HOMOTOPY PROPERTIES OF SET-VALUED MAPPINGS

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“Homotopy properties of set-valued maps”

by Wojciech Kryszewski

Abstract

The present work is the habilitation dissertation of the author written in the Faculty of Mathematics and Informatics of the Nicholas Copernicus University in Toruń, Poland.

The paper, divided into 8 Chapters, is devoted to the careful study of homotopical properties of general classes of set-valued maps and it presents two different approaches to the problem.

The first one is concerned with the existence of the so-called single-valued graph approximations of set-valued maps. In Chapter 1, some new approximation results valid for the maps with simple, from the topological viewpoint, values are given. Namely, maps whose values satisfy the so-called UV-properties are considered. Such maps appear quite naturally in applications to differential equations and inclusions.

The second method, presented in Chapter 4, can be roughly described as homotopy approximation approach and it concerns the problem of the existence of single valued maps homotopic to set-valued maps. The method based on the version of the Vietoris theorem established in Chapter 2 seems to be quite fruitful in the homotopical theory of set-valued maps. It appears that under some mild assumptions on spaces, each set-valued map from a general class which contains, for instance, finite compositions of acyclic maps, is homotopic, in a well-defined sense, to a single-valued one. Results of that type allow to encompass a large number of problems arising in applications involving explicitly or implicitly multivalued transformations such as controllability problems, bifurcation of periodic orbits of differential inclusions or equations with nonsmooth right hand sides. This approach also allows to give a consistent and self contained homotopical description and classification of some general classes of set-valued maps.

The two methods are compared and it is shown that their issues are equivalent to some extent.

Finally, some possible applications of the theory are indicated. In particular, in Chapter 5, the unified approach to the general coincidence index theory is presented in the framework of the general infinite-dimnesional cohomotopy theory from Chapter 3 and applied to different existence prob-

lems of partial differential inclusions and control theory in Chapter 6. The studied homotopy invariants are designed especially to concern problems where the effect of dimension defect occurs. For example, the bifurcation invariants introduced in Chapter 7 help to establish results concerning the behaviour of solutions of parametrized inclusions and, in particular, allow in Chapter 8 to establish the existence of continuous branches of periodic orbits of autonomous and nonautonomous differential inclusions.

The paper involves methods and techniques of algebraic and geometric topology, functional analysis and theory of differential equations.

Extensive references, list of notations and index are also provided.

Contents

This work is dedicated to Jola and Adam

INTRODUCTION

Many mathematical phenomena admit appropriate description by the use of transformations assigning to points of some space subsets of another space, i.e. the so-called multivalued transformations. The concept of a multivalued mapping, which lies at the junction of topology and nonlinear functional analysis, has been developed and studied quite intensively. Obviously, on one hand, multivalued maps can be considered as single-valued ones into the family of subsets of space (supplied with some structure), so it may appear that there is hardly a need for an independent theory of multivalued mappings. However, on the other hand, this argument helps very little in the investigation of the specific problems that arise naturally in this theory. The gradual realization of these problems and the extending collection of them have led to the formation of the theory, distinctly noticeable in the last decades. This theory has attracted attention also due to the significant extension of its applications and methods which, in some areas (such as game theory, theory of optimal control), have became generally accepted.

In the present dissertation we shall be primarily interested in the topological aspects of the theory of multivalued mappings. The fundamental notions of this branch of the theory, such as various topologies in the space of subsets, continuity concepts, the existence of continuous selections and approximations, are well-known (see e.g. [27]). Homological methods, especially needed in the fixed point issues of the theory have also been thoroughly studied (see e.g. [84]). However there are still some important gaps in the existing theory which should be filled in order to get a better understanding of the behaviour of multivalued mappings. For instance, it seems that homotopical properties of such maps are still not well recognized (except for some special cases – [31, 38, 29] and also [90, 110]; but also here there still remain some open questions). The very notion of a homotopy between multivalued maps needs a careful description and must involve a special structure that is often explicitly or implicitly inscribed into the maps.

In what follows, by a space we understand a paracompact topological space and by a map we understand a continuous transformation of spaces. By a set-valued map $\varphi : X \multimap Y$ of spaces X and Y we mean an upper semicontinuous multivalued map with compact values. By $\mathcal{A}_0$ we denote the class of acyclic set-valued maps, i.e. having trivial Čech cohomology of values.

Basically, we shall deal with maps from the class $\mathcal{A}_0^c$ of finite compositions of members of $\mathcal{A}_0$. This class has been studied in many papers – see e.g. [96, 155, 84, 163]. It is rather evident that properties of maps from $\mathcal{A}_0^c$ and $\mathcal{A}_0$ are different and, moreover, in general, the behaviour of a map admitting a factorization in the form of the composition of maps from $\mathcal{A}_0$ depends on a particular factorization (which, usually, is not unique) – see e.g. comments in [84]. Also, as mentioned above, the notion of a homotopy between set-valued maps from $\mathcal{A}_0^c$ has to involve the underlying structure of maps reflected by their particular factorizations.

In order to take these remarks into account and to get a better understanding of the class $\mathcal{A}_0^c$ one has to put it into more appropriate and convenient framework. Namely, one sees easily that an acyclic map $\varphi : X \multimap Y$ may be represented as $\varphi(x) = q_\varphi(p_\varphi^{-1}(x))$ for $x \in X$, where $X \xleftarrow{p_\varphi} W_\varphi \xrightarrow{q_\varphi} Y$, W_φ is the graph of φ and p_φ, q_φ are respective projections. Clearly, p_φ is a closed map with compact acyclic fibres. The map φ may admit other factorizations of the form $X \xleftarrow{p} W \xrightarrow{q} Y$ (i.e. $\varphi(x) = q(p^{-1}(x))$, $x \in X$) where p, q are no longer projections but p is still a Vietoris map (i.e. a perfect surjection with acyclic fibres). It also appears that any map $\varphi \in \mathcal{A}_0^c$ admits such a factorization. These observations led L. Górniewicz and A. Granas [86] to the notion of a *morphism*. Morphisms, being the equivalence classes of a certain relation between possible factorizations (p, q) of maps from $\mathcal{A}_0^c$, constitute a good tool for studying set-valued maps and have some categorical properties which justify their name. In the author's thesis [111] the notion of a morphism was modified in order to obtain some additional properties.

In Chapter 4, we again modify and generalize this notion. This allows to study the homotopical behaviour of maps which are determined by morphisms. The class $\mathcal{M}$ of these maps is more general than $\mathcal{A}_0^c$. Secondly, in Chapter 4, we study homotopies of morphisms and prove that each homotopy class of a map from a large subclass $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ (including, for example, all acyclic maps between spaces of finite dimension) contains a single-valued map. The Identification Theorems I, II and III (cf. Theorems 4.25, 4.47, 4.51; see also Propositions 4.23, 4.46, Remark 4.52 (i)) and Corollary 4.54 show that the set-valuedness of maps from the class $\widetilde{\mathcal{M}}$ may be "homotopically" killed. Therefore our results generalize results of [38, 29] and of [90, 110] where maps with additional structure of values were studied. Moreover, they constitute an answer to the old question of A. Granas whether an acyclic map $\varphi : S^n \multimap S^n$ is homotopic to a single-valued continuous one.

Proofs of these facts rely heavily on a result which may be regarded as the generalized version of the famous Vietoris theorem. The careful study of this and related problems is given in Chapter 2 where we prove, in particular, that given finite-dimensional spaces X, Y and a Vietoris map $f : X \to Y$, the induced (set) transformation $f^{\#} : [Y; P] \to [X; P]$, where P is a simply connected absolute neighborhood retract, is bijective.

Having the Identification Theorems we are also able to get generalizations of the Hopf classification and extension theorems. Recall that the remarkable Hopf classification theorem enumerates homotopy classes of maps of a compact metric space X, $\dim X \le n$, into S^n by elements of $\check{H}^n(X)$ and the extension theorem reduces the problem of the existence of an extension of a map $f : A \to S^n$, where A is a closed subset of a compact space X, $\dim X \le n + 1$, to a certain algebraic problem (see [102, 166]). These results have been extended to larger classes of spaces and polyhedra: in particular, it is known that S^n may be replaced by a simple $(n-1)$-connected polyhedron and a compact metric domain by a finite-dimensional paracompact space (comp. [166, 143] and [149] for even more general results). We show that in these classical theorems one can replace continuous maps by set-valued ones from the class $\mathcal{M}$ containing – as was said above – all acyclic maps and their finite compositions. Finally, it allows to solve another old problem whether the Hopf degree theorem holds for acyclic maps.

It has already been observed by Eilneberg and Montgomery [64] that the Vietoris Theorem also plays an important role when studying homological properties of set-valued maps necessary in the fixed point theory aspect of multivalued maps. The method started in [64] has been developed by Górniewicz in [84] and many existence results has been established. However, since this approach involves homology (or cohomology), it fails when it comes to the existence problem under the presence of a dimension defect. To see that better assume that we are to study the existence of fixed points of a parametrized inclusion of the form $x \in \varphi(\lambda, x)$ where x comes from a Banach space X and λ is a parameter from $\mathbf{R}^n$. Since the range of the problem has a nonzero codimension in the domain, the usual fixed point index (or topological degree) – even in the single-valued case – is not sufficient. However, our version of the Vietoris theorem together with Identification Theorems allow us to replace cohomology groups by cohomotopy groups which behave much better when the effect of the dimension defect occurs. Namely, following the ideas of the generalized degree due to Nirenberg, Gęba and others [146, 81, 77] we define the generalized degree (or the generalized index of coincidence with a Fredholm operator of nonnegative index) for set-valued maps. This is done is Chapter 5.

