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Ulam-Hyers stability, well-posedness and limit shadowing property of the fixed point problems in M-metric spaces

Adoon Pansuwan[a], Wutiphol Sintunavarat[a,*], Jae Young Choi[b], Yeol Je Cho[b,c,*]

[a]*Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathumthani 12121, Thailand.*

[b]*Department of Mathematics Education, Gyeongsang National University, Jinju 660-701, Korea.*

[c]*Center for General Education, China Medical University, Taichung 40402, Taiwan.*

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Abstract

In this paper, we introduce several types of Ulam-Hyers stability, well-posedness and limit shadowing property of the fixed point problem in M-metric spaces. Also, we give such results for fixed point problems of Banach and Kannan contractive mappings in M-metric spaces and provide two examples to illustrate the results presented herein.

Keywords: Fixed point, limit shadowing property, M-metric space, Ulam-Hyers stability, well-posedness.
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1. Introduction

It is well-known that the distance function or metric on arbitrary nonempty set is very useful in many branches of mathematical analysis. For instance, we recall the following classical concept of the limit in fundamental calculus:

- We say that the limit of a real-valued function $y = f(x)$ as x closed to a point $a \in \mathbb{R}$ is a real number L if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

*Corresponding author
Email addresses: pansuwan@mathstat.sci.tu.ac.th (Adoon Pansuwan), wutiphol@mathstat.sci.tu.ac.th (Wutiphol Sintunavarat), darksoul83@naver.com (Jae Young Choi), yjcho@gnu.ac.kr (Yeol Je Cho)

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Indeed, the concept of the absolute value on $\mathbb{R}$ is a special case of a metric on a nonempty set X.

In 1922, one of the very famous results obtained in a metric space was **Banach's fixed point theorem** (or **Banach's contraction principle**) which was introduced by Banach [5].

Theorem 1.1 (**Banach's fixed point theorem**). *Let (X, d) be a complete metric space and $T : X \to X$ be a contractive mapping, that is, there exists $L \in [0, 1)$ such that*

$$d(Tx, Ty) \leq Ld(x, y) \tag{BC}$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$, that is, $Tz = z$. Moreover, for each $x_0 \in X$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n$$

for all $n \in \mathbb{N} \cup \{0\}$ converges to the fixed point z of T.

Since then, because of their simplicity, usefulness and applications, it has become a very popular tools in solving the existence problems in many branches of mathematical analysis.

From the Banach contractive condition (BC), it follows that the mapping T is continuous. In fact, to show the existence of a fixed point of the mapping T, we have to use the continuity of the mapping T. Thus it is natural to consider the following question:

- To show the existence of a fixed point of the mapping T, do there exist some contractive conditions which do not force the mapping T to be continuous?

In 1968 and 1972, Kannan [11] and Chatterjea [8] gave the positive answers for this question by proving the following fixed point theorems for contractive conditions in complete metric spaces, which are called the *Kannan contraction* and *Chatterjea contraction*, respectively:

Theorem 1.2 (**Kannan's fixed point theorem**). *Let (X, d) be a complete metric space and $T : X \to X$ be a Kannan contractive mapping, i.e., there exists $k \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \tag{1.1}$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$, that is, $Tz = z$. Moreover, for each $x_0 \in X$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n$$

for all $n \in \mathbb{N} \cup \{0\}$ converges to the fixed point z of T.

Theorem 1.3 (**Chatterjea's fixed point theorem**). *Let (X, d) be a complete metric space and $T : X \to X$ be a Chatterjia contractive mapping, i.e., there exists $k \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)] \tag{1.2}$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$, that is, $Tz = z$. Moreover, for each $x_0 \in X$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n$$

for all $n \in \mathbb{N} \cup \{0\}$ converges to the fixed point z of T.

During the last few decades, the concept of a metric space has been generalized in many directions.

Especially, in 1994, Matthews [14] extended the concept of a metric to a partial metric and introduced the notion of partial metric space. Indeed, the motivation for introducing the concept of a partial metric was to obtain appropriate mathematical models in the theory of computation. Also, he obtained many results in partial metric spaces. In particular, he gave the improvement of Banach's contraction principle in the sense of partial metric spaces. Afterward, many mathematicians have studied the existence and uniqueness of a fixed point for nonlinear mappings satisfying various contractive conditions in the setting of partial metric spaces.

