Nowhere-zero 3-flows in graphs admitting solvable arc-transitive groups of automorphisms

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Abstract

Tutte’s 3-flow conjecture asserts that every 4-edge-connected graph has a nowhere-zero 3-flow. In this note we prove that, if a regular graph of valency at least four admits a solvable group of automorphisms acting transitively on its vertex set and edge set, then it admits a nowhere-zero 3-flow.

Key words: integer flow; nowhere-zero 3-flow; vertex-transitive graph; arc-transitive graph; solvable group

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1 Introduction

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph endowed with an orientation. For an integer $k \geq 2$, a $k$-flow \cite{3} in $\Gamma$ is an integer-valued function $f : E(\Gamma) \to \{0, \pm 1, \pm 2, \ldots, \pm (k-1)\}$ such that, for every $v \in V(\Gamma)$,

$$
\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)
$$

where $E^+(v)$ is the set of edges of $\Gamma$ with tail $v$ and $E^-(v)$ the set of edges of $\Gamma$ with head $v$. A $k$-flow $f$ in $\Gamma$ is called a nowhere-zero $k$-flow if $f(e) \neq 0$ for every $e \in E(\Gamma)$. Obviously, if $\Gamma$ admits a nowhere-zero $k$-flow, then $\Gamma$ admits a nowhere-zero $(k+1)$-flow. And whether a graph admits a nowhere-zero $k$-flow is independent of its orientation. The notion of nowhere-zero flows was introduced by Tutte in \cite{17, 18} who proved that a planar graph admits a nowhere-zero 4-flow if and only if the Four Color Conjecture holds. The reader is referred to Jaeger \cite{8} and Zhang \cite{22} for surveys on nowhere-zero flows and to \cite[Chapter 21]{3} for an introduction to this area.

In \cite{17, 18} Tutte proposed three celebrated conjectures on integer flows which are still open in general. One of them is the following well-known 3-flow conjecture (see e.g. \cite[Conjecture 21.16]{3}).

**Conjecture 1.1.** \textit{(Tutte’s 3-flow conjecture)} Every 4-edge-connected graph admits a nowhere-zero 3-flow.
This conjecture has been extensively studied in over four decades; see e.g. [6, 7, 9, 10, 11, 16, 20, 21]. A recent breakthrough, due to Lovász, Thomassen, Wu and Zhang [11], is the following result which refines an earlier result by Thomassen [16].

**Theorem 1.2.** ([11, Theorem 1.7]) Every 6-edge-connected graph admits a nowhere-zero 3-flow.

It is well known [19] that every vertex-transitive graph of valency $d \geq 1$ is $d$-edge-connected. Thus, when restricted to the class of vertex-transitive graphs, Conjecture 1.1 asserts that every vertex-transitive graph of valency at least four admits a nowhere-zero 3-flow. Due to Theorem 1.2 this is now boiled down to vertex-transitive graphs of valency 5. In an attempt to Tutte’s 3-flow conjecture for Cayley graphs, Potočnik, Škoviera and Škerkovski [13] proved the following result. (It is well known [2] that every Cayley graph is vertex-transitive, but the converse is not true.)

**Theorem 1.3.** ([13, Theorem 1.1]) Every Cayley graph of valency at least four on a finite abelian group admits a nowhere-zero 3-flow.

This was generalized by Nánásiová and Škoviera [12] to Cayley graphs on nilpotent groups.

**Theorem 1.4.** ([12, Theorem 4.3]) Every Cayley graph of valency at least four on a finite nilpotent group admits a nowhere-zero 3-flow.

It would be nice if one can prove Tutte’s 3-flow conjecture for all vertex-transitive graphs. As a step towards this, we prove the following result in the present paper.

**Theorem 1.5.** Let $G$ be a finite solvable group. Then every $G$-vertex-transitive and $G$-edge-transitive graph with valency at least four admits a nowhere-zero 3-flow.

In fact, we will prove a weaker version (Lemma 3.1): for any finite solvable group $G$, every $G$-vertex-transitive and $G$-edge-transitive graph with valency at least 4 and not divisible by 3 admits a nowhere-zero 3-flow. This together with Theorem 1.2 implies Theorem 1.5 immediately (as every vertex-transitive graph of valency $d \geq 1$ is $d$-edge-connected). The proof of the weaker version will be done by induction on the derived length of the solvable group $G$, using Theorem 1.3 as the base case. In view of Theorem 1.2, we could restrict to $G$-vertex-transitive and $G$-edge-transitive graphs of valency five. Nevertheless, there is no need to make such a restriction in our proof.

