SPLITTINGS OF LATTICES OF THEORIES AND UNIFICATION TYPES

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Abstract. We show that the lattice of all theories extending the equational theory of Heyting algebras is split into two parts, "upper" and "lower". Such a splitting is related to unification types: the upper part contains all theories having unitary unification type (and some of nullary type), the lower part contains all theories having finitary unification type (and some of nullary type, plus some with infinitary type, if there are such). The same splitting determines a limit between theories of constructive (upper part) and non-constructive (lower part) character.

Similar results are proved for the lattice of all theories extending the equational theory of interior algebras (or topological Boolean algebras).

Symbolic unification was introduced to logic in the pioneering paper of J. Herbrand in 1930. It was later applied in Proof Theory and Automated Deduction, in particular for resolution procedure. In 1965, J.A. Robinson published a fundamental paper in Automated Deduction containing the unification algorithm. $E$-unification is a generalization of standard unification which is related to provability modulo a particular equational theory $E$. Recently, $E$-unification algorithms are often incorporated into inferences both in term-rewriting and in theorem proving based on the paramodulation rule. In particular, an $E$-unification algorithm for Boolean algebras was used by McCune in solving the long-standing Robbins problem of axiomatizability of Boolean algebras (see e.g. [9] for references). For more about $E$-unification we refer especially to Baader and Snyder [2], see also [1], [14].

There are four unification types: unitary (or 1), finitary (or $\omega$), infinitary (or $\infty$) and nullary (or 0). Several open problems about $E$-unification types of particular equational theories are still unsolved. In the papers of S. Ghilardi [6], [7], [8], [9], unification types of some logics are determined.

We show that the lattice $\text{Ext}_H$ of all theories extending the equational theory of Heyting algebras is split into two parts: "upper" (extensions of $KC$)

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and "lower" (sub-theories of $\mathcal{G}S$) and such a splitting determines the unification type: the upper part contains all theories having unitary unification type and some of nullary type, the lower part contains all theories having finitary unification type and some of nullary type (plus some with infinitary type, if there are such). The same splitting determines a borderline between theories of constructive (lower part) and non-constructive (upper part) character, a borderline between intuitionistic and classical laws of reasoning. It is worth mentioning that in the top of the the upper part (non-constructive theories, extensions of $\mathcal{K}C$) there is a chain of theories, with unitary unification, determined by finite linearly ordered Heyting algebras which were used by Kurt Gödel to prove that intuitionistic logic is not a finitely-valued logic.

Similar results are proved for the lattice $\text{Ext} \mathcal{T}_{E\mathcal{A}}$ of all theories extending the equational theory of interior algebras (or topological Boolean algebras).

The fact that an equational theory has unitary or finitary type can be very useful for some applications in Automated Deduction, but having infinitary or nullary type may indicate that some methods cannot be used in Automated Deduction.

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1. Unification and $E$-unification

Unification (for first order terms) is concerned with finding a substitution that makes two given terms $t_1, t_2$ equal. Such a substitution is called a unifier for $t_1, t_2$. If a unifier for $t_1, t_2$ exists then $t_1, t_2$ are called unifiable.

Before formal definitions we start with some easy examples showing unifiers and $E$-unifiers. In the examples below first order terms are built up from individual variables $x, y, z, \ldots, u, v, \ldots$, individual constants $a, b$ and function symbols $f, g, h, \ldots$.

Example 1. The terms $g(f(z), x)$, and $g(y, f(f(z)))$ are unifiable; the following substitution is a unifier: $x\rightarrow f(f(z)), y\rightarrow f(z)$.

Some unifiers are more general than others; less general means that they are particular cases (instances) of the others. In some cases there is a most general unifier (modulo renaming the variables), an mgu for short.

Example 2. The terms $f(x, g(a, b))$ and $f(g(y, b), x)$ have a unifier $\delta$: $x\rightarrow g(a, b), y\rightarrow a$.

Now we consider $E$-unification, equational unification. Let $E$ be an equational theory with the axiom $f(x, y) = f(y, x)$. In Example 2 we have an additional unifier $\sigma$: $y\rightarrow a$ which is an $E$-unifier, i.e. a unifier on the ground of $E$. Moreover, $\sigma$ is more general than $\delta$, which we denote by $\delta \leq \sigma$.

