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Dynamical Behavior of Quantum Correlation Entropy Under the Noisy Quantum Channel for Multiqubit Systems

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Abstract

Quantum correlation entropy is used to measure total non-classical correlation of multiple states. It is based on a local coarse-grained measurement. Quantum noisy processes have a theoretically and experimentally important role in quantum information tasks. We study the dynamical behavior of quantum correlation entropy of output state under the effected of the $bit\text{-}flip$ channel, $phase\text{-}flip$ channel and $bit\text{-}phase$ flip channel. We find that quantum correlation entropy of output state exists the frozen phenomenon and the phenomenon of sudden death. The phenomenon of sudden death shows that under the influence of a noisy channel, the output state becomes a classically correlated state.

Keywords Observational entropy · Quantum correlation entropy · Noisy quantum channel · Frozen phenomenon · Sudden death

1 Introduction

The research on observational entropy has attracted the wide interest of many scholars, and highly significant results are obtained [1–6]. Quantum correlation entropy is defined as the difference between the infimum local observational entropy and von Neumann entropy, where the infimum is taken over all local coarse-grainings. Quantum correlation entropy, which depends on a partition into subsystems and quantifies the additional uncertainty in a multipartite system with local coarse-grained measurement, is a measure of total non-classical correlation [7]. It can be regarded as a natural generalization of entanglement entropy to mixed state and multipartite system [7]. There are numerous properties for quantum correlation entropy [7], such as additive over independent system, invariant under local unitary operation, and reduces to the entanglement entropy for bipartite pure state. In finite dimensional system, $S^{QC}_{A_1A_2\cdots A_N}(\rho) = 0$ if and only if ρ is a classically correlated state [7].

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Quantum correlation entropy of ρ with a local coarse-graining $\mathcal{C}$ is given by [7]

$$S^{QC}_{A_1A_2\cdots A_N}(\rho) = \inf_{\mathcal{C}} S_{O(\mathcal{C})}(\rho) - S(\rho). \tag{1}$$

where ρ is a N-partite quantum state associated with subsystems $A_1, A_2, \cdots, A_N$. The infimum is taken over all local coarse-grainings.

A noisy quantum channel is a noisy quantum process. It is a preserve qubit state noisy operations [8, 9], such as bit-$flip$ channel, $phase$-$flip$ channel, bit-$phase$ flip channel [8]. A noisy channel is a linear, CPTP map that maps the initial state ρ to the output state $\varepsilon(\rho)$. The dynamical behavior of quantum discord and entanglement under the noisy channels [10–18] becomes the motivation for studying the dynamical behavior of quantum correlation entropy of output state in this paper. These dynamical behaviors include the frozen phenomenon, the phenomenon of sudden death and the phenomenon of sudden revival. The reasons for these dynamical behaviors are analyzed. We give some understanding of these dynamical behaviors.

In this paper, we evaluate the local observational entropy and quantum correlation entropy of initial state and output state. The output state is generated by the bit-$flip$ channel, $phase$-$flip$ channel and bit-$phase$ flip channel acting on the initial state. The dynamical behavior of quantum correlation entropy of output state is studied. It is shown that quantum correlation entropy of output state exists the frozen phenomenon for fixed c_j. For another fixed c_j, quantum correlation entropy of output state exists the phenomenon of sudden death. This shows that the output state becomes a classically correlated state after a certain time. Since quantum correlation entropy of output state does not exist the phenomenon of sudden revival, the output state never restores to a non-classically correlated state. On the other hand, the relations between the quantum correlation entropies of different output states is studied.

The paper is organized as follows. Section 2 contains the main results of this paper. Section 3 we summarize our results and point out interesting avenues of future research. Appendix A, a number of mathematical processes are left for the Appendix.

2 Dynamical Behavior of Quantum Correlation Entropy Under Noisy Channels for Multiqubit State

2.1 Preparations

In this paper, we denote Pauli matrices as

$$\hat{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2}$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

We can verify that

$$\sigma_j \cdot \sigma_j \cdot \sigma_j = \sigma_j, \quad \sigma_j \cdot \sigma_{j'} \cdot \sigma_j = -\sigma_{j'}, \tag{4}$$

where $j, j' = 1, 2, 3$ and $j \neq j'$.

Moreover,

$$\sigma_1 \cdot \sigma_2 = -i \cdot \sigma_3, \quad \sigma_2 \cdot \sigma_1 = -i \cdot \sigma_3, \quad \sigma_3 \cdot \sigma_1 = i \cdot \sigma_2, \tag{5}$$

$$\sigma_1 \cdot \sigma_3 = -i \cdot \sigma_2, \quad \sigma_2 \cdot \sigma_1 = i \cdot \sigma_1, \quad \sigma_3 \cdot \sigma_2 = -i \cdot \sigma_1. \tag{6}$$

Consider the following N-qubit states associated with systems $A_1, A_2, \cdots, A_N$

$$\rho_N = \frac{1}{2^N}\left(\hat{I} + \sum_{j=1}^{3} c_j \sigma_j^{\otimes N}\right). \tag{7}$$

where $c_j = \mathrm{Tr}[(\sigma_j \otimes \sigma_j)\rho_N]$ and $|c_j| \leq 1$, and $\hat{I} = \hat{I}_2^{\otimes N}$.

For $N = 2$, ρ_N reduces to the well-known Bell diagonal states. For general N, these states are highly symmetric and can be considered as X-state [19–32].

In this paper, we define a class function as

$$f(x) = -\frac{1}{2}(1+x)\log_2(1+x) - \frac{1}{2}(1-x)\log_2(1-x), \tag{8}$$

where $|x| < 1$.

Let $\{\varPi_k = |k\rangle\langle k|, k = 0, 1\}$ be a standard orthogonal basis of 2-dimensional Hilbert space. We define a local coarse-graining as

$$\mathcal{C} = \mathcal{C}_{A_1} \otimes \mathcal{C}_{A_2} \otimes \cdots \otimes \mathcal{C}_{A_N} = \left\{\hat{P}_l^{A_1} \otimes \hat{P}_m^{A_2} \otimes \cdots \otimes \hat{P}_n^{A_N}\right\}, \tag{9}$$

where $l, m, \ldots, n = 0, 1$.

Denote

$$\hat{P}_k^{A_j} = V_{A_j}\varPi_k V_{A_j}^{\dagger}, \tag{10}$$

where $j = 1, 2, \ldots, N$ and $k = 0, 1$.

Denote

$$V_{A_j} = t_{A_j0}\hat{I} + t_{A_j1}\sigma_1 i + t_{A_j2}\sigma_2 i + t_{A_j3}\sigma_3 i \tag{11}$$

be a unitary matrix and $t_{A_jk} \in R$, $\sum_{k=0}^{3} t_{A_jk}^2 = 1$. Hence, the set $\{V_{A_j}\varPi_k V_{A_j}^{\dagger} : k = 0, 1\}$ constitutes the complete set of one-rank projection operators of 2-dimensional Hilbert Spaces.

Denote

$$m_{A_j1} = 2(t_{A_j1}t_{A_j3} - t_{A_j2}t_{A_j0}), \tag{12}$$

$$m_{A_j2} = 2(t_{A_j2}t_{A_j3} + t_{A_j1}t_{A_j0}), \tag{13}$$

$$m_{A_j3} = t_{A_j0}^2 + t_{A_j3}^2 - t_{A_j1}^2 - t_{A_j2}^2, \tag{14}$$

where $j = 1, 2, \ldots, N$.

We can verify that

$$m_{A_j1}^2 + m_{A_j2}^2 + m_{A_j3}^2 = 1. \tag{15}$$

From the Formula (4), we can verify that

$$\left(V_{A_j}\varPi_0 V_{A_j}^{\dagger}\right) \cdot \sigma_{j'} \cdot \left(V_{A_j}\varPi_0 V_{A_j}^{\dagger}\right) = m_{A_jj'} \cdot \sigma_{j'}, \tag{16}$$

$$\left(V_{A_j}\varPi_1 V_{A_j}^{\dagger}\right) \cdot \sigma_{j'} \cdot \left(V_{A_j}\varPi_1 V_{A_j}^{\dagger}\right) = -m_{A_jj'} \cdot \sigma_{j'}, \tag{17}$$

where $j = 0, 1, 2, \ldots, N$ and $j' = 1, 2, 3$.

2.2 Dynamical Behavior of Quantum Correlation Entropy Under Bit Flip Channel

The $bit\text{-}flip$ channel flips the state of a qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) with the degree of decoherence p [33]. This channel acting on a single qubit can be described by the following Kraus operators

$$\varGamma_0^{A_j} = \sqrt{1 - \frac{p_j}{2}}\hat{I}, \quad \varGamma_1^{A_j} = \sqrt{\frac{p_j}{2}}\sigma_1, \tag{18}$$

where A_j labels the subsystems and $p \in [0, 1]$. Here, we consider the symmetric situation in which the decoherence rate is equal, so $p_1 = p_2 = \cdots = p$.

In Appendix A, we evaluate the quantum correlation entropy with local coarse-graining $\mathcal{C}$ (9) and give the condition that quantum correlation entropy is monotonically decreasing with respect to p.

In this paper, we show the dynamical behavior of quantum correlation entropy of output state under the *bit-flip* channel for N-qubit state, where $N = 2, 3, 4$.

For $N = 2$, quantum correlation entropy of $\varepsilon(\rho_2)$ is given by (74)

$$S^{QC}_{A_1A_2}(\varepsilon(\rho_2)) = \inf_{\beta_2^\varepsilon} f(\beta_2^\varepsilon) + \sum_{i=1}^{4} \frac{\lambda_i}{4} \log_2 \lambda_i. \tag{19}$$

Equation (76) implies that the quantum correlation entropy of $\varepsilon(\rho_2)$ is monotonically decreasing with respect to p, namely,

$$2\alpha_2 \cdot \log_2 \frac{1+\beta_2^\varepsilon}{1-\beta_2^\varepsilon} < (c_2 + c_3)\log_2 \frac{\lambda_2}{\lambda_1} + (c_2 - c_3)\log_2 \frac{\lambda_4}{\lambda_3}. \tag{20}$$

For $N = 3$, quantum correlation entropy of $\varepsilon(\rho_3)$ is given by (79)

$$S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3)) = \inf_{\beta_3^\varepsilon}[f(\beta_3^\varepsilon)] + 3 - S(\varepsilon(\rho)). \tag{21}$$

Equation (85) implies that the quantum correlation entropy of $\varepsilon(\rho_3)$ is monotonically decreasing with respect to p, namely,

$$\theta\alpha_3 \cdot \log_2 \frac{1+\beta_3^\varepsilon}{1-\beta_3^\varepsilon} < \left(c_2^2 + c_3^2\right)(1-p)^3 \log_2 \frac{1+\theta}{1-\theta}. \tag{22}$$

For $N = 4$, quantum correlation entropy of $\varepsilon(\rho_4)$ is given by (90)

$$S^{QC}_{A_1A_2A_3A_4}(\varepsilon(\rho_4)) = \inf_{\beta_4^\varepsilon} f(\beta_4^\varepsilon) + \sum_{i=1}^{4} \frac{\lambda_i}{4} \log_2 \lambda_i. \tag{23}$$

Equation (92) implies that the quantum correlation entropy of $\varepsilon(\rho_4)$ is monotonically decreasing with respect to p, that is,

$$2\alpha_4 \cdot \log_2 \frac{1+\beta_4^\varepsilon}{1-\beta_4^\varepsilon} < (c_2 + c_3)\log_2 \frac{\lambda_1}{\lambda_2} + (c_2 - c_3)\log_2 \frac{\lambda_3}{\lambda_4}. \tag{24}$$

For instance, we take $c_1 = 0.8$, $c_2 = \frac{c_1}{2}$, and $c_3 = c_1 \cdot c_2$. Figure 1 shows the dynamical behavior of the quantum correlation entropy of $\varepsilon(\rho_N)$ under the action of the *bit-flip* channel, where $N = 2, 3, 4$. From Fig. 1, it shows that quantum correlation entropy of output state exists the frozen phenomenon under the action of *bit-flip* channel. This means that the quantum correlation entropy of output state is invariant and equal to $-f(c_1)$ after a certain time. The frozen phenomenon indicates that the output state never becomes a classically correlated state under the action of *bit-flip* channel.

