Coexistence of uniquely ergodic subsystems of interval mappings

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Abstract


We prove that if (I, f) has a uniquely ergodic subsystem (X, f|_X), then for every s = f^n(I, f) has a uniquely ergodic subsystem (Y, f|_Y) such that f^s = s, where I = [0, 1], f ∈ C(I, I).

1. Preliminaries

A theorem due to A.N. Sharkovskii gives a surprising answer to the following question: if f has a periodic orbit of period k, must f also have periodic orbits of other periods?

Theorem 1.1 (Sharkovskii’s theorem [20]). Order the positive integers as follows:

\[ 3 \gg 5 \gg 7 \gg 2 \cdot 3 \gg 2 \cdot 5 \gg 2 \cdot 7 \gg 2^2 \cdot 3 \gg 2^2 \cdot 5 \gg 2^2 \cdot 7 \gg 2^3 \gg 2^2 \gg 2 \gg 1 \]

If a continuous function f: \( \mathbb{R} \to \mathbb{R} \) has a periodic orbit of period k, then f has periodic orbits of all periods which follow k in the above Sharkovskii’s order.

Sharkovskii’s theorem is attractive to mathematicians because of its simple setting and surprising answer. For the original and simpler proofs see [20, 23, 6, 19]. In [14] the author generalized Sharkovskii’s theorem to special types of functions in

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higher-dimensional Euclidean space, and in [17] Matsuoka gave a 2-dimensional analogue of Sharkovskii's ordering. In spite of this, Sharkovskii's theorem is basically a one-dimensional phenomenon. In other words Sharkovskii's theorem holds for all connected ordered spaces on the order topology [21]. The reader can also find a refined Sharkovskii's theorem about orbits of periodic points and an analogue of Sharkovskii's theorem for the space $Y = \{ z \in \mathbb{C} : 0 \leq z^3 \leq 1 \}$ in [1–4].

The other surprising thing is that more complicated orbits of interval mapping, minimal sets, also coexist in a way described by some partial order [26]. The purpose of this paper is to show that uniquely ergodic subsystems of interval mapping also coexist in the same way as minimal sets do.

To do this, we give some notations in Section 2. In Section 3 we define $D$-function of a uniquely ergodic system and show its basic properties. We prove the coexistence of uniquely ergodic subsystems of interval mapping in Section 4. Finally, we give examples of uniquely ergodic systems with given $D$-functions in Section 5.

In the following we state the main results of this paper:

Let $\mathcal{Y}$ be the set of functions $s$ from $\mathbb{N}$ to $\mathbb{N}$ satisfying

(a) $s(k) \leq k$ for every $k \in \mathbb{N}$,

(b) for every $l, k \in \mathbb{N}$, if $l \mid k$ then $s(l) = s(k)$,

and $E = \{ s \in \mathcal{Y} : s(k) = (n, k) \text{ for some } n \in \mathbb{N} \text{ and all } k \in \mathbb{N} \}$ be a subset of $\mathcal{Y}$. We shall identify $E$ with $\mathbb{N}$, namely if $n \in \mathbb{N}$ then the function $s$ defined by $s(k) = (n, k)$ for $k \in \mathbb{N}$ will be identified with $n$. Denote $\mathcal{Y} = \mathcal{Y} \cup \mathcal{N}'$, where $\mathcal{N}' = \{ n' : n \in \mathbb{N} \}$.

Let $X$ be a compact metric space, $T \in C(X, X)$ and $(X, T, \mu)$ be uniquely ergodic. A $D$-function of $(X, T, \mu)$, denoted by $D$, is $n \in \mathbb{N}$ if supp$(\mu)$ is a periodic orbit of $T$ with period $n$, $n' \in \mathbb{N}'$ if supp$(\mu)$ is not a periodic orbit of $T$ but the number of the ergodic components of $T^k$ is $(n, k)$ for all $k \in \mathbb{N}$; is a function from $\mathbb{N}$ to $\mathbb{N}$ such that for every $k \in \mathbb{N}$, $T^k(\mu)(k)$ is the number of ergodic components of $T^k$ and $T^k$ is not a bounded function.

Our first main result is the following theorem.

**Theorem 1.2.** Let $X$ be a compact metric space, $T \in C(X, X)$ and $(X, T, \mu)$ be a uniquely ergodic system. Then $T \mu \in \mathcal{Y}$.

Now let $X$ be a compact, metric space and $T \in C(X, X)$. Denote $DF(X, T)$ to be the set of all $D$-functions of $(Y, T|_Y)$, where $Y \subset X$ is compact and $(Y, T|_Y)$ is uniquely ergodic. Then we have the following theorem.

**Theorem 1.3.** Let $\Sigma_M$ be a one-sided subshift of finite type with a $k \times k$ matrix $M = (m_{ij})$ satisfying $m_{01} + m_{12} + \cdots + m_{(k-1)0} = m_{00} = 1$ and $\sigma$ the left shift. Then $DF(\Sigma_M, \sigma) \supset (\mathcal{Y} \setminus E)$.

At last we prove an extension of Sharkovskii's theorem.
Theorem 1.4. (a) If \( f \in C(I, I) \), \( I = [0, 1] \), then \( DF(I, f) = \mathcal{Y}(n) \), \( n \in \mathbb{N} \cup \{ \infty \} \cup 2^\infty \).
(b) If \( n \in \mathbb{N} \cup \{ \infty \} \cup 2^\infty \), then there exists \( f \in C(I, I) \), such that \( DF(I, f) = \mathcal{Y}(n) \), where

\[
Y_i = \{ s \in \mathcal{X} : E | s(2^i) = (2^i, 2^{i+1}) \}, \quad \forall l \in \mathbb{N} \}
\]

where \( 0 \leq i < \infty \),

\[
Y_o = \{ s \in \mathcal{X} : s(2^i) = 2^i \text{ for all } l \in \mathbb{N}, \text{ all odd } p \in \mathbb{N} \},
\]

\[
Y_2 = \{ s \in \mathcal{X} : s(2^0) = 2^0 \text{ for all } l \in \mathbb{N}, \text{ all odd } p \in \mathbb{N} \},
\]

\[
\mathcal{Y}(2^\infty) = \{ 2^0, 2^1, 2^2, \ldots \}, \quad \mathcal{Y}(\infty) = Y_o \cup \mathcal{Y}(2^\infty),
\]

\[
\mathcal{Y}(n) = \begin{cases} \{ n \} \cup \{ k : n \geq k \} \cup \bigcup_{i=1}^{\infty} Y_i \cup Y^1_o \cup Y^2_o, & n = 2^0 \cdot p \quad (p \geq 3, \text{ odd}); \\ \{ 2^0, 2^1, 2^2, \ldots, 2, 1 \}, & n = 2^i. \end{cases}
\]

