On the concept of general solution for impulsive differential equations of fractional order $q \in (0, 1)$

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Abstract

In this paper, for impulsive differential equations with fractional-order $q \in (0, 1)$, we show that the formula of solutions in cited papers are incorrect. Secondly, we find out a formula of the general solution for impulsive Cauchy problem with Caputo fractional derivative $q \in (0, 1)$. Further, for a kind of impulsive fractional differential equations system with special initial value, we come to an existence result for it by applying fixed point methods.

1. Introduction

Fractional differential equations have proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. And the subject of fractional differential equations is gaining much attention. For detail, see [1–13] and the references therein.

Fractional differential equation was extended to impulsive fractional differential equations, since Agarwal and Benchohra published the first paper on the topic [14] in 2008. Recently, in [15], Fečkan and Zhou pointed out that the formula of solutions for impulsive fractional differential equations in [16–19] is incorrect and gave their correct formula. In [20,21], the authors established a general framework to find the solutions for impulsive fractional boundary value problems and obtained some sufficient conditions for the existence of the solutions to a kind of impulsive fractional differential equations respectively.

In [22], the authors illustrated their comprehensions for the counterexample in [15] and criticized the viewpoint in [15,20,21]. Next, in [23], Fečkan et al. expounded for the counterexample in [15] and provided further five explanations in the paper.

This paper is motivated from some recent papers which treated the problem of the existence of the solutions for impulsive differential equations with fractional derivate $q \in (0, 1)$ and directly or indirectly used an unfit integral equation. To prove our claim, we consider a general impulsive fractional system

$$
\begin{align*}
\frac{d^q}{dt^q} x(t) &= f(t,x(t)), \quad 0 < q < 1, \quad t \notin J, \quad t \neq t_k, \quad k = 1, 2, \ldots, m, \\
\Delta x|_{t=t_k} &= I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m, \\
x(0) &= x_0,
\end{align*}
$$

(1.1)
where $\phi D^q_{t_0}$ is Caputo fractional derivative in interval $[0, t]$. Where $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is appropriate continuous function to be specified later. $l_k : \mathbb{R} \rightarrow \mathbb{R}, (k = 1, 2, \ldots, m)$ are appropriate functions, and $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, let $J_k = (t_k, t_{k+1}], k = 1, 2, \ldots, m$, and $J_0 = [0, t_1]$. $\Delta x|_{t_k} = x(t_k^+) - x(t_k^-)$ (Here, $x(t_k^+) = \lim_{\epsilon \rightarrow 0^+} x(t_k + \epsilon)$ and $x(t_k^-) = \lim_{\epsilon \rightarrow 0^-} x(t_k + \epsilon)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively.)

For impulsive differential equations with fractional-order $q \in (0, 1)$, the approach was used in the cited papers \cite{24, 25} and some articles that quoted them, which is directly or indirectly based on the results of Eq. (1.1) is equivalent to the following integral equation

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} f(s, x(s)) \, ds, & t \in J_0, \\ x_0 + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t_{k+1}} (t_k - s)^{q-1} f(s, x(s)) \, ds + \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} f(s, x(s)) \, ds + \sum_{0 < t_k < t} l_k(x(t_k^-)), & t \in (t_k, t_{k+1}], k = 1, \ldots, m, \end{cases}$$

or

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} f(s, x(s)) \, ds, & t \in J_0, \\ x(t_k^+) + l_k(x(t_k^-)) + \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} f(s, x(s)) \, ds, & t \in (t_k, t_{k+1}], k = 1, \ldots, m, \end{cases}$$

where $q \in (0, 1)$. Then one can say that a function $x(t) \in PC^1([0, T])$ is called a solution of Eq. (1.1) if $x(t)$ satisfies Eq. (1.2) (where Eq. (1.2) is equivalence with Eq. (1.3)). So we only discuss Eq. (1.2). However, this concept of a solution is not realistic. For the system (1.1), we weaken the effect by impulse, that is, let $l_k(x(t_k^-)) \rightarrow 0 (k = 1, 2, \ldots, m)$. Then, the system (1.1) turns to the following system

$$\begin{align*}
\phi D^q_{t_0} x(t) &= f(t, x(t)), & q \in (0, 1), & t \in J = [0, T], \\
\Delta x|_{t_k} &= l_k(x(t_k^-)), & t \in (t_k, t_{k+1}], k = 1, \ldots, m, \\
x(0) &= x_0,
\end{align*}$$

and Eq. (1.4) is equivalent to the following integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, x(s)) \, ds, & q \in (0, 1), & t \in J = [0, T].$$

Correspondingly, when $l_k(x(t_k^-)) \rightarrow 0 (k = 1, 2, \ldots, m)$, the system (1.2) turns to the following system

$$\begin{align*}
x(t) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} f(s, x(s)) \, ds, & t \in J_0, \\
x(t_k^+) + l_k(x(t_k^-)) + \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} f(s, x(s)) \, ds, & t \in (t_k, t_{k+1}], k = 1, \ldots, m,
\end{align*}$$

where $q \in (0, 1)$. Then, Eq. (1.5) is equivalent to (1.6). Therefore, for $t \in J_k, k = 1, \ldots, m$, we obtain a following unsplit equation

$$\int_0^t (t - s)^{q-1} f(s, x(s)) \, ds = \sum_{0 < t_k < t} \int_{t_k}^{t_{k+1}} (t_k - s)^{q-1} f(s, x(s)) \, ds + \int_{t_k}^t (t - s)^{q-1} f(s, x(s)) \, ds. \quad (1.7)$$

where $q \in (0, 1)$. So, the solution of the system (1.1) does not satisfy Eq. (1.2).

Motivated by the above remarks, we reconsider the impulsive differential equation (1.1) with Caputo fractional derivative and seek a correct formula of the solution of the system (1.1).

2. The solution of the impulsive fractional system

In this section, we shall give an integral equation that is equivalent with the system (1.1). Firstly, we recall the definition of the Caputo fractional derivative and the fractional integral.

**Definition 2.1** \cite{2}. The fractional integral of order $q$ with the lower limit $0$ for function $f$ is defined as

$$f^q(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t - s)^{1-q}} \, ds, \quad t > 0, \quad q > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma$ is the gamma function.

**Definition 2.2** \cite{2}. The Caputo derivative of order $q$ with the lower limit $0$ for a function $f$ can be written as

$$\phi D^q_{t_0} f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{n-1-q}} \, ds = t^{n-q} f^{(n)}(t), \quad t > 0, \quad 0 \leq n - 1 < q < n.$$
According to Definition 2.2, for \( t \in J_k, k = 0, 1, \ldots, m \) in Eq. (1.2), we have
\[
\begin{align*}
_0D^\alpha_t x(t) &= _0D^\alpha_t \left[ x_0 + \frac{1}{\Gamma(q)} \left( \sum_{0 < c_1 < t} \int_{t_{c_1}}^t (t - s)^{q-1}f(s, x(s))\,ds + \int_{t_k}^t (t - s)^{q-1}f(s, x(s))\,ds \right) + \sum_{0 < c_1 < t} I_k(x(t_k)) \right) \\
&= _0D^\alpha_t \left( \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1}f(s, x(s))\,ds \right) = f(t, x(t)), \quad t \in J_k, \quad k = 0, 1, \ldots, m. \tag{2.1}
\end{align*}
\]
That is, Eq. (1.2) satisfies Caputo fractional derivative condition in the system (1.1). Therefore, for each interval \( J_k, k = 0, 1, \ldots, m \), Eq. (1.2) is only a part of the solution of system (1.1). By the above discussion, we will provide the correct solution of the system (1.1).

**Theorem 2.1.** Let \( q \in (0, 1) \) and \( h \) is a constant. The system (1.1) is equivalent with integral equation
\[
x(t) = \begin{cases} 
  x_0 + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1}f(s, x(s))\,ds, & t \in J_0, \\
  x_0 + \sum_{k=1}^n I_k(x(t_k)) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1}f(s, x(s))\,ds \\
  + \sum_{k=1}^n \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1}f(s, x(s))\,ds + \int_0^t (t - s)^{q-1}f(s, x(s))\,ds - \int_0^t (t - s)^{q-1}f(s, x(s))\,ds, & t \in J_n, \quad 1 \leq n \leq m.
\end{cases} \tag{2.2}
\]
provide that the integral in (2.2) exists.