In Fig. 1, the black line is always higher than the blue line, which means that under the operation of *bit-flip* channel, quantum correlation entropy of $\varepsilon(\rho_3)$ is not less than quantum correlation entropy of $\varepsilon(\rho_4)$. The red line always stays on top, which means quantum correlation entropy of $\varepsilon(\rho_2)$ is not less than quantum correlation entropy of $\varepsilon(\rho_3)$ and $\varepsilon(\rho_4)$. We surprise to find that $S^{QC}_{A_1A_2}(\varepsilon(\rho_2))$ is not less than $S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3))$. Meanwhile, $S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3))$ is not less than $S^{QC}_{A_1A_2A_3A_4}(\varepsilon(\rho_4))$. This shows that under the influence of

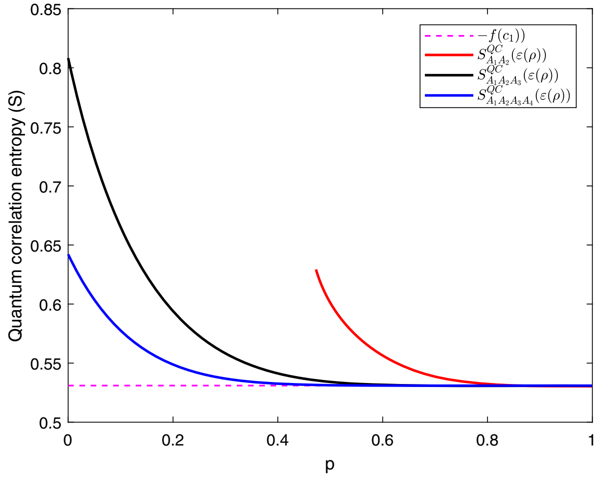

Fig. 1 The red solid line, black solid line and blue solid line represent the quantum correlation entropy of $\varepsilon(\rho_2)$, $\varepsilon(\rho_3)$, and $\varepsilon(\rho_4)$, respectively. The noisy channel is *bit-flip* channel. We take $c_1 = 0.8$, $c_2 = \frac{c_1}{2}$, and $c_3 = c_1 \cdot c_2$. The red dotted line represents the value of $-f(c_1)$. In this paper, we take $(m_{A_1 1}, m_{A_1 2}, m_{A_1 3}) = (-0.0905, -0.8613, 0.5)$, $(m_{A_2 1}, m_{A_2 2}, m_{A_2 3}) = (0.3933, -0.4368, 0.81)$, $(m_{A_3 1}, m_{A_3 2}, m_{A_3 3}) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and $(m_{A_4 1}, m_{A_4 2}, m_{A_4 3}) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

bit-flip channel, the increase in the number of subsystems does not improve the total non-classical correlation of output states. It is worth noting that when we fix the values of c_j, the eigenvalues of the output states are negative for some ranges of p. Therefore, the quantity depicted by a red line is not defined for all values of the parameter p.

Noting that c_1 is independent on time, we consider the case that $c_2 = 0.4$, and $c_1 = c_3 = 0$. Figure 2 shows the dynamical behavior of the quantum correlation entropy of output state under the action of the *bit-flip* channel. In Fig. 2, We surprise to find that $S^{QC}_{A_1A_2}(\varepsilon(\rho_2))$ is less than $S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3))$ and $S^{QC}_{A_1A_2A_3A_4}(\varepsilon(\rho_4))$, when $p < G_\diamond$. Meanwhile, $S^{QC}_{A_1A_2A_3A_4}(\varepsilon(\rho_4))$ is not less than $S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3))$, when $p \leq G_\diamond$. However, when $G_\diamond \leq p \leq 1$, we have $S^{QC}_{A_1A_2A_3A_4}(\varepsilon(\rho_4)) \leq S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3)) \leq S^{QC}_{A_1A_2}(\varepsilon(\rho_2))$. This shows that under the influence of *bit-flip* channel, the increase of the number of subsystems does not improve the total non-classical correlation of output states under a certain time.

Figure 2 also shows that quantum correlation entropy of $\varepsilon(\rho_2)$, $\varepsilon(\rho_3)$ and $\varepsilon(\rho_4)$ exist the phenomenon of sudden death. That is to say, quantum correlation entropy of output states

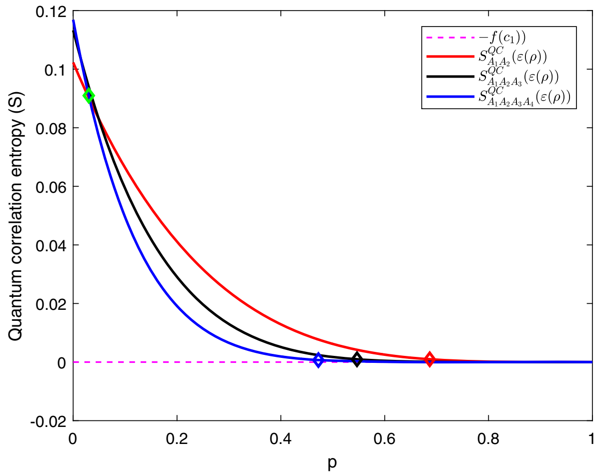

Fig. 2 The red solid line, black solid line and blue solid line represent the quantum correlation entropy of $\varepsilon(\rho_2)$, $\varepsilon(\rho_3)$, and $\varepsilon(\rho_4)$, respectively. The noisy channel is *bit-flip* channel. We take $c_2 = 0.4$, and $c_1 = c_3 = 0$. The red dotted line represents the value of $-f(c_1)$, where $f(c_1) = 0$. The green diamond shows the intersection of quantum correlation entropy of 2-qubit state, 3-qubit state and 4-qubit state. Note that the horizontal coordinates of the green diamond, red diamond, black diamond and blue diamond be $G_\diamond$, $R_\diamond$, $B_\diamond$ and $\hat{B}_\diamond$, respectively

is equal to zero, when $p \geq R_\diamond$, $p \geq B_\diamond$, and $p \geq \hat{B}_\diamond$, respectively. The phenomenon of sudden death shown in Fig. 2 is a new feature for physical dissipation [13, 16]. The zero of quantum correlation entropy of output state means that the output state becomes a classically correlated state after a certain time. Moreover, the classically correlated state never restores to a non-classically correlated state, since the quantum correlation entropy of output state does not exist the phenomenon of sudden revival.

2.3 Dynamical Behavior of Quantum Correlation Entropy Under Phase Flip Channel

The $phase\text{-}flip$ channel acting on a single qubit can be described by the following Kraus operators

$$\Gamma_0^{A_j} = \sqrt{1 - \frac{p_j}{2}}\hat{I}, \quad \Gamma_1^{A_j} = \sqrt{\frac{p_j}{2}}\sigma_3, \tag{25}$$

where A_j labels the subsystems and $p \in [0, 1]$. Here, we consider the symmetric situation in which the decoherence rate is equal, so $p_1 = p_2 = \cdots = p$.

For $N = 2$, quantum correlation entropy of $\varepsilon(\rho_2)$ is given by (102)

$$S^{QC}_{A_1A_2}(\varepsilon(\rho_2)) = \inf_{\beta_2^\varepsilon} f(\beta_2^\varepsilon) + \sum_{i=1}^{4} \frac{\lambda_i}{4} \log_2 \lambda_i, \tag{26}$$

Equation (104) implies that the quantum correlation entropy of $\varepsilon(\rho_2)$ is monotonically decreasing with respect to p, namely,

$$2\alpha_2 \cdot \log_2 \frac{1 + \beta_2^\varepsilon}{1 - \beta_2^\varepsilon} < (c_1 + c_2)\log_2 \frac{\lambda_4}{\lambda_1} + (c_1 - c_2)\log_2 \frac{\lambda_3}{\lambda_2}. \tag{27}$$

For $N = 3$, quantum correlation entropy of $\varepsilon(\rho_3)$ is given by (107)

$$S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3)) = \inf_{\beta_3^\varepsilon} \left[f(\beta_3^\varepsilon) \right] + 3 - S(\varepsilon(\rho)). \tag{28}$$

Equation (113) implies that the quantum correlation entropy of $\varepsilon(\rho_3)$ is monotonically decreasing with respect to p, namely,

$$\theta\alpha_3 \cdot \log_2 \frac{1 + \beta_3^\varepsilon}{1 - \beta_3^\varepsilon} < \left(c_1^2 + c_2^2\right)(1 - p)^3 \log_2 \frac{1 + \theta}{1 - \theta}. \tag{29}$$

For $N = 4$, quantum correlation entropy of $\varepsilon(\rho_4)$ is given by (118)

$$S^{QC}_{A_1A_2A_3A_4}(\varepsilon(\rho_4)) = \inf_{\beta_4^\varepsilon} f(\beta_4^\varepsilon) + \sum_{i=1}^{4} \frac{\lambda_i}{4} \log_2 \lambda_i, \tag{30}$$

Equation (120) implies that the quantum correlation entropy of $\varepsilon(\rho_4)$ is monotonically decreasing with respect to p, namely,

$$2\alpha_4 \cdot \log_2 \frac{1 + \beta_4^\varepsilon}{1 - \beta_4^\varepsilon} < (c_1 + c_2)\log_2 \frac{\lambda_1}{\lambda_4} + (c_1 - c_2)\log_2 \frac{\lambda_2}{\lambda_3}. \tag{31}$$

For instance, we take $c_1 = 0.8$, $c_2 = \frac{c_1}{2}$, and $c_3 = 0.5$. Figure 3 shows the dynamical behavior of the quantum correlation entropy of output state under the action of the $phase\text{-}flip$ channel. From Fig. 3, it shows that quantum correlation entropy of output state exists the frozen phenomenon under the action of $phase\text{-}flip$ channel for $N = 2, 3, 4$. Quantum correlation entropy of output state is equal to $-f(c_3)$ for $N = 3, 4$ after a certain time. But for $N = 2$, quantum correlation entropy of output state is equal to a, where $0 < a < -f(c_3)$

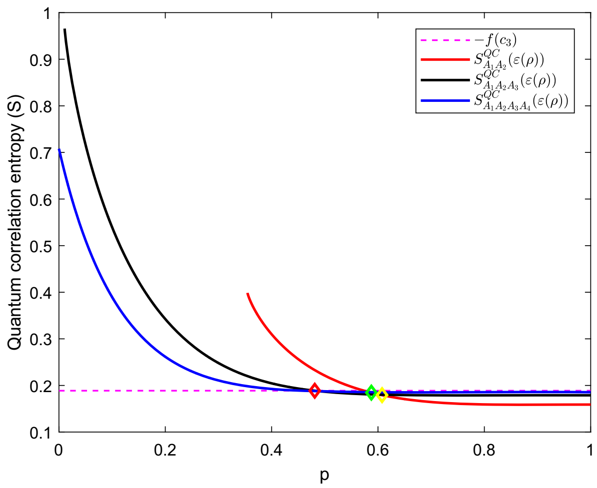

Fig. 3 The red solid line, black solid line and blue solid line represent the quantum correlation entropy of $\varepsilon(\rho_2)$, $\varepsilon(\rho_3)$, and $\varepsilon(\rho_4)$, respectively. The noisy channel is $phase\text{-}flip$ channel. We take $c_1 = 0.8$, $c_2 = 0.5$, and $c_2 = c_1 \cdot c_3$. The red dotted line represents the value of $-f(c_3)$. The red diamond is the intersection of quantum correlation entropy of $\varepsilon(\rho_3)$ and $\varepsilon(\rho_4)$. The green diamond is the intersection of quantum correlation entropy of $\varepsilon(\rho_2)$ and $\varepsilon(\rho_3)$. The yellow diamond is the intersection of quantum correlation entropy of $\varepsilon(\rho_2)$ and $\varepsilon(\rho_4)$. Let the horizontal coordinates of the red diamond, green diamond and yellow diamond be $R_\diamond$, $G_\diamond$ and $Y_\diamond$, respectively

after a certain time. The frozen phenomenon indicates that the output state never becomes a classically correlated state under the action of $phase\text{-}flip$ channel.