Remark 1.4. Theorem 1.4 can be expressed in the same way as Sharkovskii’s Theorem as follows. Give \( \mathcal{Y} \) a partial linear order:

3 \( \Rightarrow \) 5 \( \Rightarrow \) 7 \( \Rightarrow \) \( \cdots \) \( \Rightarrow \) \( Y_0 \Rightarrow \)

2 \( \Rightarrow \) 2 \( \Rightarrow \) \( \cdots \) \( \Rightarrow \) \( Y_1 \Rightarrow \)

2 \( \Rightarrow \) 3 \( \Rightarrow \) \( \cdots \) \( \Rightarrow \) \( Y_2 \Rightarrow \)

\( \Rightarrow \) \( Y^1_o \Rightarrow \) \( Y^2_o \Rightarrow \) \( \cdots \) \( \Rightarrow \) \( 2^0 \Rightarrow \) \( 2^1 \Rightarrow \) \( 2^2 \Rightarrow \) \( 2 \Rightarrow \)

If a continuous function \( f : \mathbb{R} \rightarrow \mathbb{R} \) has a uniquely ergodic subsystem \((X, f|_x, \alpha)\), then \( f \) has uniquely ergodic subsystems with \( D \)-functions which follow \( f_n \) in the above partial order. Furthermore, for every \( i \in \{ 0 \} \cup \mathbb{N} \) the existence of a uniquely ergodic subsystem \((X, f|_x, \alpha)\) with \( f_n \in Y_i \) (resp. \( f_n \in Y^1_o \)) implies the existence of a periodic orbit with period \( 2^i \cdot q \) for some odd \( q \geq 3 \) (resp. \( 2^i \cdot p \) for some \( l \in \{ 0 \} \cup \mathbb{N} \) and some odd \( p \geq 3 \)).

2. Definitions and some lemmas

Let \( X \) be a compact, metric space, \( T \in C(X, X) \). Inductively we define \( T^0 \) as the identity, \( T^n = T^{n-1} \circ T \), \( n \in \mathbb{N} \). \( O(x, T) = \{ x, T(x), T^2(x), \ldots \} \) is called the orbit of \( x \) under \( T \). A point \( x \in X \) is a periodic point of \( T \) of period \( k \in \mathbb{N} \) if \( T^k(x) = x \) but \( T^i(x) \neq x \) for \( 0 < i < k \). If \( k = 1 \), \( x \) is called a fixed point. For periodic point \( x \) of period \( k \), we call \( O(x, T) \) a periodic orbit of period \( k \). The sets of periodic points and fixed points are denoted by \( P(T), F(T) \), respectively. The \( \omega \)-limit set of \( x \) under \( T \), \( \omega(x, f) \), is the set \( \cap_{i=1}^{\infty} O(T^i(x), f) \). Two dynamical systems \((X, T), (Y, S)\) are said to be topologically conjugate if there is a homeomorphism \( h \in C(X, Y) \) such that \( h \circ T = h \circ S \).

A point \( x \in X \) is said to be almost periodic under \( T \) provided that for each neighborhood \( U \) of \( x \) there corresponds an \( m \in \mathbb{N} \) with the properties that in every set of \( m \) consecutive positive integers there appears an integer \( n \) such that \( T^n(x) \in U \). Denote the set of almost periodic points by \( AP(T) \). A subset \( A \) of \( X \) is said to be minimal under \( T \) if it is a nonempty, closed and invariant \((T(A) = A)\) subset of \( T \) and any proper subset of \( A \) does not satisfy the above three conditions.
Let \( \mathcal{B}(X) \) be the Borel \( \sigma \)-algebra generated by all open subsets of \( X \), \( M(X, T) \) be the set of \( T \)-invariant probability measures on \( \mathcal{B}(X) \) and \( E(X, T) \) be the ergodic elements of \( M(X, T) \). For \( \alpha \in M(X, T) \), \( T\alpha \in M(X, T) \) is defined by: \( T\alpha(B) = \alpha(T^{-1}B) \), \( \forall B \in \mathcal{B}(X) \). For \( \beta \in E(X, T) \), we define

\[
G(\beta) = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) = \int_X f \, d\beta, \forall f \in C(X, X) \right\}.
\]

We say that system \((X, T)\) is uniquely ergodic if \( \#M(X, T) = 1 \). Uniquely ergodic system \((X, T, \mu)\) is strictly ergodic if \( X \) is a minimal set of \( T \). That \((X, T, \mu)\) is an ergodic system means: \( \forall B \in \mathcal{B}(X) \) if \( T^{-1}B = B \) then \( \mu(B) = 0 \) or \( \mu(B) = 1 \).

For a uniquely ergodic system \((X, T, \mu)\), let

\[
\text{supp}(\mu) = \{ x \in X : \mu(V_x) > 0 \text{ for all neighborhood } V_x \text{ of } x \},
\]

then \( \text{supp}(\mu) \) is a minimal set of \( T \) [16].

Measure-preserving transformations \( T_1 \) on \((X_1, \mathcal{A}_1, \mu_1)\) and \( T_2 \) on \((X_2, \mathcal{A}_2, \mu_2)\) are spectrally isomorphic if there is a linear operator \( W : L^2(\mu_2) \to L^1(\mu_1) \) such that

1. \( W \) is invertible;
2. \( (Wf, Wg) = (f, g), \forall f, g \in L^2(\mu_2) \);
3. \( U_{T_1} W = W U_{T_2} \).

\( T_1 \) and \( T_2 \) are said to be isomorphic if there exist \( M_1, M_2 \in \mathcal{A}_1, \mathcal{A}_2 \) with \( \mu_1(M_1) = \mu_2(M_2) = 1 \) such that

1. \( T_1 M_1 \subseteq M_1, T_2 M_2 \subseteq M_2 \) and
2. there is an invertible measure-preserving transformation \( \phi : M_1 \to M_2 \) with \( \phi T_1(x) = T_2 \phi(x), \forall x \in M_1 \).

Recall that if \( X \) is a compact metric space, \( T \in C(X, X) \) and \( A \subseteq X \) is minimal, then the \( D \)-function of \( A \), \( T_A \), is \( n \), if \( A \) is a periodic orbit of \( T \) with period \( n \); is \( n' \) if \( A \) is not a periodic orbit of \( T \) but the number of distinct minimal sets for \( T^k \) contained in \( A \) is \( (n, k) \) for all \( k \in \mathbb{N} \); is a function from \( \mathbb{N} \) to \( \mathbb{N} \) such that for every \( k \in \mathbb{N} \), \( T_A(k) \) is the number of distinct minimal sets of \( T^k \) contained in \( A \) and \( T_A \) is not a bounded function.

The following lemmas are necessary in our proofs.