**Proof.** Firstly, we prove the solution of the system (1.1) satisfies Eq. (2.2). For \( t \in J_0 \), clearly, the solution of the system (1.1) satisfies Eq. (2.2). By induction \( t \in J_{n+1} \), it will be proved that the solution of the system (1.1) satisfies Eq. (2.2).

For \( t \in J_0 \), by the Definitions 2.1 and 2.2, the system (1.1) is equivalent with
\[
x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1}f(s, x(s))\,ds, \quad t \in J_0.
\]
then
\[
x(t_{1}) = x(t_{1}) + I_1(x(t_{1})) = x_0 + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1}f(s, x(s))\,ds + I_1(x(t_{1})).
\]
For \( t \in J_1 \), by the discussion for Eq. (2.1), we give a part of the solution the system (1.1) as follows
\[
\begin{align*}
\tilde{x}_1(t) &= x(t_{1}) + \frac{1}{\Gamma(q)} \int_{t_1}^t (t - s)^{q-1}f(s, x(s))\,ds \\
&= x_0 + I_1(x(t_{1})) + \frac{1}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1 - s)^{q-1}f(s, x(s))\,ds + \int_{t_1}^t (t - s)^{q-1}f(s, x(s))\,ds \right\}, \quad t \in J_1. \tag{2.3}
\end{align*}
\]
For \( t \in J_1 \), \( e_1(t) \) denotes the error between Eq. (2.3) and the correct solution of system (1.1). The solution of the system (1.1) (as \( I_1(x(t_{1})) \to 0 \)) tends to (1.5)). Therefore, by (1.5), we have
\[
\lim_{I_1(x(t_{1})) \to 0} e_1(t) = \lim_{I_1(x(t_{1})) \to 0} \{x(t) - \tilde{x}_1(t)\} = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1}f(s, x(s))\,ds - x_0 - \frac{1}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1 - s)^{q-1}f(s, x(s))\,ds + \int_{t_1}^t (t - s)^{q-1}f(s, x(s))\,ds \right\} = \frac{1}{\Gamma(q)} \left\{ \int_0^t (t - s)^{q-1}f(s, x(s))\,ds - \int_0^{t_1} (t_1 - s)^{q-1}f(s, x(s))\,ds - \int_{t_1}^t (t - s)^{q-1}f(s, x(s))\,ds \right\}. \tag{2.4}
\]
So, \( e_1(t) \) is connect with \( I_1(x(t_{1})) \) and \( \lim_{I_1(x(t_{1})) \to 0} e_1(t) \). By the structure of the (2.3) and (1.5) (as \( I_1(x(t_{1})) \to 0 \)) tends to (1.5)), we provide an assumption that
\[
\begin{align*}
e_1(t) &= \sigma(I_1(x(t_{1}))) \lim_{I_1(x(t_{1})) \to 0} e_1(t) \\
&= \frac{\sigma(I_1(x(t_{1})))}{\Gamma(q)} \left\{ \int_0^t (t - s)^{q-1}f(s, x(s))\,ds - \int_0^{t_1} (t_1 - s)^{q-1}f(s, x(s))\,ds - \int_{t_1}^t (t - s)^{q-1}f(s, x(s))\,ds \right\}. \tag{2.5}
\end{align*}
\]
where function $\sigma$ is an undetermined function and $\sigma(0) = 1$. Therefore

$$x(t) = \dot{x}(t) + e_t(t)$$

$$= x_0 + I_1(x(t_1)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds$$

$$+ \left\{ \frac{1 - \sigma(I_1(x(t_1)))}{\Gamma(q)} \right\} \left\{ \int_0^{t_1} (t_1-s)^{q-1} f(s, x(s)) ds + \int_1^t (t-s)^{q-1} f(s, x(s)) ds - \int_0^t (t-s)^{q-1} f(s, x(s)) ds \right\},$$

$t \in J_1$. \hspace{1cm} (2.6)

For (2.6), we assume $1 - \sigma(I_1(x(t_1))) = \rho(I_1(x(t_1)))$ to obtain $\rho(0) = 0$, where function $\rho$ is an undetermined function which is given later. Therefore

$$x(t) = x_0 + I_1(x(t_1)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds$$

$$+ \frac{\rho(I_1(x(t_1)))}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1-s)^{q-1} f(s, x(s)) ds + \int_1^t (t-s)^{q-1} f(s, x(s)) ds - \int_0^t (t-s)^{q-1} f(s, x(s)) ds \right\},$$

$t \in J_1$. \hspace{1cm} (2.7)

Using (2.7), we get

$$x(t_2) = x(t_2) + I_2(x(t_2))$$

$$= x_0 + \sum_{k=2} \int_k (t_2-s)^{q-1} f(s, x(s)) ds$$

$$+ \rho(I_1(x(t_2))) \left\{ \int_0^{t_1} (t_1-s)^{q-1} f(s, x(s)) ds + \int_1^t (t-s)^{q-1} f(s, x(s)) ds - \int_0^t (t-s)^{q-1} f(s, x(s)) ds \right\}. \hspace{1cm} (2.8)$$

Then, for $t \in J_2$, a part of the solution for the system (1.1) is provided

$$x(t) = x_0 + \int_0^t (t-s)^{q-1} f(s, x(s)) ds$$

$$= x_0 + \sum_{k=2} \int_k (t_2-s)^{q-1} f(s, x(s)) ds$$

$$+ \rho(I_1(x(t_2))) \left\{ \int_0^{t_1} (t_1-s)^{q-1} f(s, x(s)) ds + \int_1^t (t-s)^{q-1} f(s, x(s)) ds - \int_0^t (t-s)^{q-1} f(s, x(s)) ds \right\}. \hspace{1cm} (2.8)$$

For $t \in J_2$, $e_2(t)$ denotes the error between Eq. (2.8) and the correct solution of system (1.1). The solution of system (1.1) (as $I_k(x(t_k)) \to 0$, $k = 1, 2$) tends to Eq. (1.5). Therefore

$$\lim_{k \to \infty} e_2(t) = \lim_{k \to \infty} x(t) - x_2(t)$$

$$= \frac{1}{\Gamma(q)} \left\{ \int_0^t (t-s)^{q-1} f(s, x(s)) ds - \int_0^{t_2} (t_2-s)^{q-1} f(s, x(s)) ds - \int_t^{t_2} (t-s)^{q-1} f(s, x(s)) ds \right\}. \hspace{1cm} (2.9)$$

For $t \in J_2$, the solution of system (1.1) (as $I_1(x(t_1)) \to 0$) tends to Eq. (2.7). Therefore

$$\lim_{k \to \infty} e_2(t) = \lim_{k \to \infty} x(t) - x_2(t)$$

$$= x_0 + I_2(x(t_2)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds$$

$$- \frac{\rho(I_1(x(t_2)))}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1-s)^{q-1} f(s, x(s)) ds + \int_1^t (t-s)^{q-1} f(s, x(s)) ds \right\}$$

$$- \frac{1}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1-s)^{q-1} f(s, x(s)) ds + \int_1^t (t-s)^{q-1} f(s, x(s)) ds \right\}. \hspace{1cm} (2.10)$$

For $t \in J_2$, the solution of system (1.1) (as $I_2(x(t_2)) \to 0$) tends to Eq. (2.7). Therefore
\[ \lim_{t_2(x(t_2)) \to 0} e_2(t) = \lim_{t_2(x(t_2)) \to 0} \{x(t) - \tilde{x}_2(t)\} \]
\[ = x_0 + l_1(x(t_1)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}f(s,x(s))ds \]
\[ + \rho(l_1(x(t_1))) \left\{ \int_0^{t_1} (t_1-s)^{q-1}f(s,x(s))ds + \int_{t_1}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\} \]
\[ - x_0 - l_1(x(t_1)) - \frac{1}{\Gamma(q)} \left\{ \int_0^{t_1} (t_2-s)^{q-1}f(s,x(s))ds + \int_{t_2}^t (t-s)^{q-1}f(s,x(s))ds \right\} \]
\[ = 1 - \rho(l_1(x(t_1))) \left\{ \int_0^{t_1} (t_1-s)^{q-1}f(s,x(s))ds + \int_{t_1}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\} \]
\[ + \rho(l_1(x(t_1))) \left\{ \int_0^{t_1} (t-2)^{q-1}f(s,x(s))ds - \int_{t_1}^t (t-s)^{q-1}f(s,x(s))ds - \int_{t_1}^t (t-s)^{q-1}f(s,x(s))ds \right\}. \tag{2.11} \]