Recently, the concept of a partial metric space was extended to the concept of an M-metric space by Asadi et al. in [3]. They also studied topological properties in such spaces and established some fixed point results, which are generalization of Banach's and Kannan's fixed point theorems in the framework of partial metric spaces.

Theorem 1.4 ([3])**.** *Let (X, m) be a complete M-metric space and $T : X \to X$ be a Banach contractive mapping, i.e., there exists $k \in [0, 1)$ such that*

$$m(Tx, Ty) \leq km(x, y) \tag{1.3}$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$, that is, $Tz = z$. Moreover, for each $x_0 \in X$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n$$

for all $n \in \mathbb{N} \cup \{0\}$ converges to the fixed point z of T.

Theorem 1.5 ([3])**.** *Let (X, m) be a complete M-metric space and $T : X \to X$ be a Kannan contractive mapping, i.e., there exists $k \in [0, \frac{1}{2})$ such that*

$$m(Tx, Ty) \leq k[m(x, Tx) + m(y, Ty)] \tag{1.4}$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$, that is, $Tz = z$. Moreover, for each $x_0 \in X$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n$$

for all $n \in \mathbb{N} \cup \{0\}$ converges to the fixed point z of T.

On the other hand, there are a number of results that studied and extended the Ulam-Hyers stability for fixed point problems as Bota et al. [6], Bota-Boriceanu and Petrusel [7]. Also, the notion of the *well-posedness* and the *limit shadowing property* of the fixed point problem have evoked much interest to many researchers, for example, De Blassi and Myjak [10], Reich and Zaslavski [15], Lahiri and Das [12].

In this paper, first, we define various types of the Ulam-Hyers stability, the well-posedness and the limit shadowing property of the fixed point problem in M-metric spaces which are generalizations of the well-known concepts in metric spaces. Second, we deal with the Ulam-Hyers stability, the well-posedness and the limit shadowing property of the fixed point problem for Banach and Kannan contractive mappings in M-metric spaces. Finally, we furnish two suitable examples to demonstrate the validity of the hypotheses of our main results in this paper.

2. Preliminaries

Throughout this work, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, nonnegative real numbers and real numbers, respectively.

The following definitions, notations and lemma are needed in the sequel:

Definition 2.1 ([14])**.** Let X be a nonempty set and a function $p : X \times X \to \mathbb{R}_+$ satisfies the following conditions: for all $x, y, z \in X$,

(P1) $p(x, x) = p(y, y) = p(x, y) \Longleftrightarrow x = y$;

(P2) $p(x, x) \leq p(x, y)$;

(P3) $p(x, y) = p(y, x)$;

(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then p is said to be a *partial metric* and a pair (X, p) is called a *partial metric space*.

It is easy to see that a metric is also a partial metric, but the converse is not true in general case. Now, we give some examples to show that a partial metric need not to be necessarily a metric.

Example 2.2. Let $X = [0, \infty)$ and $p : X \times X \to \mathbb{R}_+$ be defined by

$$p(x, y) = \max\{x, y\}$$

for all $x, y \in X$. Then p is a partial metric on X, but it is not a metric on X. Indeed, for any $x > 0$, we have $p(x, x) = x \neq 0$.

Example 2.3. Let $X = \{[a, b] : a, b \in \mathbb{R},\ a \leq b\}$ and $p : X \times X \to \mathbb{R}_+$ be defined by

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$$

for all $[a, b], [c, d] \in X$. Then p is a partial metric on X, but it is not a metric on X. Indeed, $p([1, 2], [1, 2]) = 1$.

For a nonempty set X and a function $m : X \times X \to \mathbb{R}_+$, the following notation are useful in the sequel:

(1) $m_{xy} := \min\{m(x, x), m(y, y)\}$;

(2) $M_{xy} := \max\{m(x, x), m(y, y)\}$.

Definition 2.4 ([3])**.** Let X be a nonempty set and $m : X \times X \to \mathbb{R}_+$ be a function satisfying the following conditions: for all $x, y, z \in X$,

(M1) $m(x, x) = m(y, y) = m(x, y)$ if and only if $x = y$;

(M2) $m_{xy} \leq m(x, y)$;

(M3) $m(x, y) = m(y, x)$;

(M4) $m(x, y) - m_{xy} \leq [m(x, z) - m_{xz}] + [m(z, y) - m_{zy}]$.

Then m is said to be an *m-metric* and a pair (X, m) is called an *M-metric space*.