In Fig. 3, if $p \leq R_\diamond$, the black line is always higher than the blue line, which means that under the operation of $phase\text{-}flip$ channel, quantum correlation entropy of $\varepsilon(\rho_3)$ is not less than quantum correlation entropy of $\varepsilon(\rho_4)$. And vice versa, if $p \geq R_\diamond$. If $p \leq G_\diamond$, the red line always stays on top, which means quantum correlation entropy of $\varepsilon(\rho_2)$ is not less than quantum correlation entropy of $\varepsilon(\rho_3)$ and $\varepsilon(\rho_4)$. However, if $p \geq G_\diamond$, quantum correlation entropy of $\varepsilon(\rho_2)$ is not larger than quantum correlation entropy of $\varepsilon(\rho_3)$. If $p \geq Y_\diamond$, quantum correlation entropy of $\varepsilon(\rho_2)$ is not larger than quantum correlation entropy of $\varepsilon(\rho_4)$. This shows that under the influence of $phase\text{-}flip$ channel, the increase of the number of subsystems can improve the total non-classical correlation of output states under a certain time. It is worth noting that when we fix the values of c_j, the eigenvalues of the output states are negative for some ranges of p. Therefore, the quantities depicted by a black and red line are not defined for all values of the parameter p.

Noting that c_3 is independent on time, we consider the case that $c_1 = 0.8$, and $c_2 = c_3 = 0$. Figure 4 shows the dynamical behavior of the quantum correlation entropy of output state under the action of the $phase\text{-}flip$ channel for $N = 2, 3, 4$. In Fig. 4, we surprise to find that quantum correlation entropy of output state exists the phenomenon of sudden death. Quantum correlation entropy of output state is equal to zero, when $p \geq R_\diamond$, $p \geq B_\diamond$, and $p \geq \hat{B}_\diamond$, respectively. The phenomenon of sudden death indicates that the output state becomes a classically correlated state after a certain time. Moreover, the classically correlated state never restores to a non-classically correlated state, since the quantum correlation entropy of output state does not exists the phenomenon of sudden revival. We also observer that under the influence of $phase\text{-}flip$ channel, the increase of the number of subsystems can not improve the total non-classical correlation of output states under a certain time.

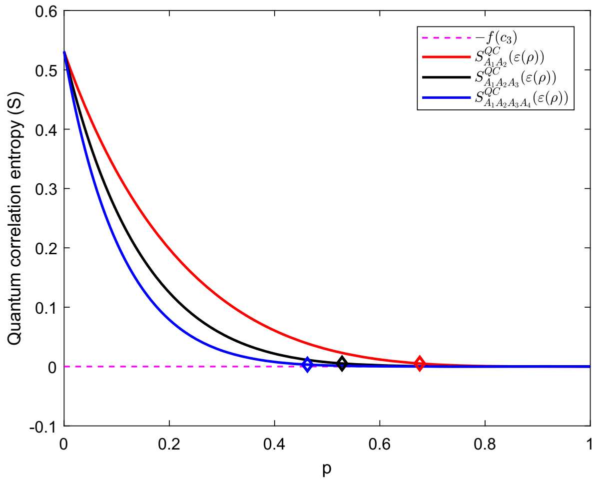

Fig. 4 The red solid line, black solid line and blue solid line represent the quantum correlation entropy of $\varepsilon(\rho_2)$, $\varepsilon(\rho_3)$, and $\varepsilon(\rho_4)$, respectively. The noisy channel is *phase-flip* channel. We take $c_1 = 0.8$, and $c_2 = c_3 = 0$. The red dotted line represents the value of $-f(c_3)$ and $f(c_3) = 0$. Let the horizontal coordinates of the red diamond, black diamond and blue diamond be $R_\diamond$, $B_\diamond$ and $\hat{B}_\diamond$, respectively

2.4 Dynamical Behavior of Quantum Correlation Entropy Under Bit-Phase Flip Channel

The *bit-phase* flip channel acting on a single qubit can be described by the following Kraus operators

$$\Gamma_0^{A_j} = \sqrt{1 - \frac{p_j}{2}}\hat{I}, \quad \Gamma_1^{A_j} = \sqrt{\frac{p_j}{2}}\sigma_2, \tag{32}$$

where A_j labels the subsystems and $p \in [0, 1]$. Here, we consider the symmetric situation in which the decoherence rate is equal, so $p_1 = p_2 = \cdots = p$.

For $N = 2$, quantum correlation entropy of $\varepsilon(\rho_2)$ is given by (130)

$$S^{QC}_{A_1A_2}(\varepsilon(\rho_2)) = \inf_{\beta_2^\varepsilon} f(\beta_2^\varepsilon) + \sum_{i=1}^{4} \frac{\lambda_i}{4} \log_2 \lambda_i. \tag{33}$$

Equation (132) implies that the quantum correlation entropy of $\varepsilon(\rho_2)$ is monotonically decreasing with respect to p, namely,

$$2\alpha_2 \cdot \log_2 \frac{1 + \beta_2^\varepsilon}{1 - \beta_2^\varepsilon} < (c_1 + c_3) \log_2 \frac{\lambda_3}{\lambda_1} + (c_1 - c_3) \log_2 \frac{\lambda_4}{\lambda_2}. \tag{34}$$

For $N = 3$, quantum correlation entropy of $\varepsilon(\rho_3)$ is given by (135)

$$S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3)) = \inf_{\beta_3^\varepsilon}[f(\beta_3^\varepsilon)] + 3 - S(\varepsilon(\rho)). \tag{35}$$

Equation (141) implies that the quantum correlation entropy of $\varepsilon(\rho_3)$ is monotonically decreasing with respect to p, namely,

$$\theta\alpha_3 \cdot \log_2 \frac{1 + \beta_3^\varepsilon}{1 - \beta_3^\varepsilon} < (c_1^2 + c_3^2)(1 - p)^3 \log_2 \frac{1 + \theta}{1 - \theta}. \tag{36}$$

For $N = 4$, quantum correlation entropy of $\varepsilon(\rho_4)$ is given by (146)

$$S^{QC}_{A_1A_2A_3A_4}(\varepsilon(\rho_4)) = \inf_{\beta_4^\varepsilon} f(\beta_4^\varepsilon) + \sum_{i=1}^{4} \frac{\lambda_i}{4} \log_2 \lambda_i, \tag{37}$$

Equation (148) implies that the quantum correlation entropy of $\varepsilon(\rho_4)$ is monotonically decreasing with respect to p, namely,

$$2\alpha_4 \cdot \log_2 \frac{1+\beta_4^{\varepsilon}}{1-\beta_4^{\varepsilon}} < (c_1+c_3)\log_2 \frac{\lambda_1}{\lambda_4} + (c_1-c_3)\log_2 \frac{\lambda_2}{\lambda_3}. \tag{38}$$

For instance, we take $c_2 = 0.8$, $c_1 = 0.5$, and $c_3 = c_2 \cdot c_1$. Figure 5 shows the dynamical behavior of quantum correlation entropy under the action of *bit-phase* flip channel. From Fig. 5, it shows that quantum correlation entropy of output state exists the frozen phenomenon under the action of *bit-phase* flip channel for $N = 2, 3, 4$. Quantum correlation entropy of output state is equal to A_N for $N = 3, 4$ after a certain time, where $0 < A_N < -f(c_2)$, and $A_3 < A_4$. But for $N = 2$, quantum correlation entropy of output state is equal to A_2 in a short time, where $0 < A_2 < A_3$. The frozen phenomenon indicates that the output state never becomes a classically correlated state under the action of *bit-phase* flip channel.

In Fig. 5, if $p \leq R_\diamond$, the black line is always higher than the blue line, which means that under the operation of *bit-phase* flip channel, quantum correlation entropy of $\varepsilon(\rho_3)$ is not less than quantum correlation entropy of $\varepsilon(\rho_4)$. And vice versa, if $p \geq R_\diamond$. If $p \leq G_\diamond$, the red line always stays on top, which means that quantum correlation entropy of $\varepsilon(\rho_2)$ is not less than quantum correlation entropy of $\varepsilon(\rho_3)$ and $\varepsilon(\rho_4)$. However, if $p \geq G_\diamond$, quantum correlation entropy of $\varepsilon(\rho_2)$ is not larger than quantum correlation entropy of $\varepsilon(\rho_4)$. If $p \geq Y_\diamond$, quantum correlation entropy of $\varepsilon(\rho_2)$ is not larger than quantum correlation entropy of $\varepsilon(\rho_3)$. It is worth noting that when we fix the values of c_j, the eigenvalues of the output states are negative for some ranges of p. Therefore, the quantities depicted by a black and red line are not defined for all values of the parameter p.

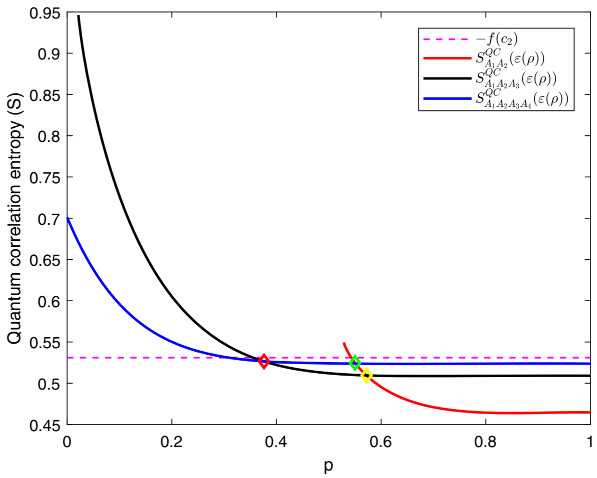

Fig. 5 The red solid line, black solid line and blue solid line represent the quantum correlation entropy of $\varepsilon(\rho_2)$, $\varepsilon(\rho_3)$, and $\varepsilon(\rho_4)$, respectively. The noisy channel is *bit-phase* flip channel. We take $c_2 = 0.8$, $c_1 = 0.5$, and $c_3 = c_2 \cdot c_1$. The red dotted line represents the value of $-f(c_2)$. The red diamond is the intersection of quantum correlation entropy of $\varepsilon(\rho_3)$ and $\varepsilon(\rho_4)$. The green diamond is the intersection of quantum correlation entropy of $\varepsilon(\rho_2)$ and $\varepsilon(\rho_4)$. The yellow diamond is the intersection of quantum correlation entropy of $\varepsilon(\rho_2)$ and $\varepsilon(\rho_3)$. Let the horizontal coordinates of the red diamond, green diamond and yellow diamond be $R_\diamond$, $G_\diamond$ and $Y_\diamond$, respectively

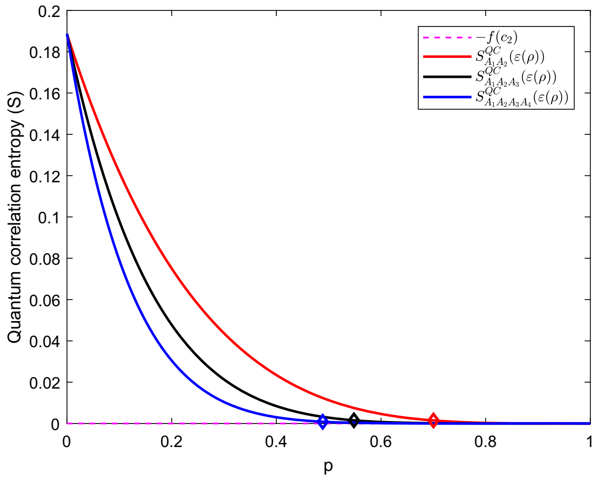

Fig. 6 The red solid line, black solid line and blue solid line represent the quantum correlation entropy of $\varepsilon(\rho_2)$, $\varepsilon(\rho_3)$, and $\varepsilon(\rho_4)$, respectively. The noisy channel is *bit-phase* flip channel. We take $c_1 = 0.5$, and $c_2 = c_3 = 0$. The red dotted line represents the value of $-f(c_2)$ and $f(c_2) = 0$. Let the horizontal coordinates of the red diamond, black diamond and blue diamond be $R_\diamond$, $B_\diamond$ and $\hat{B}_\diamond$, respectively

Note that c_2 is independent on time, we consider the case that $c_1 = 0.5$, and $c_2 = c_3 = 0$. Figure 6 shows the dynamical behavior of the quantum correlation entropy of output state under the action of the *phase-flip* channel for $N = 2, 3, 4$. In Fig. 6, we surprise to find that quantum correlation entropy of output state exists the phenomenon of sudden death. Quantum correlation entropy of output state is equal to zero, when $p \geq R_\diamond$, $p \geq B_\diamond$, and $p \geq \hat{B}_\diamond$, respectively. The phenomenon of sudden death indicates that the output state becomes a classically correlated state after a certain time. Moreover, the classically correlated state never restores to a non-classically correlated state, since the quantum correlation entropy of output state does not exists the phenomenon of sudden revival.

