of the union and intersection operations of rough fuzzy approximation pairs are not proved;
(2) from rough fuzzy sets we cannot obtain Stone algebra (in fact, we can only obtain weak Stone algebra).

Definition 6.1 [13,18,24]. Let \( U \) be a universe and \( R \) be an equivalence relation on \( U \). The lower and upper approximations of the fuzzy set \( A \in F(U) \), denoted \( RA \) and \( RA \), respectively, are defined as fuzzy sets in \( U \) such that
\[
\begin{align*}
(RA)(x) &= \bigvee_{y \in U} (R(x, y) \land A(y)), \quad x \in U; \\
(RA)(x) &= \bigwedge_{y \in U} ((1 - R(x, y)) \lor A(y)), \quad x \in U.
\end{align*}
\]

The pair \( RA = (RA, RA) \) is referred to as a rough fuzzy set.

Proposition 6.1 18. Let \( U \) be a universe and \( R \) be an equivalence relation on \( U \). Then for \( x \in U \)
\[
(RA)(x) = \bigwedge_{y \mid \downarrow x} A(y), \quad (RA)(x) = \bigvee_{y \mid \uparrow x} A(y).
\]

Theorem 6.1. Let \( U \) be a universe and \( R \) be an equivalence relation on \( U \). Denote \( RF(U) = \{(RA, RA) | A \in F(U)\} \), define \( \cup \) and \( \cap \) by:
\[
\begin{align*}
(RA, RA) \cup (RB, RB) &= (RA \cup RB, RA \cup RB), \quad \forall A, B \in F(U); \\
(RA, RA) \cap (RB, RB) &= (RA \cap RB, RA \cap RB), \quad \forall A, B \in F(U).
\end{align*}
\]

Then \( (RF(U), \cup, \cap) \) is a lattice.

Proof. We only prove that \( \cup \) and \( \cap \) are closed in \( RF(U) \).

For every equivalence classes of \( R \), select a representative element, denote the set of the all representative elements by \( I \). For any \( A, B \in F(U) \), we define a fuzzy set \( Z \in F(U) \) by: for all \( x \in U \),
By Proposition 6.1, we have

Example 6.1. That is, 

This means that

where \( a \lor b \) denote \( \max(a, b) \). It is easy to verify that \( (\overline{RA})(x) = (\overline{RA})(x) = A(x) \) and \( (\overline{RB})(x) = (\overline{RB})(x) = B(x) \) when \( [x]_R - \{x\} = \emptyset \), and we can get that

This means that

That is, \( \cup \) is closed in \( RF(U) \). Moreover, for any \( A, B \in F(U) \), we define a fuzzy set \( W \in F(U) \) by: for all \( x \in U \),

where \( a \land b \) denote \( \min(a, b) \). We can get that

This means that

That is, \( \cap \) is closed in \( RF(U) \). \( \square \)

Example 6.1. Let \( U = \{a, b, c, d\} \) and

Then \( R \) is an equivalent relation on \( U \) and \( U/R = \{\{a\}, \{b, c, d\}\} \). Define fuzzy set \( A, B \) in \( U \) (we use a vector to denote fuzzy set) as follows:

\[
A = (0.7, 0.6, 0.8, 0.3), \quad B = (0.4, 0.5, 0.9, 0.46).
\]

By Proposition 6.1, we have

\[
\overline{RA} = (0.7, 0.3, 0.3, 0.3), \quad \overline{RA} = (0.7, 0.8, 0.8, 0.8);
\]

\[
\overline{RB} = (0.4, 0.46, 0.46, 0.46), \quad \overline{RB} = (0.4, 0.9, 0.9, 0.9).
\]

Let \( I = \{a, b\} \) be the set of the all representative elements of equivalence classes (it is not unique). Then, by the proof of Theorem 6.1, we have fuzzy sets \( Z = (0.7, 0.46, 0.9, 0.9) \), \( W = (0.4, 0.3, 0.8, 0.8) \). It is easy to verify that

\[
(\overline{RZ}, \overline{RW}) = (\overline{RA} \cup \overline{RB}, \overline{RA} \cup \overline{RB}) = (\overline{RA}, \overline{RA}) \cup (\overline{RB}, \overline{RB}),
\]

\[
(\overline{RW}, \overline{RW}) = (\overline{RA} \cap \overline{RB}, \overline{RA} \cap \overline{RB}) = (\overline{RA}, \overline{RA}) \cap (\overline{RB}, \overline{RB}).
\]

Note that, Theorem 4.2 in [18], the closeness of \( \cup \) and \( \cap \) are not proved. By the above Theorem 6.1 we know that it is not obvious.