Now we state formal definitions. Given an equational theory $E$ and a finite set of pairs of terms (i.e. an $E$-unification problem)

$$(\Pi) : \ (s_1, t_1), \ldots, (s_n, t_n),$$
a unifier (a solution) for (II) is a substitution \( \sigma \) such that

\[ E \vdash \sigma(s_1) = \sigma(t_1), \ldots, \sigma(s_n) = \sigma(t_n). \]

(II) is called unifiable if there exists at least one unifier.

A substitution \( \sigma \) is more general than a substitution \( \tau \), in symbols \( \tau \preceq \sigma \), if there is a substitution \( \theta \) such that \( E \vdash \theta \circ \sigma = \tau \). \( \preceq \) is reflexive and transitive, it is called a subsumption preorder. Two substitutions \( \sigma, \tau \) are \( E \)-equivalent, if \( \sigma \preceq \tau \) and \( \tau \preceq \sigma \). \( \sigma \) is a most general unifier (mgu) for (II) if it is a unifier for (II) and for any other unifier \( \tau \) for (II), if \( \sigma \preceq \tau \) then \( \sigma \) and \( \tau \) are \( E \)-equivalent.

After J. Robinson's result mentioned at the beginning, if \( E = \emptyset \), then unifiable terms always have an mgu. If \( E \neq \emptyset \), then there are four cases of unification types, according to the number of maximal unifiers w.r.t. \( \preceq \) for the "worst examples" of unifiable terms: unitary (or 1), i.e. an mgu always exists, finitary (or \( \omega \)), i.e. there are always finitely many maximal unifiers, infinitary (or \( \infty \)), i.e. there are infinitely many maximal unifiers (for some terms), and nullary (or 0), i.e. for some terms there are no maximal unifiers.

Since there is a well known one-to-one correspondence between equational theories and varieties of algebras, cf. Burris and Sankappanavar [3], unification types of varieties of algebras are the same as types of the corresponding equational theories.

**Example 3.** The equational theory of Boolean algebras: unifiable (II) always has the mgu, the theory (the variety) of BA has unitary unification or unification type 1; the best case.

**Example 4.** The variety of Heyting algebras cannot have unitary unification, it has unification type \( \omega \) (finitary, not unitary), i.e. if terms are unifiable, then there exist finitely many "best", i.e. maximal unifiers w.r.t. \( \preceq \), see Ghiardi [7] (1999). **Abelian groups** also have type \( \omega \).

Other cases: if, for some terms, there exist infinitely many maximal unifiers w.r.t. \( \preceq \) then the unification type is infinitary (\( \infty \)). For example, semigroups (Plotkin), groups and rings have infinitary type.

If, for some terms, there are no maximal unifiers w.r.t. \( \preceq \) then the unification is nullary (or type 0), very bad. This is the case for lattices and distributive lattices.

Sometimes we omit the word "type" and talk on unitary unification, nullary unification etc.

By a result of S. Burris (1992), all discriminator varieties have unitary unification.

Unification is filtering if for every two unifiers (for given terms) there is another unifier which is more general than both of them (cf. [9]). If unification is filtering for a given theory \( T \) then \( T \) has either unitary or nullary unification type. The following theorem is proved in [9].
Theorem 1. Unification for a theory $T$ is filtering iff the direct product of two finitely presented and projective $T$-algebras is (finitely presented and) projective.

2. Splittings

Splittings of a lattice were used already by Whitman but R. McKenzie in [10] (1972) introduced and developed splittings of lattices of equational theories (in particular splittings of lattices of theories of lattices).

Let $D$ be a complete lattice. An element $p_0 \in D$ splits $D$ if there exists $p_1 \in D$ such that $D$ is divided by $(p_0, p_1)$ into two disjoint parts $\{d \in D : p_1 \leq d\}$ and $\{d \in D : d \leq p_0\}$. The element $p_1$ is uniquely determined by $p_0$, and $p_1$ is called the splitting companion of $p_0$, or $p_1$ co-splits $D$. $p_1$ is denoted by $D/p_0$. The pair $(p_0, p_1)$ is called a splitting pair. If $p_1$ is a splitting companion of some $p_0$ then it is a splitting of $D$. If $D$ is a lattice of theories and an element $p_0 \in D$ which splits $D$ is determined by an algebra $C$ then it is said that the algebra $C$ splits $D$.