3 Conclusion

We study the dynamical behavior of quantum correlation entropy under the noisy channels. These noisy channels are *bit-flip* channel, *phase-flip* channel and *bit-phase* flip channel.

Under the action of *bit-flip* channel, the quantum correlation entropy of output state exists the frozen phenomenon, when we set $c_1 = 0.8$, $c_2 = \frac{c_1}{2}$, and $c_3 = c_1 \cdot c_2$. This indicates that the quantum correlation entropy of output state is invariant after a certain time. Meanwhile, the output state remains a non-classically correlated state, which shows the influence of *bit-flip* channel. However, when we set $c_2 = 0.4$, and $c_1 = c_3 = 0$. The quantum correlation entropy of output state exists the phenomenon of sudden death. This means that the quantum correlation entropy of output state is equal to zero, and the output state becomes a classically correlated state. Since the quantum correlation entropy does not exist the phenomenon of sudden revival, the output state never restores to a non-classically correlated state.

Under the action of *phase-flip* channel, the quantum correlation entropy of output state exists the frozen phenomenon, when we set $c_1 = 0.8$, $c_2 = \frac{c_1}{2}$, and $c_3 = 0.5$. This indicates that the quantum correlation entropy of output state is invariant after a certain time. Meanwhile, the output state remains a non-classically correlated state, which shows the influence of *phase-flip* channel. However, when we set $c_1 = 0.8$, and $c_2 = c_3 = 0$. The quantum correlation entropy of output state exists the phenomenon of sudden death. This means that the quantum correlation entropy of output state is equal to zero, and the output state becomes a classically correlated state. Since the quantum correlation entropy does not exist the phenomenon of sudden revival, the output state never restores to a non-classically correlated state.

Under the action of *bit-phase* flip channel, the quantum correlation entropy of output state exists the frozen phenomenon, when we set $c_2 = 0.8$, $c_1 = 0.5$, and $c_3 = c_2 \cdot c_1$. This indicates that the quantum correlation entropy of output state is invariant after a certain time. Meanwhile, the output state remains a non-classically correlated state, which shows the influence of *bit-phase* flip channel. However, when we set $c_1 = 0.5$, and $c_2 = c_3 = 0$. The quantum correlation entropy of output state exists the phenomenon of sudden death. This means that the quantum correlation entropy of output state is equal to zero, and the output state become a classically correlated state. Since the quantum correlation entropy does not exist the phenomenon of sudden revival, the output state never restores to a non-classically correlated state.

We give the results of the dynamical behavior of quantum correlation entropy for N-qubit state, but we only analyze the case for N is equal to 2, 3, and 4. A discussion of higher dimensions might be an interesting question. Our results may highlight further investigations on quantum correlation entropy and their applications in quantum information processing.

Appendix

A.1: Observational Entropy and Quantum Correlation Entropy for Multiqubit State

For the initial state ρ_N (7), and under the action of local coarse-graining $\mathcal{C}$ (9), we obtain the final state as the ensemble $\{\rho_{lm\dots n}, p_{lm\dots n}\}$ with

$$\rho_{lm\dots n} = \frac{1}{p_{lm\dots n}} \left(\hat{P}_l^{A_1} \otimes \hat{P}_m^{A_2} \otimes \cdots \otimes \hat{P}_n^{A_N}\right) \rho_N \left(\hat{P}_l^{A_1} \otimes \hat{P}_m^{A_2} \otimes \cdots \otimes \hat{P}_n^{A_N}\right), \tag{39}$$

and

$$p_{lm\dots n} = \mathrm{Tr}\left[(\hat{P}_l^{A_1} \otimes \hat{P}_m^{A_2} \otimes \cdots \otimes \hat{P}_n^{A_N})\rho_N\right]. \tag{40}$$

For instance, we take $N = 3$. We can verify that

$$\rho_{000} = \frac{1}{p_{000}} \left(\hat{P}_0^{A_1} \otimes \hat{P}_0^{A_2} \otimes \hat{P}_0^{A_3})\rho_3(\hat{P}_0^{A_1} \otimes \hat{P}_0^{A_2} \otimes \hat{P}_0^{A_3}\right) = \hat{P}_0^{A_1} \otimes \hat{P}_0^{A_2} \otimes \hat{P}_0^{A_3},\ p_{000} = \frac{1}{8}(1+\beta_3), \tag{41}$$

$$\rho_{001} = \frac{1}{p_{001}} \left(\hat{P}_0^{A_1} \otimes \hat{P}_0^{A_2} \otimes \hat{P}_1^{A_3})\rho_3(\hat{P}_0^{A_1} \otimes \hat{P}_0^{A_2} \otimes \hat{P}_1^{A_3}\right) = \hat{P}_0^{A_1} \otimes \hat{P}_0^{A_2} \otimes \hat{P}_1^{A_3},\ p_{001} = \frac{1}{8}(1-\beta_3), \tag{42}$$

$$\rho_{010} = \frac{1}{p_{010}} \left(\hat{P}_0^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_0^{A_3}\right) \rho_3 \left(\hat{P}_0^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_0^{A_3}\right) = \hat{P}_0^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_0^{A_3},\ p_{010} = \frac{1}{8}(1-\beta_3), \tag{43}$$

$$\rho_{011} = \frac{1}{p_{011}} \left(\hat{P}_0^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_1^{A_3}\right) \rho_3 \left(\hat{P}_0^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_1^{A_3}\right) = \hat{P}_0^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_1^{A_3},\ p_{011} = \frac{1}{8}(1+\beta_3), \tag{44}$$

$$\rho_{100} = \frac{1}{p_{100}}\left(\hat{P}_1^{A_1}\otimes\hat{P}_0^{A_2}\otimes\hat{P}_0^{A_3}\right)\rho_3\left(\hat{P}_1^{A_1}\otimes\hat{P}_0^{A_2}\otimes\hat{P}_0^{A_3}\right) = \hat{P}_1^{A_1}\otimes\hat{P}_0^{A_2}\otimes\hat{P}_0^{A_3},\ p_{100} = \frac{1}{8}(1-\beta_3), \tag{45}$$

$$\rho_{101} = \frac{1}{p_{101}}\left(\hat{P}_1^{A_1}\otimes\hat{P}_0^{A_2}\otimes\hat{P}_1^{A_3}\right)\rho_3\left(\hat{P}_1^{A_1}\otimes\hat{P}_0^{A_2}\otimes\hat{P}_1^{A_3}\right) = \hat{P}_1^{A_1}\otimes\hat{P}_0^{A_2}\otimes\hat{P}_1^{A_3},\ p_{101} = \frac{1}{8}(1+\beta_3), \tag{46}$$

$$\rho_{110} = \frac{1}{p_{110}}\left(\hat{P}_1^{A_1}\otimes\hat{P}_1^{A_2}\otimes\hat{P}_0^{A_3}\right)\rho_3\left(\hat{P}_1^{A_1}\otimes\hat{P}_1^{A_2}\otimes\hat{P}_0^{A_3}\right) = \hat{P}_1^{A_1}\otimes\hat{P}_1^{A_2}\otimes\hat{P}_0^{A_3},\ p_{110} = \frac{1}{8}(1+\beta_3), \tag{47}$$

$$\rho_{111} = \frac{1}{p_{111}}\left(\hat{P}_1^{A_1}\otimes\hat{P}_1^{A_2}\otimes\hat{P}_1^{A_3}\right)\rho_3\left(\hat{P}_1^{A_1}\otimes\hat{P}_1^{A_2}\otimes\hat{P}_1^{A_3}\right) = \hat{P}_1^{A_1}\otimes\hat{P}_1^{A_2}\otimes\hat{P}_1^{A_3},\ p_{111} = \frac{1}{8}(1-\beta_3), \tag{48}$$

where $\sum_{l=0}^{1}\sum_{m=0}^{1}\sum_{n=0}^{1} p_{lmn} = 1$, and $\beta_3 = \sum_{j=1}^{3} c_j m_{A_1 j} m_{A_2 j} m_{A_3 j}$, $|\beta_3| \le 1$.

The local observational entropy of ρ_3 is given by

$$\begin{aligned} S_{O(\mathcal{C})}(\rho_3) &= -\sum_{lmn} p_{lmn}\log_2\frac{p_{lmn}}{V_{lmn}} \\ &= -4\cdot\frac{1+\beta_3}{8}\log_2\frac{1+\beta_3}{8} - 4\cdot\frac{1-\beta_3}{8}\log_2\frac{1-\beta_3}{8} \\ &= -\frac{1+\beta_3}{2}\log_2(1+\beta_3) - \frac{1-\beta_3}{2}\log_2(1-\beta_3) + 3 \\ &= f(\beta_3) + 3, \end{aligned} \tag{49}$$

where $V_{lmn} = \mathrm{Tr}\left(\hat{P}_l^{A_1}\otimes\hat{P}_m^{A_2}\otimes\hat{P}_n^{A_3}\right)$ and $\sum_{l=0}^{1}\sum_{m=0}^{1}\sum_{n=0}^{1} V_{lmn} = 8$.

The quantum correlation entropy of ρ_3 is given by

$$S_{A_1A_2A_3}^{QC}(\rho_3) = \inf_{\beta_3}[f(\beta_3)] + 3 - S(\rho_3), \tag{50}$$

where $S(\rho_3) = -tr[\rho_3\log_2\rho_3]$ is the von Neumann entropy of ρ_3.

Under the operation of a quantum channel, we assume that ρ_3 be mapped to $\varepsilon(\rho_3)$ by this quantum channel $\varepsilon(\cdot)$.