**Lemma 2.1** (Walters [24]). If \( T \) is a continuous transformation of compact metric space, then

1. \( E(X, T) \neq \emptyset \);
2. \( G(\alpha) \cap G(\beta) = \emptyset \), if \( \alpha \neq \beta \in E(X, T) \).

**Lemma 2.2** (Gottshalk [9]). If \( X \) is a metric space, \( f \in C(X, X) \) and \( x \in AP(f) \) then \( \omega(x, f) \) is a minimal set under \( f \); if \( X \) is a locally compact space, \( f \in C(X, X) \) and \( A \) is a minimal set of \( f \) then \( A \subseteq AP(f) \).
Lemma 2.3 (Ye [26]). Let $X$ be a compact metric space, $f \in C(X, X)$ and $A$ be a minimal set of $f$. Then

1. $f_A \neq \emptyset$;
2. $x \in \omega(f(x), f^k)$ for some $x \in A$ and some $k \in \mathbb{N}$ if and only if $f_A(k) = 1$;
3. $x \notin \omega(f^k(x), f^k)$ for some $x \in A$ and all $0 < i < k$ if and only if $f_A(k) = k$;
4. $T_A(nm) = T_A(n)(T_m)_A(m)$, where $A \subset X$ is a minimal set of $T$, $A_1 \subset A$ is a minimal set of $T^m$.

Lemma 2.4. Let $f \in C(I, I)$, $I = [0, 1]$. Suppose the period of every periodic orbit of $f$ is not an odd number except 1, $A$ is a minimal set of $f$ with $\# A > 2$, then there exists a fixed point $x_0 \in (\inf A, \sup A)$ such that $f^2(A_1) = A_1, f^2(A_2) = A_2$, where $A_1 = \{ x \in X: x < x_0 \}$, $A_2 = \{ x \in X: x > x_0 \}$.

Lemma 2.4 is a direct consequence of the results of [25, 15]. See [26, Corollary 4.2] for details.

3. D-function of uniquely ergodic system

To generalize Sharkovskii’s theorem on the coexistence of periodic orbits of interval mapping to uniquely ergodic subsystems, the first difficulty we meet is to describe these subsystems with some useful isomorphic invariant. We find that the $D$-function of a uniquely ergodic system which we will define below is such an invariant. To give the definition, we first present the following theorem.

Theorem 3.1. Let $X$ be a compact metric space, $T \in C(X, X)$ and $(X, T, \mu)$ be uniquely ergodic. Then for every $k \in \mathbb{N}$ there exists an integer $n$ dividing $k$ such that $\forall x \in E(X, T^k)$, $E(x, T^k) = \{ x, T^k x, \ldots, T^{n-1} x \}$ and $T^i x \neq T^j x$ if $1 \leq i < j \leq n - 1$.

Proof. By Lemma 2.1 $E(X, T^k) \neq \emptyset$. Let $x \in E(X, T^k)$ then $T^k x = x$. Now we suppose that $\beta \in E(X, T^k)$ and want to show that $\beta = T^i x$ for some $0 \leq i_0 \leq k - 1$. Because

$$\mu = \frac{1}{k} (\gamma + T \gamma + \cdots + T^{k-1} \gamma), \quad \forall \gamma \in E(X, T^k),$$

we have

$$\mu \left( \bigcup_{i=0}^{k-1} G(T^i x) \right) = \mu \left( \bigcup_{i=0}^{k-1} G(T^i \beta) \right) = 1$$

($T^j x \in E(X, T^k), \forall \gamma \in E(X, T^j)$). So there are $0 \leq j_1, j_2 \leq k - 1$ such that $G(T^{j_1} x) \cap G(T^{j_2} \beta) \neq \emptyset$. This implies that $\beta = T^i x$, $0 \leq i_0 \leq k - 1$.

Let $n = \min \{ 0 \leq i \in \mathbb{N}: T^i x = x \}$. It is obvious that $n \mid k$ and $E(x, T^k) = \{ x, T^i x, \ldots, T^{n-1} x \}$. □
**Definition 3.2.** A D-function of a uniquely ergodic system \((X, T, \mu)\), \(T\mu\): is \(n\) if \(\text{supp}(\mu)\) is a periodic orbit of \(T\) with period \(n\); is \(n'\) if \(\text{supp}(\mu)\) is not a periodic orbit of \(T\) but the number of ergodic components of \(T^k\) is \((n, k)\) for all \(k \in \mathbb{N}\); is a function from \(\mathbb{N}\) to \(\mathbb{N}\) such that \(T^k\mu(k)\) is the number of ergodic components of \(T^k\) for all \(k \in \mathbb{N}\) and \(T\mu\) is not a bounded function.

**Remark 3.3.** By Lemma 2.1 the number of ergodic components for \(T^k\) is equal to the integer \(n\) obtained in Theorem 3.1. So we see that \(T\mu\) is well defined. For convenience we also write \(T\mu(k) = (n, k)\), \(\forall k \in \mathbb{N}\) if \(T\mu = n'\).

**Remark 3.4.** A function \(s : \mathbb{N} \to \mathbb{N}\) belongs to \(\mathcal{S}\) if and only if there exists a function \(t\) from the set of all prime numbers to the set \(\mathbb{N} \cup \{0, \infty\}\), such that for every \(k \in \mathbb{N}\), if \(p_1^{a_1}p_2^{a_2} \cdots p_n^{a_n}\) is the decomposition of \(k\) into prime factors, then

\[
s(k) = \prod_{1}^{n} p_i^{\min\{a_i, t(p_i)\}}
\]

or equivalently \(s \in \mathcal{S}\) if and only if: (1) \(s(mn) = s(m)s(n)\), if \((m, n) = 1\); (2) for prime number \(p\) either \(s(p^l) = p^l\) for all \(l \in \mathbb{N}\) or there exists \(l_0 \in \mathbb{N} \cup \{0\}\) such that \(s(p^l) = (p^l, p^{l_0})\) for all \(l \in \mathbb{N}\). In the future we will use any equivalent statement of \(\mathcal{S}\) for convenience.

**Theorem 3.5.** \(T\mu \in \mathcal{S}\).

**Proof.** The proof of this theorem is similar to the proof of Theorem 3.8 of [26].

**Theorem 3.6.** Let \((X_i, T_i, \mu_i)\) be uniquely ergodic systems, \(i = 1, 2\). If \(T_1\) is spectrally isomorphic to \(T_2\), then \(T_{1 \mu_1} = T_{2 \mu_2}\).