So, to contrast the structure of (2.9)–(2.11) and the structure of Eq. (2.7), we obtain
\[ e_2(t) = \left( 1 - \rho(l_1(x(t_1))) - \rho(l_2(x(t_2))) \right) \frac{1}{\Gamma(q)} \]
\[ \times \left\{ \int_0^{t_1} (t-2)^{q-1}f(s,x(s))ds - \int_{t_1}^t (t-s)^{q-1}f(s,x(s))ds - \int_{t_1}^t (t-s)^{q-1}f(s,x(s))ds \right\} \]
\[ + \frac{\rho(l_1(x(t_1)))}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1-s)^{q-1}f(s,x(s))ds + \int_{t_1}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\} \]
\[ + \frac{\rho(l_2(x(t_2)))}{\Gamma(q)} \left\{ \int_0^{t_1} (t_2-s)^{q-1}f(s,x(s))ds + \int_{t_2}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\}. \tag{2.12} \]

So, for \( t \in j_2 \), we have
\[ x(t) = \tilde{x}_2(t) + e_2(t) \]
\[ = x_0 + \sum_{k \leq 2} l_k(x(t_k)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}f(s,x(s))ds \]
\[ + \rho(l_1(x(t_1))) \left\{ \int_0^{t_1} (t_1-s)^{q-1}f(s,x(s))ds + \int_{t_1}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\} \]
\[ + \frac{\rho(l_1(x(t_1)))}{\Gamma(q)} \left\{ \int_0^{t_1} (t_2-s)^{q-1}f(s,x(s))ds + \int_{t_2}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\}, \tag{2.13} \]

\[ t \in j_2. \]

Letting \( t_2 \to t_1 \), we have
\[ \lim_{t_2 \to t_1} \left\{ \begin{array}{ll}
\partial_0^D_1 x(t) = f(t, x(t)), t \in \bigcup_{i=0}^{1.2} j_i, t \neq t_1, t_2, \\
\partial_0^D_2 x(t) = f(t, x(t)), t \in \bigcup_{i=0}^{1.2} j_i, t \neq t_1, t_2,
\end{array} \right. \]
\[ \left\{ \Delta x_{i=t_i} = l_k(x(t_k)), k = 1, 2 \right\} \rightarrow \Delta x_{i=t_1} = l_1(x(t_1)) + l_2(x(t_2)), \]
\[ x(0) = x_0. \tag{2.14} \]

By (2.13) and (2.7), for \( \forall l_1(x(t_1)) \in \mathbb{R} \) and \( \forall l_2(x(t_2)) \in \mathbb{R} \), we obtain
\[ \rho(l_1(x(t_1)) + l_2(x(t_2))) = \rho(l_1(x(t_1))) + \rho(l_2(x(t_2))). \tag{2.15} \]

Therefore \( \rho(x) = hx, \forall x \in \mathbb{R} \) (where \( h \) is a constant). So, for \( t \in j_1 \) and \( t \in j_2 \), it is obtained respectively
\[ x(t) = x_0 + l_1(x(t_1)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}f(s,x(s))ds \]
\[ + l_2(x(t_2)) \int_0^{t_1} (t_1-s)^{q-1}f(s,x(s))ds + \frac{\rho(l_1(x(t_1)))}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1-s)^{q-1}f(s,x(s))ds + \int_{t_1}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\}, \tag{2.16} \]

\[ t \in j_1, \]

\[ x(t) = x_0 + \sum_{k \leq 2} l_k(x(t_k)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}f(s,x(s))ds \]
\[ + l_1(x(t_1)) \int_0^{t_1} (t_1-s)^{q-1}f(s,x(s))ds + \frac{\rho(l_1(x(t_1)))}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1-s)^{q-1}f(s,x(s))ds + \int_{t_1}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\} \]
\[ + l_2(x(t_2)) \int_0^{t_2} (t_2-s)^{q-1}f(s,x(s))ds + \frac{\rho(l_2(x(t_2)))}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2-s)^{q-1}f(s,x(s))ds + \int_{t_2}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\}, \tag{2.17} \]