A simple example of m-metric is arbitrary partial metric (see in [3]). Hence the class of m-metrics is larger than the class of partial metrics. Then we obtain the following relation:

$$\boxed{\text{metric}} \Longrightarrow \boxed{\text{partial metric}} \Longrightarrow \boxed{m\text{-metric}}.$$

Remark 2.5. If m is an m-metric on a nonempty set X, then two functions $m^w, m^s : X \times X \to \mathbb{R}_+$ defined by

$$m^w(x, y) := m(x, y) - 2m_{xy} + M_{xy}$$

and

$$m^s(x, y) := \begin{cases} m(x, y) - m_{xy}, & x \neq y, \\ 0, & x = y, \end{cases}$$

are metrics on X.

Now, we give some examples which show that an m-metric is a proper real generalization of a partial metric.

Example 2.6. Let $X = [0, \infty)$ and $m : X \times X \to \mathbb{R}_+$ be a function defined by

$$m(x, y) = \frac{x + y}{2}$$

for all $x, y \in X$. Then m is an m-metric, but it is not a p-metric. Indeed, $m(3, 3) = 3 > 2 = m(1, 3)$.

Example 2.7. Let $X = \{1, 2, 3\}$ and $m : X \times X \to \mathbb{R}_+$ be a function defined by

$$m(x, y) = \begin{cases} 1, & x = y = 1, \\ 9, & x = y = 2, \\ 5, & x = y = 3, \\ 10, & x, y \in \{1, 2\} \text{ and } x \neq y, \\ 7, & x, y \in \{1, 3\} \text{ and } x \neq y, \\ 8, & x, y \in \{2, 3\} \text{ and } x \neq y. \end{cases}$$

Then m is an m-metric but it is not a p-metric. Indeed, $m(2, 2) = 9 > 8 = m(2, 3)$.

Next, we give the concepts of convergent sequence, m-Cauchy sequence and completeness in M-metric spaces.

Definition 2.8. Let (X, m) be an m-metric space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to point $x \in X$ if

$$\lim_{n\to\infty} [m(x_n, x) - m_{x_n x}] = 0; \tag{2.1}$$

(2) A sequence $\{x_n\}$ in X is called an *m-Cauchy sequence* if

$$\lim_{n,m\to\infty} [m(x_n, x_m) - m_{x_n x_m}], \quad \lim_{n,m\to\infty} [M_{x_n x_m} - m_{x_n x_m}] \tag{2.2}$$

exist and are finite;

(3) A space X is said to be *complete* if every m-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that

$$\lim_{n\to\infty} [m(x_n, x) - m_{x_n x}] = 0, \quad \lim_{n\to\infty} [M_{x_n x} - m_{x_n x}] = 0.$$

Lemma 2.9 ([3])**.** *Let (X, m) be an m-metric space. Then the following assertions hold.*

(1) *$\{x_n\}$ is an m-Cauchy sequence in (X, m) if and only if it is a Cauchy sequence in the metric space (X, m^w);*

(2) *(X, m) is complete if and only if the metric space (X, m^w) is complete. Furthermore, for a sequence $\{x_n\}$ in X and $x \in X$, we have*

$$\lim_{n\to\infty} m^w(x_n, x) = 0 \Longleftrightarrow \lim_{n\to\infty} [m(x_n, x) - m_{x_n x}] = 0, \quad \lim_{n\to\infty} [M_{x_n x} - m_{x_n x}] = 0.$$

Moreover, two above assertions hold for m^s.

Example 2.10. Let $X = [0, \infty)$ and $m : X \times X \to \mathbb{R}_+$ be a function defined by

$$m(x, y) = \frac{x + y}{2}$$

for all $x, y \in X$. Then (X, m) is a complete M-metric space since $(X, m^w) = ([0, \infty), \frac{3}{2}|\cdot|)$ is a complete metric space.

3. On Fixed Point Problem for Banach's Contractive Mappings in M-Metric Spaces

In this section, we introduce the concepts of the Ulam-Hyers stability, the well-posedness and the limit shadowing property of the fixed point problem in M-metric spaces. Also, we study the Ulam-Hyers stability, the well-posedness and the limit shadowing property results for the fixed point problem of Banach contractive mappings in M-metric spaces.