For lattice \( (RF(U), \cup, \cap) \), we define complement \( \sim \) by:

\[
\sim (\overline{RA}, \overline{RA}) = (\overline{RA})', \quad \forall A \in F(U).
\]

Table 1
The closeness of relation based rough approximation operators.

<table>
<thead>
<tr>
<th>Relation</th>
<th>Closeness</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R \cup \overline{R} ), when ( R ) is an equivalent relation</td>
<td>✔️</td>
</tr>
<tr>
<td>( R \cup \overline{R} ), when ( R ) is a preorder (quasi-order) relation</td>
<td>✔️</td>
</tr>
<tr>
<td>( R \cup \overline{R} ), when ( R ) is a tolerance relation</td>
<td>❌</td>
</tr>
<tr>
<td>( (T_\tau, T_\tau) ), when ( \tau ) is a tolerance relation</td>
<td>? (unknown)</td>
</tr>
</tbody>
</table>
According to Theorem 4.3 in [18]. 

\((RF(U), \cup, \cap, \sim, (\emptyset, \emptyset), (U, U))\) is a Stone algebra. However, the following example shows that it is not true. Firstly, we recall some concepts.

**Definition 6.2** [2, 15]. Let \((L, \land, \lor, 0)\) be a lattice with the least element 0. An element \(x^*\) is a pseudocomplement of \(x \in L\), if \(x \land x^* = 0\) and for all \(a \in L, x \land a = 0\) implies \(a \leq x^*\). A lattice is pseudocomplemented if each element has a pseudocomplement.

**Definition 6.3** [2, 15]. Let \((L, \land, \lor, 0, 1)\) be a bounded lattice with the least element 0 and the greatest element 1. If \(L\) is a pseudocomplemented distributive lattice satisfying the identity

\[x^* \lor x^{**} = 1\]

then \(L\) is called a Stone lattice, \((L, \land, \lor, *, 0, 1)\) is called a Stone algebra.

**Example 6.2.** Let \(U = \{a, b, c\}\) and \(R = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}\). Then \(R\) is an equivalence relation on \(U\). Define fuzzy set \(X\) in \(U\) (we use a vector to denote fuzzy set) as follows:

\[X = (0.7, 0.4, 0.9)\].

By Definition 6.1, we have \(RX = (0.7, 0.4, 0.4)\), \(\overline{RX} = (0.7, 0.9, 0.9)\). Thus \((RX)' = (0.3, 0.6, 0.6), (\overline{RX})' = (0.3, 0.1, 0.1)\). It is easy to verify that

\[\land (RX, \overline{RX}) \cap (RX, \overline{RX}) \neq (\emptyset, \emptyset);\]
\[\lor (RX, \overline{RX}) \cup (RX, \overline{RX}) \neq (U, U)\].

Therefore, \((RF(U), \cup, \cap, \sim)\) is not a pseudocomplemented lattice, and \((RF(U), \cup, \cap, \sim, (\emptyset, \emptyset), (U, U))\) is not a Stone algebra.

Moreover, for lattice \((RF(U), \cup, \cap, \sim)\), we define a new complement \(\sim_1\) by:

\[\sim_1(RA, \overline{RA}) = \begin{cases} (U, U), & \text{if } (RA, \overline{RA}) = (\emptyset, \emptyset); \\ (\emptyset, \emptyset), & \text{otherwise.} \end{cases}\]

Then we can verify that \(RF(U)\) is a Stone lattice with pseudocomplemented \(\sim_1\) (please see [7] for general discussion).

By the proof of Theorem 4.2 in [18] we have.

**Theorem 6.2.** Let \(U\) be a universe and \(R\) be an equivalence relation on \(U\). Then \((RF(U), \cup, \cap, \sim, (\emptyset, \emptyset), (U, U))\) is a bounded distributive lattice. Moreover, for the complement \(\sim_1\) (as defined above), \((RF(U), \cup, \cap, \sim_1, (\emptyset, \emptyset), (U, U))\) is a Stone algebra.

### 7. Summary

For rough sets and various generalized rough sets, the closeness of union and intersection operations of rough approximation pairs (lower approximation, upper approximation) is a basic problem; however, it is often overlooked. In this paper, we investigated into this problem for classical rough sets and generalized rough sets based on tolerance, preorder, covering (twenty-two classes), fuzzy approximation space. We present that the union and intersection operations of classical rough approximation pairs, two kinds of covering based rough approximation pairs and rough fuzzy approximation pairs are
closed, and we also pointed out some inverse cases (including fuzzy rough sets and twenty kinds of covering based rough sets) by some counterexamples. The main results can be represented by Tables 1 and 2. For the open problem (the “unknown” in Table 1), we are computing by mathematical software, and it will be solved in the forthcoming paper.

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References