\[ t \in j_2. \]
By (2.17), we have
\[
x(t_j^+) = x(t_j^-) + I_1(x(t_j^-)) + \frac{1}{\Gamma(q)} \int_{t_j}^{t_j^+} (t_j - s)^{q-1} f(s, x(s)) ds
\]
\[
+ \frac{h_1(x(t_j^-))}{\Gamma(q)} \left\{ \int_0^{t_j} (t_j - s)^{q-1} f(s, x(s)) ds + \int_{t_j}^{t_1} (t_1 - s)^{q-1} f(s, x(s)) ds - \int_0^{t_1} (t_1 - s)^{q-1} f(s, x(s)) ds \right\}
\]
\[
+ \frac{h_2(x(t_j^-))}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2 - s)^{q-1} f(s, x(s)) ds + \int_{t_2}^{t_1} (t_1 - s)^{q-1} f(s, x(s)) ds - \int_0^{t_1} (t_1 - s)^{q-1} f(s, x(s)) ds \right\}.
\tag{2.18}
\]
For \( t \in J_3 \), in the same way as the process \( t \in J_2 \), a part of the solution the system (1.1) is provided
\[
x_3(t) = x(t_j^+) + \frac{1}{\Gamma(q)} \int_{t_j}^{t} (t - s)^{q-1} f(s, x(s)) ds
\]
\[
= x_0 + \sum_{k \leq 3} I_k(x(t_k^-)) + \frac{1}{\Gamma(q)} \left\{ \int_0^{t_j} (t_j - s)^{q-1} f(s, x(s)) ds + \int_{t_j}^{t_1} (t_1 - s)^{q-1} f(s, x(s)) ds \right\}
\]
\[
+ \frac{h_1(x(t_j^-))}{\Gamma(q)} \left\{ \int_0^{t_j} (t_j - s)^{q-1} f(s, x(s)) ds + \int_{t_j}^{t_1} (t_1 - s)^{q-1} f(s, x(s)) ds - \int_0^{t_1} (t_1 - s)^{q-1} f(s, x(s)) ds \right\}
\]
\[
+ \frac{h_2(x(t_j^-))}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2 - s)^{q-1} f(s, x(s)) ds + \int_{t_2}^{t_1} (t_1 - s)^{q-1} f(s, x(s)) ds - \int_0^{t_1} (t_1 - s)^{q-1} f(s, x(s)) ds \right\}.
\tag{2.19}
\]
For \( t \in J_3 \), \( e_3(t) \) denotes the error between Eq. (2.19) and the correct solution of system (1.1). The solution of system (1.1) (as \( I_k(x(t_k^-)) \to 0, k = 1, 2, 3 \)) tends to the (1.5). Therefore
\[
\lim_{I_k(x(t_k^-)) \to 0, k = 1, 2, 3} e_3(t) = \lim_{I_k(x(t_k^-)) \to 0, k = 1, 2, 3} \left\{ \left| x(t) - \tilde{x}_3(t) \right| \right\}
\]
\[
= \frac{1}{\Gamma(q)} \left\{ \int_0^{t} (t - s)^{q-1} f(s, x(s)) ds - \int_0^{t_1} (t_1 - s)^{q-1} f(s, x(s)) ds - \int_{t_1}^{t} (t - s)^{q-1} f(s, x(s)) ds \right\}
\tag{2.20}
\]
For \( t \in J_3 \), the solution of system (1.1) (as \( I_1(x(t_1^-)) \to 0 \)) satisfies Eq. (2.17). Therefore
\[
\lim_{I_1(x(t_1^-)) \to 0} e_3(t) = \lim_{I_1(x(t_1^-)) \to 0} \left\{ \left| x(t) - \tilde{x}_3(t) \right| \right\}
\]
\[
= x_0 + I_2(x(t_2^-)) + I_3(x(t_3^-)) + \frac{1}{\Gamma(q)} \int_0^{t} (t - s)^{q-1} f(s, x(s)) ds
\]
\[
+ \frac{h_2(x(t_2^-))}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2 - s)^{q-1} f(s, x(s)) ds + \int_{t_2}^{t} (t - s)^{q-1} f(s, x(s)) ds - \int_0^{t} (t - s)^{q-1} f(s, x(s)) ds \right\}
\]
\[
+ \frac{h_3(x(t_3^-))}{\Gamma(q)} \left\{ \int_0^{t_3} (t_3 - s)^{q-1} f(s, x(s)) ds + \int_{t_3}^{t} (t - s)^{q-1} f(s, x(s)) ds - \int_0^{t} (t - s)^{q-1} f(s, x(s)) ds \right\}
\]
\[
- x_0 - I_2(x(t_2^-)) - I_3(x(t_3^-)) - \frac{1}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2 - s)^{q-1} f(s, x(s)) ds + \int_{t_2}^{t} (t - s)^{q-1} f(s, x(s)) ds - \int_0^{t} (t - s)^{q-1} f(s, x(s)) ds \right\}
\]
\[
- \frac{h_2(x(t_2^-))}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2 - s)^{q-1} f(s, x(s)) ds + \int_{t_2}^{t} (t - s)^{q-1} f(s, x(s)) ds - \int_0^{t} (t - s)^{q-1} f(s, x(s)) ds \right\}
\]
\[
= 1 - \frac{h_2(x(t_2^-))}{\Gamma(q)} \times \left\{ \int_0^{t} (t - s)^{q-1} f(s, x(s)) ds - \int_0^{t_1} (t_1 - s)^{q-1} f(s, x(s)) ds - \int_{t_1}^{t} (t - s)^{q-1} f(s, x(s)) ds \right\}
\]
\[
+ \frac{h_2(x(t_2^-))}{\Gamma(q)} \left\{ \int_0^{t} (t - s)^{q-1} f(s, x(s)) ds - \int_0^{t_2} (t_2 - s)^{q-1} f(s, x(s)) ds - \int_{t_2}^{t} (t - s)^{q-1} f(s, x(s)) ds \right\}.
\tag{2.21}
\]
\[
\lim_{t \to x(t_2)} e_3(t) = \lim_{t \to x(t_2)} x(t) - \bar{x}_3(t)
\]
\[
= x_0 + I_1(x(t_1)) + I_2(x(t_2)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds
\]
\[
+ \frac{h_1(x(t_1))}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1-s)^{q-1} f(s, x(s)) ds + \int_{t_1}^t (t-s)^{q-1} f(s, x(s)) ds - \int_0^t (t-s)^{q-1} f(s, x(s)) ds \right\}
\]
\[
+ \frac{h_2(x(t_2))}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2-s)^{q-1} f(s, x(s)) ds + \int_{t_2}^t (t-s)^{q-1} f(s, x(s)) ds - \int_0^t (t-s)^{q-1} f(s, x(s)) ds \right\}
\]
\[
- x_0 - l_1(x(t_1)) - l_2(x(t_2)) - \frac{1}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1-s)^{q-1} f(s, x(s)) ds + \int_{t_1}^t (t-s)^{q-1} f(s, x(s)) ds - \int_0^t (t-s)^{q-1} f(s, x(s)) ds \right\}
\]
\[
= 1 - h_1(x(t_1)) - h_2(x(t_2))
\]
\[
\times \left\{ \int_0^t (t-s)^{q-1} f(s, x(s)) ds - \int_0^{t_1} (t_1-s)^{q-1} f(s, x(s)) ds - \int_{t_1}^t (t-s)^{q-1} f(s, x(s)) ds \right\}
\]
\[
+ \frac{h_1(x(t_1))}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1-s)^{q-1} f(s, x(s)) ds + \int_{t_1}^t (t-s)^{q-1} f(s, x(s)) ds - \int_0^t (t-s)^{q-1} f(s, x(s)) ds \right\}
\]
\[
+ \frac{h_2(x(t_2))}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2-s)^{q-1} f(s, x(s)) ds + \int_{t_2}^t (t-s)^{q-1} f(s, x(s)) ds - \int_0^t (t-s)^{q-1} f(s, x(s)) ds \right\}.
\]
Therefore
\[ x(t) = \dot{x}_{1}(t) + e(t) \]
\[ = x_0 + \sum_{k \leq n} l_k(x(t_k)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}f(s,x(s))ds \]
\[ + \frac{h_1(x(t_{t_1}))}{\Gamma(q)} \left\{ \int_0^{t_{t_1}} (t_1-s)^{q-1}f(s,x(s))ds + \int_{t_1}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\} \]
\[ + \frac{h_2(x(t_{t_2}))}{\Gamma(q)} \left\{ \int_0^{t_{t_2}} (t_2-s)^{q-1}f(s,x(s))ds + \int_{t_2}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\} \]
\[ + \frac{h_3(x(t_{t_3}))}{\Gamma(q)} \left\{ \int_0^{t_{t_3}} (t_3-s)^{q-1}f(s,x(s))ds + \int_{t_3}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\} \]
\[ t \in J_3. \]  
(2.25)

For \( t \in J_n \), suppose
\[ x(t) = x_0 + \sum_{k \leq n} l_k(x(t_{t_k})) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}f(s,x(s))ds \]
\[ + \frac{h_1(x(t_{t_1}))}{\Gamma(q)} \left\{ \int_0^{t_{t_1}} (t_1-s)^{q-1}f(s,x(s))ds + \int_{t_1}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\} \]
\[ + \frac{h_2(x(t_{t_2}))}{\Gamma(q)} \left\{ \int_0^{t_{t_2}} (t_2-s)^{q-1}f(s,x(s))ds + \int_{t_2}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\} \]
\[ \vdots \]
\[ + \frac{h_n(x(t_{t_n}))}{\Gamma(q)} \left\{ \int_0^{t_{t_n}} (t_n-s)^{q-1}f(s,x(s))ds + \int_{t_n}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\}, \]
\[ t \in J_n. \]  
(2.26)

When \( t \in J_{n+1} \), in the same way as the above process \( t \in J_3 \), by (2.26), a part of the solution the system (1.1) is given that
\[ \tilde{x}_{n+1}(t) = x(t_{n+1}) + \frac{1}{\Gamma(q)} \int_{t_{n+1}}^t (t-s)^{q-1}f(s,x(s))ds \]
\[ = x_0 + \sum_{k \leq n+1} l_k(x(t_{t_k})) + \frac{1}{\Gamma(q)} \left\{ \int_0^{t_{n+1}} (t_{n+1}-s)^{q-1}f(s,x(s))ds + \int_{t_{n+1}}^t (t-s)^{q-1}f(s,x(s))ds \right\} \]
\[ + \frac{h_1(x(t_{t_1}))}{\Gamma(q)} \left\{ \int_0^{t_{t_1}} (t_1-s)^{q-1}f(s,x(s))ds + \int_{t_1}^{t_{n+1}} (t_{n+1}-s)^{q-1}f(s,x(s))ds - \int_0^{t_{n+1}} (t_{n+1}-s)^{q-1}f(s,x(s))ds \right\} \]
\[ + \frac{h_2(x(t_{t_2}))}{\Gamma(q)} \left\{ \int_0^{t_{t_2}} (t_2-s)^{q-1}f(s,x(s))ds + \int_{t_2}^{t_{n+1}} (t_{n+1}-s)^{q-1}f(s,x(s))ds - \int_0^{t_{n+1}} (t_{n+1}-s)^{q-1}f(s,x(s))ds \right\} \]
\[ \vdots \]
\[ + \frac{h_n(x(t_{t_n}))}{\Gamma(q)} \left\{ \int_0^{t_{t_n}} (t_n-s)^{q-1}f(s,x(s))ds + \int_{t_n}^{t_{n+1}} (t_{n+1}-s)^{q-1}f(s,x(s))ds - \int_0^{t_{n+1}} (t_{n+1}-s)^{q-1}f(s,x(s))ds \right\}. \]  
(2.27)

For \( t \in J_{n+1} \), \( e_{n+1}(t) \) denotes the error between Eq. (2.27) and the correct solution of system (1.1). The solution of system (1.1) (as \( l_k(x(t_{t_k})) \rightarrow 0, k = 1, 2, \ldots, n+1 \)) tends to Eq. (1.5). Therefore
\[ \lim_{q \to 1, k \to 0} e_{n+1}(t) = \lim_{q \to 1, k \to 0} \{x(t) - \tilde{x}_{n+1}(t)\} \]
\[ = \frac{1}{\Gamma(q)} \left\{ \int_0^t (t-s)^{q-1}f(s,x(s))ds - \int_{t_{n+1}}^t (t_{n+1}-s)^{q-1}f(s,x(s))ds \right\}. \]  
(2.28)