Definition 3.1. Let (X, m) be an M-metric space and $T : X \to X$ be a mapping. The fixed point problem

$$x = Tx \tag{3.1}$$

is said to be *Ulam-Hyers stable* if there exists $c > 0$ such that, for any $\epsilon > 0$ and for each $w^* \in X$ which is an ϵ-solution of the fixed point problem (3.1), i.e., w^* satisfies the inequality

$$m(w^*, Tw^*) \leq \epsilon, \tag{3.2}$$

there exists a solution $x^* \in X$ of the equation (3.1) such that

$$m(x^*, w^*) \leq c\epsilon.$$

Definition 3.2. Let (X, m) be an M-metric space and $T : X \to X$ be a mapping. The fixed point problem of T is said to be *well-posed* if the following conditions hold:

(1) T has a unique fixed point x^* in X;

(2) for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} m(x_n, Tx_n) = 0$, we have $\lim_{n\to\infty} m(x_n, x^*) = 0$.

Definition 3.3. Let (X, m) be an M-metric space and $T : X \to X$ be a mapping. The fixed point problem of T is said to have the *limit shadowing property* in X if, for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} m(x_n, Tx_n) = 0$, it follows that there exists $z \in X$ such that

$$\lim_{n\to\infty} m(T^n z, x_n) = 0.$$

Theorem 3.4. *Let (X, m) be a complete M-metric space and $T : X \to X$ be a Banach contractive mapping with constant $k \in [0, 1)$. Then the following assertions hold:*

(1) *the fixed point problem of T is Ulam-Hyers stable;*

(2) *the fixed point problem of T is well-posed;*

(3) *the fixed point problem of T has the limit shadowing property in X.*

Proof. From Theorem 1.4, it follows that T has a unique fixed point and so let x^* be a unique fixed point of T.

Next, we claim that the fixed point problem of T is Ulam-Hyers stable. Let us $\epsilon > 0$ and $w^* \in X$ be a solution of (3.2), i.e.,

$$m(w^*, Tw^*) \leq \epsilon.$$

From (M_4), we obtain that

$$\begin{aligned}
m(x^*, w^*) &\leq [m(x^*, Tw^*) - m_{x^*(Tw^*)}] + [m(Tw^*, w^*) - m_{(Tw^*)w^*}] + m_{x^*w^*} \\
&= [m(Tx^*, Tw^*) - m_{x^*(Tw^*)}] + [m(Tw^*, w^*) - m_{(Tw^*)w^*}] + m_{x^*w^*} \\
&\leq [km(x^*, w^*) - m_{x^*(Tw^*)}] + m(Tw^*, w^*) + [m_{x^*w^*} - m_{(Tw^*)w^*}] \\
&= [km(x^*, w^*) - m_{x^*(Tw^*)}] + m(Tw^*, w^*) \\
&\quad + [\min\{m(x^*, x^*), m(w^*, w^*)\} - \min\{m(Tw^*, Tw^*), m(w^*, w^*)\}] \\
&= [km(x^*, w^*) - m_{x^*(Tw^*)}] + m(Tw^*, w^*) \\
&\quad + [\min\{m(x^*, x^*), m(w^*, w^*)\} - m(w^*, w^*)] \\
&\leq km(x^*, w^*) + \epsilon,
\end{aligned}$$

which implies that

$$m(x^*, w^*) \le c\epsilon,$$

where $c := \frac{1}{1-k} > 0$. Therefore, the fixed point problem of T is Ulam-Hyers stable.

Next, we prove that the fixed point problem of T is well-posed. Assume that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} m(x_n, Tx_n) = 0$. Here, we show that $\lim_{n\to\infty} m(x^*, x_n) = 0$. By (M4), we obtain that

$$\begin{aligned} m(x^*, x_n) &\le [m(x^*, Tx_n) - m_{x^*(Tx_n)}] + [m(Tx_n, x_n) - m_{(Tx_n)x_n}] + m_{x^*x_n} \\ &= [m(Tx^*, Tx_n) - m_{x^*(Tx_n)}] + [m(Tx_n, x_n) - m_{(Tx_n)x_n}] + m_{x^*x_n} \\ &\le [km(x^*, x_n) - m_{x^*(Tx_n)}] + m(Tx_n, x_n) + [m_{x^*x_n} - m_{(Tx_n)x_n}] \\ &= [km(x^*, x_n) - m_{x^*(Tx_n)}] + m(Tx_n, x_n) \\ &\quad + [\min\{m(x^*, x^*), m(x_n, x_n)\} - \min\{m(Tx_n, Tx_n), m(x_n, x_n)\}] \\ &= [km(x^*, x_n) - m_{x^*(Tx_n)}] + m(Tx_n, x_n) \\ &\quad + [\min\{m(x^*, x^*), m(x_n, x_n)\} - m(x_n, x_n)] \\ &\le km(x^*, x_n) + m(Tx_n, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$, which implies that