But again since we are also dealing with infinite-dimensional problems we were forced to prepare the infinite-dimesional version of stable cohomotopy theory. In Chapter 3, inspired by results of [76, 79], we have introduced the functor of infinite dimensional (stable) cohomotopy theory in order to investigate the essentiality of compact perturbations of some given proper maps (for example, Fredholm operators of nonnegative index). In particular, we get counterparts of results from [168]. We could not directly use the theory of Gęba and Granas because our maps do not act between necessarily linear (normed) spaces and their subsets.

Having developed the unified framework for the index or degree theory we use it in Chapter 7 in order to study bifurcation phenomena for inclusions of the above type. We introduce homotopy invariants responsible for the appearance of bifurcations and compare them with some existing invariants, for instance – the Alexander invariant.

Finally, the index machinery makes it possible to give some concrete applications of the theory. For instance in Chapter 8, we establish some results concerning the existence of continuous branches of nontrivial periodic orbits of autonomous and nonautonomous differential inclusions (or equations with merely continuous right-hand sides). In Chapter 6 we use the presented methods to study some general abstract and concrete boundary value problems for ordinary and partial differential inclusions. We also show how can one apply the topological techniques to controllability problems.

We also deal with some approximation aspects of the theory of multivalued maps. Since the early paper of von Neumann [145], approximation methods proved to be useful in many applications involving set-valued maps: game theory [18], differential equations and inclusions [17] and others. These methods, apart from the homological ones, play an important role in the fixed point theory of set-valued maps, too. In Chapter 1 we give some new results concerning the existence of graph approximations of set-valued maps whose values satisfy some conditions even stronger than acyclicity. Namely, we state and prove several results concerning approximability (see Theorems 1.22, 1.23, 1.38, 1.39 and Corollary 1.36) which form quite far going generalization of the most existing results. Our viewpoint yields sufficient and adequate generality also when the domain of a map is not compact (compactness was an essential component of hypotheses of most previous results). This chapter is somewhat separated from others. However, in the last section of Chapter 4, we compare implications of Identification and approximation Theorems showing that these two, apparently different, approaches give the same results – see Theorems 4.57, 4.59 and Corollary 4.58. The important is, however, that approximation methods are sometimes easier to apply.

About the paper

The paper ([1]) is organized as follows. After this Introduction and a chapter devoted to some preliminary notations and definitions it consists of two parts. The first part contains results of more theoretical character:

[1] The paper is to some extent an extension and continuation of the author's doctoral dissertation [111]. Some parts of its contents has already been published: the main results of Chapter 1 are contained in [115]; the contents of Chapter 2 appears in [114]; the substantial part of Chapter 4 has been published in [112, 113]; results of Section 7.A. corresponds to the published paper [91] and generalize results contained in it; a part of Section 6.F. appeared in [118].

Other results are presented here for the first time.

[111] W. Kryszewski, *Topological and approximation methods in the degree theory of set-valued maps*, Diss. Math. **336** (1994), 1–102;

[115] W. Kryszewski, *Graph-approximation of set-valued maps on noncompact domains*, to appear in Topology & Appl.;

[114] W. Kryszewski, *Remarks to the Vietoris Theorem*, to appear in Top. Meth. in Nonlinear Anal.;

[112] W. Kryszewski, *Some homotopy classification and extension theorems for the class of compositions of acyclic set-valued maps*, Bull. Sci. Math. **119** (1995), 21–48;

[113] W.Kryszewski, *The fixed-point index for the class of compositions of acyclic set-valued maps on* ANR*'s*, Bull. Sci. Math. **120** (1996), 129–151;

[91] L. Górniewicz, W. Kryszewski, *Bifurcation invariants for acyclic mappings*, Reports on Math. Phys. **31** (1992), 217–239;

[118] W. Kryszewski, PL. Zezza, *Remarks on the relay controllability of control systems*, J. Math. Anal. Appl. **188** (1994), 45–65.

- In Chapter 1 we study sets having various UV-properties and deal with the existence of graph-approximations. The main results here are approximability Theorems 1.22, 1.23, 1.34, 1.38 and 1.39. We show, in particular, that an arbitrary neighborhood of the graph of a map whose values satisfy a certain UV-property contains the graph of a continuous single-valued map. This is true provided the domain is either a finite-dimensional metric space, a locally finite dimensional polyhedron or an absolute neighborhood retract.

- Chapter 2 deals with various extensions of the celebrated Vietoris theorem. Generalized Vietoris Theorems 2.17, 2.18 and 2.20 (the last one is just a relative version of the result essentially due to G. Kozlowski) are the main results. This theorems constitute homotopy counterparts of the classical Vietoris theorem which is stated in terms of cohomology.

- In Chapter 3 we study cohomotopical implications of the generalized Vietoris theorems mentioned above. Following the ideas of K. Gęba, we introduce here the infinite dimensional stable cohomotopy theory for arbitrary pair from the so-called generalized Leray-Schauder category. The main result here are Theorems 3.28 and 3.35.

- Chapter 4 is devoted to the study of topological and homotopical properties of the class of set-valued maps determined by morphisms. We develop and refine the notion of a morphism due to L. Górniewicz and A. Granas here. The main results of this chapter are Identification Theorems 4.25, 4.47 and 4.51. Moreover, some generalizations of the Hopf Classification and Extension Theorems in the context of set-valued maps are presented. In the last section of this chapter we compare graph approximation results from Chapter 1 with the mentioned above – Theorems 4.57, 4.59.

The second part is devoted to applications of results from Part I. Namely:

- In Chapter 5, the homotopical framework of Chapters 3 and 4 is used to give a unified approach for the topological degree (or the coincidence index) theory valid for both single- and set-valued maps under the presence of the dimension defect. The main result here is the construction of the degree (or index) itself.

- Chapter 6 is concerned with applications of the degree theory developed in Chapter 5 and the study of some abstract and concrete boundary value problems. We introduce an abstract setting (in spirit of Olech, Lasota and Pruszko [156]) which allows to deal with boundary value problems arising in (partial) differential inclusions. In Theorem 6.27 we establish abstract existence criteria sufficient to show in Theorem 6.43 the existence of solutions to a general boundary value problem for an elliptic partial differential inclusion generalizing some known results due to Nirenberg [146]. In this chapter we also give a self contained review and prerequisities concerning Sobolev spaces. Finally in Chapter 6, some concepts of controllability of nonlinear systems governed by differential inclusions or equations are studied. We develop the notion of the so-called relay controllability giving some simple criteria of complete controllability by means of such controls.

- In Chapter 7 homotopy invariants responsible for the existence of bifurcation phenomena of general parametrized inclusions in Banach spaces are dealt with. First the finite-dimensional bifurcation of zeros of a parametrized inclusion is studied. In Theorems 7.4, 7.7 the structure of its solutions is described. Next, an invariant closely related to the well-known J. C. Alexander invariant γ is introduced and studied.

- In Chapter 8, we apply the results of the previous chapters to the bifurcation of periodic solutions to an autonomous differential inclusion. We first give Theorem 8.10 which forms a generalization of the famous Hopf bifurcation theorem. Next, we apply the guiding function technique in order to establish the existence of branching of periodic orbits in case of a nonautonomous differential inclusion – see Theorem 8.20.

The paper ends with References, Glossary of Notations, Index and an information about the author.

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Preliminaries

In what follows by a *space* we understand a *paracompact* (*Hausdorff*) *topological* space; by a *pair* of spaces – a pair (X, A) where X is a space and A is a *closed* subset of X; a space X is identified with the pair $(X, \emptyset)$. If $B \subset X$, then $\operatorname{cl} B$, $\operatorname{int} \mathrm{B}$ and $\operatorname{bd} \mathrm{B}$ denote the *closure*, the *interior* and the *boundary* of B, respectively. The *free* (or *disjoint*) union of spaces X, Y is denoted by $X \oplus Y$ (or $X \vee Y$).

By a *map* $f : X \to Y$ of spaces we understand a continuous transformation and by a *map of pairs* $f : (X, A) \to (Y, B)$ we mean a map $f : X \to Y$ such that $f(A) \subset B$. We use standard notation, in particular: $f : f_0 \simeq f_1 : (X, A) \to (Y, B)$ means that the map $f : (X \times I, A \times I) \to (Y, B)$ (where $I := [0, 1]$) provides a *homotopy* joining the maps f_0 with f_1; the symbol $[f]$ denotes the *homotopy class* of a map $f : (X, A) \to (Y, B)$; $[X, A; Y, B]$ stands for the set of all homotopy classes of maps $(X, A) \to (Y, B)$ (if $A = B = \emptyset$, then we write $[X; Y]$) and $\mathcal{C}(X, A; Y, B)$ — for the set of all such maps. If $g : (Y, B) \to (Y', B')$, then the transformation $g_\# : [X, A; Y, B] \to [X, A; Y', B']$ induced by g is defined by $g_\#[f] = [g \circ f]$ for $[f] \in [X, A; Y, B]$ and, if $g : (X', A') \to (X, A)$, then $g^\# : [X, A; Y, B] \to [X', A'; Y, B]$ is defined by $g^\#[f] = [f \circ g]$ for $[f] \in [X, A; Y, B]$.

By AE (resp. ANE) we denote the collection of all spaces which are *absolute* (resp. *neighborhood*) *extensors* for the class of paracompact spaces ([2]). By AR (resp. ANR) we denote the collection of *metric absolute* (resp. *neighborhood*) *retracts*.