For \( t \in J_{n+1} \), the solution of system (1.1) (as \( l_1(x(t_{t_1})) \rightarrow 0 \)) satisfies Eq. (2.26). Therefore
\[
\lim_{t \downarrow (t_{k+1})} e_{n+1}(t) = \lim_{t \downarrow (t_{k+1})} \{x(t) - \tilde{x}_{n+1}(t)\}
\]

\[
= x_0 + \sum_{0 \leq k < n+1} I_k(x(t_k)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}f(s,x(s))ds \\
+ \frac{h_l(x(t_2))}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2-s)^{q-1}f(s,x(s))ds + \int_{t_2}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\} \\
+ \frac{h_l(x(t_3))}{\Gamma(q)} \left\{ \int_0^{t_3} (t_3-s)^{q-1}f(s,x(s))ds + \int_{t_3}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\} \\
+ \frac{h_l(x(t_4))}{\Gamma(q)} \left\{ \int_0^{t_4} (t_4-s)^{q-1}f(s,x(s))ds + \int_{t_4}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\} \\
\vdots \\
+ \frac{h_l(x(t_n))}{\Gamma(q)} \left\{ \int_0^{t_n} (t_n-s)^{q-1}f(s,x(s))ds + \int_{t_n}^t (t-s)^{q-1}f(s,x(s))ds - \int_0^t (t-s)^{q-1}f(s,x(s))ds \right\} \\
= x_0 - \sum_{0 \leq k < n+1} I_k(x(t_k)) - \frac{1}{\Gamma(q)} \left\{ \int_0^{t_{n+1}} (t_{n+1}-s)^{q-1}f(s,x(s))ds + \int_{t_{n+1}}^t (t-s)^{q-1}f(s,x(s))ds \right\}
\]

In the same way as the process of (2.29), for \( t \in J_{n+1} \) and each \( 1 \leq j \leq n \), we obtain

\[
\frac{1 - h}{\Gamma(q)} \sum_{0 \leq k < n+1} h_x(t_k)) \left\{ \int_0^{t_k} (t_k-s)^{q-1}f(s,x(s))ds - \int_0^{t_{k+1}} (t_{k+1}-s)^{q-1}f(s,x(s))ds - \int_{t_k}^t (t-s)^{q-1}f(s,x(s))ds \right\}
\]

\[
+ \frac{h_l(x(t_2))}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2-s)^{q-1}f(s,x(s))ds - \int_0^{t_{n+1}} (t_{n+1}-s)^{q-1}f(s,x(s))ds - \int_{t_2}^t (t-s)^{q-1}f(s,x(s))ds \right\} \\
+ \frac{h_l(x(t_3))}{\Gamma(q)} \left\{ \int_0^{t_3} (t_3-s)^{q-1}f(s,x(s))ds - \int_0^{t_{n+1}} (t_{n+1}-s)^{q-1}f(s,x(s))ds - \int_{t_3}^t (t-s)^{q-1}f(s,x(s))ds \right\} \\
\vdots \\
+ \frac{h_l(x(t_n))}{\Gamma(q)} \left\{ \int_0^{t_n} (t_n-s)^{q-1}f(s,x(s))ds - \int_0^{t_{n+1}} (t_{n+1}-s)^{q-1}f(s,x(s))ds - \int_{t_n}^t (t-s)^{q-1}f(s,x(s))ds \right\}.
\]

In the same way as the above process, for \( t \in J_{n+1} \), as \( I_{n+1}(x(t_{n+1})) \rightarrow 0 \), we get
\[
\lim_{t_n \to t} a_{n+1}(t) = \lim_{t_n \to t} \{x(t) - \bar{x}_{n+1}(t)\}
\]
\[
= x_0 + \sum_{k=1}^{n} \frac{h_k(x(t_k^+))}{k} \int_0^t (t-s)^{q-1} f(s,x(s))ds
\]
\[
+ \frac{h_1(x(t_1^-))}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1-s)^{q-1} f(s,x(s))ds + \int_{t_1}^{t} (t-s)^{q-1} f(s,x(s))ds - \int_{t_1}^{t} (t-s)^{q-1} f(s,x(s))ds \right\}
\]
\[
+ \frac{h_2(x(t_1^+))}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2-s)^{q-1} f(s,x(s))ds + \int_{t_2}^{t} (t-s)^{q-1} f(s,x(s))ds - \int_{t_2}^{t} (t-s)^{q-1} f(s,x(s))ds \right\}
\]
\[
\vdots
\]
\[
+ \frac{h_n(x(t_n^-))}{\Gamma(q)} \left\{ \int_0^{t_n} (t_n-s)^{q-1} f(s,x(s))ds + \int_{t_n}^{t} (t-s)^{q-1} f(s,x(s))ds - \int_{t_n}^{t} (t-s)^{q-1} f(s,x(s))ds \right\}
\]
\[
- x_0 - \sum_{k=1}^{n} \frac{h_k(x(t_k^-))}{k} \left\{ \int_{t_n}^{t_n+1} (t_n+1-s)^{q-1} f(s,x(s))ds + \int_{t_n+1}^{t} (t-s)^{q-1} f(s,x(s))ds \right\}
\]
\[
\vdots
\]
\[
+ \frac{h_1(x(t_1^-))}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1-s)^{q-1} f(s,x(s))ds + \int_{t_1}^{t_1+1} (t_1+1-s)^{q-1} f(s,x(s))ds - \int_{t_1}^{t_1+1} (t_1+1-s)^{q-1} f(s,x(s))ds \right\}
\]
\[
+ \frac{h_2(x(t_1^+))}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2-s)^{q-1} f(s,x(s))ds + \int_{t_2}^{t_2+1} (t_2+1-s)^{q-1} f(s,x(s))ds - \int_{t_2}^{t_2+1} (t_2+1-s)^{q-1} f(s,x(s))ds \right\}
\]
\[
\vdots
\]
\[
+ \frac{h_n(x(t_n^-))}{\Gamma(q)} \left\{ \int_0^{t_n} (t_n-s)^{q-1} f(s,x(s))ds + \int_{t_n}^{t_n+1} (t_n+1-s)^{q-1} f(s,x(s))ds - \int_{t_n}^{t_n+1} (t_n+1-s)^{q-1} f(s,x(s))ds \right\}.
\]

So, to contrast (2.28)–(2.31) (for each \(j, 1 \leq j \leq n\)) and the structure of (2.26) and (2.27), for \(t \in J_{n+1}\) we have
\[
e_{n+1}(t) = \frac{1 - h \sum_{k=1}^{n+1} \frac{h_k(x(t_k ))}{\Gamma(q)}}{1 - h \sum_{k=1}^{n} \frac{h_k(x(t_k ))}{\Gamma(q)}}
\times \left\{ \int_0^t (t-s)^{q-1} f(s,x(s))ds - \int_{t_n+1}^{t} (t-s)^{q-1} f(s,x(s))ds \right\}
\]
\[
+ \frac{h_1(x(t_1^-))}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1-s)^{q-1} f(s,x(s))ds + \int_{t_1}^{t_1+1} (t_1+1-s)^{q-1} f(s,x(s))ds - \int_{t_1}^{t_1+1} (t_1+1-s)^{q-1} f(s,x(s))ds \right\}
\]
\[
+ \frac{h_2(x(t_1^+))}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2-s)^{q-1} f(s,x(s))ds + \int_{t_2}^{t_2+1} (t_2+1-s)^{q-1} f(s,x(s))ds - \int_{t_2}^{t_2+1} (t_2+1-s)^{q-1} f(s,x(s))ds \right\}
\]
\[
\vdots
\]
\[
+ \frac{h_n(x(t_n^-))}{\Gamma(q)} \left\{ \int_0^{t_n} (t_n-s)^{q-1} f(s,x(s))ds + \int_{t_n}^{t_n+1} (t_n+1-s)^{q-1} f(s,x(s))ds - \int_{t_n}^{t_n+1} (t_n+1-s)^{q-1} f(s,x(s))ds \right\}.
\]
Therefore, for \( t \in J_{n+1} \), we obtain

\[
x(t) = x_0 + \sum_{k=0}^{n+1} I_k(x(t_k)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))ds
+ \frac{h_1(x(t_1))}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1-s)^{q-1} f(s, x(s))ds + \int_{t_1}^{t} (t-s)^{q-1} f(s, x(s))ds - \int_0^{t} (t-s)^{q-1} f(s, x(s))ds \right\}
+ \frac{h_2(x(t_2))}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2-s)^{q-1} f(s, x(s))ds + \int_{t_2}^{t} (t-s)^{q-1} f(s, x(s))ds - \int_0^{t} (t-s)^{q-1} f(s, x(s))ds \right\}
+ \cdots
+ \frac{h_{n+1}(x(t_{n+1}))}{\Gamma(q)} \left\{ \int_0^{t_{n+1}} (t_{n+1}-s)^{q-1} f(s, x(s))ds + \int_{t_{n+1}}^{t} (t-s)^{q-1} f(s, x(s))ds - \int_0^{t} (t-s)^{q-1} f(s, x(s))ds \right\},
\]

\( t \in J_{n+1} \).