For instance, we evaluate the observational entropy and quantum correlation entropy of $\varepsilon(\rho_3)$ (78), where $\varepsilon(\rho_3)$ is the output state of ρ_3 (77) under the operation of *bit-flip* channel. Let local coarse-graining $\mathcal{C}$ be performed on $\varepsilon(\rho_3)$ (78). The $\varepsilon(\rho_3)$ will be updated to final states as the ensemble $\left\{\rho_{lmn}^{\varepsilon}, p_{lmn}^{\varepsilon}\right\}$ with $\rho_{lmn}^{\varepsilon} := \frac{1}{p_{lmn}}\left(\hat{P}_l^{A_1}\otimes\hat{P}_m^{A_2}\otimes\hat{P}_n^{A_3}\right)\varepsilon(\rho_3)\left(\hat{P}_l^{A_1}\otimes\hat{P}_m^{A_2}\otimes\hat{P}_n^{A_3}\right)$ and $p_{lmn}^{\varepsilon} = \mathrm{Tr}\left[\left(\hat{P}_l^{A_1}\otimes\hat{P}_m^{A_2}\otimes\hat{P}_n^{A_3}\right)\varepsilon(\rho)\left(\hat{P}_l^{A_1}\otimes\hat{P}_m^{A_2}\otimes\hat{P}_n^{A_3}\right)\right]$. We can verify that

$$\rho_{000}^{\varepsilon} = \frac{1}{p_{000}^{\varepsilon}}\left(\hat{P}_0^{A_1}\otimes\hat{P}_0^{A_2}\otimes\hat{P}_0^{A_3}\right)\varepsilon(\rho_3)\left(\hat{P}_0^{A_1}\otimes\hat{P}_0^{A_2}\otimes\hat{P}_0^{A_3}\right) = \hat{P}_0^{A_1}\otimes\hat{P}_0^{A_2}\otimes\hat{P}_0^{A_3},\ p_{000}^{\varepsilon} = \frac{1}{8}(1+\beta_3^{\varepsilon}), \tag{51}$$

$$\rho_{001}^{\varepsilon} = \frac{1}{p_{001}^{\varepsilon}}\left(\hat{P}_0^{A_1}\otimes\hat{P}_0^{A_2}\otimes\hat{P}_1^{A_3}\right)\varepsilon(\rho_3)\left(\hat{P}_0^{A_1}\otimes\hat{P}_0^{A_2}\otimes\hat{P}_1^{A_3}\right) = \hat{P}_0^{A_1}\otimes\hat{P}_0^{A_2}\otimes\hat{P}_1^{A_3},\ p_{001}^{\varepsilon} = \frac{1}{8}(1-\beta_3^{\varepsilon}), \tag{52}$$

$$\rho_{010}^{\varepsilon} = \frac{1}{p_{010}^{\varepsilon}}\left(\hat{P}_0^{A_1}\otimes\hat{P}_1^{A_2}\otimes\hat{P}_0^{A_3}\right)\varepsilon(\rho_3)\left(\hat{P}_0^{A_1}\otimes\hat{P}_1^{A_2}\otimes\hat{P}_0^{A_3}\right) = \hat{P}_0^{A_1}\otimes\hat{P}_1^{A_2}\otimes\hat{P}_0^{A_3},\ p_{010}^{\varepsilon} = \frac{1}{8}(1-\beta_3^{\varepsilon}), \tag{53}$$

$$\rho_{011}^{\varepsilon} = \frac{1}{p_{011}^{\varepsilon}} \left(\hat{P}_0^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_1^{A_3}\right) \varepsilon(\rho_3) \left(\hat{P}_0^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_1^{A_3}\right) = \hat{P}_0^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_1^{A_3},\ p_{011}^{\varepsilon} = \frac{1}{8}(1+\beta_3^{\varepsilon}), \tag{54}$$

$$\rho_{100}^{\varepsilon} = \frac{1}{p_{100}^{\varepsilon}} \left(\hat{P}_1^{A_1} \otimes \hat{P}_0^{A_2} \otimes \hat{P}_0^{A_3}\right) \varepsilon(\rho_3) \left(\hat{P}_1^{A_1} \otimes \hat{P}_0^{A_2} \otimes \hat{P}_0^{A_3}\right) = \hat{P}_1^{A_1} \otimes \hat{P}_0^{A_2} \otimes \hat{P}_0^{A_3},\ p_{100}^{\varepsilon} = \frac{1}{8}(1-\beta_3^{\varepsilon}), \tag{55}$$

$$\rho_{101}^{\varepsilon} = \frac{1}{p_{101}^{\varepsilon}} \left(\hat{P}_1^{A_1} \otimes \hat{P}_0^{A_2} \otimes \hat{P}_1^{A_3}\right) \varepsilon(\rho_3) \left(\hat{P}_1^{A_1} \otimes \hat{P}_0^{A_2} \otimes \hat{P}_1^{A_3}\right) = \hat{P}_1^{A_1} \otimes \hat{P}_0^{A_2} \otimes \hat{P}_1^{A_3},\ p_{101}^{\varepsilon} = \frac{1}{8}(1+\beta_3^{\varepsilon}), \tag{56}$$

$$\rho_{110}^{\varepsilon} = \frac{1}{p_{110}^{\varepsilon}} \left(\hat{P}_1^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_0^{A_3}\right) \varepsilon(\rho_3) \left(\hat{P}_1^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_0^{A_3}\right) = \hat{P}_1^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_0^{A_3},\ p_{110}^{\varepsilon} = \frac{1}{8}(1+\beta_3^{\varepsilon}), \tag{57}$$

$$\rho_{111}^{\varepsilon} = \frac{1}{p_{111}^{\varepsilon}} \left(\hat{P}_1^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_1^{A_3}\right) \varepsilon(\rho_3) \left(\hat{P}_1^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_1^{A_3}\right) = \hat{P}_1^{A_1} \otimes \hat{P}_1^{A_2} \otimes \hat{P}_1^{A_3},\ p_{111}^{\varepsilon} = \frac{1}{8}(1-\beta_3^{\varepsilon}), \tag{58}$$

where $\sum_{l=0}^{1}\sum_{m=0}^{1}\sum_{n=0}^{1} p_{lmn}^{\varepsilon} = 1$ and $\beta_3^{\varepsilon} = c_1 m_{A_1 1} m_{A_2 1} m_{A_3 1} + c_2 m_{A_1 2} m_{A_2 2} m_{A_3 2}(1-p)^3 + c_3 m_{A_1 3} m_{A_2 3} m_{A_3 3}(1-p)^3$, $|\beta_3^{\varepsilon}| \le 1$.

Local observational entropy of $\varepsilon(\rho_3)$ can be given by

$$S_{O(\mathcal{C})}(\varepsilon(\rho_3)) = f(\beta_3^{\varepsilon}) + 3. \tag{59}$$

Quantum correlation entropy of $\varepsilon(\rho_3)$ is given by

$$\begin{aligned} S_{A_1A_2A_3}^{QC}(\varepsilon(\rho_3)) &= \inf_{\mathcal{C}} S_{O(\mathcal{C})}(\varepsilon(\rho_3)) - S(\varepsilon(\rho_3)) \\ &= \inf_{\beta_3^{\varepsilon}}[f(\beta_3^{\varepsilon})] + 3 - S(\varepsilon(\rho_3)). \end{aligned} \tag{60}$$

In general, for N-qubit state, we can verify that the local observational entropy of ρ_N (7) is given by

$$S_{O(\mathcal{C})}(\rho_N) = f(\beta_N) + N. \tag{61}$$

where $\beta_N = \sum_{j=1}^{3} c_j m_{A_1 j} m_{A_2 j} \cdots m_{A_N j}$, $|\beta_N| \le 1$.

Hence, the quantum correlation entropy of ρ_N (7) is given by

$$S_{A_1A_2\cdots A_N}^{QC}(\rho_N) = \inf_{\beta_N}[f(\beta_N)] + N - S(\rho_N), \tag{62}$$

where $S(\rho_N) = -tr[\rho_N \log_2 \rho_N]$ is the von Neumann entropy of ρ_N.

Meanwhile, We also take $\varepsilon(\rho_N)$ as the output state of ρ_N under the operation of a *bit-flip* channel. We can verify that the local observational entropy of $\varepsilon(\rho_N)$ is given by

$$S_{O(\mathcal{C})}(\varepsilon(\rho_N)) = f(\beta_N^{\varepsilon}) + N. \tag{63}$$

where $\beta_N^{\varepsilon} = c_1 m_{A_1 1} m_{A_2 1} \cdots m_{A_N 1} + c_2 m_{A_1 2} m_{A_2 2} \cdots m_{A_N 2}(1-p)^N + c_3 m_{A_1 3} m_{A_2 3} \cdots m_{A_N 3}(1-p)^N$, $|\beta_N^{\varepsilon}| \le 1$.

Hence, the quantum correlation entropy of $\varepsilon(\rho_N)$ is given by

$$S_{A_1A_2\cdots A_N}^{QC}(\rho_N) = \inf_{\beta_N^{\varepsilon}}[f(\beta_N^{\varepsilon})] + N - S(\varepsilon(\rho_N)), \tag{64}$$

where $S(\varepsilon(\rho_N)) = -tr[\varepsilon(\rho_N) \log_2 \varepsilon(\rho_N)]$ is the von Neumann entropy of $\varepsilon(\rho_N)$.

A.2: Quantum Correlation Entropy Under Bit Flip Channel

The *bit-flip* channel flips the state of a qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) with the degree of decoherence p [33]. This channel acting on a single qubit can be described by the following Kraus operators

$$\Gamma_0^{A_j} = \sqrt{1 - \frac{p_j}{2}}\hat{I}, \quad \Gamma_1^{A_j} = \sqrt{\frac{p_j}{2}}\sigma_1, \tag{65}$$

where A_j labels the subsystems and $p \in [0, 1]$. Here, we consider the symmetric situation in which the decoherence rate is equal, so $p_1 = p_2 = \cdots = p$.

We can verify that

$$\sum_{l,\ldots,n} \Gamma_l^{A_1} \cdots \Gamma_n^{A_N} \sigma_1^{\otimes N} \Gamma_n^{A_N\dagger} \cdots \Gamma_l^{A_1\dagger} = \sigma_1^{\otimes N}, \tag{66}$$

$$\sum_{l,\ldots,n} \Gamma_l^{A_1} \cdots \Gamma_n^{A_N} \sigma_2^{\otimes N} \Gamma_n^{A_N\dagger} \cdots \Gamma_l^{A_1\dagger} = (1-p)^N \sigma_2^{\otimes N}, \tag{67}$$

$$\sum_{l,\ldots,n} \Gamma_l^{A_1} \cdots \Gamma_n^{A_N} \sigma_3^{\otimes N} \Gamma_n^{A_N\dagger} \cdots \Gamma_l^{A_1\dagger} = (1-p)^N \sigma_3^{\otimes N}, \tag{68}$$

where $l = \cdots = n = 0, 1$.

Consider the following 2-qubit state associated with subsystems A_1, A_2

$$\rho_2 = \frac{1}{2^2}(\hat{I} + c_1\sigma_1^{\otimes 2} + c_2\sigma_2^{\otimes 2} + c_3\sigma_3^{\otimes 2}). \tag{69}$$

Under the operation of *bit-flip* channel, the output state $\varepsilon(\rho_2)$ is given by

$$\varepsilon(\rho_2) = \frac{1}{4}\left(\hat{I} + c_1\sigma_1^{\otimes 2} + (1-p)^2 c_2\sigma_2^{\otimes 2} + (1-p)^2 c_3\sigma_3^{\otimes 2}\right). \tag{70}$$

The nonzero and positive eigenvalues of $\varepsilon(\rho_2)$ are written as $\lambda_1 = 1 - c_1 - (1-p)^2 c_2 - (1-p)^2 c_3$, $\lambda_2 = 1 - c_1 + (1-p)^2 c_2 + (1-p)^2 c_3$, $\lambda_3 = 1 + c_1 - (1-p)^2 c_2 + (1-p)^2 c_3$, $\lambda_4 = 1 + c_1 + (1-p)^2 c_2 - (1-p)^2 c_3$. The derivative of λ_i with respect to p can be cast as

$$\lambda_1' = \frac{\partial \lambda_1}{\partial p} = 2(c_2 + c_3)(1-p), \quad \lambda_2' = \frac{\partial \lambda_2}{\partial p} = -2(c_2 + c_3)(1-p), \tag{71}$$

$$\lambda_3' = \frac{\partial \lambda_3}{\partial p} = 2(c_2 - c_3)(1-p), \quad \lambda_4' = \frac{\partial \lambda_4}{\partial p} = -2(c_2 - c_3)(1-p). \tag{72}$$

From (63), the local observational entropy of $\varepsilon(\rho_2)$ is given by

$$S_{O(C)}(\rho_2) = f(\beta_2^\varepsilon) + 2. \tag{73}$$

where $\beta_2^\varepsilon = c_1 m_{A_1 1} m_{A_2 1} + \alpha_2(1-p)^2$, $|\beta_2^\varepsilon| \leq 1$ and $\alpha_2 = \sum_{j=2}^{3} c_j m_{A_1 j} m_{A_2 j}$.