Recall that any pair (X, A) has the *homotopy extension property* (HEP)

[2]That is: for a space Y, one writes $Y \in$ AE (resp. $\in$ ANE) provided given a pair (X, A) and a map $f : A \to Y$ there is an extension $F : X \to Y$ (resp. there is a neighborhood U of A and an extension $F : U \to Y$) of f onto X (resp. onto U).

– see e.g. [166] – with respect to any ANE.

Observe that a metrizable AE (resp. ANE) is an AR (resp. ANR) and a complete AR (resp. ANR) is an AE (resp. ANE). Moreover, any AR (resp. ANR) is an absolute (resp. neighborhood) extensor for the class of compact spaces. For further details – see e.g. [34, 101].

If $A \subset X$ is a closed subset and $g : A \to Y$, then the space obtained by attaching X to Y by means of g (i.e. the quotient space obtained from the disjoint union of X and Y by identifying each point $x \in A$ with $g(x) \in Y$) is denoted by $X \cup_g Y$ ([3]). If $f : X \to Y$, then by the *cylinder* $Z = Z(f)$ of f we mean the space $X \times [0,1] \cup_g Y$ where $g : X \times \{1\} \to Y$ is given by $g(x,1) = f(x)$. The element of Z corresponding to $(x,t) \in X \times [0,1]$ under the identification is denoted by $[x,t]$ and $[y] \in Z$ denotes an element corresponding to $y \in Y$ (hence $[x,1] = [f(x)]$ for $x \in X$). There are embeddings $X \ni x \mapsto [x,0] \in Z$ and $Y \ni y \mapsto [y] \in Z$, hence both spaces X, Y are regarded as closed subsets of Z. Moreover, there is a strong deformation retraction $r : Z \to Y$ such that $r \circ i = f$ where $i : X \to Z$ is the inclusion. The pair (Z, X) has the homotopy extension property with respect to any space (see [166]).

By $\check{H}^*(X, A; G)$ we denote the *Čech cohomology* (*graded*) *group* of a pair (X, A) with *coefficients in a group* G; $\check{H}^*(X, A) := \check{H}^*(X, A; \mathbf{Z})$ (see [166]). A space X is G-*acyclic* provided it has G-cohomology of a point. In particular, if $G = \mathbf{Z}$, then we say that X is *acyclic*.

If (X, d) is a metric space, $\varepsilon > 0$ and $A \subset X$, then by the ε-*neighborhood* of A in X we mean the set

$$B^X(A, \varepsilon) := \{x \in X \mid d(x, A) < \varepsilon\}$$

where $d(x, A) = \inf_{a \in A} d(x, a)$ is the distance of a point $x \in X$ from the set A. In particular, $B^X(a, \varepsilon) = \{x \in X \mid d(x, a) < \varepsilon\}$ (resp. $D^X(a, \varepsilon) = \{x \in X \mid d(x, a) \le \varepsilon\}$) is the open *ball* (closed *disk*) of radius ε centered at $a \in X$ (if it does not lead to an ambiguity we usually surpress the superscript X from the above notation).

The (real) Euclidean n-dimensional space, $n \geq 1$, is denoted by $\mathbf{R}^n$; the standard norm of $x \in \mathbf{R}^n$ is denoted by $|x|$ and the inner product

[3] Recall that, in general, if R is a closing equivalence relation in a space X (i.e. the natural quotient projection $X \to X/R$ is closed), then X/R is paracompact; in particular $X \cup_g Y$ is paracompact provided g is a closed map.

of $x, y \in \mathbf{R}^n$, by $x \cdot y$. An open ball (resp. a closed disk) of radius $r > 0$ centered in $x \in \mathbf{R}^n$ is denoted by $B^n(x, r)$ (resp. $D^n(x, r)$) and $S^{n-1}(x, r) = \mathrm{bd}\, \mathrm{D^n(x, r)}$. Moreover, $B^n_r := B^n(0, r)$, $D^n_r := D^n(0, r)$ and $S^{n-1}_r := S^{n-1}(0, r)$. As usual $S^{n-1} := S^{n-1}_1$, $D^n := D^n_1$ and $B^n := B^n_1$.

Given sets X, Y, a *multivalued map* or *transformation* (denoted $\varphi : X \multimap Y$) is a transformation $\varphi : X \to 2^Y \setminus \{\emptyset\}$. By a *set-valued map* of a space X into a space Y we understand an *upper semicontinuous* multivalued transformation $\varphi : X \multimap Y$ (i.e. such that, for any open $U \subset Y$, the set $\varphi^{-1}(U) := \{x \in X \mid \varphi(x) \subset U\}$ is open (in X)) with *compact* values. The multivalued map φ is *lower semicontinuous* if, for any open $U \subset Y$, the set $\{x \in X \mid \varphi(x) \cap U \neq \emptyset\}$ is open. We say that φ is *continuous* when it is both lower and upper semicontinuous. See [27] or [84] for more details concerning multivalued and set-valued maps.

If f (resp. φ) is a map (resp. multivalued map) of spaces, then Gr(f) (resp. Gr(φ)) stands for the *graph* of f (resp. of φ), i.e.

$$\begin{array}{rcl} \mathrm{Gr(f)} & := & \{(x, f(x)) \mid x \in X\}; \\ \mathrm{Gr}(\varphi) & := & \{(x, y) \in X \times Y \mid y \in \varphi(x)\}. \end{array}$$

We say that a multivalued map φ of spaces X, Y is *closed* if its graph $\mathrm{Gr}(\varphi)$ is closed in $X \times Y$; it is *compact* if the closure of its image $\varphi(X) = \bigcup_{x \in X} \varphi(x)$ is compact. Note that if φ is a set-valued map, then it is closed; if a multivalued map φ is compact and closed, then it is upper semicontinuous and has compact values, i.e. φ is a set-valued map.

Given sets X, Y and Z, transformations $\varphi : X \multimap Y$ and $\psi : Y \multimap Z$, we define the *composition* $\psi \circ \varphi : X \multimap Z$ by $\psi \circ \varphi(x) := \bigcup_{y \in \varphi(x)} \psi(y)$ for $x \in X$. If X, Y, Z are spaces and φ, ψ are set-valued maps, then $\psi \circ \varphi$ is also a set-valued map.

We write $\varphi : (X, A) \multimap (Y, B)$ if $\varphi : X \multimap Y$ and $\varphi(A) \subset B$.

Let E be a Banach space. Recall [51, Cor. III.2.3] that a subset $X \subset E$ is *relatively weakly compact* (i.e. its weak closure is compact in the weak topology) if and only if it is *relatively weakly sequentially compact* (i.e. each sequence (x_n) in X possesses a weakly convergent subsequence). If E is reflexive, then X is relatively weakly compact if and only if it is bounded (the Eberlein-Shmulyan theorem – see [51, Th. III.4.1]).

PART I: THEORY

Chapter 1.

APPROXIMABILITY OF SET-VALUED MAPS

As stated in Introduction, approximation methods prove to be very useful when studying multivalued maps. The idea is simple: one approximates (in a well-defined sense) a given multimap by a single-valued one and, then, applying a limiting process, investigates to what extent properties of single valued approximations are inherited by the original map. For example, the very well-known approximation result due to Cellina asserts that any upper semicontinuous map with convex closed values from a paracompact space to a normed one admits arbitarily close approximations. This result implies that convex-valued maps share most of characteristic properties of single-valued ones (see also e.g. [47]).

The assumption concerning the convexity of values is not topological; therefore it makes sense to ask what kind of assumptions, more general than convexity, also lead to similar statements. In this chapter we try to give some results in this direction.

1.A. Graph-approximations

The so-called *graph-approximations* of set-valued maps are the main object of our interest. Recall that by a space we understand a paracompact topological space.

Let $\varphi : X \multimap Y$ be a set-valued map from a space X into a space Y and let $A \subset X$.

1.1 Definition Given a neighborhood $\mathcal{U}$ of the graph $\mathrm{Gr}(\varphi)$ in $X \times Y$ ([1]), we say that a map $f : A \to Y$ is a *$\mathcal{U}$-approximation of φ over A* provided

$$\mathrm{Gr(f)} \subset \mathcal{U}. \tag{1.1}$$

We say that φ is *approximable* if, for each neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there exists a $\mathcal{U}$-approximation $f : X \to Y$ of φ (over X).

In case of metrizable spaces X and Y one may use epsilons instead of neigborhoods in (1.1). Given a continuous function $\varepsilon : X \to (0, +\infty)$ we say that a map $f : A \to Y$ is an *ε-approximation* of φ over A if

$$\forall x \in A \quad f(x) \in B(\varphi(B(x, \varepsilon(x))), \varepsilon(x)) \tag{1.2}$$

In case A is compact, one may replace ε-functions in (1.2) by positive constants and thus arrive the traditional notion of a graph-approximation.

1.2 Proposition *Let X, Y be metrizable spaces, $A \subset X$ and $\varphi : X \multimap Y$ be a set-valued map.*

(i) *For each neighborhood $\mathcal{U}$ of* $\mathrm{Gr}(\varphi)$, *there is a continuous function $\varepsilon : X \to (0, +\infty)$ such that any ε-approximation of φ over A is a $\mathcal{U}$-approximation of φ over A.*

(ii) *Conversely, given a continuous function $\varepsilon : X \to (+\infty)$, there is a neighborhood $\mathcal{U}$ of* $\mathrm{Gr}(\varphi)$ *such that any $\mathcal{U}$-approximation of φ over A is an ε-approximation of φ over A.*

Proof (i) The upper semicontinuity of φ implies that, for any $x \in X$, there exists a number $r(x) > 0$ such that

$$B(x, r(x)) \times B(\varphi(B(x, 2r(x))), r(x)) \subset \mathcal{U}.$$

[1] In the sequel, we always speak of neighborhoods of $\mathrm{Gr}(\varphi)$ in $X \times Y$.