**Proof.** By the properties of \(\mathcal{S}\) it is enough for us to show that if \(T_{2 \mu_2}(n) = n\) for some \(n \in \mathbb{N}\), then \(T_{1 \mu_1}(n) = n\). Because \(T_1\) is spectrally isomorphic to \(T_2\), there is an isomorphic from \(L^2(\mu_2)\) to \(L^2(\mu_1)\) satisfying \(U_{T_1} \cdot W = W \cdot U_{T_2}\). Suppose for some \(n \in \mathbb{N}\), \(T_{2 \mu_2}(n) = n\). Let \(\alpha \in E(X_2, T_2)\), \(B_i = G(T_{2 \alpha}^i), 0 \leq i \leq n - 1\) then \(B_i \in \mathcal{B}(X_2)\) and

\[
T_2(B_i) = B_{i+1}, \quad T^{-1}_2(B_{i+1}) = B_i, \quad B_i \cap B_j = \emptyset \quad \text{if} \ 1 \leq i < j \leq n - 1.
\]

By the condition satisfied by \(W\) we get \(U_{T_2}^*W = WU_{T_2}^*\). Hence

\[
W(B_i) = W(U_{T_2}^*B_i) = (U_{T_1}^*W)(B_i) = W(B_i)T_{T_1}^*.
\]

Therefore,

\[
W(B_i) = C_j \text{ a.e. for every } \gamma_j \in E(X_1, T_1), 1 \leq j \leq T_{1 \mu_1}(n) = k.
\]
From

\[ \langle W(1_{B_i}), W(1_{B_j}) \rangle = \langle 1_{B_i}, 1_{B_j} \rangle = \frac{1}{n}, \]

\[ \langle W(1_{B_i}), W(1_{B_j}) \rangle = \langle 1_{B_i}, 1_{B_j} \rangle = 0, \quad 1 \leq i < j \leq n - 1, \]

\[ \mu_1 = \frac{1}{k}(\gamma_1 + \cdots + \gamma_k), \]

we have

\[ \sum_{j=1}^{k} |C_{ij}|^2 = \frac{k}{n}, \quad 0 \leq i \leq n - 1, \]

\[ \sum_{j=1}^{k} C_{ij} = 0, \quad 0 \leq i_1 < i_2 \leq n - 1. \]

If \( k < n \), it contradicts the fact that if we denote

\[ a_i = \sum_{j=1}^{n} C_{ij}, \]

then \( |a_i| = 1, \ a_i \tilde{a}_j = 0, \quad 0 \leq i < j \leq n - 1. \) Because \( k \leq n \) we get \( k = n \). \( \square \)

Remark 3.7. Because \( \{ e^{2\pi ij/T_{n,n}} : n \in \mathbb{N}, 0 \leq j \leq T_{n,n}(n) - 1 \} \) is only a subset of the spectrum of the uniquely ergodic system \((X, T, \mu)\), the inverse of Theorem 3.6 is not true in the general case.

Let \( \mathcal{Y} \) be the set defined in the introduction. We naturally ask the following question: Given \( s \in \mathcal{Y} \) does there exist a compact metric space \( (X, T, \mu) \) such that \((X, T, \mu)\) is uniquely ergodic and \( T_{\mu} = s \)? The answer is positive. We have the following theorems that we will prove in Section 5.

Theorem 3.8. For every \( s \in \mathcal{Z} \setminus E \), there exists a strictly ergodic system \((A, \sigma, \mu_s)\) which is a subsystem of \((\Sigma_M, \sigma)\) such that \( \sigma_{\mu_s} = s \).

Theorem 3.9. For every \( n \in \mathbb{N} \), there exists a strictly ergodic system \((A_n, \sigma, \mu_n)\) which is a subsystem of \((\Sigma_M, \sigma)\) such that \( \sigma_{\mu_n} = n' \).

To prove the existence of a uniquely ergodic system with a given \( D \)-function of interval mapping we need the following theorem.

Theorem 3.10. Let \( X, Y \) be compact metric spaces, \( T \in C(X, X) \) and \( S \in C(Y, Y) \). There is a measurable transformation \( \varphi : X \rightarrow Y \) satisfying \( S \varphi = \varphi T \) and \( \varphi(B) \in \mathcal{B}(Y) \) for every \( B \in \mathcal{B}(X) \). \( \varphi \) is one-to-one except for a countable set on which \( \varphi \) is countable-to-one. If \((Y, S)\) is a uniquely ergodic system and \( \text{supp}(\gamma) \) is not a periodic orbit, then \((X, T)\) is also a uniquely ergodic system and \( T_{\mu} = S_{\gamma} \), where \( \{\mu\} = M(X, T) \), \( \{\gamma\} = M(Y, S) \).

Proof. It is easy to see that \( \gamma(B) = \mu(\varphi^{-1}(B)) \), for every \( B \in \mathcal{B}(Y) \) and \( \mu \in M(X, T) \). Hence for every \( B \in \mathcal{B}(X) \), \( \mu \in M(X, T) \) we have \( \mu(B) = \mu(\varphi^{-1}(\varphi(B))) = \gamma(\varphi(B)) \). This
implies that $\# M(X, T) = 1$. Suppose $M(X, T) = \{\mu\}$, then by the properties of $D$-function we only need to prove:

1. if $S_\mu(n) = n$, then $T_\mu(n) = n$;
2. if $T_\mu(n) = n$, then $S_\mu(n) = n$.

We only show that (1) and (2) are similar.

Suppose $S_\mu(n) = n$ for some $n \in \mathbb{N}$. Then for $\alpha \in E(Y, S^n)$, $E(Y, S^n) = \{x, S(x), \ldots, S^{n-1}(x)\}$ by Theorem 3.1. Let $G(i) = G(S'(x))$, $0 \leq i \leq n - 1$, then

$$G(i) \in \mathfrak{B}(Y) \text{ and } G(i) \cap G(j) = \emptyset \text{ if } i \neq j.$$ 

Furthermore

$$S(G(i)) = G(i + 1), S^{-1}(G(i)) = G(i - 1).$$ 

Let $X_i = \varphi^{-1}(G(i))$, then $X_i \in \mathfrak{B}(X)$ and $T^{-1}(X_i) = X_{i-1}$. Hence $\mu(X_i) = 1/n$, $0 \leq i \leq n - 1$.

Let $\beta \in E(X, T^n)$. Then $E(X, T^n) = \{\beta, T(\beta), \ldots, T^{n-1}(\beta)\}$ and $\beta = T^k(\beta)$. That is to say for every $B \in \mathfrak{B}(X)$, $\beta(B) = \beta(T^{-k}(B))$. Hence $\beta(X_i) = \beta(T^{-k}(X_i))$ for every $0 \leq i \leq n - 1$. Because $T^{-n}(X_i) = X_i$, we have $\beta(X_i) = 0$, or 1, $0 \leq i \leq n - 1$. Let $\beta(X_{i_0}) = 1$, then $1 = \beta(X_{i_0}) = \beta(T^{-k}(X_{i_0})) = (X_{i_0 - k (mod n)}).$ By the fact that $\beta(X) = 1$, $\beta(X_i \cap X_j) = 0$ if $i \neq j$, we get $k = n$.