(2.33)

So, the solution of the system (1.1) satisfies Eq. (2.2).

Secondly, we will prove Eq. (2.2) satisfies the system (1.1).

Taking Caputo fractional derivative to Eq. (2.2) on the both sides, for \( t \in J_0 \) we have

\[
0D_t^q x(t) = 0D_t^q \left\{ x_0 + \sum_{k=0}^{n} I_k(x(t_k)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))ds \right\} = f(t, x(t)), \quad t \in J_0.
\]

For \( t \in J_n, 1 \leq n \leq m \), we get

\[
0D_t^q x(t) = 0D_t^q \left\{ x_0 + \sum_{k=0}^{n} I_k(x(t_k)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))ds \right\}
+ \frac{h_1(x(t_1))}{\Gamma(q)} \left\{ \int_0^{t_1} (t_1-s)^{q-1} f(s, x(s))ds + \int_{t_1}^{t} (t-s)^{q-1} f(s, x(s))ds - \int_0^{t} (t-s)^{q-1} f(s, x(s))ds \right\}
+ \frac{h_2(x(t_2))}{\Gamma(q)} \left\{ \int_0^{t_2} (t_2-s)^{q-1} f(s, x(s))ds + \int_{t_2}^{t} (t-s)^{q-1} f(s, x(s))ds - \int_0^{t} (t-s)^{q-1} f(s, x(s))ds \right\}
+ \cdots
+ \frac{h_{n+1}(x(t_{n+1}))}{\Gamma(q)} \left\{ \int_0^{t_{n+1}} (t_{n+1}-s)^{q-1} f(s, x(s))ds + \int_{t_{n+1}}^{t} (t-s)^{q-1} f(s, x(s))ds - \int_0^{t} (t-s)^{q-1} f(s, x(s))ds \right\}
= 0D_t^q \left\{ \frac{1 - h\sum_{k=0}^{n} I_k(x(t_k))}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))ds \right\}
+ \frac{h_1(x(t_1))}{\Gamma(q)} \int_{t_1}^{t} (t-s)^{q-1} f(s, x(s))ds
+ \cdots
+ \frac{h_{n+1}(x(t_{n+1}))}{\Gamma(q)} \int_{t_{n+1}}^{t} (t-s)^{q-1} f(s, x(s))ds
= \left\{ \frac{1 - h\sum_{k=0}^{n} I_k(x(t_k))}{\Gamma(q)} f(t, x(t)) \right\}_{t \in \bigcup_{k=0}^{l} H_k}
+ \frac{h_1(x(t_1))}{\Gamma(q)} f(t, x(t))_{t \in J_1}
+ \cdots
+ \frac{h_{n+1}(x(t_{n+1}))}{\Gamma(q)} f(t, x(t))_{t \in J_n}
= f(t, x(t))_{t \in J_n}
\]

Then, for \( t \in J_n, 0 \leq n \leq m \), we have \( 0D_t^q x(t) = f(t, x(t)), t \in J_n \). So, Eq. (2.2) satisfies Caputo fractional derivative condition in the system (1.1).

For \( \forall t_n \in J_{n-1}, n \geq 2 \), by (2.2), we have
\[ x(t^n_+) - x(t^n_-) = x_0 + \sum_{k=1}^m I_k(x(t^n_k)) + \frac{1}{\Gamma(q)} \int_0^{t^n} (t - s)^{q-1} f(s,x(s)) \, ds \]
\[ + \frac{h_1(x(t^n_1))}{\Gamma(q)} \left\{ \int_0^{t^n_1} (t^n_1 - s)^{q-1} f(s,x(s)) \, ds + \int_{t^n_1}^{t^n} (t^n - s)^{q-1} f(s,x(s)) \, ds - \int_0^{t^n} (t^n - s)^{q-1} f(s,x(s)) \, ds \right\} \]
\[ + \frac{h_2(x(t^n_2))}{\Gamma(q)} \left\{ \int_0^{t^n_2} (t^n_2 - s)^{q-1} f(s,x(s)) \, ds + \int_{t^n_2}^{t^n} (t^n - s)^{q-1} f(s,x(s)) \, ds - \int_0^{t^n} (t^n - s)^{q-1} f(s,x(s)) \, ds \right\} \]
\[ \vdots \]
\[ + \frac{h_{n-1}(x(t^n_{n-1}))}{\Gamma(q)} \left\{ \int_0^{t^n_{n-1}} (t^n_{n-1} - s)^{q-1} f(s,x(s)) \, ds + \int_{t^n_{n-1}}^{t^n} (t^n - s)^{q-1} f(s,x(s)) \, ds - \int_0^{t^n} (t^n - s)^{q-1} f(s,x(s)) \, ds \right\} \]
\[ = I_n(x(t^n_n)) \]

Then, for \( t \in J_n, 1 \leq n \leq m \), we have \( x(t^n_+) - x(t^n_-) = I_n(x(t^n_n)), 1 \leq n \leq m \). So, Eq. (2.2) satisfies impulsive condition in the system (1.1).

By above the proof process, we know that the system (1.1) is equivalent with integral equation (2.2).

The proof is now completed. \( \square \)

**Remark 2.1.** From Theorem 2.1, it is come to that Eq. (2.2) is the general solution of the (1.1). However, letting \( q \to 1 \), we have

\[
\lim_{q \to 1} \left\{ \begin{array}{l}
\sigma \frac{d\varphi}{dt}(t) = f(t,\varphi(t)), \quad 0 < q < 1, \quad t \in J = [0,T], \quad t \neq t_k, \quad k = 1, 2, \ldots, m,

\Delta \varphi|_{t=t_k} = I_k(\varphi(t_k)), \quad k = 1, 2, \ldots, m,

\varphi(0) = \varphi_0.
\end{array} \right.
\]

(2.34)

From (2.2), the solution of system (2.34) is given as follows

\[
x(t) = \lim_{q \to 1} \left\{ x_0 + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s,\varphi(s)) \, ds + \frac{1}{\Gamma(q)} \sum_{k=1}^n I_k(\varphi(t_k)) \right\}
\]
\[
\times \left\{ \int_0^{t_k} (t_k - s)^{q-1} f(s,\varphi(s)) \, ds + \int_{t_k}^t (t - s)^{q-1} f(s,\varphi(s)) \, ds - \int_0^t (t - s)^{q-1} f(s,\varphi(s)) \, ds \right\} \right\}
\]

(2.35)

\[
x(t) = \lim_{q \to 1} \left\{ \int_0^t (t - s)^{q-1} f(s,\varphi(s)) \, ds + \int_{t_k}^t (t - s)^{q-1} f(s,\varphi(s)) \, ds - \int_0^t (t - s)^{q-1} f(s,\varphi(s)) \, ds \right\} = 0.
\]

(2.36)

Therefore,

\[
x(t) = x_0 + \sum_{k=1}^m I_k(x(t_k)) + \int_0^t f(s,\varphi(s)) \, ds.
\]

(2.37)
Hence, for the first-order impulsive differential equation (2.34), (2.37) is decided by initial value $x_0$ that it is an exact solution. But, for impulsive fractional system (1.1), it need give more condition to decide the constant $h$ than the first-order impulsive one.

So, we will consider the existence of solution for a class of impulsive fractional system as follows

$$\begin{cases}
0D_t^\alpha x(t) = f(t, x(t)), & 0 < q < 1, \ t \in J = [0, T], \ t \neq t_k, \ k = 1, 2, \ldots, m, \\
\Delta x|_{t_k} = l_k(x(t_k^+)), & k = 1, 2, \ldots, m, \\
x(0) = x_0, \\
x(t_1^-) - x(t_1^+) = l_1(x(t_1^+)) + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} f(s, x(s)) \, ds + \frac{1-\Gamma(q)}{\Gamma(q)} f(s, x(s))ds \\
& \times \left\{ \int_0^1 (t_2 - s)^{q-1} f(s, x(s)) \, ds - \int_0^1 (t_1 - s)^{q-1} f(s, x(s)) \, ds - \int_1^t (t_2 - s)^{q-1} f(s, x(s)) \, ds \right\}.
\end{cases}$$

(2.38)

3. Existence of solutions for the impulsive fractional differential system

Let us now begin to establish the theory of the existence of solutions for the system (2.38). Firstly, we give some basic definitions, notations and lemmas.