$$m(x^*, x_n) \le \frac{1}{1-k} m(Tx_n, x_n) \tag{3.3}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in (3.3), we have $\lim_{n\to\infty} m(x_n, x^*) = 0$ and hence the fixed point problem of T is well-posed.

Finally, we prove that T has the limit shadowing property. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} m(x_n, Tx_n) = 0$. By the similar method as in (3.3), we have $\lim_{n\to\infty} m(x_n, x^*) = 0$. Since x^* is a fixed point of T, we have

$$\lim_{n\to\infty} m(x_n, T^n x^*) = \lim_{n\to\infty} m(x_n, x^*) = 0.$$

Therefore, T has the limit shadowing property. This completes the proof. □

Now, we give two examples to illustrate Theorem 3.4.

Example 3.5. Let $X = [0, \infty)$ and $m : X \times X \to \mathbb{R}_+$ be a function defined by

$$m(x, y) = \frac{x + y}{2}$$

for all $x, y \in X$. Then (X, m) is a complete M-metric space. Define a mapping $T : X \to X$ by $Tx = \frac{x}{2}$ for all $x \in X$. For each $x, y \in X$, we obtain

$$m(Tx, Ty) = \frac{1}{2}\left(\frac{x}{2} + \frac{y}{2}\right) = \frac{1}{2} m(x, y).$$

It follows that T is a Banach contractive mapping.

First, we claim that the fixed point problem of T is Ulam-Hyers stable. Assume that $\epsilon > 0$ and $w^* \in X$ is an ϵ-solution of the fixed point problem of T, that is,

$$m(w^*, Tw^*) \le \epsilon \implies \frac{1}{2}\left(w^* + \frac{w^*}{2}\right) \le \epsilon \implies \frac{w^*}{2} \le \frac{2}{3}\epsilon.$$

It is easy to see that $x^* = 0$ is a solution of the fixed point of T and

$$m(x^*, w^*) = m(0, w^*) = \frac{w^*}{2} \le \frac{2}{3}\epsilon$$

and so the fixed point problem of T is Ulam-Hyers stable.

Next, we prove that the fixed point problem of T is well-posed. We can see that $x^* = 0$ is a unique fixed point of T.

Now, we assume that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} m(x_n, Tx_n) = 0$, that is,

$$\lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{x_n}{2}\right) = 0 \implies \lim_{n\to\infty} x_n = 0.$$

Then we obtain

$$\lim_{n\to\infty} m(x_n, x^*) = \lim_{n\to\infty} m(x_n, 0) = \lim_{n\to\infty} \frac{x_n}{2} = 0$$

and so the fixed point problem of T is well-posed.

Finally, we show that the fixed point problem of T has the limit shadowing property in X. Suppose that $\{x_n\}$ is any sequence in X such that $\lim_{n\to\infty} m(x_n, Tx_n) = 0$. It follows that $\lim_{n\to\infty} x_n = 0$. We can see that there is $z = 0 \in X$ such that

$$\lim_{n\to\infty} m(T^n z, x_n) = \lim_{n\to\infty} m(0, x_n) = \lim_{n\to\infty} \frac{x_n}{2} = 0,$$

which implies that the fixed point problem of T has the limit shadowing property in X.

Example 3.6. Let $X = [0, 1]$ and $m : X \times X \to \mathbb{R}_+$ be a function defined by

$$m(x, y) = \frac{x + y}{2}$$

for all $x, y \in X$. Then (X, m) is a complete M-metric space. Define a mapping $T : X \to X$ by $Tx = \frac{x^2}{2}$ for all $x \in X$. For each $x, y \in X$, we have

$$m(Tx, Ty) = \frac{1}{2}\left(\frac{x^2}{2} + \frac{y^2}{2}\right) \leq \frac{1}{2}\left(\frac{x}{2} + \frac{y}{2}\right) = \frac{1}{2} m(x, y).$$

This implies that T is a Banach contractive mapping.