4. The proof of Theorem 4.5

In this section we will prove the coexistence of uniquely ergodic subsystems of interval mapping. At first we show the following theorem.

**Lemma 4.1.** Let $(X, T, \mu)$ be a uniquely ergodic system. If for some $n_0 \in \mathbb{N}$, $T_{supp(\alpha)}(n_0) = n_0$, then $T_\mu(n_0) = n_0$. Furthermore $T_{supp(\alpha)}(n) | T_\mu(n)$ for every $n \in \mathbb{N}$.

**Proof.** Obviously every ergodic component of $T^k$ is contained in some minimal components of $T^k$. So if $T_{supp(\alpha)}(n_0) = n_0$, then $T_\mu(n_0) = n_0$. Hence it is enough for us to show

$$T_{supp(\alpha)}(n) | T_\mu(n) \text{ for every } n \in \mathbb{N}. \quad (*)$$

Let $p$ be a prime number. If $T_{supp(\alpha)}(p^l) = p^l$ for every $l \in \mathbb{N}$, then $(*)$ is true. Hence we may assume that there exists $l_0 \in \mathbb{N} \cup \{0\}$ such that $T_{supp(\alpha)}(p^l) = (p^l, p^{l_0})$ for every $l \in \mathbb{N}$. Then

$$T_\mu(p^l) = T_{supp(\alpha)}(p^l) = p^l \text{ if } l \leq l_0; \quad T_{supp(\alpha)}(p^l) = p^{l_0} | T_\mu(p^l) \text{ if } l > l_0 + 1.$$

This implies that for every $n = p_1^{m_1} \cdots p_m^{m_m} \in \mathbb{N}$,

$$T_{supp(\alpha)}(n) = \prod_{i=1}^{m} T_{supp(\alpha)}(p_i^{m_i}) | \prod_{i=1}^{m} T_\mu(p_i^{m_i}) = T_\mu(n),$$

where $p_1 < \cdots < p_m$ are prime numbers. □
Lemma 4.2. For a uniquely ergodic system \((X, T, \mu)\), \(n, m \in \mathbb{N}\), if \(T_{\text{supp}(\mu)}(n) = n\) we have \(T_{\mu}(nm) = T_{\mu}(n)(T^n)_{\gamma} (m)\), where \(\gamma \in M(A_1, T^n)\), and \(A_1 \subset A\) is a minimal set of \(T^n\).

Proof. Because \(T_{\text{supp}(\mu)}(n) = n\) we know that \(T_{\mu}(n) = n\). Let

\[
E(X, T^n) = \{x, T^n x, \ldots, T^{n-1} x\},
\]

\[
E(X, T^{nm}) = \{\beta, T^n \beta, \ldots, T^{n-1} \beta\},
\]

where \(k = T_{\mu}(nm)\) and \(m_0 = \min \{i : T^i \beta = \beta, 0 < i \leq m\}\). Obviously

\[
(1/m_0) \{\beta + T^n \beta + \ldots + T^{n(m_0-1)} \beta\} \in E(X, T^n).
\]

Let \(C_0 = \{\beta, T^n \beta, \ldots, T^{n(m_0-1)} \beta\}\). Then \(E(X, T^{nm}) = \bigcup_{i=0}^{n-1} T^i C_0\). It implies that \(T_{\mu}(nm) = k = nm_0 = T_{\mu}(n)m_0\). By the definition of \(m_0\), \(m_0 = (T^n)_{\gamma}(m)\), \(\gamma \in E(A_1, T^n)\), \(A_1 \subset A\) is a minimal set of \(T^n\).

Lemma 4.3. Let \(X\) be a compact metric space, \(A \subset X\) be a minimal set of \(f \in C(X, X)\), \(A = A_1 \cup A_2\), \(A_1 \cap A_2 = \emptyset\), \(A_i\) be minimal set of \(f^2\), and \(i = 1, 2\). If \((A_i, f^2)\) is strictly ergodic for some \(i \in \{1, 2\}\), then \((A, f)\) is also strictly ergodic.

Proof. Without loss of generality, we suppose that \((A_1, f^2)\) is strictly ergodic. At first we show that \((A_2, f^2)\) is also strictly ergodic. If this is not the case, then there exist \(\mu_1, \mu_2 \in E(A_2, f^2)\) such that \(\mu_1(C) = 1, \mu_2(C) = 0\), where \(C = G(\mu_1) \subset A_2, f^{-2}(C) = C\). Because \(\mu_1\) is regular, there exists a closed set \(C_1 \subset C\) such that \(\mu_1(C_1) \geq 1/2\) [24].

For \(B \in \mathcal{B}(A_1)\), let \(\tilde{\mu}_i(B) = \mu_i(f^{-1}(B))\) then \(\tilde{\mu}_i \in M(A_1, f^2)\), \(i = 1, 2\). As \(f(C_1) \in \mathcal{B}(A_1)\), therefore

\[
\tilde{\mu}_1(f(C_1)) = \mu_1(f^{-1}f(C_1)) \geq \mu_1(C_1) \geq \frac{1}{2},
\]

\[
\tilde{\mu}_2(f(C_1)) = \mu_2(f^{-1}f(C_1)) \leq \mu_2(C) = 0,
\]

where \(f^{-1}f(C_1) \subset f^{-1}f(C) = f^{-1}f^{-1}(C) = f^{-2}(C) = C\). This is a contradiction to the strict ergodicity of \((A_1, f^2)\). Hence \((A_2, f^2)\) is also strictly ergodic. Now let \(x_1, x_2 \in M(A, f)\), then \(x_i(B) = 2x_i(A_j \cap B) \in M(A_j, f^2)\), \(1 \leq i, j \leq 2\). Therefore for \(B \in \mathcal{B}(A)\)

\[
x(B) = x_1(B \cap A_1) + x_1(B \cap A_2) + \frac{1}{2} x_1^2(B) + \frac{1}{2} x_2^2(B) = \frac{1}{2} x_1^2(B) + \frac{1}{2} x_2^2(B).
\]

This implies that \((A, f)\) is strictly ergodic.

Lemma 4.4. Let \(f \in C(I, I)\) and any period of the periodic orbit of \(f\) be a power of 2. Then for any uniquely ergodic subsystem \((X, f, \mu)\) we have \(f_n \in Y^2_{\infty}\) or \(f_n = 2^n\) for some \(n \in \mathbb{N} \cup \{0\}\).