Let $\{\lambda_j\}_{j\in J}$ be a partition of unity inscribed into the cover $\{B(x,r(x))\}_{x\in X}$. Hence, for each $j \in J$, there is $x_j \in X$ such that $\operatorname{supp}\lambda_j \subset B(x_j, r(x_j))$. Let $r_j := r(x_j)$ and define

$$\varepsilon(x) = \sum_{j\in J} \lambda_j(x) r_j, \quad x \in X.$$

Suppose that $f : A \to Y$ is an ε-approximation of φ. Given $x \in A$, there is $i \in J$ such that $\lambda_i(x) > 0$ (therefore $x \in B(x_i, r_i)$) and $\varepsilon(x) \le r_i$. Since $f(x) \in B(\varphi(B(x, \varepsilon(x))), \varepsilon(x))$, there is $x' \in B(x, \varepsilon(x))$ and $y' \in \varphi(x')$ such that $f(x) \in B(y', \varepsilon(x))$. Therefore $y' \in \varphi(B(x_i, 2r_i))$ and $f(x) \in B(\varphi(B(x_i, 2r_i)), r_i)$. Altogether

$$(x, f(x)) \in B(x_i, r_i) \times B(\varphi(B(x_i, 2r_i)), r_i) \subset \mathcal{U}.$$

(ii) For each $(x, y) \in \mathrm{Gr}(\varphi)$, let

$$U(x,y) = \left[\varepsilon^{-1}\left(\frac{\varepsilon(x)}{2}, +\infty\right) \cap B\left(x, \frac{\varepsilon(x)}{2}\right)\right] \times B\left(y, \frac{\varepsilon(x)}{2}\right)$$

and

$$\mathcal{U} = \bigcup \{U(x,y) \mid x \in X, y \in \varphi(x)\}.$$

Clearly $\mathcal{U}$ is an open neighborhood of $\mathrm{Gr}(\varphi)$. Assuming that a continuous map $f : A \to Y$ is a $\mathcal{U}$-approximation of φ, we easily gather that (1.2) is also satisfied. □

Along with approximability, we shall study the following notions.

1.3 Definition Let $\varphi : X \multimap Y$ be a set-valued map between spaces X and Y. Let A, B be closed subsets of X such that $A \subset \operatorname{int} \mathrm{B}$. We say that φ is:

(i) *relatively approximable over* A if, for any neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ such that any $\mathcal{V}$-approximation $f : A \to Y$ of φ extends to a $\mathcal{U}$-approximation $F : X \to Y$;

(ii) *weakly relatively approximable over* A if, for every neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ with the following property: if a $\mathcal{V}$-approximation $f : A \to Y$ of φ extends to a map $f' : N \to Y$ where N is a neighborhood of A in X, then there is a $\mathcal{U}$-approximation $F : X \to Y$ of φ such that $F|A = f$;

(iii) *relatively approximable over* (A, B) if, for every neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ with the following property: if $f : B \to Y$ is a $\mathcal{V}$-approximation of φ, then there is a $\mathcal{U}$-approximation $F : X \to Y$ of φ such that $F|A = f|A$.

Observe that in general, if φ is relatively approximable over A, then it is weakly relatively approximable over A. If Y is a neighborhood extensor with respect to the pair (X, A), then clearly the weak relative approximability of φ over A implies relative approxiambility. This holds, for instance, when A is a neighborhood retract in X. If φ is weakly relatively approximable over A, then it is relatively approximable over (A, B) for any closed neighborhood B of A.

Let us now pose the main problem.

1.4 Question *Suppose that* $\varphi : X \multimap Y$ *is a set-valued map between spaces, and* A, B *are closed subsets of* X *such that* $A \subset \mathrm{int\,B}$. *Under what conditions* (*on* φ, X *and* A, B) *is* φ *approximable* (*resp.* (*weakly*) *relatively approximable over* A *or over* (A, B))*? In particular, what may be said with regard to this problem if* φ *is a* UV^n*-valued map* ([2]), $0 \leq n \leq \omega$, *and*

(a) $X \setminus A$ *is finite-dimensional;*
(b) $X \setminus A$ *is countable dimensional;*
(c) X *is countable dimensional;*
(d) X *is an* ANR*?*

Moreover, under what conditions does relative approximability over (A, B) *imply relative approximability over* A*?*

Regarding graph-approximations as tools for studying properties of set-valued maps (in particular, the existence of their fixed points via homotopy invariants such as e.g. the fixed point index – see [110]), the following concept and problem seems to be of importance.

1.5 Definition Let $\varphi : X \to Y$ be a set-valued map between spaces. We say that φ is *homotopy approximable* if, for each neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ (in $X \times Y$), there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ such that any $\mathcal{V}$-approximations $f, g : X \to Y$ of φ are joined by a homotopy $h : X \times [0,1] \to Y$ such that $h_t = h(\cdot, t)$ is a $\mathcal{U}$-approximation of φ for every $t \in [0,1]$.

1.6 Question *Under what conditions is* φ *homotopy approximable? In particular, what may be said with regard to this problem if* φ *is a* UV^n*-valued map,* $0 \leq n \leq \omega$, *and*

[2] this concept will be defined in Section B.

(a) $X \setminus A$ *is finite-dimensional;*
(b) $X \setminus A$ *is countable dimensional;*
(c) X *is countable dimansional;*
(d) X *is an* ANR*?*

Of course one may generalize the notion of homotopy approximability in the same way approximability was generalized.

1.B. UV-sets

It is clear that if no conditions are imposed on the values of a map φ, then the problem of the existence of sufficiently close approximations has hardly an answer.

In the present chapter we shall deal with set-valued maps whose values satisfy some of the so-called UV-properties. For the convenience of the reader we now recall these well-known concepts.

1.7 Definition Let $V \subset U$ be subsets of a space Y and let $n \geq 0$ be an integer. The inclusion $V \hookrightarrow U$ is *homotopy n-trivial* if, for any integer $0 \leq k \leq n$, each singular k-sphere in V is inessential (i.e. null-homotopic) in U. Equivalently: for any k-dimensional simplex Δ, $0 < k \leq n+1$, and a continuous map $f : \partial\Delta \to V$, there is a continuous map $F : \Delta \to U$ such that $f(x) = F(x)$ for $x \in \partial\Delta$ ($\partial\Delta$ denotes the geometric boundary of Δ).

Let A be a compact subset of a space Y.

1.8 Definition (comp. [13] or [122]) We say that the inclusion $A \hookrightarrow Y$ has:

- *UV^n-property* ($n \geq 0$ is an integer) if each open neighborhood U of A in Y contains a neighborhood V of A such that the inclusion $V \hookrightarrow U$ is homotopy n-trivial;
- *UV^ω-property* if it has UV^n-property for each $n \geq 0$;
- *UV^∞-property* if each neighborhood U of A (in Y) contains a neighborhood V of A such that V is contractible in U.

For instance, the inclusion $A \hookrightarrow Y$ has UV^0-property if and only if, for each neighborhood U of A, there is a neighborhood $V \subset U$ of A such that any two points from V are joined by a path in U.

We recommend [122] or [3] as the best source of information and results concerning sets having various types of UV-properties (although spaces in [3] are assumed to be separable metric).

Let us collect several simple observations:

1.9 Remark

(i) It is clear that properties defined above are properties of the *embedding* of a given compact set in the ambient space and not of the set itself. For instance, a point $x \in Y$ has properties UV^n, $0 \leq n < \infty$, (resp. UV^ω, UV^∞) if and only if the ambient space Y is locally n-connected (resp. locally ∞-connected, locally contractible) at x. The inclusion $S^{n+1} \hookrightarrow \mathbf{R}^{n+2}$, $n \geq 0$, has UV^n-property but not UV^{n+1}-property.

(ii) For an integer $n \geq 0$, the inclusion $A \hookrightarrow Y$ has UV^n-property if and only if each neighborhood U of A in Y contains a neighborhood V of A such that any map $P \to V$, where P is a polyhedron with $\dim P \leq n$, is null-homotopic in U (in other words the map $j_\# : [P; V] \to [P; U]$, induced by the inclusion $j : V \to U$, is trivial).

If the inclusion $A \hookrightarrow Y$ has UV^∞-property, then A is contractible in each of its neighborhoods; moreover, each neighborhood U of A in Y contains a neighborhood V of A such that $j_\# : [Z; V] \to [Z; U]$ is trivial for any space Z.

(iii) Property UV^ω is, in general, weaker than UV^∞-property. It is easy to show examples of sets having UV^ω-property which do not possess UV^∞-property.

The above properties of the inclusion $A \hookrightarrow Y$ of a compact set A in a space Y have absolute versions, as well. Recall that a space Y is an ANE provided it is an absolute neighborhood extensor for the class of all (i.e. paracompact) spaces.

1.10 Definition We say that a compact space X has *UV^n-property*, $0 \leq n \leq \infty$ or $n = \omega$, if one of the following equivalent conditions is satisfied:

- there is an embedding $e : X \to Y$ into an ANE Y such that $e(X) \hookrightarrow Y$ has UV^n-property ([3]);
- for every embedding $e : X \to Y$ into an ANE Y, the inclusion $e(X) \hookrightarrow Y$ has UV^n-property.

[3]Recall that a compact space X always admit an embedding into a Tychonoff cube (of the appropriate weight) being an AE.

To make it simpler we write also $X \in UV^n$, $0 \leq n \leq \infty$ or $n = \omega$.