None of Theorems 1.5 and 1.4 is implied by the other, because not every Cayley graph is edge-transitive, and on the other hand a vertex-transitive and edge-transitive graph is not necessarily a Cayley graph (see e.g. [2, 14]).

Note that any $G$-arc-transitive graph is necessarily $G$-vertex-transitive and $G$-edge-transitive. On the other hand, any $G$-vertex-transitive and $G$-edge-transitive graph with odd valency is $G$-arc-transitive. (This result is due to Tutte, and its combinatorial proof given in [4, Proposition 1.2] can be easily extended from $G = \text{Aut}(\Gamma)$ to a subgroup $G$ of $\text{Aut}(\Gamma)$.) Note also that any
regular graph with even valency admits a nowhere-zero 2-flow. Therefore, Theorem 1.5 and its corollary as follows are equivalent.

**Corollary 1.6.** Every regular graph of valency at least four admitting a solvable arc-transitive group of automorphisms admits a nowhere-zero 3-flow.

At this point we would like to mention the following result due to Alspach, Liu and Zhang [1] which also involves solvable groups.

**Theorem 1.7.** ([1, Corollary 2.3]) Every Cayley graph of valency at least two on a finite solvable group admits a nowhere-zero 4-flow.

It would be pleasing if one can remove the condition of \(G\)-edge-transitivity in Theorem 1.5. As an intermediate step towards this, one may try to prove that every Cayley graph of valency at least four on a finite solvable group admits a nowhere-zero 3-flow, thus generalizing Theorems 1.4 and 1.7 simultaneously.

### 2 Preparations

We follow [3] and [5, 15] respectively for graph- and group-theoretic terminology and notation. The derived subgroup of a group \(G\) is defined as \(G' := [G, G]\), the subgroup of \(G\) generated by all commutators \(x^{-1}y^{-1}xy, \ x, y \in G\). Define \(G^{(0)} := G, G^{(1)} := G'\) and \(G^{(i)} := (G^{(i-1)})'\) for \(i \geq 1\). A group \(G\) is *solvable* if there exists an integer \(n \geq 0\) such that \(G^{(n)} = 1\); in this case the least integer \(n\) with \(G^{(n)} = 1\) is called the *derived length* of \(G\). Solvable groups with derived length 1 are precisely nontrivial abelian groups.

All definitions in the next three paragraphs can be found in [2, Part Three] or [14].

Let \(G\) be a group acting on a set \(\Omega\). We use \(\alpha^G := \{\alpha^g : g \in G\}\) to denote the \(G\)-orbit on \(\Omega\) containing \(\alpha\) (where \(\alpha^g\) is the image of \(\alpha\) under \(g\)), and \(G_\alpha := \{g \in G : \alpha^g = \alpha\}\) the stabilizer of \(\alpha\) in \(G\). We say that \(G\) is *semiregular* on \(\Omega\) if \(G_\alpha = 1\) is the trivial subgroup of \(G\) for every \(\alpha \in \Omega\), *transitive* on \(\Omega\) if \(\alpha^G = \Omega\) for some (and hence all) \(\alpha \in \Omega\), and *regular* on \(\Omega\) if it is both transitive and semiregular on \(\Omega\). The group \(G\) is *intransitive* on \(\Omega\) if it is not transitive on \(\Omega\). A partition \(\mathcal{P}\) of \(\Omega\) is *\(G\)-invariant* if \(P^g := \{\alpha^g : g \in G\} \in \mathcal{P}\) for any \(P \in \mathcal{P}\) and \(g \in G\), and *nontrivial* if \(1 < |P| < |\Omega|\) for every \(P \in \mathcal{P}\).