Proof. Without loss of generality, we suppose that \(\text{supp}(\mu)\) is not a periodic orbit of \(f\). By [18] \((X, f, \mu)\) is isomorphic to the generalized adding machine \((X_\mathcal{Q}, D)\), where \(X_\mathcal{Q} = \{0, 1\}^{\mathcal{Q}}\). So \(f_n \in Y^2_{\infty}\) by Theorems 5.2 and 5.4.
Theorem 4.5. (a) If \( f \in \mathcal{C}(I, I) \), \( I = [0, 1] \), then \( DF(I, f) = \emptyset(n) \), \( n \in \mathbb{N} \cup \{ \infty \} \cup 2^\infty \),
(b) If \( n \in \mathbb{N} \cup \{ \infty \} \cup 2^\infty \), then there exists \( f \in \mathcal{C}(I, I) \), such that \( DF(I, f) = \emptyset(n) \).

Proof. We will prove Theorem 4.5(a) in the following steps:

1. If \( f \) has a uniquely ergodic subsystem \((X, f, \mu)\) with \( f_\mu \in \mathcal{Y}_i \) for some \( 0 \leq i < \infty \), then \( f \) must have a periodic orbit with period \( 2^i q \) \((q \geq 3\) is odd). If this is not the case, then the biggest period of the periodic orbit of \( f \) according to Sharkovskii’s order is \( 2^{i+1} q \) \((q \geq 3\) is odd and \( j \geq 1\), or any period of the periodic orbit of \( f \) is a power of 2. By Lemma 4.4 the latter case is impossible. Repeatedly using Lemma 2.4 we get \( f_\mu(2^{i+1}) = 2^{i+1} \) for every minimal set \( A \) of \( f \) with \( \#A \geq 2^{i+1} \). By Lemma 4.1, \( f_\mu(2^{i+1}) = f_\mu(2^{i+1}) = 2^{i+1} \). This implies that \( f_\mu \notin \mathcal{Y}_i \). Hence \( f \) must have a periodic orbit with period \( 2^i q \) \((q \geq 3\) is odd).

2. If the biggest period of the periodic orbit of \( f \) according to Sharkovskii’s order is \( q \) \((q \geq 3\), odd\), then for every \( s \in \mathcal{Y} \setminus E \), there exists a uniquely ergodic system \((X, f, \mu)\) such that \( f_\mu = s \).

3. If the biggest period of the periodic orbits of \( f \) according to Sharkovskii’s order is \( 2^i q \) \((q \geq 3\), odd\), then for every \( s \in \mathcal{Y}_i \) there exists a uniquely ergodic subsystem \((X, f, \mu)\) such that \( f_\mu = s \).

Let \( g = f^2 \), then \( g \) has a periodic orbit of period \( q \). By step (2) there exists a uniquely ergodic subsystem \((X, g, \mu)\) such that \( g_\mu = s' \), where \( s = s' \cdot s_i \) and \( s_i \in \mathbb{Y}_i \), \( s_i : \mathbb{N} 
\Rightarrow \mathbb{N} \) satisfies \( s_i(2^j m) = (2^j, 2^j) \), \( j \geq 0, m \) odd. Let \( X = \bigcup_{j=0}^{2^j} f^j(supp(\mu)) \). By Lemma 4.3, \((X, f)\) is strictly ergodic. Let \( \{ \mu \} = E(X, f) \). If \( p \) is odd, then

\[ f_\mu(p)f_\mu(2^i) = f_\mu(p2^i) = f_\mu(2^i)g_\mu(p) \]

This implies \( f_\mu(p) = g_\mu(p) \). At the same time

\[ f_\mu(2^{i+1}) = f_\mu(2^i)g_\mu(2) = f_\mu(2^i) = 2^{i+1} \]

(by Lemmas 4.2 and 2.4). Hence \( f_\mu = s \).

4. If the periods of periodic orbits are not all power of 2, then for every \( s \in \mathcal{Y}_\infty \cup \mathcal{Y}_2^\infty \), there exists a uniquely ergodic subsystem \((X, f, \mu)\) such that \( f_\mu = s \).

Let \( 2^q \) \((q \geq 3\), odd\) be the period of some periodic orbit of \( f \) and \( g = f^{2^q} \). Then for every \( s \in \mathcal{Y}_\infty \cup \mathcal{Y}_2^\infty \), there exists a uniquely ergodic subsystem \((X, g, \mu)\) such that \( g_\mu = s \). Let \( X = \bigcup_{j=0}^{2^q} f^j(supp(\mu)) \), then \((X, f)\) is the uniquely ergodic subsystem that we need.

5. If \( f \) has a uniquely ergodic subsystem \((X, f, \mu)\) such that \( f_\mu \in \mathcal{Y}_2^\infty \), then for every \( n \in \mathbb{N} \cup \{ 0 \} \), \( f \) has a periodic orbit of period \( 2^n \).
If this is not the case, then there exists \( n_0 \) such that any period of the periodic orbit of \( f \) is an element of set \( \{ 2^i : 0 \leq i \leq n_0 \} \) (Sharkovskii's order). Let \( g = f^{2^{n_0}} \), then \( P(f) - P(g) - F(g) = F(f) \) [7]. This implies \( \text{supp}(\mu) \subset P(f) = P(f) \). Hence \( f_\mu(2^{n_0+1}) = 2^{n_0} \). This contradicts \( f_\mu \in Y_\omega^2 \).

(6) If \( f \) has a uniquely ergodic subsystem \((X, f, \mu)\) with \( f_\mu \in Y_n \), then there exists a uniquely ergodic subsystem \((X, f, \mu)\) such that \( f_\mu \in Y_\omega^2 \).

At first we claim that period orbits of \( f \) are not all power of 2. If this is not the case, by Lemma 4.4 we know that \( \text{supp}(\mu) \subset P(f) = P(f) \). Hence \( f_\mu(2^m) = 2^m \). This contradicts \( f_\mu \in Y_\omega^2 \).

By step (4), (6) holds.

Combining (1)-(6) and using Sharkovskii's theorem we conclude that \( D(1, f) = \mathcal{Y}(n), n \in \mathbb{N} \cup \{ \infty \} \cup 2^\infty \).

Proof of Theorem 4.5 (b). It is a well-known fact that for every \( n \in \mathbb{N} \), there exists \( f \in C(I, I) \) such that \( \{ K : k \text{ is a period of the periodic orbit of } f \} = \{ n \} \cup \{ k : n \cdot r \leq k, k \in \mathbb{N} \} \) or \( \{ 2^l : l = 0, 1, \ldots, n-1 \} \), see [1]. So by the above argument, it remains to show that there exists \( f \in C(I, I) \) such that \( D(1, f) = \mathcal{Y}(n) \) or \( \mathcal{Y}(2^\infty) \). Any chaotic function in the sense of Li–York with zero topological entropy is the function with \( D(1, f) = \mathcal{Y}(\infty) \) [22]. Any function which satisfies \( \bar{P}(f) = P(f) \) and \( \mathcal{Y}(2^\infty) \subset D(1, f) \) is an example that \( D(1, f) = \mathcal{Y}(2^\infty) \).