The equivalence of these conditions may be established as in [122, p. 499]; proofs follow easily from basic properties of ANE's.

1.11 Remark Let X be a compact space.

(i) Clearly, if $X \in UV^\infty$, then $X \in UV^\omega$.

(ii) Let X be an ANE and $n \geq 0$; then $X \in UV^n$ if and only if X is n-connected and $X \in UV^\infty$ if and only if X is contractible.

(iii) Compact spaces having UV^∞-property are also called *cell-like* sets (see [122, 141]).

(iv) If X is a compact metric space, then it admits an embedding into a complete ANR being an ANE. Hence a compact metric space $X \in UV^n$, $0 \leq n \leq \infty$ or $n = \omega$, if and only if there is an ANR Y and an embedding $e : X \to Y$ such that the inclusion $e(X) \hookrightarrow Y$ has UV^n-property.

Now we collect some facts concerning the above notions. All of them are well-known (see e.g. [122]) or belong to mathematical folklore; we recall them in order to help the reader to get a better understanding of sets having various UV-properties.

After [122] (see also [13], [104]) we have the following simple result.

1.12 Proposition *For a compact metric space X the following conditions are equivalent:*

1. *X is an R_δ-set (i.e. it can be represented as the intersection of a decreasing sequence of compact AR-spaces – see [16]);*
2. *X has UV^∞-property (is a cell-like set);*
3. *X may be represented as the intersection of a decreasing sequence of compact contractible spaces (see [104]);*
4. *For any embedding of X into an* ANE*, X is contractible in each of its neighborhoods (i.e. X is approximatively contractible – see [35]);*
5. *X has the shape of a point (see [35]).*

Recall the following definition

1.13 Definition (See [150] or [59]) The *deformation dimension* $\operatorname{def}\dim \mathrm{X}$ of a space X is the smallest integer m such that any map $f : X \to Q$ from X into a CW-complex Q is homotopic to a map into the m-dimensional skeleton Q^m of Q.

Recall (cf. [59]) that

$$\operatorname{def}\dim \mathrm{X} \leq \dim \mathrm{X}.$$

We have the following

1.14 Proposition *Suppose that a compact metric space X has finite deformation dimension:* $\operatorname{def}\dim \mathrm{X} < \infty$. *If $X \in UV^\omega$, then $X \in UV^\infty$. More precisely, if* $\operatorname{def}\dim \mathrm{X} \leq \mathrm{n} < \infty$ *and $X \in UV^n$, then $X \in UV^\infty$.*

Proof We may suppose that X is a compact subset of a complete ANR Y. Let U be a neighborhood of A in Y. It is enough to show that X is contractible in U.

There is a neighborhood V of X such that $V \subset U$ and each map of a polyhedron of dimension $\leq n$ into V is null-homotopic in U.

It is well-known that there exists a compact polyhedron P and maps $s : X \to P$, $r : P \to V$ such that $j \simeq r \circ s : X \to V$ where $j : X \to V$ is the inclusion ([4]). By our assumption concerning $\operatorname{def}\dim$, $s \simeq s' : X \to P^n$. Since $r' := r|P^n$ is null-homotopic in U, we gather that j is null-homotopic, i.e. X is contractible in U. □

Let us also note the following simple results.

1.15 Proposition

(i) *If A is a compact subset of a space Y and $A \hookrightarrow Y$ has UV^∞-property, then $[A; Z] = 0$ provided Z is an* ANE *or* ANR. *A compact space X such that $[X; Z] = 0$ for any* ANE *Z has UV^∞-property. Consequently*

[4] The polyhedron P may be realized as the nerve of a sufficiently fine (finite) open covering of X, s is then the canonical map and the existence of r follows since V is an ANR.

$X \in UV^\infty$ provided there is a space Y and an embedding $e : X \to Y$ such that the inclusion $e(X) \hookrightarrow Y$ has UV^∞-property.

(ii) *Let A be a compact subset of an* ANR *Y. If each open neighborhood U of A in Y contains a neighborhood V of A such that the inclusion $j : V \to U$ induces the trivial transformation $j_\# : [P; V] \to [P; U]$ for every compact polyhedron P, then $A \in UV^\infty$.*

Proof (i) Take a map $f : A \to Z$. If Z is an ANE, then there is a neighborhood U of A in Y and an extension $F : U \to Z$ of f. Since A is contractible in U, we obtain that $f = F \circ i$, where $i : A \to U$ is the inclusion, is null-homotopic. If Z is an ANR, then there is a complete ANR Z' being homotopy equivalent to Z and since $[A, Z'] = 0$, we get that also $[A, Z] = 0$.

Conversely: we may suppose that $X \subset Y$ where Y is an ANE. By the hypothesis, $[X, U] = 0$ for any open neighborhood U of X in Y because U is an ANE.

(ii) Without loss of generality we may assume that Y is a complete ANR. Take any neighborhood U of A in Y and choose V provided by the assumption. There are maps $s : A \to P$ and $r : P \to V$, where P is a compact polyhedron, such that $r \circ s$ is homotopic to the inclusion $i : A \to V$. Since r is null-homotopic in U we see that A is contractible in U. □

Now we shall compare UV-notions with acyclicity with respect to $\check{H}^*(\cdot\,; G)$ where G is an abelian group. By *pt* we denote a one-point space.

1.16 Theorem *Let X be a compact metric space.*

(i) *$X \in UV^0$ if and only if $\check{H}^0(X; G) = G$.*

(ii) *If $X \in UV^m$, $m \geq 0$, then $\check{H}^n(X; G) = \check{H}^n(pt; G)$ for $0 \leq n \leq m$.*

(iii) *If $X \in UV^1$ and $\check{H}^n(X; G) = \check{H}^n(pt; G)$ for $0 \leq n \leq m+1$ ($m \geq 0$), then $X \in UV^m$.*

These statements follow almost immediately from results of [122].

1.17 Corollary *Let X be a compact metric space.*

(i) *If $X \in UV^\omega$, then X is G-acyclic. In particular, if $X \in UV^\infty$, then X is G-acyclic.*

(ii) *If $X \in UV^1$ and is G-acyclic, then $X \in UV^\omega$. In particular, if $\dim X < \infty$, X is G-acyclic and $X \in UV^1$, then $X \in UV^\infty$.*

The author is not sure whether Theorem 1.16 and Corollary 1.17 hold true for (an arbitrary) compact space X. However we also have

1.18 Remark It is clear that any compact space $X \in UV^\infty$ is G-acyclic.

The class of acyclic compact spaces is essentially larger than that of cell-like sets.

1.19 Example

(i) Kahn [107] gives an example of an acyclic compact metric space X, $\dim X = \infty$, admitting an essential map $g : X \to P = S^3$. Clearly X does not have UV^∞-property (see Proposition 1.15 (i)).

(ii) Let Σ be the Alexander horned sphere in S^3. By the Alexander duality (see e.g. [166]), $S^3 \setminus \Sigma$ has two components A, B each with (singular) homology of a point. Take the component A having the nontrivial fundamental group and let $X = \Sigma \cup A$. Then $\dim X \leq 3$ and $\check{H}^q(X) = H_{3-q}(S^3, B)$ for any $q \geq 0$; hence X is acyclic. Since Σ is a neighborhood retract of S^3, one verifies easily that X is an ANR. Since $\pi_1(X) \neq 0$ (and is not abelian) we see that $X \notin UV^1$ (see Remark 1.11 (ii)).

1.C. Existence of approximations

In the rest of this chapter we are going to study set-valued maps whose values satisfy some properties introduced above. Hence the following definition.

1.20 Definition Let $0 \leq n \leq \infty$ or $n = \omega$. A set-valued map $\varphi : X \to Y$ is a *UV^n-valued map* if, for each $x \in X$, the inclusion $\varphi(x) \hookrightarrow Y$ has UV^n-property.

There are several results concerning the existence of graph-approximations; some of them overlap and were obtained independently by different authors.

1.21 Theorem *Let $\varphi : X \multimap Y$ be a set-valued map of spaces.*

(i) (Cellina [46]) *If Y is a normed space and the values of φ are convex,* *then φ is approximable* ([5]).

(ii) (Cannon [45]) *If X is a locally compact separable metric space, Y is an* ENR (*Euclidean Neighborhood Retract*) *and φ is a UV^∞-valued map, then it is relatively approximable over any closed subset $A \subset X$.*

(iii) (Ancel [3]) *Let $\varphi : X \multimap Y$ is a UV^∞-valued map, X be metrizable and let A be a closed subset of X. If*

- $\dim(X \setminus A) < \infty$*; or*
- *X is countable dimensional,*

then φ is weakly relatively approximable over A;

(iv) ([90, 20]) *If (X, A) is a finite polyhedral pair,* $\dim X \leq n+1$, (*resp. X and A are compact* ANR*s*) *and φ is a UV^n-valued map* (*resp. UV^ω-valued map*)*, then φ is relatively approximable over A* ([6]).

Some of the above mentioned results were generalized for maps being finite compositions of maps from above classes ([92, 116] and [45]) and applied to construct the fixed point index theory on compact ANRs ([90, 116]) and arbitrary ones ([21]).

Result 1.21 (iii) due to Ancel is by all means the most general one concerning UV^∞-valued maps. It appeared while the author investigated general properties of the so-called cell-like maps from the strictly topological viewpoint (extending ideas of [13, 45, 122, 97] and others).

Some later results of, e.g. [10, 23, 11] (where UV^∞-valued maps on convex compact subsets of normed spaces were studied) are implied by Ancel's. However they were obtained independently, the starting point to these investigations was different and they were addressed to analysts rather.