First, we prove that the fixed point problem of T is Ulam-Hyers stable. Suppose that $\epsilon > 0$ and $w^* \in X$ is an ϵ-solution of the fixed point problem of T, that is,

$$m(w^*, Tw^*) \leq \epsilon \implies \frac{1}{2}\left(w^* + \frac{(w^*)^2}{2}\right) \leq \epsilon \implies \frac{w^*}{2} + \frac{(w^*)^2}{4} \leq \epsilon.$$

We can show that $x^* = 0$ is a solution of the fixed point of T and hence

$$m(x^*, w^*) = m(0, w^*) = \frac{w^*}{2} \leq \frac{w^*}{2} + \frac{(w^*)^2}{4} \leq \epsilon.$$

This finishes our claim.

Next, we prove that the fixed point problem of T is well-posed. It is easy to see that $x^* = 0$ is a unique fixed point of T. Now, we assume that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} m(x_n, Tx_n) = 0$. For each $n \in \mathbb{N}$, we obtain

$$m(x_n, x^*) = m(x_n, 0) = \frac{x_n}{2} \leq \frac{1}{2}\left(x_n + \frac{x_n}{2}\right) = m(x_n, Tx_n).$$

Since $\lim_{n\to\infty} m(x_n, Tx_n) = 0$, we also have $\lim_{n\to\infty} m(x_n, x^*) = 0$ and so the fixed point problem of T is well-posed.

Finally, we prove that the fixed point problem of T has the limit shadowing property in X. Suppose that $\{x_n\}$ is any sequence in X so that $\lim_{n\to\infty} m(x_n, Tx_n) = 0$. Let us $z := 0 \in X$. It follows from

$$m(T^n z, x_n) = m(0, x_n) = \frac{x_n}{2} \leq \frac{1}{2}\left(x_n + \frac{x_n}{2}\right) = m(x_n, Tx_n) \to 0$$

as $n \to \infty$. Thus the fixed point problem of T has the limit shadowing property in X.

4. On Fixed Point Problem for Kannan's Contractive Mappings in M-Metric Spaces

The purpose of this section is to introduce another type of the Ulam-Hyers stability, the well-posedness and the limit shadowing property of the fixed point problem in M-metric spaces. By using these concept, we give the second main results for the fixed point problem of Kannan contractive mappings in M-metric spaces.

Definition 4.1. Let (X, m) be an M-metric space and $T : X \to X$ be a mapping. The fixed point problem

$$x = Tx \tag{4.1}$$

is said to be *Ulam-Hyers stable type* (K) if there exists $c > 0$ such that, for each $\epsilon > 0$, for each $w^* \in X$ which is an ϵ-solution of the fixed point equation (4.1), i.e., w^* satisfies the inequality

$$m(w^*, Tw^*) \leq \epsilon, \tag{4.2}$$

there exists a solution $x^* \in X$ of the equation (4.1) such that

$$m(x^*, w^*) - cm(x^*, x^*) \leq c\epsilon.$$

Remark 4.2. It is easy to see that the Ulam-Hyers stability of the fixed point problem implies the Ulam-Hyers stability type (K).

Definition 4.3. Let (X, m) be an M-metric space and $T : X \to X$ be a mapping. The fixed point problem of T is said to be *well-posed type* (K) if the following conditions hold:

(1) T has a unique fixed point x^* in X;

(2) there exists $c > 0$ such that, for any sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} m(x_n, Tx_n) = 0$, we have $\lim_{n\to\infty} m(x_n, x^*) = cm(x^*, x^*)$.

Definition 4.4. Let (X, m) be an M-metric space and $T : X \to X$ be a mapping. The fixed point problem of T is said to have the *limit shadowing property type* (K) in X if there exists $c > 0$ such that, for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} m(x_n, Tx_n) = 0$, it follows that there exists $z \in X$ such that $\lim_{n\to\infty} m(T^n z, x_n) = cm(z, z)$.

Theorem 4.5. *Let (X, m) be a complete M-metric space and $T : X \to X$ be a Kannan contractive mapping with constant $k \in [0, 1/2)$. Then the following assertions hold:*

(1) *the fixed point problem of T is Ulam-Hyers stable type (K);*

(2) *the fixed point problem of T is well-posed type (K);*

(3) *the fixed point problem of T has the limit shadowing property type (K) in X.*

Proof. From Theorem 1.5, it follows that T has a unique fixed point and so let x^* be a unique fixed point of T.