The proof is complete.

5. Uniquely ergodic system with a given D-function

In order to prove Theorem 4.5, we give examples of uniquely ergodic systems with given \( D \)-functions in a subshift of finite type \( \Sigma_M \) (see [10] for the construction of strictly ergodic systems with given entropy). Let

\[
M = (m_{ij}) = \begin{pmatrix}
1 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}_{p \times p}
\]

\( \Sigma_M = \{ x = (x_1, x_2, \ldots) : m_{x_j x_{j+1}} = 1, 0 \leq x_j \leq p - 1, \text{ for all } j \in \mathbb{N} \} \),

\( \sigma : \Sigma_M \to \Sigma_M \) is the shift defined by \( (\sigma(x))_j = x_{j+1} \) for \( x \in \Sigma_M, j \in \mathbb{N} \).

Recall that \( \Sigma_M \) with product topology is a compact metrizable space. Let \( P = \{ 0, 1, \ldots, p - 1 \} \). For \( B \in P^r \) we call \( B \) a block over \( P \) of length \( r \) and write \( l(B) = r \). If \( A = (a_1 \cdots a_n), B = (b_1 \cdots b_m) \) are two blocks, then we denote \( AB = (a_1 \cdots a_n b_1 \cdots b_m), A^\infty = AAA \cdots = (a_1 \cdots a_n a_1 \cdots a_n \cdots) \).
**Definition 5.1.** Let \( Q = \{ q_i \} \) \( i = 1, \ldots, q_i \geq 2 \) \( i = 1, \ldots, q_i \geq 2 \) and \( X_Q = \prod_{i=1}^{\infty} X_{q_i} \). A transformation \( D \) from \( X_Q \) into \( X_Q \) is defined as follows: \( D(q_1 - 1, q_2 - 1, \ldots, q_n - 1, x_{n+1}, \ldots) = (0, 0, \ldots, 0, x_{n+1} + 1, \ldots) \) if \( x_{n+1} < q_{n+1} - 1 \), \( n \in \mathbb{N} \setminus \{ 0 \} \). In particular \( D(q_1 - 1, q_2 - 1, \ldots) = (0, 0, \ldots) \). The couple \( (X_Q, D) \) is called a generalized adding machine (GAM for short). It is a well-known fact that \( (X_Q, D) \) is strictly ergodic.

**Theorem 5.2.** Let \( (X_Q, D) \), \( Q = \{ q_i \} \) be a GAM. If \( q_i = p_i^{a_i(1)} \cdot p_i^{a_i(2)} \cdots p_i^{a_i(r_i)} \) is the decomposition \( q_i \) into prime factors, then for every prime number \( p = p_j \), \( D_{X_Q}(p') = p' \) for all \( i \in \mathbb{N} \) if \( \sum_{i=1}^{\infty} a_i(j) = \infty \), or, \( D_{X_Q}(p') = (p', p^c) \) if \( \sum_{i=1}^{\infty} a_i(j) = l_0 \). More simply, for every \( n \in \mathbb{N} \), \( D_{X_Q} \) is the limit of \( (q_1, \ldots, q_j) \), where \( p_1 < p_2 < \cdots \) are all prime numbers \( a_i(r_i) > 0 \).

**Proof.** Note that for each \( j \), \( D \) permutes cyclically all \( \prod_{i=1}^{j} \) cylinders of length \( j \). That is to say \( D_{X_Q}(q_1 \cdots q_j) = q_1 \cdots q_j \) for every \( j \in \mathbb{N} \). So if \( n | q_1 \cdots q_j \) for some \( n, j \in \mathbb{N} \) then \( D_{X_Q}(n) = (n, D_{X_Q}(q_1 \cdots q_j)) = n \).

Let \( p \) be a prime number such that \( p^l \mid q_1 \cdots q_j \) for \( j \) large enough and \( p^l + 1 \notq_1 \cdots q_j \) for all \( j \in \mathbb{N} \). Then \( (p, q_1 \cdots q_j / p^c) = 1 \). So for \( j \) large enough there are \( k_j \) such that \( q_1 \cdots q_j / p^c = 1 + k_j p \). This implies \( q_1 \cdots q_j = p^c + k_j p^{l+1} \). Because \( D^{q_1 \cdots q_j}((0, 0, \ldots)) = (0, 0, \ldots, 0, 10, \ldots) \) we conclude that \( 0 \in \omega(D^{p^c + k_j p^{l+1}}) \). Consequently, \( D_{X_Q}(p^{c+1}) = p^c \).

By combining the results obtained above, we get the theorem. \( \square \)

**Lemma 5.3.** \( (X_Q, D) \) is topologically conjugate to \( (X_Q', D) \) if and only if they have the same \( D \)-functions.

**Proof.** See [2, p. 417] and use Theorem 5.2. \( \square \)

**Lemma 5.4.** Let \( M(X_Q, D) = \{ \mu \} \); then \( D_{X_Q} = D_{\mu} \).

**Proof.** Let \( p \) be a prime number. If \( D_{X_Q}(p') = p' \), \( \forall \in \mathbb{N} \) then \( D_{\mu}(p') = D_{X_Q}(p') = p' \). So without loss of generality we suppose there is \( l_0 \) such that \( D_{X_Q}(p') = (p', p^c), \forall \in \mathbb{N} \). By Lemma 4.2 \( D_{\mu}(p') = D_{X_Q}(p') = p', \forall \in \mathbb{N} \). So it is enough to say that \( D_{\mu}(p^{c+1}) = p^c \). Let \( A = \omega(0, D^{p^{c+1}}), \) where \( 0 = (0, 0, \ldots) \). By [8] \( D^{p^{c+1}} \) is a rotation on the compact metrizable group and \( (A, D^{p^{c+1}}) \) is minimal. Therefore it is strictly ergodic [24, p. 162]. This implies that \( D_{\mu}(p^{c+1}) = p^c \). \( \square \)

Now we are ready to prove Theorem 3.8.

**Proof of Theorem 3.8.** Let \( s \in \mathcal{E} \cap E \), and \( p_{l_0} \) be a prime number such that \( s(p_{l_0}) = p_{l_0} > p \). Let \( s_1 : \mathbb{N} \to \mathbb{N} \) satisfying \( s_1(p') = s(p') \), if \( p \neq p_{l_0} \) or \( s(p_{l_0}') = p_{l_0}', \forall l \in \mathbb{N} \); \( s_1(p_{l_0}') = (p_{l_0}', p_{l_0}' - 1) \) if \( s(p_{l_0}) = (p_{l_0}', p_{l_0}' - 1), \forall l \in \mathbb{N} \). Suppose \( A \) is a GAM such that \( D_{X_Q} = s_1 \).
By the theorem of Jewett and Krieger [24, p. 161] there exists $A_1 \subset \Sigma_2$ such that $(A_1, \sigma)$ is strictly ergodic and is isomorphic to $(X_0, D)$. So $\sigma_\mu = s_1, \{\mu\} = M(A_1, \sigma)$.