In [90], the authors were interested in the fixed-point theory implications of the existence of graph-approximations; moreover assumptions concerning maps in [90] (or [116, 21]) are weaker than those in [3] (UV^∞-property is stronger than UV^ω).

In what follows we are going to establish some existence results for UV^ω-valued maps and thus give some partial positive answers to Question 1.4. The proofs will follow in the next section.

[5] It is sufficient here to assume that values of φ are merely convex and closed.

[6] For contractible-valued maps defined on polyhedra or compact convex subsets of a normed space – see [137, 10]; for starshaped-valued maps – see [23].

1.22 Theorem *Let $0 \leq n < \infty$ and let $\varphi : X \multimap Y$ be a UV^n-valued map from a metric space X to a locally n-connected metric space Y. Then, for every neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ with the following property: if A is a closed subset of X such that $\dim(X \setminus A) \leq n+1$ and $f : A \to Y$ is a $\mathcal{V}$-approximation of φ, then f extends to a $\mathcal{U}$-approximation $F : X \to Y$ of φ.*

Recall that a simplicial complex K is *locally finite dimensional* if, for each vertex $v \in K$, $\sup\{\dim \sigma \mid v \in \sigma \prec K\} < \infty$.

1.23 Theorem *Let $\varphi : X \multimap Y$ be a UV^ω-valued map from a locally finite dimensional polyhedron X (with the Whitehead topology) to a space Y. Then, for any neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ in $X \times Y$, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ in $X \times Y$ with the following property: if A is a closed subpolyhedron of X and $f : A \to Y$ is a $\mathcal{V}$-approximation of φ, then f extends to a $\mathcal{U}$-approximation $F : X \to Y$ of φ. In particular φ is approximable.*

Theorem 1.22 also implies that φ is approximable provided $\dim X \leq n+1$ (see Corollary 1.33); it is enough to set $A = \emptyset$; in this case the assumptions that X, Y are metrizable and Y is locally n-connected are superfluous.

It is clear that Theorems 1.22 and 1.23 imply that φ is relatively approximable over A. Moreover, in the situation of Theorems 1.22 and 1.23 the relative approximability over A is equivalent to weak relative approximability of φ over A. This is because the hypotheses of these theorems guarantee that every map $f : A \to Y$ has a neighborhood extension $g : B \to Y$. In particular, the assumption of 1.22 that Y is locally n-connected combined with the result of [65] or [120] provides such an extension. In Theorem 1.23 a closed subpolyhedron A is a neighborhood retract of X and such an extension exists for trivial reasons.

1.D. Proofs

In order to prove these theorems we shall need some notation, auxiliary concepts and lemmas.

1.24 Notation If $A \subset X$ and $\mathfrak{A}$ is a collection of sets in X, then $\mathrm{st}\,(\mathrm{A}, \mathfrak{A}) := \bigcup\{U \in \mathfrak{A} \mid U \cap A \neq \emptyset\}$ is the *star* of A with respect to $\mathfrak{A}$. Recall that if $\mathcal{U}$ is a *relation* in $X \times Y$, i.e. $\mathcal{U} \subset X \times Y$, then $\mathcal{U}^{-1} := \{(y, x) \in Y \times X \mid (x, y) \in \mathcal{U}\}$; if $\mathcal{V} \subset Y \times Z$ is another relation, then $\mathcal{V} \circ \mathcal{U} := \{(x, z) \in X \times Z \mid \exists\, y \in Y\ (x, y) \in \mathcal{U},\ (y, z) \in \mathcal{V}\}$; for $A \subset X$, $\mathcal{U}(A) := \{y \in Y \mid \exists\, x \in A\ (x, y) \in \mathcal{U}\}$. Δ_X denotes the *diagonal* of $X \times X$, i.e. $\Delta_X := \{(x, x) \mid x \in X\}$.

Given a simplicial complex K, $|K|$ denotes the *geometric realization* (i.e. the space) of K endowed with the *Whitehead topology*; we write $v \in \sigma \prec K$ to denote that v is a *vertex* of a *simplex* σ in K; for $\sigma \prec K$, $|\sigma|$ (resp. $\langle\sigma\rangle$) denotes the *closed* (resp. *open*) simplex in $|K|$ spanned by σ and $\partial\sigma = \bigcup\{|\tau| \mid \tau \subset \sigma, \tau \neq \sigma\}$ is the *geometric boundary* of σ.

We shall make use of the following versions of some results from [3].

1.25 Lemma *Let $\varphi : X \multimap Y$ be a set-valued map between spaces.*

(i) (Ancel [3, Lemma A.8]) *Suppose that, for each $x \in X$, N_x is a neighborhood of $\varphi(x)$ in Y and let $\{U_x\}_{x \in X}$ be an open cover of X such that $x \in U_x$ for all $x \in X$. Then there are an open cover $\{L_x\}_{x \in X}$ of X and a neighborhood $\mathcal{U}$ of* $\mathrm{Gr}(\varphi)$ *such that $L_x \subset U_x$ for all $x \in X$ and $\mathcal{U}(L_x) \subset N_x$.*

(ii) ([3, Lemma A.10]) *Given a proper map $f : Z \to X$ and a neighborhood $\mathcal{U}$ of* $\mathrm{Gr}(\varphi \circ \mathrm{f})$ *in $Z \times Y$, there are neighborhoods $\mathcal{W}$ of* $\mathrm{Gr}(\varphi)$ *and $\mathcal{N}$ of* $\mathrm{Gr}(\mathrm{f})$ *such that $\mathcal{W} \circ \mathcal{N} \subset \mathcal{U}$. In particular, $\{(z, y) \in Z \times Y \mid (f(z), y) \in \mathcal{W}\} \subset \mathcal{U}$.*

The following is the key notion.

1.26 Definition Let $\varphi : X \multimap Y$ be a set-valued map and let $0 \leq n \leq \omega$. By an *n-nest for φ* we mean a sequence $\{(\mathcal{U}_i, \mathfrak{A}_i)\}_{i \geq 0}$ such that:

(a) $\mathcal{U}_i \supset \mathcal{U}_{i+1}$, $i \geq 0$, are (open) neighborhoods of $\mathrm{Gr}(\varphi)$ in $X \times Y$;

(b) $\mathfrak{A}_i$, $i \geq 0$, are open covers of X;

(c) for each $i \geq 1$ and each $U \in \mathfrak{A}_i$, there is a member $U^\varphi \in \mathfrak{A}_{i-1}$ such that:

- $\mathrm{st}\,(\mathrm{U}, \mathfrak{A}_\mathrm{i}) \subset \mathrm{U}^\varphi$ (i.e. $\mathfrak{A}_i$ is a star refinement of $\mathfrak{A}_{i-1}$);
- for any $0 \leq k \leq \min\{i-1, n\}$ ([7]), any singular k-sphere in $\mathcal{U}_i(\mathrm{st}\,(\mathrm{U}, \mathfrak{A}_\mathrm{i}))$ is null-homotopic in $\mathcal{U}_{i-1}(U^\varphi)$.

[7] i.e. if $n = \omega$, for $0 \leq k \leq i-1$.

1.27 Lemma *If $0 \le n \le \omega$, $\varphi : X \multimap Y$ is a UV^n-valued map between spaces and $\mathcal{U}$ is a neighborhood of* $\mathrm{Gr}(\varphi)$, *then there exists an n-nest $\{(\mathcal{U}_i, \mathfrak{A}_i)\}_{i\ge 0}$ of φ such that*

$$\forall\ U \in \mathfrak{A}_0\ \ \forall\ x \in U\ \ \mathcal{U}_0(U) \subset \mathcal{U}(x). \tag{1.3}$$

Proof Since $\mathrm{Gr}(\varphi \circ 1_{\mathrm{X}} \circ 1_{\mathrm{X}}) = \mathrm{Gr}(\varphi)$ and $\mathrm{Gr}(\varphi) \subset \mathcal{U}$, in virtue of Lemma 1.25 (ii), there are neighborhoods $\mathcal{U}_0$ of $\mathrm{Gr}(\varphi)$ and $\mathcal{M}$ of $\Delta_X = \mathrm{Gr}(1_{\mathrm{X}})$ in $X \times X$ such that $\mathcal{U}_0 \circ \mathcal{M} \circ \mathcal{M}^{-1} \subset \mathcal{U}$. Let $\mathfrak{A}_0 := \{\mathcal{M}(x) \mid x \in X\}$. Then $\mathfrak{A}_0$ is an open cover of X and if $x \in \mathcal{M}(x')$ for some $x' \in X$, then $x' \in \mathcal{M}^{-1}(x)$. Hence $\mathcal{U}_0(\mathcal{M}(x')) \subset \mathcal{U}_0 \circ \mathcal{M} \circ \mathcal{M}^{-1}(x) \subset \mathcal{U}(x)$.