Quantum correlation entropy of $\varepsilon(\rho_2)$ is given by

$$S_{A_1A_2}^{QC}(\varepsilon(\rho_2)) = \inf_{\beta_2^\varepsilon} f(\beta_2^\varepsilon) + \sum_{i=1}^{4} \frac{\lambda_i}{4} \log_2 \lambda_i, \tag{74}$$

Denote $F(p) = f(\beta_2^\varepsilon) + \sum_{i=1}^{4} \frac{\lambda_i}{4} \log_2 \lambda_i$. The derivative of $F(p)$ is given by

$$\frac{dF(p)}{dp} = -\frac{\beta_2^{\varepsilon'}}{2} \log_2 \frac{1 + \beta_2^\varepsilon}{1 - \beta_2^\varepsilon} - \frac{\lambda_1'}{4} \log_2 \frac{\lambda_2}{\lambda_1} - \frac{\lambda_3'}{4} \log_2 \frac{\lambda_4}{\lambda_3}, \tag{75}$$

where $\beta_2^{\varepsilon'} = \frac{\partial \beta_2^{\varepsilon}}{\partial p} = -2(1-p)\cdot\alpha_2$.

In order to show $\frac{dF(p)}{dp} < 0$, the following inequality must hold

$$2\alpha_2 \cdot \log_2 \frac{1+\beta_2^{\varepsilon}}{1-\beta_2^{\varepsilon}} < (c_2+c_3)\log_2 \frac{\lambda_2}{\lambda_1} + (c_2-c_3)\log_2 \frac{\lambda_4}{\lambda_3}. \tag{76}$$

Therefore, (76) implies that $S^{QC}_{A_1A_2}(\varepsilon(\rho_2))$ is monotonically decreasing with respect to p.

Consider the following 3-qubit states associated with subsystems A_1, A_2 and A_3,

$$\rho_3 = \frac{1}{2^3}\left(\hat{I} + \sum_{j=1}^{3} c_j \sigma_j \otimes \sigma_j \otimes \sigma_j\right). \tag{77}$$

Under the action of *bit-flip* channel, the initial state ρ_3 be updated to $\varepsilon(\rho_3)$ as

$$\varepsilon(\rho_3) = \frac{1}{8}\left(\hat{I} + c_1\sigma_1^{\otimes 3} + (1-p)^3 c_2\sigma_2^{\otimes 3} + (1-p)^3 c_3\sigma_3^{\otimes 3}\right). \tag{78}$$

From (64), quantum correlation entropy of $\varepsilon(\rho_3)$ is given by

$$S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3)) = \inf_{\beta_3^{\varepsilon}}[f(\beta_3^{\varepsilon})] + 3 - S(\varepsilon(\rho_3)). \tag{79}$$

where $\beta_3^{\varepsilon} = c_1 m_{A_1 1} m_{A_2 1} m_{A_3 1} + \alpha_3(1-p)^3$, $|\beta_3^{\varepsilon}| \le 1$ and $\alpha_3 = \sum_{j=2}^{3} c_j m_{A_1 j} m_{A_2 j} m_{A_3 j}$.

Set $\theta = \sqrt{c_1^2 + (1-p)^6 c_2^2 + (1-p)^6 c_3^2}$, we have

$$S(\varepsilon(\rho_3)) = f(\theta) + 3. \tag{80}$$

Hence, we have

$$S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3)) = \inf_{\beta_3^{\varepsilon}}[f(\beta_3^{\varepsilon})] - f(\theta). \tag{81}$$

Since $\theta > 0$, this implies that $\log_2 \frac{1+\theta}{1-\theta} > 0$.

We have

$$\theta' = \frac{\partial \theta}{\partial p} = -\frac{3(c_2^2+c_3^2)(1-p)^5}{\theta}. \tag{82}$$

For $0 < (1-p)^5 < 1$, then $\theta' < 0$.

We assume that $S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3))$ is monotonically decreasing with respect to p, i.e., $\frac{\partial S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3))}{\partial p} < 0$.

Denote

$$g(p) = f(\beta_3^{\varepsilon}) - f(\theta). \tag{83}$$

We have

$$\frac{\partial g(p)}{\partial p} = -\frac{\beta_3^{\varepsilon'}}{2}\log_2 \frac{1+\beta_3^{\varepsilon}}{1-\beta_3^{\varepsilon}} + \frac{\theta'}{2}\log_2 \frac{1+\theta}{1-\theta}. \tag{84}$$

where $\beta_3^{\varepsilon'} = \frac{\partial \beta_3^{\varepsilon}}{\partial p} = -3\alpha_3(1-p)^2$.

In order to show $\frac{\partial g(p)}{\partial p} < 0$, the following inequality must hold

$$\theta\alpha_3 \cdot \log_2 \frac{1+\beta_3^{\varepsilon}}{1-\beta_3^{\varepsilon}} < (c_2^2+c_3^2)(1-p)^3 \log_2 \frac{1+\theta}{1-\theta}. \tag{85}$$

The c_1, c_2, c_3 and p satisfying (85) follows that $g(p)$ is monotonically decreasing with respect to p, i.e., $S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3))$ is monotonically decreasing with respect to p.

For 4-qubit state, the state ρ_4 associated with subsystems A_1, A_2, A_3, A_4 is given by

$$\rho_4 = \frac{1}{16}\left(\hat{I} + c_1\sigma_1^{\otimes 4} + c_2\sigma_2^{\otimes 4} + c_3\sigma_3^{\otimes 4}\right). \tag{86}$$

Under the operation of *bit-flip* channel, the output state $\varepsilon(\rho_4)$ is given by

$$\varepsilon(\rho_4) = \frac{1}{16}\left(\hat{I} + c_1\sigma_1^{\otimes 4} + (1-p)^4 c_2\sigma_2^{\otimes 4} + (1-p)^4 c_3\sigma_3^{\otimes 4}\right). \tag{87}$$

The nonzero and positive eigenvalues of $\varepsilon(\rho)$ are written as $\lambda_1 = 1 + c_1 + (1-p)^4c_2 + (1-p)^4c_3$, $\lambda_2 = 1 + c_1 - (1-p)^4c_2 - (1-p)^4c_3$, $\lambda_3 = 1 - c_1 - (1-p)^4c_2 + (1-p)^4c_3$, $\lambda_4 = 1 - c_1 + (1-p)^4c_2 - (1-p)^4c_3$. The derivative of λ_i with respect to p can be cast as

$$\lambda_1' = \frac{\partial\lambda_1}{\partial p} = -4(c_2+c_3)(1-p)^3, \quad \lambda_2' = \frac{\partial\lambda_2}{\partial p} = 4(c_2+c_3)(1-p)^3, \tag{88}$$

$$\lambda_3' = \frac{\partial\lambda_3}{\partial p} = -4(c_2-c_3)(1-p)^3, \quad \lambda_4' = \frac{\partial\lambda_4}{\partial p} = 4(c_2-c_3)(1-p)^3. \tag{89}$$

From (64), quantum correlation entropy of $\varepsilon(\rho_4)$ is given by

$$S^{QC}_{A_1A_2A_3A_4}(\varepsilon(\rho_4)) = \inf_{\beta_4^\varepsilon} f(\beta_4^\varepsilon) + \sum_{i=1}^{4}\frac{\lambda_i}{4}\log_2\lambda_i, \tag{90}$$

where $\beta_4^\varepsilon = c_1 m_{A_1 1} m_{A_2 1} m_{A_3 1} m_{A_4 1} + \alpha_4(1-p)^4$, $|\beta_4^\varepsilon| \le 1$ and $\alpha_4 = \sum_{j=2}^{3} c_j m_{A_1 j} m_{A_2 j} m_{A_3 j} m_{A_4 j}$.

Denote $h(p) = f(\beta_4^\varepsilon) + \sum_{i=1}^4 \frac{\lambda_i}{4}\log_2\lambda_i$. The derivative of $h(p)$ is given by

$$\frac{\partial h(p)}{\partial p} = -\frac{\beta_4^{\varepsilon'}}{2}\log_2\frac{1+\beta_4^\varepsilon}{1-\beta_4^\varepsilon} + \frac{\lambda_1'}{4}\log_2\frac{\lambda_1}{\lambda_2} + \frac{\lambda_3'}{4}\log_2\frac{\lambda_3}{\lambda_4}, \tag{91}$$

where $\beta_4^{\varepsilon'} = \frac{\partial\beta_4^\varepsilon}{\partial p} = -4(1-p)^3\cdot\alpha_4$.

In order to show $\frac{\partial h(p)}{\partial p} < 0$, the following inequality must hold

$$2\alpha_4\cdot\log_2\frac{1+\beta_4^\varepsilon}{1-\beta_4^\varepsilon} < (c_2+c_3)\log_2\frac{\lambda_1}{\lambda_2} + (c_2-c_3)\log_2\frac{\lambda_3}{\lambda_4}. \tag{92}$$

Therefore, (92) implies that $S^{QC}_{A_1A_2A_3A_4}(\varepsilon(\rho_4))$ is monotonically decreasing with respect to p.

A.3: Quantum Correlation Entropy Under Phase Flip Channel

The *phase flip* channel acting on a single qubit can be described by the following Kraus operators

$$\Gamma_0^{A_j} = \sqrt{1-\frac{p_j}{2}}\hat{I}, \quad \Gamma_1^{A_j} = \sqrt{\frac{p_j}{2}}\sigma_3, \tag{93}$$

where A_j labels the subsystems and $p \in [0, 1]$. Here, we consider the symmetric situation in which the decoherence rate is equal, so $p_1 = p_2 = \cdots = p$.

We can verify that

$$\sum_{l,\ldots,n}\Gamma_l^{A_1}\cdots\Gamma_n^{A_N}\sigma_1^{\otimes N}\Gamma_n^{A_N\dagger}\cdots\Gamma_l^{A_1\dagger} = (1-p)^N\sigma_1^{\otimes N}, \tag{94}$$

$$\sum_{l,\ldots,n} \Gamma_l^{A_1} \cdots \Gamma_n^{A_N} \sigma_2^{\otimes N} \Gamma_n^{A_N \dagger} \cdots \Gamma_l^{A_1 \dagger} = (1-p)^N \sigma_2^{\otimes N}, \tag{95}$$

$$\sum_{l,\ldots,n} \Gamma_l^{A_1} \cdots \Gamma_n^{A_N} \sigma_3^{\otimes N} \Gamma_n^{A_N \dagger} \cdots \Gamma_l^{A_1 \dagger} = \sigma_3^{\otimes N}, \tag{96}$$

where $l = \cdots = n = 0, 1$.