Now let $\theta$ be a bijection from $\{0, 1, \ldots, p-1\}$ to $\{0, 1, \ldots, p-1\}^{p_0}$ such that

$$\theta(0) = 0 \cdots 0 \cdots 0, \theta(1) = 0 \cdots 0 \cdots (p-1).$$

Let $x \in A_1$. The induced transformation $\theta_* : A_1 \to \omega(\theta_*(x), \sigma^{p_0})$ is defined by $\theta_*(y) = \theta(Y_1) \theta(Y_2) \cdots$ if $y \in A_1$ and $y = Y_1 Y_2 \cdots, l(Y_i) = p_\omega, 1 \leq i < \infty$. $\theta_*$ is one-to-one and continuous. It is easy to check that $\theta_* \sigma = \sigma^{p_0} \theta_*$. So by Lemma 4.2 and Theorem 3.6 we immediately get $\sigma_{A_1} = s$, where $A_1 = \omega(\theta(x), \sigma)$.

**Proof of Theorem 3.9.** For $n \in \mathbb{N}$ choose an irrational number $\beta$ such that $0 < \beta < 1/p$.

We define

$$x = \sum_{i=1}^{p_1} x_i^{-1}$$

such that $x_i = 0$ if $i \beta \in [k(i)(1 + \beta), k(i)(1 + \beta) + 1)$; $x_i = 1$ if $i \beta \in [k(i)(1 + \beta) + 1, (k(i) + 1)(1 + \beta))$, $k(i) \in \mathbb{N} \cup \{0\}$. Then $\omega(x, \sigma)$ is the Sturmian dynamical system defined by Hedlund [11, 13]. By the choice of $\beta$ we know $x = C_1 C_2 \cdots$, where $C_i = 0 \cdots 01$ and $l(C_i) = k$ or $k+1$, $k > p$.

Let $\varphi$ be the bijection from $\{0, 1, \ldots, p-1\}^k$ onto $\{0, 1, \ldots, p-1\}$ satisfying

$$\varphi(0 \cdots 01 0 \cdots 0) = \begin{cases} 0 \cdots 0 \cdots (p-1)0 \cdots 0, & \text{if } i > p-1, \\ (p-1) \cdots (p-1)0 \cdots 0 \cdots (p-1), & \text{if } i \leq p-1, \\ (0 \cdots 01 \cdots (p-2)). & \end{cases}$$

Denote $\tilde{x} = \varphi_*(x)$, where $\varphi_*$ is the induced transformation. Then $\varphi_*$ is a topological conjugation between $(\omega(x, \sigma), \sigma)$ and $(\omega(\tilde{x}, \sigma), \sigma)$. Hence $(\omega(x, \sigma), \sigma)$ is a strictly ergodic system and $\sigma_{\mu_\theta} = 1'$ by the fact that $(\omega(x, \sigma), \sigma)$ is a strictly ergodic system and $\sigma_{\mu_\mu} = 1'$, where $\{\mu_\mu\} = M(\omega(y, \sigma), \sigma)$, $y = x$ or $\tilde{x}$.

Now let $\psi = \psi_n$ be a mapping from $\{0, 1, \ldots, p-1\}$ into $\{0, 1, \ldots, p-1\}^n$ satisfying

$$\psi(0) \psi(1) \cdots \psi(p-1) = 0 \cdots 0 \cdots (p-1).$$

We have the following commutative graph:

$$\begin{array}{ccc}
\omega(x, \sigma) & \xrightarrow{\sigma} & \omega(x, \sigma) \\
\downarrow \psi_* & & \downarrow \psi_* \\
\omega(x', \sigma^n) & \xrightarrow{\sigma^n} & \omega(x', \sigma^n)
\end{array}$$

where $x' = \psi_*(\tilde{x})$. 
We prove that $\psi_*$ is one-to-one. Let $z \neq y \in \omega(\tilde{x}, \sigma)$, then there exists $i_0 \in \mathbb{N}$ such that $z_i = y_i, 1 \leq i \leq i_0$, $z_{i_0 + 1} \neq y_{i_0 + 1}$. By the construction of $\psi_*(\tilde{x})$ there are two possibilities:

1. $z_{i_0 + 1} = 1, y_{i_0 + 1} = 0$;
2. $z_{i_0 + 1} = 0, y_{i_0 + 1} = 1$.

In the first case $z_{i_0 + i} = y_{i_0 + i}, 1 \leq i \leq p - 1$. If $y_{i_0 + i} = 0, 1 \leq i \leq p - 1$, then $\psi_*(z) \neq \psi_*(y)$. If $y_{i_0 + i}$ for some $2 \leq i \leq p$, then

$$
(\psi_*(z))_{n(i_0 + p - 1)} = p - 1 \neq (\psi_*(y))_{n(i_0 + p - 1)}.
$$

That is, $\psi_*(z) \neq \psi_*(y)$. Case (2) is similar to (1). So we conclude that $\psi_*$ is one-to-one, onto and satisfies $\psi_* \sigma = \sigma^n \psi_*$, and so is a topological conjugation between $(\omega(\tilde{x}, \sigma), \sigma)$ and $(\omega(x', \sigma^n), \sigma^n)$.

Hence $(\sigma^n)_{\mu} = \sigma_{\mu'} = 1'$, where $\{\mu'\} = M(\omega(x', \sigma^n), \sigma^n)$, where $A_n = \omega(x', \sigma^n)$, then $(A_n, \sigma)$ is a strictly ergodic system. So we have

$$
\sigma_{\mu}(m) = (\sigma^n)_{\mu'}(m)\sigma_{\mu}(n) = \sigma_{\mu}(n) = m
$$

for every $m \in \mathbb{N}$ by the fact $\sigma_{A_n}(n) = n$ and Lemma 4.2 and Theorem 3.6, where $\{\mu'\} = M(A_n, \sigma)$. This implies that $\sigma_{\mu}(m) = (m, n)$ for all $m \in \mathbb{N}$. In other words $\sigma_{\mu} = n'$.

To sum up we get the following theorem.

**Theorem 5.5.** Let $\Sigma_M$ be a one-sided subshift of finite type with a $k \times k$ matrix $M = (m_{ij})$ satisfying $m_{01} = m_{12} = \cdots = m_{k-1,0} = m_{00} = 1$ and $\sigma$ the left shift. Then $DF(\Sigma_M, \sigma) = (E \setminus \emptyset)$.

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Uniquely ergodic subsystems