Suppose that \(\Gamma\) is a graph admitting \(G\) as a group of automorphisms. That is, \(G\) acts on \(V(\Gamma)\) (not necessarily faithfully) such that, for any \(\alpha, \beta \in V(\Gamma)\) and \(g \in G\), \(\alpha\) and \(\beta\) are adjacent in \(\Gamma\) if and only if \(\alpha^g\) and \(\beta^g\) are adjacent in \(\Gamma\). (If \(K\) is the kernel of the action of \(G\) on \(V(\Gamma)\), namely, the subgroup of all elements of \(G\) that fix every vertex of \(\Gamma\), then \(G/K\) is isomorphic to a subgroup of the automorphism group \(\text{Aut}(\Gamma)\) of \(\Gamma\).) We say that \(\Gamma\) is *\(G\)-vertex-transitive* if \(G\) is transitive on \(V(\Gamma)\), and *\(G\)-edge-transitive* if \(G\) is transitive on the set of edges of \(\Gamma\). If \(\Gamma\) is \(G\)-vertex-transitive such that \(G\) is also transitive on the set of arcs of \(\Gamma\), then \(\Gamma\) is called *\(G\)-arc-transitive*, where an *arc* is an ordered pair of adjacent vertices. A \(G\)-vertex-transitive graph may not be \(G\)-edge-transitive, and a \(G\)-edge-transitive graph may not be \(G\)-vertex-transitive.
Let $\Gamma$ be a graph and $\mathcal{P}$ a partition of $V(\Gamma)$. The *quotient graph* of $\Gamma$ with respect to $\mathcal{P}$, denoted by $\Gamma_{\mathcal{P}}$, is the graph with vertex set $\mathcal{P}$ in which $P,Q \in \mathcal{P}$ are adjacent if and only if there exists at least one edge of $\Gamma$ joining a vertex of $P$ and a vertex of $Q$. For blocks $P,Q \in \mathcal{P}$ adjacent in $\Gamma_{\mathcal{P}}$, denote by $\Gamma[P,Q]$ the bipartite subgraph of $\Gamma$ with vertex set $P \cup Q$ whose edges are those of $G$ between $P$ and $Q$. In the case when all blocks of $\mathcal{P}$ are independent sets of $\Gamma$ and $\Gamma[P,Q]$ is a $t$-regular bipartite graph for each pair of adjacent $P,Q \in \mathcal{P}$, where $t \geq 1$ is an integer independent of $(P,Q)$, we say that $\Gamma$ is a *multicover* of $\Gamma_{\mathcal{P}}$. A multicover with $t=1$ is thus a topological cover in the usual sense. In the proof of Theorem 1.5, we will use the following lemma in the case when $k = 3$.

**Lemma 2.1.** Let $k \geq 2$ be an integer. If a graph admits a nowhere-zero $k$-flow, then its multicovers all admit a nowhere-zero $k$-flow.

**Proof.** Using the notation above, let $\Gamma$ be a multicover of $\Sigma := \Gamma_{\mathcal{P}}$. Suppose that $\Sigma$ admits a nowhere-zero $k$-flow $f$ (with respect to some orientation). For each oriented edge $(P,Q)$ of $\Sigma$, orient the edges of the $t$-regular bipartite graph $\Gamma[P,Q]$ in such a way that they all have tails in $P$ and heads in $Q$, and then assign $f(P,Q)$ to each of them. Denote this nowhere-zero function on the oriented edges of $\Gamma$ by $g$, and denote the oriented edge of $\Gamma$ with tail $\alpha$ and head $\beta$ by $(\alpha, \beta)$. It can be verified that, for any $P \in \mathcal{P}$ and $\alpha \in P$, $\sum_{(\alpha, \beta) \in E^+_\Gamma(\alpha)} g(\alpha, \beta) = \sum_{(P,Q) \in E^+_\Gamma(P)} t \cdot f(P,Q)$ and $\sum_{(\beta, \alpha) \in E^-_\Gamma(\alpha)} g(\beta, \alpha) = \sum_{(Q,P) \in E^-_\Gamma(P)} t \cdot f(Q,P)$. Since $f$ is a nowhere-zero $k$-flow in $\Sigma$, we have $\sum_{(P,Q) \in E^+_\Gamma(P)} f(P,Q) = \sum_{(Q,P) \in E^-_\Gamma(P)} f(Q,P)$ for every $P \in \mathcal{P}$. Therefore, $\sum_{(\alpha, \beta) \in E^+_\Gamma(\alpha)} g(\alpha, \beta) = \sum_{(\beta, \alpha) \in E^-_\Gamma(\alpha)} g(\beta, \alpha)$ for every $\alpha \in V(\Gamma)$ and so $g$ is a nowhere-zero $k$-flow in $\Gamma$. \hfill $\Box$

If $\Gamma$ is a $G$-vertex-transitive graph, then for any normal subgroup $N$ of $G$, the set $\mathcal{P}_N$ of $N$-orbits on $V(\Gamma)$ is a $G$-invariant partition of $V(\Gamma)$, called a $G$-normal partition of $V(\Gamma)$ [14]. Denote the corresponding quotient graph by $\Gamma_N := \Gamma_{\mathcal{P}_N}$. The following observations are known in the literature (see e.g. [14]). We give their proof for the completeness of the paper.