Consider the following 2-qubit state with subsystems A_1, A_2

$$\rho_2 = \frac{1}{2^2}\left(\hat{I} + c_1\sigma_1^{\otimes 2} + c_2\sigma_2^{\otimes 2} + c_3\sigma_3^{\otimes 2}\right). \tag{97}$$

Under the operation of *phase-flip* channel, the output state $\varepsilon(\rho_2)$ is given by

$$\varepsilon(\rho_2) = \frac{1}{4}\left(\hat{I} + (1-p)^2 c_1\sigma_1^{\otimes 2} + (1-p)^2 c_2\sigma_2^{\otimes 2} + c_3\sigma_3^{\otimes 2}\right). \tag{98}$$

The nonzero and positive eigenvalues of $\varepsilon(\rho_2)$ are written as $\lambda_1 = 1-(1-p)^2c_1-(1-p)^2c_2-c_3$, $\lambda_2 = 1-(1-p)^2c_1+(1-p)^2c_2+c_3$, $\lambda_3 = 1+(1-p)^2c_1-(1-p)^2c_2+c_3$, $\lambda_4 = 1+(1-p)^2c_1+(1-p)^2c_2-c_3$. The derivative of λ_i with respect to p can be cast as

$$\lambda_1' = \frac{\partial\lambda_1}{\partial p} = 2(c_1+c_2)(1-p), \quad \lambda_2' = \frac{\partial\lambda_2}{\partial p} = 2(c_1-c_2)(1-p), \tag{99}$$

$$\lambda_3' = \frac{\partial\lambda_3}{\partial p} = -2(c_1-c_2)(1-p), \quad \lambda_4' = \frac{\partial\lambda_4}{\partial p} = -2(c_1+c_2)(1-p). \tag{100}$$

From (63), the local observational entropy of $\varepsilon(\rho_2)$ is given by

$$S_{O(\mathcal{C})}(\rho_2) = f(\beta_2^{\varepsilon}) + 2. \tag{101}$$

where $\beta_2^{\varepsilon} = \alpha_2(1-p)^2 + c_3 m_{A_1 3} m_{A_2 3}$, $|\beta_2^{\varepsilon}| \le 1$ and $\alpha_2 = \sum_{j=1}^{2} c_j m_{A_1 j} m_{A_2 j}$.

Quantum correlation entropy of $\varepsilon(\rho_2)$ is given by

$$S_{A_1A_2}^{QC}(\varepsilon(\rho_2)) = \inf_{\beta_2^{\varepsilon}} f(\beta_2^{\varepsilon}) + \sum_{i=1}^{4} \frac{\lambda_i}{4}\log_2\lambda_i, \tag{102}$$

Denote $F(p) = f(\beta_2^{\varepsilon}) + \sum_{i=1}^{4} \frac{\lambda_i}{4}\log_2\lambda_i$. The derivative of $F(p)$ is given by

$$\frac{\partial F(p)}{\partial p} = -\frac{\beta_2^{\varepsilon'}}{2}\log_2\frac{1+\beta_2^{\varepsilon}}{1-\beta_2^{\varepsilon}} + \frac{\lambda_1'}{4}\log_2\frac{\lambda_1}{\lambda_4} + \frac{\lambda_2'}{4}\log_2\frac{\lambda_2}{\lambda_3}, \tag{103}$$

where $\beta_2^{\varepsilon'} = \frac{\partial\beta_2^{\varepsilon}}{\partial p} = -2(1-p)\cdot\alpha_2$.

In order to show $\frac{\partial F(p)}{\partial p} < 0$, the following inequality must hold

$$2\alpha_2 \cdot \log_2\frac{1+\beta_2^{\varepsilon}}{1-\beta_2^{\varepsilon}} < (c_1+c_2)\log_2\frac{\lambda_4}{\lambda_1} + (c_1-c_2)\log_2\frac{\lambda_3}{\lambda_2}. \tag{104}$$

Therefore, (104) implies that $S_{A_1A_2}^{QC}(\varepsilon(\rho))$ is monotonically decreasing with respect to p.

Consider the following 3-qubit states associated with subsystems A_1, A_2 and A_3,

$$\rho_3 = \frac{1}{2^3}\left(\hat{I} + \sum_{j=1}^{3} c_j\sigma_j \otimes \sigma_j \otimes \sigma_j\right), \tag{105}$$

Under the action of *phase-flip* channel, the output state of ρ_3 is given by

$$\varepsilon(\rho_3) = \frac{1}{8}\left(\hat{I} + (1-p)^3 c_1\sigma_1^{\otimes 3} + (1-p)^3 c_2\sigma_2^{\otimes 3} + c_3\sigma_3^{\otimes 3}\right). \tag{106}$$

From (64), quantum correlation entropy of $\varepsilon(\rho_3)$ is given by

$$S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3)) = \inf_{\beta_3^\varepsilon}[f(\beta_3^\varepsilon)] + 3 - S(\varepsilon(\rho_3)). \tag{107}$$

where $\beta_3^\varepsilon = \alpha_3(1-p)^3 + c_3 m_{A_1 3} m_{A_2 3} m_{A_3 3}$, $|\beta_3^\varepsilon| \le 1$ and $\alpha_3 = \sum_{j=1}^{2} c_j m_{A_1 j} m_{A_2 j} m_{A_3 j}$.

Set $\theta = \sqrt{(1-p)^6 c_1^2 + (1-p)^6 c_2^2 + c_3^2}$, we have

$$S(\varepsilon(\rho_3)) = f(\theta) + 3. \tag{108}$$

Hence, we have

$$S^{QC}_{A_1A_2A_3}\varepsilon(\rho_3) = \inf_{\beta_3^\varepsilon}[f(\beta_3^\varepsilon)] - f(\theta) \tag{109}$$

Since $\theta > 0$, this implies that $\log_2 \frac{1+\theta}{1-\theta} > 0$.
We have

$$\theta' = \frac{\partial \theta}{\partial p} = -\frac{3(c_1^2 + c_2^2)(1-p)^5}{\theta}, \tag{110}$$

For $0 < (1-p)^5 < 1$, then $\theta' < 0$.

We assume that $S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3))$ is monotonically decreasing with respect to p, i.e., $\frac{\partial S^{QC}_{A_1A_2A_3}(\varepsilon(\rho))}{\partial p} < 0$.
Denote

$$g(p) = f(\beta_3^\varepsilon) - f(\theta). \tag{111}$$

We have

$$\frac{\partial g(p)}{\partial p} = -\frac{\beta_3^{\varepsilon'}}{2} \log_2 \frac{1+\beta_3^\varepsilon}{1-\beta_3^\varepsilon} + \frac{\theta'}{2} \log_2 \frac{1+\theta}{1-\theta}. \tag{112}$$

In order to show $\frac{\partial g(p)}{\partial p} < 0$, the following inequality must hold

$$\theta \alpha_3 \cdot \log_2 \frac{1+\beta_3^\varepsilon}{1-\beta_3^\varepsilon} < (c_1^2 + c_2^2)(1-p)^3 \log_2 \frac{1+\theta}{1-\theta}. \tag{113}$$

The c_1, c_2, c_3 and p satisfying (113) follows that $g(p)$ is monotonically decreasing with respect to p, i.e., $S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3))$ is monotonically decreasing.

For 4-qubit state, the state ρ associated with subsystems A_1, A_2, A_3, A_4 is given by

$$\rho_4 = \frac{1}{16}\left(\hat{I} + c_1\sigma_1^{\otimes 4} + c_2\sigma_2^{\otimes 4} + c_3\sigma_3^{\otimes 4}\right). \tag{114}$$

Under the operation of *phase-flip* channel, the output state $\varepsilon(\rho_4)$ is given by

$$\varepsilon(\rho_4) = \frac{1}{16}\left(\hat{I} + (1-p)^4 c_1\sigma_1^{\otimes 4} + (1-p)^4 c_2\sigma_2^{\otimes 4} + c_3\sigma_3^{\otimes 4}\right). \tag{115}$$

The nonzero and positive eigenvalues of $\varepsilon(\rho_4)$ are written as $\lambda_1 = 1 + (1-p)^4 c_1 + (1-p)^4 c_2 + c_3$, $\lambda_2 = 1 + (1-p)^4 c_1 - (1-p)^4 c_2 - c_3$, $\lambda_3 = 1 - (1-p)^4 c_1 + (1-p)^4 c_2 - c_3$, $\lambda_4 = 1 - (1-p)^4 c_1 - (1-p)^4 c_2 + c_3$. The derivative of λ_i with respect to p can be cast as

$$\lambda_1' = \frac{\partial \lambda_1}{\partial p} = -4(c_1 + c_2)(1-p)^3, \quad \lambda_2' = \frac{\partial \lambda_2}{\partial p} = -4(c_1 - c_2)(1-p)^3, \tag{116}$$

$$\lambda_3' = \frac{\partial \lambda_3}{\partial p} = 4(c_1 - c_2)(1-p)^3, \quad \lambda_4' = \frac{\partial \lambda_4}{\partial p} = 4(c_1 + c_2)(1-p)^3. \tag{117}$$

From (64), quantum correlation entropy of $\varepsilon(\rho_4)$ is given by

$$S^{QC}_{A_1A_2A_3A_4}(\varepsilon(\rho_4)) = \inf_{\beta_4^\varepsilon} f(\beta_4^\varepsilon) + \sum_{i=1}^{4} \frac{\lambda_i}{4} \log_2 \lambda_i, \tag{118}$$

where $\beta_4^\varepsilon = \alpha_4(1-p)^4 + c_3 m_{A_1 3} m_{A_2 3} m_{A_3 3} m_{A_4 3}$, $|\beta_4^\varepsilon| \leq 1$ and $\alpha_4 = \sum_{j=1}^{2} c_j m_{A_1 j} m_{A_2 j} m_{A_3 j} m_{A_4 j}$.

Denote $h(p) = f(\beta_4^\varepsilon) + \sum_{i=1}^4 \frac{\lambda_i}{4}\log_2 \lambda_i$. The derivative of $h(p)$ is given by

$$\frac{\partial h(p)}{\partial p} = -\frac{\beta_4^{\varepsilon'}}{2} \log_2 \frac{1+\beta_4^\varepsilon}{1-\beta_4^\varepsilon} + \frac{\lambda_1'}{4} \log_2 \frac{\lambda_1}{\lambda_4} + \frac{\lambda_2'}{4} \log_2 \frac{\lambda_2}{\lambda_3}, \tag{119}$$

where $\beta_4^{\varepsilon'} = \frac{\partial \beta_4^\varepsilon}{\partial p} = -4(1-p)^3 \cdot \alpha_4$.

In order to show $\frac{\partial h(p)}{\partial p} < 0$, the following inequality must hold

$$2\alpha_4 \cdot \log_2 \frac{1+\beta_4^\varepsilon}{1-\beta_4^\varepsilon} < (c_1+c_2)\log_2 \frac{\lambda_1}{\lambda_4} + (c_1-c_2)\log_2 \frac{\lambda_2}{\lambda_3}. \tag{120}$$

Therefore, (120) implies that $S^{QC}_{A_1A_2A_3A_4}(\varepsilon(\rho))$ is monotonically decreasing with respect to p.

A.4: Quantum Correlation Entropy Under Bit Phase Flip Channel

The *bit-phase* flip channel acting on a single qubit can be described by the following Kraus operators

$$\Gamma_0^{A_j} = \sqrt{1-\frac{p_j}{2}}\hat{I}, \quad \Gamma_1^{A_j} = \sqrt{\frac{p_j}{2}}\sigma_2, \tag{121}$$

where A_j labels the subsystems and $p \in [0, 1]$. Here, we consider the symmetric situation in which the decoherence rate is equal, so $p_1 = p_2 = \cdots = p$.

We can verify that

$$\sum_{l,\ldots,n} \Gamma_l^{A_1} \cdots \Gamma_n^{A_N} \sigma_1^{\otimes N} \Gamma_n^{A_N \dagger} \cdots \Gamma_l^{A_1 \dagger} = (1-p)^N \sigma_1^{\otimes N}, \tag{122}$$

$$\sum_{l,\ldots,n} \Gamma_l^{A_1} \cdots \Gamma_n^{A_N} \sigma_2^{\otimes N} \Gamma_n^{A_N \dagger} \cdots \Gamma_l^{A_1 \dagger} = \sigma_2^{\otimes N}, \tag{123}$$

$$\sum_{l,\ldots,n} \Gamma_l^{A_1} \cdots \Gamma_n^{A_N} \sigma_3^{\otimes N} \Gamma_n^{A_N \dagger} \cdots \Gamma_l^{A_1 \dagger} = (1-p)^N \sigma_3^{\otimes N}, \tag{124}$$

where $l, \ldots, n = 0, 1$.