Example 5. The equational theory of the pentagon $N_5$ splits the lattice of equational theories of lattices. The theory of modular lattices is its splitting companion.

Let $T^0$ be an equational theory, $\text{Ext}T^0$ denotes the lattice of all equational theories containing $T^0$.

The following result of McKenzie [10] is an example showing how splittings can be useful: Given two theories $T_0 \subset T_1$, $T_1$ co-splits the lattice $\text{Ext}T_0$ iff $T_1$ is independently axiomatized over $T_0$ with exactly one axiom.

$Th(C)$ denotes the equational theory of an algebra $C$, that is all equations which are valid in $C$, i.e. $Th(C) = \{p \approx q : C \models p \approx q\}$ (for explanation of $\models$ see e.g. §11 of [3]). For a given equational theory $T$, if $C \models T$ then $C$ is called a $T$-algebra.

3. Theories of Heyting algebras

An algebra $(A, \land, \lor, \rightarrow, 0, 1)$ is a Heyting algebra if it satisfies the equational axioms H11 – H15 from Burris and Sankappanavar [3]. We will use the equivalent notation $(A, \land, \lor, \rightarrow, \neg)$, where $\neg x \approx x \rightarrow 0$.

$\mathbb{H}$ denotes the equational theory of all Heyting algebras. In this part, $T$ will be an equational theory such that $\mathbb{H} \subseteq T$.

By $\mathcal{F}_T(\underline{x})$ we will denote a free algebra generated by a finite set of generators (variables) $\underline{x} = \{x_1, \ldots, x_n\}$ in the variety of all $T$-algebras. $\mathcal{F}_T(\underline{x})/t$ denotes the algebra $\mathcal{F}_T(\underline{x})$ factored by the congruence generated by the condition $t \approx 1$, where $t$ is a term with variables in $\underline{x}$. A $T$-algebra $A$ is finitely presented if it is isomorphic to $\mathcal{F}_T(\underline{x})/t$ for some $t$ and some $\underline{x}$, cf. [9], [6]. $\mathcal{F}_T(\underline{x})/t$ is called a finite presentation of $A$. 

Lemma 2. The direct product of two finitely presented $T$-algebras is finitely presented.

Proof. Assume that $A_1$ and $A_2$ are two finitely presented $T$-algebras with the presentations $F(x_1)/t_1$ and $F(x_2)/t_2$ respectively, where $t_i$ is a term with variables in $x_i$, $i = 1, 2$. We can choose the variables in such a way that the finite sets $x_1$ and $x_2$ are disjoint, and that $v \not\in x_1 \cup x_2$.

Let $t_3(x_1, x_2, v) = (t_1 \land x_2 \land v) \lor (t_2 \land x_1 \land \neg v)$, where $\land x$ is the meet of $x$. Now define two functions $\sigma_1 : x_1 \cup x_2 \cup \{v\} \to x_1$ and $\sigma_2 : x_1 \cup x_2 \cup \{v\} \to x_2$ as follows: $\sigma_1(y) = y$ for $y \in x_1$, $\sigma_1(y) = 1$ for $y \in x_2 \cup \{v\}$, and similarly for $\sigma_2$. Consider the product function:

$$
\langle \sigma_1, \sigma_2 \rangle : F(x_1, x_2, v)/t_3 \to F(x_1)/t_1 \times F(x_2)/t_2.
$$

Observe that $t_1 \to \sigma_1(t_3) \approx 1$ and $t_2 \to \sigma_2(t_3) \approx 1$, hence $\sigma_1, \sigma_2$ are well defined.

$\langle \sigma_1, \sigma_2 \rangle$ is onto: for each $([q_1], [q_2]) \in A_1 \times A_2$ we find $s = [(q_1 \land v) \lor (q_2 \land \neg v)]$ such that $\langle \sigma_1, \sigma_2 \rangle(s) = ([q_1], [q_2])$.