**Lemma 2.2.** ([14]) Let $\Gamma$ be a connected $G$-vertex-transitive graph, and $N$ a normal subgroup of $G$ that is intransitive on $V(\Gamma)$. Then the following hold:

(a) $\Gamma_N$ is $G/N$-vertex-transitive under the induced action of $G/N$ on $\mathcal{P}_N$;

(b) for $P,Q \in \mathcal{P}_N$ adjacent in $\Gamma_N$, $\Gamma[P,Q]$ is a regular subgraph of $\Gamma$;

(c) if in addition $\Gamma$ is $G$-edge-transitive, then $\Gamma_N$ is $G/N$-edge-transitive and $\Gamma$ is a multicover of $\Gamma_N$.

**Proof.** (a) The induced action of $G/N$ on $\mathcal{P}_N$ is defined by $P^Ng := P^g$ for $P \in \mathcal{P}_N$ and $Ng \in G/N$. It is straightforward to verify that this is a well-defined action that preserves
the adjacency and non-adjacency relations of $\Gamma_N$. Moreover, since $G$ is transitive on $V(\Gamma)$, it is easy to see that $G/N$ is transitive on $P_N$.

(b) Let $P, Q \in P_N$ be adjacent in $\Gamma_N$, and let $\alpha \in P$ and $\beta \in Q$. Then $P = \alpha^N$ and $Q = \beta^N$ as $P_N$ is normal. Let $\beta_1, \ldots, \beta_d$ be the neighbours of $\alpha$ in $Q$. Then, for any $g \in N$, $\beta_1^g, \ldots, \beta_d^g \in Q$ are neighbours of $\alpha^g$. Since this holds for any $\alpha \in P$, and since for a fixed $\alpha$, $\alpha^g$ is running over all vertices of $P$ when $g$ is running over all elements of $N$, all vertices of $P$ must have the same valency in $\Gamma[P, Q]$. Similarly, all vertices of $Q$ have the same valency in $\Gamma[P, Q]$. Since $|P| = |Q|$, it follows that $\Gamma[P, Q]$ is regular.

(c) Suppose that $\Gamma$ is $G$-edge-transitive. We first prove that each $P \in P_N$ is an independent set of $\Gamma$. In fact, since $\Gamma$ is connected and $N$ is intransitive on $V(\Gamma)$, $\Gamma_N$ is connected with at least two vertices. Thus $P$ has at least one neighbour in $\Gamma_N$, say, $Q$, and so there is an edge $\{\alpha, \beta\}$ of $\Gamma$ with $\alpha \in P$ and $\beta \in Q$. If $P$ is not independent, say, $\{\gamma, \delta\}$ is an edge of $\Gamma$ with $\gamma, \delta \in P$, then there exists $g \in G$ such that $(\gamma, \delta)^g = (\alpha, \beta)$ or $(\beta, \alpha)$, contradicting the fact that $P_N$ is $G$-invariant.

Let $\{P, Q\}$ and $\{P', Q'\}$ be edges of $\Gamma_N$. Then there exist $\alpha \in P, \beta \in Q, \alpha' \in P', \beta' \in Q'$ such that $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$ are edges of $\Gamma$. Since $\Gamma$ is $G$-edge-transitive, there exists $g \in G$ such that $\{\alpha, \beta\}^g = \{\alpha', \beta'\}$. Since $P_N$ is $G$-invariant, we have $\{P, Q\}^g = \{P', Q'\}$. Therefore, $\Gamma_N$ is $G/N$-edge-transitive. It follows that the valency of $\Gamma[P, Q]$ is independent of the choice of adjacent blocks $P, Q \in P_N$. This together with (b) implies that $\Gamma$ is a multicover of $\Gamma_N$. 

3 Proof of Theorem 1.5

As mentioned in the introduction, to prove Theorem 1.5 it suffices to prove the following lemma.

**Lemma 3.1.** Let $G$ be a finite solvable group. Then every $G$-vertex-transitive and $G$-edge-transitive graph with valency at least four and not divisible by three admits a nowhere-zero 3-flow.

We will use the following fact [2, Lemma 16.3]: a graph is isomorphic to a Cayley graph if and only if its automorphism group contains a subgroup that is regular on the vertex set. Denote by $\text{val}(\Gamma)$ the valency of a regular graph $\Gamma$.

**Proof of Lemma 3.1.** Without loss of generality we may assume that the solvable group $G$ is faithful on the vertex set of the graph under consideration for otherwise we can replace $G$ by its quotient group (which is also solvable) by the kernel of $G$ on the vertex set. Under this assumption $G$ is isomorphic to a subgroup of the automorphism group of the graph. We may also assume that the graph under consideration is connected (for otherwise we consider its components). We make induction on the derived length $n(G)$ of $G$.