Now let $i \ge 1$ and suppose that a neighborhood $\mathcal{U}_{i-1}$ of $\mathrm{Gr}(\varphi)$ and an open cover $\mathfrak{A}_{i-1}$ of X have been constructed. Let $x \in X$. Choose $U_x \in \mathfrak{A}_{i-1}$ such that $x \in U_x$. Then $\varphi(x) \subset \mathcal{U}_{i-1}(U_x)$. Since $\varphi(x) \hookrightarrow Y$ has UV^n-property, there is a neighborhood N_x of $\varphi(x)$ in Y such that $N_x \subset \mathcal{U}_{i-1}(U_x)$ and, for $0 \le k \le \min\{i-1, n\}$, every singular k-sphere in N_x is null-homotopic in $\mathcal{U}_{i-1}(U_x)$. Lemma 1.25 (i) provides an open cover $\{L_x\}_{x\in X}$ of X and a neighborhood $\mathcal{U}_i$ of $\mathrm{Gr}(\varphi)$ such that $\mathcal{U}_i \subset \mathcal{U}_{i-1}$ and, for each $x \in X$, $L_x \subset U_x$ and $\mathcal{U}_i(L_x) \subset N_x$. Let $\mathfrak{A}_i$ be a star-refinement of $\{L_x\}$. Then $(\mathcal{U}_i, \mathfrak{A}_i)$ satisfies condition (c) from the definition of n-nest. □

1.28 Definition Let K be a simplicial complex. By a *level function* for K we mean a function $\lambda : K \to \mathbf{N} := \{1, 2, 3, \ldots\}$ with the following properties:

(a) If $\sigma \prec K$, then $\lambda(\sigma) > \dim \sigma$;

(b) If $\sigma, \tau \prec K$, $\sigma \subset \tau$ and $\sigma \ne \tau$, then $\lambda(\sigma) > \lambda(\tau)$.

1.29 Lemma *Every locally finite dimensional simplicial complex admits a level function λ such that if K is finite dimensional, then $\lambda(\sigma) \le 2\dim K + 1$ for any $\sigma \prec K$.*

Proof Define $\lambda : K \to \mathbf{N}$ as follows. If $v \in K$ is a vertex, then $\lambda(v) := 2\max\{\dim \sigma \mid v \in \sigma \prec K\} + 1$. If a simplex $\sigma \prec K$ is not a vertex, then $\lambda(\sigma) := \min\{\lambda(v) \mid v \in \sigma\} - \dim \sigma$.

Let $\sigma \prec K$. If $v \in \sigma$, then $\lambda(v) \ge 2\dim \sigma + 1$. Hence $\lambda(\sigma) \ge 2\dim \sigma + 1 - \dim \sigma > \dim \sigma$. This establishes condition (a).

To establish condition (b), suppose that $\sigma \subset \tau \prec K$, $\sigma \ne \tau$. Since

$\min\{\lambda(v) \mid v \in \sigma\} \geq \min\{\lambda(v) \mid v \in \tau\}$ and $\dim \sigma < \dim \tau$, we see that $\lambda(\sigma) > \lambda(\tau)$.

Finally, if $\dim K < \infty$, then clearly $\lambda(\sigma) \leq 2 \dim K + 1$ for each $\sigma \prec K$. □

The following lemma formalizes the principle of inductive extension of approximations over skelata of increasing dimension.

1.30 Lemma *Suppose that $\varphi : X \multimap Y$ is a UV^n-valued map between paracompact spaces, $0 \leq n \leq \omega$, and let $\{(\mathcal{U}_i, \mathfrak{A}_i)\}_{i \geq 0}$ be an n-nest for φ. Suppose that K is a locally finite dimensional complex such that $\dim K \leq n+1$ if $n < \infty$, and $\lambda : K \to \mathbf{N}$ is a level function for K. Suppose that, for each $\sigma \prec K$, an element $U_\sigma \in \mathfrak{A}_{\lambda(\sigma)}$ has been selected so that if $\sigma \subset \tau \prec K$, then $U_\sigma \cap U_\tau \neq \emptyset$. If L is a subcomplex of K and $f : |L| \to Y$ is a map such that $f(|\sigma|) \subset \mathcal{U}_{\lambda(\sigma)-1}(U^\varphi_\sigma)$ ([8]) for each $\sigma \prec L$, then f extends to a map $F : |K| \to Y$ such that $F(|\sigma|) \subset \mathcal{U}_{\lambda(\sigma)-1}(U^\varphi_\sigma)$ for each $\sigma \prec K$.*

Proof We define g inductively on skeleta of K of increasing dimension.

Let $k \geq 0$ and assume that F has been defined on the $(k-1)$-dimensional skeleton $K^{(k-1)}$ of K ([9]). Let σ be a k-simplex of K. Thus F is already defined on $\partial\sigma$. If $\sigma \prec L$, set $F||\sigma| = f||\sigma|$. Now assume that $\sigma \not\prec L$. If $k = 0$, i.e. $\sigma = \{v\}$, $v \in K$, then choose $F(v) \in \mathcal{U}_{\lambda(\sigma)-1}(U^\varphi_\sigma)$. Now suppose that $k \geq 1$. Then $\partial\sigma = |\tau_0| \cup |\tau_1| \cup \ldots \cup |\tau_k|$ where $\dim \tau_j = k-1$ for $j = 0, 1, \ldots, k$. Hence $U_{\tau_j} \cap U_\sigma \neq \emptyset$ for all $0 \leq j \leq k$. For such j, we have $U_{\tau_j} \subset U^\varphi_{\tau_j} \in \mathfrak{A}_{\lambda(\tau_j)-1}$ and $\mathfrak{A}_{\lambda(\tau_j)-1}$ refines $\mathfrak{A}_{\lambda(\sigma)}$ because $\lambda(\tau_j) - 1 \geq \lambda(\sigma)$. Therefore $\bigcup_{j=0}^k U^\varphi_{\tau_j} \subset \mathrm{st}\,(\mathrm{U}_\sigma, \mathfrak{A}_{\lambda(\sigma)})$. By inductive hypothesis, $F(|\tau_j|) \subset \mathcal{U}_{\lambda(\tau_j)-1}(U^\varphi_{\tau_j})$ for all $0 \leq j \leq k$. Thus $F(\partial\sigma) \subset \mathcal{U}_{\lambda(\sigma)}(\mathrm{st}\,(\mathrm{U}^\varphi_\sigma, \mathfrak{A}_{\lambda(\sigma)}))$. Since $\dim \sigma \leq \min\{\lambda(\sigma)-1, n\}$, we gather that $F|\partial\sigma : \partial\sigma \to \mathcal{U}_{\lambda(\sigma)}(\mathrm{st}\,(\mathrm{U}_\sigma, \mathfrak{A}_{\lambda(\sigma)}))$ admits an extension onto $|\sigma|$ with values in $\mathcal{U}_{\lambda(\sigma)-1}(U^\varphi_\sigma)$. This shows that F may be extended onto the k-skeleton of K. □

Now we are in a position to present:

Proof of Theorem 1.23 Assume that K is a locally finite dimensional simplicial complex such that $|K| = X$ and let $\mathcal{U}$ be a neighborhood of $\mathrm{Gr}(\varphi)$.

Lemma 1.27 provides an ω-nest $\{(\mathcal{U}_i, \mathfrak{A}_i)\}_{i \geq 0}$ for φ such that, for each

[8]the open set $U^\varphi_\sigma \in \mathfrak{A}_{\lambda(\sigma)-1}$ corresponds to U_σ according to Definition 1.26 (c).

[9]understanding that $K^{(-1)} = \emptyset$.

$x \in U \in \mathfrak{A}_0$, $\mathcal{U}_0(U) \subset \mathcal{U}(x)$. Let λ be the level function for K given by Lemma 1.29. For each $k \geq 1$, set

$$C_k := \bigcup\{|\sigma| \mid \sigma \prec K,\, \lambda(\sigma) \geq k\}.$$

Then $X = C_1 \supset C_2 \supset C_3 \supset \ldots$ are closed subsets of X.

Claim 1. $\bigcap_{k=1}^{\infty} C_k = \emptyset$ and hence $\bigcup_{k=1}^{\infty}(X \setminus C_k) = X$.

To see it, let $\sigma \prec K$ and set $m = \max\{\lambda(v) \mid v \in \sigma\}$. It is sufficient to show that $|\sigma| \cap C_{m+1} = \emptyset$. Suppose to the contrary that there is $\tau \prec K$ such that $\lambda(\tau) \geq m+1$ and $|\sigma| \cap |\tau| \neq \emptyset$. Hence σ and τ have a common vertex v. But then $m \geq \lambda(v) \geq \lambda(\tau) \geq m+1$, a contradiction. This completes the proof of Claim 1.

For each $k \geq 1$, let $\mathfrak{B}_k := \{U \setminus C_k \mid U \in \mathfrak{A}_{2k}\}$ and $\mathfrak{B} := \bigcup_{k=1}^{\infty} \mathfrak{B}_k$. Clearly $\mathfrak{B}$ is an open cover of X. Let K' be a subdivision of K refining $\mathfrak{B}$.

For each $\sigma \prec K'$, let σ^+ denote the unique simplex of K such that $\langle\sigma\rangle \subset \langle\sigma^+\rangle$.

Claim 2. There is a level function $\mu : K' \to \mathbf{N}$ such that $\mu(\sigma) < 2\lambda(\sigma^+)$ for each $\sigma \prec K'$.

Define $\mu : K' \to \mathbf{N}$ by the formula:

$$\mu(\sigma) := \lambda(\sigma^+) + \dim\sigma^+ - \dim\sigma$$

for any $\sigma \prec K'$.

Condition (a) of Definition 1.28 is evident since $\mu(\sigma) \geq \lambda(\sigma^+) > \dim\sigma^+ \geq \dim\sigma$ for any $\sigma \prec K'$.

To establish condition (b) of Definition 1.28, suppose that different simplices $\sigma \subset \tau$ in K' are given. Then $\sigma^+ \subset \tau^+$. Hence $\lambda(\sigma^+) + \dim\sigma^+ = \min\{\lambda(v) \mid v \in \sigma^+\} \geq \min\{\lambda(v) \mid v \in \tau^+\} = \lambda(\tau^+) + \dim\tau^+$. Since $\dim\sigma < \dim\tau$, we get that $\mu(\sigma) > \mu(\tau)$.