To show that $\langle \sigma_1, \sigma_2 \rangle$ is one-to-one it is enough to prove that if $\langle \sigma_1, \sigma_2 \rangle(d) = (1, 1)$ then $d = 1$. Let $[t(x_1, x_2, v)]$ be such that $\sigma_1(t) = 1$ and $\sigma_2(t) = 1$. Then $t_1 \to t(x_1, 1, 1) \approx 1$ (i.e. 1 is substituted for $x_2$ and $v$) and $t_2 \to t(1, t_2, 0) \approx 1$ (i.e. 0 is substituted for $x_1$ and $v$). By replacing equivalent parts we get $(t_1 \land x_2 \land v) \to t(x_1, x_2, v) \approx 1$ and $(t_2 \land \neg x_1 \land \neg v) \to t(x_1, x_2, v) \approx 1$.

Finally we get $t_3(x_1, x_2, v) \to t(x_1, x_2, v) \approx 1$, i.e. $t(x_1, x_2, v) = 1$. □

Let $BA$ denote the equational theory of Boolean algebras. A theory $T$ of Heyting algebras is called semi-constructive if, for any two terms $t_1, t_2$, whenever $(t_1 \lor t_2) \approx 1$ in $T$ then $(t_1 \approx 1)$ or $(t_2 \approx 1)$ is in $BA$.

Lemma 3. If a theory of Heyting algebras is semi-constructive then it cannot have unitary unification.

Proof. Let $T$ be a semi-constructive theory. The terms $x \lor \neg x$ and 1 are unifiable in $T$. There are two unifiers $\sigma_1 : x \mapsto 1$ and $\sigma_2 : x \mapsto 0$.

Now assume that $\sigma$ is a unifier in $T$ for $x \lor \neg x$ and 1, i.e. $\sigma(x) \lor \neg \sigma(x) \approx 1$. Since $T$ is semi-constructive we have either $\sigma(x) \approx 1$ in $BA$ and then $\sigma$ cannot be more general than $\sigma_2$ (since $\theta(\sigma(x))$ is in $BA$, for any $\theta$). or $\neg \sigma(x) \approx 1$ in $BA$ and then $\sigma$ is equivalent to $\sigma_2$ (and $\sigma$ cannot be more general then $\sigma_1$). Hence there is no unifier more general then $\sigma_1$ and $\sigma_2$. □
Let \( GS = Th(C_0) \), i.e. \( GS \) (greatest semi-constructive) is the equational theory of the Heyting algebra \( C_0 = (2 \times 2) \oplus 1 \) given by the following diagram:

\[
\begin{array}{c}
\bullet \\
/ \ \\
\bullet \\
/ \\
\bullet \\
\end{array}
\]

\( C_0 \)

By the duality between (finite) Heyting algebras and (finite) Kripke frames, represented by posets, it is known that \( GS = Th(f_2) \), where \( f_2 \), the “fork”, is a Kripke frame \( \{a, b, c\}, \leq \) , where the (reflexive) partial order \( \leq \) is such that \( a \leq b, a \leq c \) (see the diagram below). By the duality it is known that the validity (satisfiability) of an equation in finite Heyting algebras is equivalent to the validity (satisfiability) in the corresponding finite Kripke frames.

The fork \( f_2 \).

**Lemma 4.** Every theory included in the theory \( GS \) is semi-constructive.

**Proof.** Suppose, a contrario, that neither \( t_1 \approx 1 \) nor \( t_2 \approx 1 \) is in \( BA \), for some terms \( t_1 \) and \( t_2 \) with the variables contained in \( x \). Then, by the completeness of one-element Kripke frames with respect to \( BA \), there are two Kripke models \( v_1 \) on \( K_1 = \{b\} \), \( v_2 \) on \( K_2 = \{c\} \), \( b \neq c \), \( v_1 : \{b\} \to P(x) \), \( v_2 : \{c\} \to P(x) \) (\( x_i \in v_1(b) \) means that \( x_i \) is true in \( b \) under \( v_1 \)), such that \( v_1 \) falsifies \( t_1 \approx 1 \) in \( b \) and \( v_2 \) falsifies \( t_2 \approx 1 \) in \( c \). Define a new model on the fork \( f_2 \), \( v_3 : \{a, b, c\} \to P(x) \) which is an extension of \( v_1 \) and \( v_2 \), i.e. such that \( v_3(b) = v_1(b) \) and \( v_3(c) = v_2(c) \). Then \( v_3 \) cannot verify \( (t_1 \lor t_2) \approx 1 \) in \( a \) (since we would have that \( v_3 \) verifies \( t_1 \approx 1 \) in \( a \) or \( v_3 \) verifies \( t_2 \approx 1 \) in \( a \), impossible), i.e. \( (t_1 \lor t_2 \approx 1) \notin Th(f_2) = GS \), hence \( (t_1 \lor t_1 \approx 1) \notin T \), for any theory \( T \subseteq GS \).