Consider the following 2-qubit state associated with subsystems A_1, A_2

$$\rho_2 = \frac{1}{2^2}\left(\hat{I} + c_1\sigma_1^{\otimes 2} + c_2\sigma_2^{\otimes 2} + c_3\sigma_3^{\otimes 2}\right). \tag{125}$$

Under the operation of *bit-phase* flip channel, the output state $\varepsilon(\rho_2)$ is given by

$$\varepsilon(\rho_2) = \frac{1}{4}\left(\hat{I} + (1-p)^2 c_1\sigma_1^{\otimes 2} + c_2\sigma_2^{\otimes 2} + (1-p)^2 c_3\sigma_3^{\otimes 2}\right). \tag{126}$$

The nonzero and positive eigenvalues of $\varepsilon(\rho_2)$ are written as $\lambda_1 = 1-(1-p)^2c_1-c_2-(1-p)^2c_3$, $\lambda_2 = 1-(1-p)^2c_1+c_2+(1-p)^2c_3$, $\lambda_3 = 1+(1-p)^2c_1-c_2+(1-p)^2c_3$, $\lambda_4 = 1+(1-p)^2c_1+c_2-(1-p)^2c_3$. The derivative of λ_i with respect to can be cast as

$$\lambda_1' = \frac{\partial\lambda_1}{\partial p} = 2(c_1+c_3)(1-p), \quad \lambda_2' = \frac{\partial\lambda_2}{\partial p} = 2(c_1-c_3)(1-p), \tag{127}$$

$$\lambda_3' = \frac{\partial\lambda_3}{\partial p} = -2(c_1+c_3)(1-p), \quad \lambda_4' = \frac{\partial\lambda_4}{\partial p} = -2(c_1-c_3)(1-p). \tag{128}$$

From (63), the local observational entropy of $\varepsilon(\rho_2)$ is given by

$$S_{O(C)}(\rho_2) = f(\beta_2^\varepsilon) + 2. \tag{129}$$

where $\beta_2^\varepsilon = \alpha_2(1-p)^2 + c_2 m_{A_1 2} m_{A_2 2}$, $|\beta_2^\varepsilon| \le 1$ and $\alpha_2 = c_1 m_{A_1 1} m_{A_2 1} + c_3 m_{A_1 3} m_{A_2 3}$.

Quantum correlation entropy of $\varepsilon(\rho_2)$ is given by

$$S_{A_1A_2}^{QC}(\varepsilon(\rho_2)) = \inf_{\beta_2^\varepsilon} f(\beta_2^\varepsilon) + \sum_{i=1}^{4} \frac{\lambda_i}{4}\log_2\lambda_i, \tag{130}$$

Denote $F(p) = f(\beta_2^\varepsilon) + \sum_{i=1}^4 \frac{\lambda_i}{4}\log_2\lambda_i$. The derivative of $F(p)$ is given by

$$\frac{\partial F(p)}{\partial p} = -\frac{\beta_2^{\varepsilon'}}{2}\log_2\frac{1+\beta_2^\varepsilon}{1-\beta_2^\varepsilon} + \frac{\lambda_1'}{4}\log_2\frac{\lambda_1}{\lambda_3} + \frac{\lambda_2'}{4}\log_2\frac{\lambda_2}{\lambda_4}, \tag{131}$$

where $\beta_2^{\varepsilon'} = \frac{\partial\beta_2^\varepsilon}{\partial p} = -2(1-p)\cdot\alpha_2$.

In order to show $\frac{\partial F(p)}{\partial p} < 0$, it must satisfy that

$$2\alpha_2\cdot\log_2\frac{1+\beta_2^\varepsilon}{1-\beta_2^\varepsilon} < (c_1+c_3)\log_2\frac{\lambda_3}{\lambda_1} + (c_1-c_3)\log_2\frac{\lambda_4}{\lambda_2}. \tag{132}$$

Therefore, (132) implies that $S_{A_1A_2}^{QC}(\varepsilon(\rho_2))$ is monotonically decreasing with respect to p.

Consider the following 3-qubit states associated with subsystems A_1, A_2 and A_3,

$$\rho_3 = \frac{1}{2^3}\left(\hat{I} + \sum_{j=1}^{3} c_j\sigma_j\otimes\sigma_j\otimes\sigma_j\right), \tag{133}$$

Under the action of *phase-flip* channel $\varepsilon(\cdot)$, the output state of ρ_3 is given by

$$\varepsilon(\rho_3) = \frac{1}{8}\left(\hat{I} + (1-p)^3c_1\sigma_1^{\otimes 3} + c_2\sigma_2^{\otimes 3} + (1-p)^3c_3\sigma_3^{\otimes 3}\right). \tag{134}$$

From (64), quantum correlation entropy of $\varepsilon(\rho_3)$ is given by

$$S_{A_1A_2A_3}^{QC}(\varepsilon(\rho_3)) = \inf_{\beta_3^\varepsilon}[f(\beta_3^\varepsilon)] + 3 - S(\varepsilon(\rho_3)). \tag{135}$$

where $\beta_3^\varepsilon = \alpha_3(1-p)^3 + c_2 m_{A_1 2} m_{A_2 2} m_{A_3 2}$, $|\beta_3^\varepsilon| \le 1$ and $\alpha_3 = c_1 m_{A_1 1} m_{A_2 1} m_{A_3 1} + c_3 m_{A_1 3} m_{A_2 3} m_{A_3 3}$.

Set $\theta = \sqrt{(1-p)^6c_1^2 + c_2^2 + (1-p)^6c_3^2}$, we have

$$S(\varepsilon(\rho_3)) = f(\theta) + 3. \tag{136}$$

Hence, we have

$$S_{A_1A_2A_3}^{QC}(\varepsilon(\rho_3)) = \inf_{\beta_3^\varepsilon}[f(\beta_3^\varepsilon)] - f(\theta). \tag{137}$$

Since $\theta > 0$, this implies that $\log_2 \frac{1+\theta}{1-\theta} > 0$.

We have

$$\theta' = \frac{\partial \theta}{\partial p} = -\frac{3(c_1^2 + c_3^2)(1-p)^5}{\theta}, \tag{138}$$

For $0 < (1-p)^5 < 1$, then $\theta' < 0$.

We assume that $S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3))$ is monotonically decreasing with respect to p, i.e., $\frac{\partial S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3))}{\partial p} < 0$.

Denote

$$g(p) = f(\beta_3^{\varepsilon}) - f(\theta). \tag{139}$$

We have

$$\frac{\partial g(p)}{\partial p} = -\frac{\beta_3^{\varepsilon'}}{2}\log_2 \frac{1+\beta_3^{\varepsilon}}{1-\beta_3^{\varepsilon}} + \frac{\theta'}{2}\log_2 \frac{1+\theta}{1-\theta}. \tag{140}$$

In order to show $\frac{\partial g(p)}{\partial p} < 0$, the following inequality must hold

$$\theta\alpha_3 \cdot \log_2 \frac{1+\beta_3^{\varepsilon}}{1-\beta_3^{\varepsilon}} < (c_1^2 + c_3^2)(1-p)^3 \log_2 \frac{1+\theta}{1-\theta}. \tag{141}$$

The c_1, c_2, c_3 and p satisfying (141) follows that $g(p)$ is monotonically decreasing with respect to p, i.e., $S^{QC}_{A_1A_2A_3}(\varepsilon(\rho_3))$ is monotonically decreasing.

For 4-qubit state, the state ρ associated with subsystems A_1, A_2, A_3, A_4 is given by

$$\rho_4 = \frac{1}{16}\left(\hat{I} + c_1\sigma_1^{\otimes 4} + c_2\sigma_2^{\otimes 4} + c_3\sigma_3^{\otimes 4}\right). \tag{142}$$

Under the operation of *bit-phase* flip channel, the output state $\varepsilon(\rho_4)$ is given by

$$\varepsilon(\rho_4) = \frac{1}{16}(\hat{I} + (1-p)^4 c_1\sigma_1^{\otimes 4} + c_2\sigma_2^{\otimes 4} + (1-p)^4 c_3\sigma_3^{\otimes 4}). \tag{143}$$

The nonzero and positive eigenvalues of $\varepsilon(\rho_4)$ are written as $\lambda_1 = 1 + (1-p)^4 c_1 + c_2 + (1-p)^4 c_3$, $\lambda_2 = 1 + (1-p)^4 c_1 - c_2 - (1-p)^4 c_3$, $\lambda_3 = 1 - (1-p)^4 c_1 - c_2 + (1-p)^4 c_3$, $\lambda_4 = 1 - (1-p)^4 c_1 + c_2 - (1-p)^4 c_3$. The derivative of λ_i with respect to p can be cast as

$$\lambda_1' = \frac{\partial \lambda_1}{\partial p} = -4(c_1 + c_3)(1-p)^3, \quad \lambda_2' = \frac{\partial \lambda_2}{\partial p} = -4(c_1 - c_3)(1-p)^3, \tag{144}$$

$$\lambda_3' = \frac{\partial \lambda_3}{\partial p} = 4(c_1 - c_3)(1-p)^3, \quad \lambda_4' = \frac{\partial \lambda_4}{\partial p} = 4(c_1 + c_3)(1-p)^3. \tag{145}$$

From (64), quantum correlation entropy of $\varepsilon(\rho_4)$ is given by

$$S^{QC}_{A_1A_2A_3A_4}(\varepsilon(\rho_4)) = \inf_{\beta_4^{\varepsilon}} f(\beta_4^{\varepsilon}) + \sum_{i=1}^{4} \frac{\lambda_i}{4}\log_2 \lambda_i, \tag{146}$$

where $\beta_4^{\varepsilon} = \alpha_4(1-p)^4 + c_2 m_{A_12} m_{A_22} m_{A_32} m_{A_42}$, $|\beta_4^{\varepsilon}| \leq 1$ and

$$\alpha_4 = c_1 m_{A_11} m_{A_21} m_{A_31} m_{A_41} + c_3 m_{A_13} m_{A_23} m_{A_33} m_{A_43}$$

.

Denote $h(p) = f(\beta_4^{\varepsilon}) + \sum_{i=1}^{4} \frac{\lambda_i}{4}\log_2 \lambda_i$. The derivative of $h(p)$ with respect to p is given by

$$\frac{\partial h(p)}{\partial p} = -\frac{\beta_4^{\varepsilon'}}{2}\log_2 \frac{1+\beta_4^{\varepsilon}}{1-\beta_4^{\varepsilon}} + \frac{\lambda_1'}{4}\log_2 \frac{\lambda_1}{\lambda_4} + \frac{\lambda_2'}{4}\log_2 \frac{\lambda_2}{\lambda_3}, \tag{147}$$

where $\beta_4^{\varepsilon'} = -4(1-p)^3 \cdot \alpha_4$.

In order to show $\frac{\partial h(p)}{\partial p} < 0$, the following inequality must hold

$$2\alpha_4 \cdot \log_2 \frac{1+\beta_4^{\varepsilon}}{1-\beta_4^{\varepsilon}} < (c_1+c_3)\log_2 \frac{\lambda_1}{\lambda_4} + (c_1-c_3)\log_2 \frac{\lambda_2}{\lambda_3}. \tag{148}$$

Therefore, (148) implies that $S^{QC}_{A_1A_2A_3A_4}(\varepsilon(\rho_4))$ is monotonically decreasing with respect to p.

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Data Availability The datasets analysed during the current study are available from the corresponding author on reasonable request.

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