Suppose that $n(G) = 1$ and $\Gamma$ is a $G$-vertex-transitive and $G$-edge-transitive graph with $\text{val}(\Gamma) \geq 4$. Then $G$ is abelian and so is regular on $V(\Gamma)$. (A transitive abelian group must be
regular.) Since $G$ is isomorphic to a subgroup of $\text{Aut}(\Gamma)$, it follows that $\Gamma$ is isomorphic to a Cayley graph on $G$. Thus, by Theorem 1.3, $\Gamma$ admits a nowhere-zero 3-flow.

Assume that, for some integer $n \geq 1$, the result holds for any finite solvable group of derived length at most $n$. Let $G$ be a finite solvable group with derived length $n(G) = n + 1$. Let $\Gamma$ be a connected $G$-vertex-transitive and $G$-edge-transitive graph such that $\text{val}(\Gamma) \geq 4$ and $\text{val}(\Gamma)$ is not divisible by 3. If $\text{val}(\Gamma)$ is even, then $\Gamma$ admits a nowhere-zero 2-flow and hence a nowhere-zero 3-flow. So we assume that $\text{val}(\Gamma) \geq 5$ is odd. Since 3 does not divide $\text{val}(\Gamma)$ by our assumption, every prime factor of $\text{val}(\Gamma)$ is no less than 5. Since $G$ is solvable, it contains an abelian normal subgroup $N$ such that the quotient group $G/N$ has derived length at most $n(G) - 1 = n$. Note that $G/N$ is solvable (as any quotient group of a solvable group is solvable) and $N \neq 1$ (for otherwise $G/N \cong G$ would have derived length $n(G)$). If $N$ is transitive on $V(\Gamma)$, then it is regular on $V(\Gamma)$ as $N$ is abelian. In this case $\Gamma$ is isomorphic to a Cayley graph on $N$ and so admits a nowhere-zero 3-flow by Theorem 1.3.

In what follows we assume that $N$ is intransitive on $V(\Gamma)$. By Lemma 2.2, $\Gamma_N$ is a connected $G/N$-vertex-transitive and $G/N$-edge-transitive graph, and $\Gamma$ is a multicover of $\Gamma_N$. Thus $\text{val}(\Gamma_N)$ is a divisor of $\text{val}(\Gamma)$ and so is not divisible by 3. If $\text{val}(\Gamma_N) = 1$, then $\Gamma$ is a regular bipartite graph of valency at least two and so admits a nowhere-zero 3-flow [3]. Assume that $\text{val}(\Gamma_N) > 1$. Then $\text{val}(\Gamma_N) \geq 5$ and every prime factor of $\text{val}(\Gamma_N)$ is no less than 5. Thus, since $G/N$ is solvable of derived length at most $n$, by the induction hypothesis, $\Gamma_N$ admits a nowhere-zero 3-flow. Since $\Gamma$ is a multicover of $\Gamma_N$, by Lemma 2.1, $\Gamma$ admits a nowhere-zero 3-flow.

\[ \square \]

4 Remarks

A graph $\Gamma$ is called $G$-locally primitive [14] if it is $G$-arc-transitive and for some (and hence all) $\alpha \in V(\Gamma)$, the neighbourhood of $\alpha$ in $\Gamma$ does not admit any nontrivial $G_\alpha$-invariant partition. It is known [14, Theorem 10.2] that, for any $G$-locally primitive graph $\Gamma$ and any normal subgroup $N$ of $G$ transitive on $V(\Gamma)$, either (a) $\Gamma$ is bipartite, with bipartition formed by the two $N$-orbits on $V(\Gamma)$, and $\Gamma_N \cong K_2$; or (b) $\Gamma_N$ is a connected $G/N$-locally primitive graph, and $\Gamma$ is a cover of $\Gamma_N$. Using this and by a similar induction on the derived length as in the proof of Lemma 3.1, one can prove the following: For any finite solvable group $G$, every $G$-locally primitive graph of valency at least four admits a nowhere-zero 3-flow. Note that, although this result is implied by Theorem 1.5, its proof does not require Theorem 1.2. Since $(G, 2)$-arc-transitive graphs are $G$-locally primitive [14], we obtain without involving Theorem 1.2 that, for any finite solvable group $G$, every $(G, 2)$-arc-transitive graph with valency at least four admits a nowhere-zero 3-flow. (A graph is $(G, 2)$-arc-transitive if it admits $G$ as a group of automorphisms acting transitively on the set of oriented paths of length two.)

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References