Let $C(J, \mathbb{R})$ be a Banach space of all continuous functions from $J$ into $\mathbb{R}$ with norm $|x|_C := \sup \{|x| : t \in J\}$ for $x \in C(J, \mathbb{R})$. We also introduce the Banach space

$$PC(J, \mathbb{R}) = \{ x : J \rightarrow \mathbb{R} : x \in C(J_k, \mathbb{R}), \ \text{and there exist} \ x(t_k^-) \ \text{and} \ x(t_k^+) \ \text{with} \ x(t_k^+) = x(t_k) \}$$

where $k = 0, 1, \ldots, w$, with norm $|x|_{PC} := \sup \{|x| : t \in J\}$. Denote

$PC^k(J, \mathbb{R}) = \{ x \in PC(J, \mathbb{R}) : x' \in PC(J, \mathbb{R}) \}$ with $|x|_{PC^k} = |x|_{PC} + |x'|_{PC}$.

Clearly, $(PC^k(J, \mathbb{R}), |.|_{PC^k})$ is also a Banach space.

For measurable functions $\delta : J \rightarrow \mathbb{R}$, define the norm

$$\|\delta\|_{L^p(J)} = \begin{cases}
\left( \int_J |\delta(t)|^p \, dt \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\
\inf \{ \sup_{t \in J} |\delta(t)| \}, & p = \infty,
\end{cases}$$

where $|J|$ is the Lebesgue measure on $J$. Let $L^p(J, \mathbb{R})$ be the Banach space of all Lebesgue measurable functions $\delta : J \rightarrow \mathbb{R}$ with $\|\delta\|_{L^p(J)} < \infty$.

**Theorem 3.1** (Hölder’s inequality). Assume that $l, p > 1$, and $\frac{1}{l} + \frac{1}{p} = 1$. If $\delta \in L^l(J, \mathbb{R}), \lambda \in L^p(J, \mathbb{R})$, then for $1 \leq p < \infty, \delta\lambda \in L^1(J, \mathbb{R})$ and

$$\|\delta\lambda\|_{L^1(J)} \leq \|\delta\|_{L^l(J)} \|\lambda\|_{L^p(J)}.$$

**Theorem 3.2** (Krasnoselskii [26, pp. 31]). Let $C$ be a nonempty closed convex of a Banach space $(\mathbb{X}, |.|)$. Suppose that $P$ and $Q$ map $C$ into $\mathbb{X}$ such that

(i) $PX + QY \subset C$ where $X, Y \subset C$;
(ii) $P$ is compact and continuous;
(iii) $Q$ is a contraction mapping.

Then there exist $Z \subset C$ such that $Z = PZ + QZ$.

To prove our main results, we list the following basic assumptions of this paper:

(H1) for almost all $t \in J$, the function $f(t, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and for each $x \in \mathbb{R}$, the function $f(\cdot, x) : J \rightarrow \mathbb{R}$ is strongly measurable.

(H2) there exists a constant $q_1 \in (0, q) \subset (0, 1)$ and $\delta(\cdot) \in L^{l_1}(J, \mathbb{R}^+) \subset L^{l}(J, \mathbb{R}^+)$ such that $|f(t, x)| \leq \delta(t) |x|_{PC} + 1$ for all $x \in \mathbb{R}$ and all most $t \in J$.

(H3) there exists a constant $q_2 \in (0, q) \subset (0, 1)$ and $\lambda(\cdot) \in L^{l_2}(J, \mathbb{R}^+) \subset L^{l}(J, \mathbb{R}^+)$ such that $|f(t, x) - f(t, y)| \leq \lambda(t)|x - y|_{PC}$ for all $x, y \in \mathbb{R}$ and all most $t \in J$.

(H4) the function $l_k : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exists $\Lambda > 0$ such that

$$\Lambda = \max_{t \in J, x \in \mathbb{X}} |l_k(x)|.$$
where $PC_r = \{x \in PC(f, \mathbb{R}) : \|x\| \leq r, r > 0\}$. The set $PC_r$ is clearly a bounded closed convex set in $PC(f, \mathbb{R})$ for each $r$ and for each $x \in PC_r$.

**Theorem 3.3.** Suppose that the conditions (H1)–(H4) are satisfied. If

$$r \geq \|x_0\|_{PC} + m\Lambda + \frac{(1 + 3m\Lambda)(r + 1)}{1 + \psi_1} \left( \frac{d}{d_{\alpha,\beta}} \right)^{\frac{1}{\psi_1}} \left( \|\delta\|_{L^p(f_{\alpha,\beta})} \right) \left( \frac{1}{1 + \beta} \right)^{\frac{1}{\psi_1}} \|x_0\|_{PC} + \frac{1}{\psi_1} \left( \frac{1}{1 + \beta} \right)^{\frac{1}{\psi_1}} \|\delta\|_{L^p(f_{\alpha,\beta})} \|x_0\|_{PC} + \frac{1}{\psi_1} \left( \frac{1}{1 + \beta} \right)^{\frac{1}{\psi_1}} \|\delta\|_{L^p(f_{\alpha,\beta})} < 1$$

(where $\alpha = \frac{q - 1}{q_1}, \beta = \frac{q - 1}{q_2}$ and $\Lambda = \max_{1 \leq k \leq m, x \in PC_r} |f_k(x)|$), then the system (2.38) has at least one mild solution on $J$.

**Proof.** Let $\Pi_1 : PC_r \rightarrow PC_r$ and $\Pi_2 : PC_r \rightarrow PC_r$ be defined as

$$\Pi_1(x)(t) = \begin{cases} x_0, & t \in J_0, \\ x_0 + \sum_{k \in n} I_k(x(t_k)) + \frac{i_1(x(t_1))}{\Gamma(q)} \int_0^{t_1} (t - s)^{q-1} f(s, x(s))ds + \int_{t_1}^{t} (t - s)^{q-1} f(s, x(s))ds - \int_0^{t} (t - s)^{q-1} f(s, x(s))ds, & t \in J_n, 1 \leq n \leq m. \end{cases}$$

and

$$\Pi_2(x)(t) = \frac{1}{\Gamma(q)} \int_0^{t} (s - t)^{q-1} f(s, x(s))ds, \quad t \in J_n, 0 \leq n \leq m.$$
By the assumption, we know $\Pi_1 x + \Pi_2 x' \in PC_r$.

Step 2. We prove that the map $\Pi_1$ is continuous on $PC_r$.

Let $\{x_t\}_{t=0}^{\infty} \subseteq PC_r$ with $x_t \to x$ in $PC_r$. Then for $t \in I_n, n = 0, 1, \ldots, m$, we have

$$
\left| (\Pi_1 x') (t) - (\Pi_1 x) (t) \right| \leq \frac{I_1 (x_t (t_1))}{\Gamma (q)} \left\{ \int_0^{t_1} (t_1 - s)^{q-1} f (s, x(s)) ds + \int_{t_1}^{t} (t - s)^{q-1} f (s, x(s)) ds - \int_0^t (t - s)^{q-1} f (s, x(s)) ds \right\}
$$

Step 4. The map $\Pi_1$ is equicontinuous.

Let $x, y \in I_n = (t_n, t_{n+1}], t_n \leq x < y \leq t_{n+1}, n = 0, 1, \ldots, m, x \in PC_r$, we obtain

$$
\left| (\Pi_1 x) (t) - (\Pi_1 y) (t) \right| \leq \frac{I_1 (x_t (t_1))}{\Gamma (q)} \left\{ \int_0^{t_1} (t_1 - s)^{q-1} f (s, x(s)) ds + \int_{t_1}^{t} (t - s)^{q-1} f (s, x(s)) ds - \int_0^t (t - s)^{q-1} f (s, x(s)) ds \right\}
$$
is a contraction mapping. Hence, by Krasnoselskii’s fixed point theorem, we can conclude that the system (2.38) has at least one solution on $J$. Therefore, $\lim_{t \to 0} (\Pi_1 x)(t) = (\Pi_1 x)(0) = 0$, which implies that $\Pi_1(\mathcal{PC})$ is equicontinuous. Finally, combining Step 1 to Step 4 together with Ascoli’s theorem, we come to the conclusion that the operator $\Pi_1$ is compact.