Let \( KC \) be the theory of “weak excluded middle” \( \mathbb{H} + \neg x \lor \neg \neg x \approx 1 \).

The following theorem is based on theorems of W. Rautenberg (cf. [12]) originally stated for intermediate logics.
Theorem 5. The Heyting algebra $C_0 = (2 \times 2) \oplus 1$ splits the lattice of theories $\text{Ext } H$, and $\text{Ext } H/C_0 = \mathcal{K}C$. The pair $(\mathcal{G}S, \mathcal{K}C)$ is a splitting pair of the lattice $\text{Ext } H$, i.e. for every theory $T$ from $\text{Ext } H$ either $\mathcal{K}C \subseteq T$ or $T \subseteq \mathcal{G}S$.

The following corollary was first proved by S. Ghilardi [7] in an entirely different way.

Corollary 6. If a theory $T \in \text{Ext } H$ has unitary unification, then $\mathcal{K}C \subseteq T$.

Proof. If $T$ is a theory with unitary unification then, by the previous two lemmas, it cannot be semi-constructive, hence cannot be included in $\mathcal{G}S$ thus $\mathcal{K}C \subseteq T$, by the theorem. □

Theorem 7. Every theory extending $\mathcal{K}C$ has either unitary or nullary unification.

Proof. Let $T$ be any theory extending $\mathcal{K}C$. We will show that $T$ has filtering unification and hence has either unitary or nullary unification. By Theorem 1 and Lemma 2, all we have to show is that given two finitely generated free $T$-algebras $\mathcal{F}_T(x_1)$ and $\mathcal{F}_T(x_2)$, their direct product $\mathcal{F}_T(x_1) \times \mathcal{F}_T(x_2)$ is projective (if $A$ is a retract of $\mathcal{F}$ and $B$ is a retract of $\mathcal{F}'$ then $A \times B$ is a retract of $\mathcal{F} \times \mathcal{F}'$, i.e. it is a retract of a free $T$-algebra. Hence we need to show that for some $\mathcal{F}_T(x_1, x_2, v)$ there are two homomorphisms $m : \mathcal{F}_T(x_1) \times \mathcal{F}_T(x_2) \to \mathcal{F}_T(x_1, x_2, v)$ and (a quotient) $q : \mathcal{F}_T(x_1, x_2, v) \to \mathcal{F}_T(x_1) \times \mathcal{F}_T(x_2)$, such that $q \circ m = \text{id}$.

Let us choose the variables in such a way that the finite sets $x_1$ and $x_2$ are disjoint, and that $v \not\in x_1 \cup x_2$, $t_i$ is a term with variables in $x_i$, $i = 1, 2$. Define the following functions:

\[
\begin{align*}
(m) & \quad m([t_1(x_1)], [t_2(x_2)]) = [(t_1(x_1) \land \neg v) \lor (t_2(x_2) \land \neg v)]; \\
(q) & \quad q([t_3(x_1, x_2, v)]) = ([t_3(x_1, x_2, 1, v/1)], [t_3(x_1/0, x_2, v/0)])
\end{align*}
\]