Now, we claim that the fixed point problem of T is Ulam-Hyers stable type (K). Let $\epsilon > 0$ and $w^* \in X$ be a solution of (3.2), that is,

$$m(w^*, Tw^*) \leq \epsilon.$$

From (M4), we obtain

$$\begin{aligned}
m(x^*, w^*) &\leq [m(x^*, Tw^*) - m_{x^*(Tw^*)}] + [m(Tw^*, w^*) - m_{(Tw^*)w^*}] + m_{x^*w^*} \\
&= [m(Tx^*, Tw^*) - m_{x^*(Tw^*)}] + [m(Tw^*, w^*) - m_{(Tw^*)w^*}] + m_{x^*w^*} \\
&\leq m(Tx^*, Tw^*) + m(Tw^*, w^*) + m_{x^*w^*} \\
&\leq k[m(x^*, Tx^*) + m(w^*, Tw^*)] + m(Tw^*, w^*) + m(x^*, x^*) \\
&= k[m(x^*, x^*) + m(w^*, Tw^*)] + m(Tw^*, w^*) + m(x^*, x^*) \\
&= (k+1)m(x^*, x^*) + (k+1)\epsilon,
\end{aligned}$$

which implies that

$$m(x^*, w^*) - cm(x^*, x^*) \leq c\epsilon,$$

where $c := k + 1 > 0$. Thus the fixed point problem of T is Ulam-Hyers stable type (K).

Next, we prove that the fixed point problem of T is well-posed type (K). Assume that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} m(x_n, Tx_n) = 0$. From (M4), it follows that

$$\begin{aligned} m(x^*, x_n) &\leq [m(x^*, Tx_n) - m_{x^*(Tx_n)}] + [m(Tx_n, x_n) - m_{(Tx_n)x_n}] + m_{x^*x_n} \\ &= [m(Tx^*, Tx_n) - m_{x^*(Tx_n)}] + [m(Tx_n, x_n) - m_{(Tx_n)x_n}] + m_{x^*x_n} \\ &\leq m(Tx^*, Tx_n) + m(Tx_n, x_n) + m_{x^*x_n} \\ &\leq k[m(x^*, Tx^*) + m(x_n, Tx_n)] + m(Tx_n, x_n) + m(x^*, x^*) \\ &= k[m(x^*, x^*) + m(x_n, Tx_n)] + m(Tx_n, x_n) + m(x^*, x^*) \\ &= (k+1)[m(x^*, x^*) + m(x_n, Tx_n)] \end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} m(x_n, x^*) = (k+1)m(x^*, x^*) \tag{4.3}$$

and so the fixed point problem of T is well-posed type (K).

Finally, we prove that the fixed point problem of T has the limit shadowing property type (K). Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} m(x_n, Tx_n) = 0$. Since x^* is a fixed point of T, from (4.3), it follows that

$$\lim_{n\to\infty} m(x_n, T^n x^*) = \lim_{n\to\infty} m(x_n, x^*) = (k+1)m(x^*, x^*).$$

Therefore, the fixed point problem of T has the limit shadowing property type (K). This completes the proof. □

5. Conclusions and Open Problems

In this paper, based on the fixed point results of Asadi et al. [3], we have studied the Ulam-Hyers stability, well-posedness and limit shadowing property results for the fixed point problems of Banach and Kannan contractive mappings in M-metric spaces. Also, we give two examples to illustrate the validity of the hypotheses and degree of utility of our results.

However, several fixed point results established in M-metric spaces and other spaces have been studied by many mathematicians, for example, see Asadi's results in [2] and Abodayeh et al. [1]. Therefore, in the next paper, we will study the Ulam-Hyers stability, well-posedness and limit shadowing of fixed point problems for various kinds of nonlinear mappings in many distance spaces.

Finally, we propose the following problems:

(1) Can we define other types of the Ulam-Hyers stability, well-posedness and limit shadowing property of the fixed point problem in M-metric spaces?

(2) Can we extend the results in this paper to some other spaces, for example, b-metric spaces [9], complex-valued metric spaces [4], C^*-algebra-valued metric spaces [13] and others?

(3) Can we define new contractive condition in M-metric spaces and prove new Ulam-Hyers stability, well-posedness and limit shadowing property of the fixed point problems in M-metric spaces?

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