Step 5. The map $\Pi_2$ is a contraction mapping. Let $x, x^* \in \mathcal{PC}$ and $t \in J_n = [0, 1]$. We have

$$||\Pi_2 x(t) - \Pi_2 x^*(t)|| = \frac{1}{\Gamma(q)} \left| \int_0^t \left( (t-s)^{q-1} f(s, x(s)) ds - \int_0^t (t-s)^{q-1} f(s, x^*(s)) ds \right) \right|$$

$$\leq \frac{1}{\Gamma(q)} \left| \int_0^t (t-s)^{q-1} f(s, x(s)) ds \right| \leq \frac{1}{\Gamma(q)} \left| \int_0^t (t-s)^{q-1} \|x(s) - x^*(s)\|_\mathcal{PC} ds \right|$$

$$\leq \frac{1}{\Gamma(q)} \left| \int_0^t (t-s)^{q-1} \|x(s) - x^*(s)\|_\mathcal{PC} ds \right| \leq \frac{1}{\Gamma(q)} \left| \int_0^t (t-s)^{q-1} \|x(s) - x^*(s)\|_\mathcal{PC} ds \right|$$

So, we obtain that $\Pi_2$ is a contraction mapping. Hence, by Krasnoselskii’s fixed point theorem, we can conclude that the system (2.38) has at least one solution on $J$. 

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**Note:** The text continues with more mathematical details and proofs, which are not fully transcribed here due to the complexity and length of the content.
4. Examples

In this section, some examples are given to illustrate the usefulness of the results in this paper.

**Example 1.** Let us consider the general solution of the impulsive fractional system

\[
\begin{align*}
  \alpha D^\gamma_t x(t) &= t, \quad t \in [0,2] \setminus \{1\}, \\
  x(1) &= x(1^-) + l(x(1^-)), \\
  x(0) &= 0.
\end{align*}
\]

By the **Theorem 2.1**, the general solution is obtained that

\[
 x(t) = \frac{1}{\Gamma(\frac{\gamma}{2})} \int_0^t (t-s)^{\frac{\gamma-1}{2}} ds = \frac{16}{5 \Gamma(\frac{\gamma}{2})} t^\gamma, \quad t \in [0,1].
\]

and

\[
 x(t) = I(x(1^-)) + \frac{1}{\Gamma(\frac{\gamma}{2})} \int_0^t (t-s)^{\frac{\gamma-1}{2}} ds \\
 + \frac{hI(x(1^-))}{\Gamma(\frac{\gamma}{2})} \left[ \int_0^1 (1-s)^{\frac{\gamma-1}{2}} ds + \int_1^t (t-s)^{\frac{\gamma-1}{2}} ds - \int_0^t (t-s)^{\frac{\gamma-1}{2}} ds \right] \\
= I(x(1^-)) + \frac{16}{5 \Gamma(\frac{\gamma}{2})} t^\gamma + \frac{4hl(x(1^-))}{\Gamma(\frac{\gamma}{2})} \left[ 4 + (t-1)^{\frac{\gamma}{2}}(4t+1) - 4t^\gamma \right], \quad t \in (1,2].
\]

where \( h \) is a constant.

**Remark 4.1.** For \( t \in (1,2] \), there exist two different functions in the solution (4.5) which the function \( (t-1)^{\frac{\gamma}{2}}(4t+1) \) (where \( t \in (1,2] \)) is piecewise function on \((1,2]\), but the function \( t^\gamma \) (where \( t \in (1,2] \)) is a part of \( t^\gamma \) on \( [0,2] \). It is due to that the two functions are decided by \( \frac{1}{\Gamma(\frac{\gamma}{2})} \int_0^t (t-s)^{\frac{\gamma-1}{2}} ds \) and \( \frac{1}{\Gamma(\frac{\gamma}{2})} \int_0^t (t-s)^{\frac{\gamma-1}{2}} ds \) respectively. That is

\[
 \frac{1}{\Gamma(\frac{\gamma}{2})} \int_1^t (t-s)^{\frac{\gamma-1}{2}} ds = \frac{4}{5 \Gamma(\frac{\gamma}{2})} (t-1)^{\frac{\gamma}{2}}(4t+1)
\]

and

\[
 \frac{1}{\Gamma(\frac{\gamma}{2})} \int_0^t (t-s)^{\frac{\gamma-1}{2}} ds = \frac{16}{5 \Gamma(\frac{\gamma}{2})} t^\gamma.
\]

Therefore, for \( t \in (1,2] \), we have

\[
 \alpha D^\gamma_t \left( \frac{4}{5 \Gamma(\frac{\gamma}{2})} (t-1)^{\frac{\gamma}{2}}(4t+1) \right) = \alpha D^\gamma_t \left( \frac{1}{\Gamma(\frac{\gamma}{2})} \int_1^t (t-s)^{\frac{\gamma-1}{2}} ds \right) = \frac{1}{\Gamma(\frac{\gamma}{2})} \alpha D^\gamma_t \left( \int_1^t (t-s)^{\frac{\gamma-1}{2}} ds \right) = t_{t\in(1,2]}
\]

but

\[
 \alpha D^\gamma_t \left( \frac{16}{5 \Gamma(\frac{\gamma}{2})} t^\gamma \right) = \alpha D^\gamma_t \left( \frac{1}{\Gamma(\frac{\gamma}{2})} \int_0^t (t-s)^{\frac{\gamma-1}{2}} ds \right) = \frac{1}{\Gamma(\frac{\gamma}{2})} \alpha D^\gamma_t \left( \int_0^t (t-s)^{\frac{\gamma-1}{2}} ds \right) = t_{t\in[0,2]}|_{t\in(1,2]}
\]

Then,

\[
 \alpha D^\gamma_t x(t) = \alpha D^\gamma_t \left( I(x(1^-)) + \frac{16}{5 \Gamma(\frac{\gamma}{2})} t^\gamma + \frac{4hl(x(1^-))}{\Gamma(\frac{\gamma}{2})} \left[ 4 + (t-1)^{\frac{\gamma}{2}}(4t+1) - 4t^\gamma \right] \right) = t, \quad t \in (1,2].
\]

**Example 2.** Let us the existence of the solution for the following system

\[
\begin{align*}
  \alpha D^\gamma_t x(t) &= \frac{e^{\alpha x(t)}}{1 + \alpha x(t)}, \quad q \in (0,1), \quad t \in [0,T] \setminus \{t_1, t_2\}, \\
  \Delta x|_{t=t_1} &= \Delta x|_{t=t_2} = 1, \\
  x(0) &= 0, \\
  x(t_2) - x(t_1) &= 1 + \frac{1}{\Gamma(\gamma)} \int_{t_1}^{t_2} (t_2 - s)^{\gamma-1} \frac{e^{\alpha x(s)}}{1 + \alpha x(s)} ds,
\end{align*}
\]

where \( \alpha > 0 \) is a constant.

For \( t \in [0,T] \), we have
such that ten language, which have been very useful for improving the quality of this paper.

For $t \in [0, T)$ and some $p \in (0, q)$, let $\delta(t) = \frac{\varepsilon_{at}}{t} \in L^2([0, T], \mathbb{R})$. Choosing some $\alpha > 0$ large enough and suitable $p \in (0, q)$ such that

$$\frac{\Gamma(q+1-p)\|\dot{\delta}\|_{\mathcal{L}^p(\mathcal{J}_p^{0, a} \delta)}}{\Gamma(q)(1+x)^{q-p}} < 1$$

and

$$\|x_0\|_{\mathcal{L}} + m\Lambda + \frac{(1+3m\Lambda)(r+1)\Gamma(1-q)}{\Gamma(q)(1+x)^{q-p}} \|\delta\|_{\mathcal{L}^p(\mathcal{J}_p^{0, a} \delta)} = 2 + \frac{7(r+1)^{1-q}}{2} \frac{\|e^{-at}\|_{\mathcal{L}^p(\mathcal{J}_p^{0, a} \delta)}}{\Gamma(q)(1+x)^{q-p}} \leq r.$$

Then, all the assumptions in Theorem 3.3 are satisfied, our result can be applied in the above system.

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References