where $x_2/1$ and $x_1/0$ mean the results of substituting 1 for every $y_i \in x_2$ in $t_3$ and 0 for every $x_i \in x_1$ in $t_3$, respectively.
To show that the definitions (m) (and (q)) above are correct, in particular that $m$ and $q$ are homomorphisms, we prove that: if (m) holds for $p_1, q_1$ and $p_2, q_2$ then
\[ m([p_1(x_1)], [q_1(x_2)]) * m([p_2(x_1)], [q_2(x_2)]) = s([p_1(x_1) * p_2(x_1)], [q_1(x_2) * q_2(x_2)]), \]
for $* = \lor, \land, \to$, and
\[ \neg m([p(x_1)], [q(x_2)]) = m([-p(x_1)], [-q(x_2)]). \]
First observe that in $\mathcal{K}C$ the following holds
\[ (s') \quad (t_1(x_1) \land \neg v) \lor (t_2(x_2) \land \neg v) \approx (t_1(x_1) \lor \neg v) \land (t_2(x_2) \lor \neg v). \]
The DeMorgan Laws hold in $\mathcal{K}C$, hence, by $(s')$, we have proved the last condition (for $\neg$). The proofs for $* = \lor, \land$ are analogous. In the case of $\to$ the proof requires $\neg z \lor \neg \neg z$ of $\mathcal{K}C$. We omit the easy proof for $(q)$.

Finally,
\[ (q \circ m)([t_1(x_1)], [t_2(x_2)]) = ([(t_1(x_1) \land 1) \lor (t_2 \land 0)], [(t_1 \land 0) \lor (t_2(x_2) \land 1)]) \]
\[ = ([t_1(x_1)], [t_2(x_2)]). \]
Hence $q \circ m = id$, the identity morphism, thus the product is projective. $\square$

**Remark.** A Heyting algebra is called a **linear** or a **Gödel algebra** if $(x \to y) \lor (y \to x)$ holds. Linearly ordered $n$-element Heyting algebras $\mathfrak{G}_n$ were introduced by Gödel in his proof that intuitionistic logic is not a finitely-valued logic. The theory of Gödel algebras contains the theory $\mathcal{K}C$ and it has, together with all $\text{Th}(\mathfrak{G}_n), n \leq \omega$, unitary unification (cf. [4], [6]).

Nothing is known about theories of Heyting algebras having infinitary unification type. Taking together the previous facts we get the theorem:

**Theorem 8.** All extensions of $\mathcal{K}C$ coincide with all theories having unitary unification type plus some having nullary unification type. All subtheories of $\mathcal{G}S (= \text{Th}(\mathcal{C}_0))$ coincide with all theories having unification type $\omega$ plus some having nullary unification type (plus all theories having infinitary unification type, if there are such).

4. **THEORIES OF INTERIOR ALGEBRAS (OR TOPOLOGICAL BOOLEAN ALGEBRAS)**

Now we consider unification types of some theories of topological Boolean algebras (called also interior algebras or, dually, closure algebras). The lattice of theories of Heyting algebras can be embedded into the lattice of theories of topological Boolean algebras, as was proved by C.C. McKinsey and A. Tarski but later it was shown that the former is much more complicated. We will see that there is some similarity with unification types of theories of Heyting algebras.
A topological Boolean algebra \((A, \wedge, \vee, -, I)\) (or interior algebra) is a Boolean algebra \((A, \wedge, \vee, -)\) with 0 and 1 and with the operator \(I\) (interior) i.e. such that \(I x \leq x, II x = I x, I(x \wedge y) = I x \wedge I y, I 1 = 1\), cf. [11]; \(T_{BA}\) denotes the equational theory of all such algebras. In this part \(T\) will be an equational theory such that \(T_{BA} \subseteq T\).

A topological Boolean algebra is **extremally disconnected**, if \(Cl x \leq IC x\), for \(x \in A\), where \(Cx \overset{def}{=} -I - x\). This definition comes from topology: A topological space \(X\) is extremally disconnected if a closure of any open set is open, see R. Engelking [5]. It can be shown that this is equivalent to: \(Cl int A \subseteq int cl A\), for \(A \subseteq X\). The equational theory of all extremally disconnected topological Boolean algebras will be denoted by \(\mathcal{ED}\). Hence \(\mathcal{ED} = T_{BA} + CI x \leq IC x\). We will use the abbreviation “topological BA” for “topological Boolean algebra”.

Let \(Tr\) denotes the equational theory of trivial topological Boolean algebras, i.e. such that \(I x \approx x\) is in the theory. We say that a theory \(T\) has the **Weak Disjunction Property** if, for any two terms \(t_1, t_2\), whenever \((It_1 \vee It_2) \approx 1\) is in \(T\) then \((It_1 \approx 1)\) or \((It_2 \approx 1)\) is in \(Tr\).

**Lemma 9.** If a theory of topological BA's has the Weak Disjunction Property then it cannot have unitary unification.

**Proof.** Let \(T\) be a theory with the Weak Disjunction Property. The terms \(I x \vee I - x\) and 1 are unifiable in \(T\). There are two unifiers \(\sigma_1 : x \mapsto 1\) and \(\sigma_2 : x \mapsto 0\). Now assume that \(\sigma\) is a unifier in \(T\) for \(I x \vee I - x\) and 1, i.e. \(I \sigma(x) \vee I - \sigma(x) \approx 1\). Since \(T\) has the Weak Disjunction Property we have either \(I \sigma(x) \approx 1\) in \(Tr\) and then \(\sigma\) cannot be more general then \(\sigma_2\) (since \(\theta(\sigma(x))\) is in \(Tr\), for any \(\theta\)), or \(I - \sigma(x) \approx 1\) in \(Tr\) and then \(\sigma\) is equivalent to \(\sigma_2\) and \(\sigma\) cannot be more general then \(\sigma_1\). Hence there is no unifier more general then \(\sigma_1\) and \(\sigma_2\).

Let \(D_0\) be the topological BA given by the following diagram and let \(\mathcal{VD} = Th(D_0)\). An element \(a\) is marked as a circle \(\circ\) if \(a\) is open, \(Ia = a\).

![Diagram](image)

In an analogous way as for Heyting algebras we can prove
Lemma 10. Every theory included in the theory $WD = Th(D_0)$ has the Weak Disjunction Property.

By modification of the theorem of W. Rautenberg [13] we get the following:

Theorem 11. The topological $BA D_0$ splits the lattice of theories $Ext T_{BA}$ of topological Boolean algebras, and $Ext T_{BA}/D_0 = E\mathcal{D}$. The pair $(WD, E\mathcal{D})$ is a splitting pair of the lattice $Ext T_{BA}$, i.e. for every $T$, $E\mathcal{D} \subseteq T$ or $T \subseteq WD$.

From the theorem and two previous lemmas we have:

Corollary 12. If a theory $T \in Ext T_{BA}$ has unitary unification then $E\mathcal{D} \subseteq T$.

The next theorem can be obtained by application of the following result for modal logics by Ghilardi and Sacchetti [9]: For a normal modal logic $L$ extending $K4$ the following conditions are equivalent:

(i) unification in $L$ is filtering,

(ii) $\vdash_L \Diamond^+ \Box^+ A \rightarrow \Box^+ \Diamond^+ A$, where $\Diamond^+ A = A \lor \Diamond A$ and $\Box^+ A = A \land \Box A$.

Note that the formula $\Diamond^+ \Box^+ A \rightarrow \Box^+ \Diamond^+ A$ translated to topological BA's means "extremally disconnected". Hence from Ghilardi and Sacchetti [9] it follows that:

Theorem 13. Every theory extending $E\mathcal{D}$ has either unitary or nullary unification.

Remark. A topological Boolean algebra is called a monadic algebra if $I - Ix = -Ix$ holds. Monadic algebras were introduced by Paul Halmos in his algebraization of 1-st order logic. The equational theory of monadic algebras contains the theory $E\mathcal{D}$ and it has, together with all its extensions, unitary unification, see e.g. [4].

From the the previous facts we get the theorem:

Theorem 14. All extensions of $E\mathcal{D}$ coincide with all theories having unitary unification type plus some having nullary unification type. All subtheories of $G\mathcal{D}$ coincide with all theories having unification type $\omega$ plus some having
nullary unification type (plus all theories having infinitary unification type, if there are such).

Note that by the result of Ghilardi and Sacchetti [9], $\mathcal{ED}$ has unitary unification. Despite similarities of the final theorems for the lattice of theories of Heyting algebras and the lattice of theories of topological BA's the latter is more complicated.

Open problem: give an example of a theory of topological BA's (or Heyting algebras) having infinitary unification type.

REFERENCES


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