Finally, $\mu(\sigma) \leq \lambda(\sigma^+) + \dim\sigma^+ < 2\lambda(\sigma^+)$. This completes the proof of Claim 2.

Claim 3. For every $\sigma \prec K'$, there is a member $U_\sigma \in \mathfrak{A}_{\mu(\sigma)}$ such that $|\sigma| \subset U_\sigma$.

Indeed, let $\sigma \prec K'$. Since K' refines $\mathfrak{B}$, there is $k \geq 1$ and $U \in \mathfrak{A}_{2k}$ such that $|\sigma| \subset U \setminus C_k$. Hence $|\sigma^+| \not\subset C_k$ and, therefore, $\lambda(\sigma^+) < k$. So, by Claim 2, $\mu(\sigma) < 2k$. Thus $\mathfrak{A}_{2k}$ refines $\mathfrak{A}_{\mu(\sigma)}$. Consequently, there is $U_\sigma \in \mathfrak{A}_{\mu(\sigma)}$ such that $|\sigma| \subset U \subset U_\sigma$.

Claim 3 implies, in particular, that if $\sigma \subset \tau \prec K'$, then $U_\sigma \cap U_\tau \neq \emptyset$. We need this in order to apply Lemma 1.30.

Now set

$$\mathcal{V} := \bigcup_{k=1}^{\infty} \mathcal{U}_{2k} \cap ((X \setminus C_k) \times Y).$$

Then $\mathcal{V}$ is a neighborhood of $\mathrm{Gr}(\varphi)$.

Let L be a subcomplex of K, L' be a subcomplex of K' such that $|L'| = |L|$ and let $f : |L| \to Y$ be a $\mathcal{V}$-approximation of φ. We are to construct a $\mathcal{U}$-approximation $F : X \to Y$ of φ such that $F||L| = f$.

Let $\sigma \prec L'$. In order to use Lemma 1.30, we have to check that $f(|\sigma|) \subset \mathcal{U}_{\mu(\sigma)-1}(U_\sigma^\varphi)$ where U_σ^φ corresponds to U_σ as in Definition 1.26. Since $|\sigma| \subset |\sigma^+| \subset C_{\lambda(\sigma^+)} \subset C_k$ for $1 \leq k \leq \lambda(\sigma^+)$, we get that $|\sigma| \cap (X \setminus C_k) = \emptyset$ and $\mathrm{Gr(f||\sigma|)} \cap ((\mathrm{X} \setminus \mathrm{C_k}) \times \mathrm{Y}) = \emptyset$ for $1 \leq k \leq \lambda(\sigma^+)$. But $\mathrm{Gr(f)} \subset \mathcal{V}$, so

$$\mathrm{Gr(f||\sigma|)} \subset \bigcup_{\mathrm{k}=\lambda(\sigma^+)+1}^{\infty} \mathcal{U}_{2\mathrm{k}} \cap ((\mathrm{X} \setminus \mathrm{C_k}) \times \mathrm{Y}).$$

Since $\mu(\sigma) < 2\lambda(\sigma^+)$, then $\mathcal{U}_{2k} \subset \mathcal{U}_{2\lambda(\sigma^+)} \subset \mathcal{U}_{\mu(\sigma)}$ for all $k > \lambda(\sigma^+)$. Hence $\mathrm{Gr(f||\sigma|)} \subset \mathcal{U}_{\mu(\sigma)}$. Consequently, by Claim 3, $f(|\sigma|) \subset \mathcal{U}_{\mu(\sigma)}(|\sigma|) \subset \mathcal{U}_{\mu(\sigma)-1}(U_\sigma^\varphi)$.

Now Lemma 1.30 provides an extension $F : |K'| = X \to Y$ such that $F(|\sigma|) \subset \mathcal{U}_{\mu(\sigma)-1}(U_\sigma^\varphi)$ for each $\sigma \prec K'$. For such σ, U_σ^φ is contained in an element U_σ^* of $\mathfrak{A}_0$. Therefore $F(|\sigma|) \subset \mathcal{U}_0(U_\sigma^*)$ for each $\sigma \prec K'$. Now if $x \in |\sigma| \subset U_\sigma^*$ for some $\sigma \prec K'$, then $F(x) \in \mathcal{U}_0(U_\sigma^*) \subset \mathcal{U}(x)$ and this means that $\mathrm{Gr(F)} \subset \mathcal{U}$. The proof is completed. □

1.31 Corollary *Let $\varphi : X \multimap Y$ be as in Theorem 1.23. If A is a closed subset of X, then φ is weakly relatively approximable over A.*

Proof For any neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ satisfying the assertion of Theorem 1.23. Suppose that $f : A \to Y$ is a $\mathcal{V}$-approximation of φ admitting an extension $f' : N \to Y$ onto a neighborhood N of A in X. Using the continuity of f' we may assume, without loss of generality, that f' is a $\mathcal{V}$-approximation of φ, too. There is a closed subpolyhedron Q in X such that $A \subset Q \subset N$. This completes the proof if we invoke Theorem 1.23. □

The proof of Theorem 1.22 relies on the theorem of Kuratowski [120] and its extension from [65]. The following lemma formulates this result along with some aspects of its proof as given by Hu [101, p. 51–55, 150–155] or by Borsuk [34, p. 80–83].

1.32 Lemma *Let A be a closed subset of a metric space X such that $\dim(X \setminus A) \leq n+1$, $0 \leq n < \infty$. For every open cover $\mathfrak{B}$ of $X \setminus A$, there is an $(n+1)$-dimensional simplicial complex K and a map $\beta : X \setminus A \to |K|$ with the following properties:*

(a) *$\{\beta^{-1}(|\sigma|) \mid \sigma \prec K\}$ refines $\mathfrak{A}$;*

(b) *for every vertex $v \in K$, $\beta^{-1}(v) \neq \emptyset$;*

(c) *if Y is a locally n-connected metric space and $f : A \to Y$ is a map, then there is a subcomplex L of K and a map $\gamma : |L| \to Y$ such that:*

- *$A \subset \mathrm{int}\,(\mathrm{A} \cup \beta^{-1}(|\mathrm{L}|))$;*
- *if $a \in A$ and H is a neighborhood of $f(a)$ in Y, then there is a neighborhood V of a in X such that if $\sigma \prec L$ and $\beta^{-1}(|\sigma|) \subset V$, then $\gamma(|\sigma|) \subset H$.*

Only property (b) requires a comment. In e.g. [34], one chooses a so-called canonical locally finite open cover $\mathfrak{K}'$ of $X \setminus A$ refining $\mathfrak{B}$ and defines β to be the canonical map from $X \setminus A$ to the nerve of $\mathfrak{K}'$. To arrive condition (b), one may "reduce" $\mathfrak{K}'$ by means of transfinite induction to a locally finite open cover $\mathfrak{K}$ of $X \setminus A$ with the property that no member of $\mathfrak{K}$ is covered by other elements of $\mathfrak{K}$. Thus if K is the nerve of $\mathfrak{K}$ and a "new" $\beta : X \setminus A \to |K|$ is the canonical map, then β has property (b) stated above.

Observe that condition (b) implies that $\beta^{-1}(|\sigma|) \neq \emptyset$ for any $\sigma \prec K$. Moreover, condition (c) implies that the function $f^+ : A \cup \beta^{-1}(|L|) \to Y$ defined by

$$f^+(x) = \begin{cases} f(x) & \text{if } x \in A \\ \gamma \circ \beta(x) & \text{if } x \in \beta^{-1}(|L|) \end{cases} \tag{1.4}$$

is continuous.

Proof of Theorem 1.22 Let $\mathcal{U}$ be a neighborhood of $\mathrm{Gr}(\varphi)$. Lemma 1.27 provides an n-nest $\{(\mathcal{U}_i, \mathfrak{A}_i)\}_{i\geq 0}$ for φ such that, for each $x \in U \in \mathfrak{A}_0$, $\mathcal{U}_0(U) \subset \mathcal{U}(x)$.

Let $\mathcal{V} = \mathcal{U}_{2n+3}$, take a closed subset A of X such that $\dim(X \setminus A) \leq n+1$ and suppose that $f : A \to Y$ is a $\mathcal{V}$-approximation of φ. We are to prove that there is a $\mathcal{U}$-approximation $F : X \to Y$ of φ such that $F|A = f$.

Let $\mathfrak{B}$ be a canonical covering (see [34, p. 69]) of $X \setminus A$ inscribed into $\mathfrak{A}_{2n+3}$. The important here is that $\mathfrak{B}$ has the following property:

($*$) for any $a \in A$ and every neigbhborhood V of a in X, there is a neighborhood W of a such that $\mathrm{st}\,(\mathrm{W}, \mathfrak{B}) \subset \mathrm{V}$.

In virtue of Lemma 1.32, there are an $(n+1)$-dimensional complex K, a map $\beta : X \setminus A \to |K|$ satisfying conditions (a), (b) from Lemma 1.32, a subcomplex L of K and a map $\gamma : |L| \to Y$ such that condition (c) from Lemma 1.32 holds.

Let $a \in A$ and choose $U_a \in \mathfrak{A}_{2n+3}$ such that $a \in U_a$. Since $H_a := \mathcal{V}(U_a)$ is a neighborhood of $f(a)$ in Y, Lemma 1.32 (c) provides a neighborhood V_a of a in X such that $V_a \subset U_a \cap (A \cup \beta^{-1}(|L|))$ and if $\sigma \prec L$, $\beta^{-1}(|\sigma|) \subset V_a$, then $\gamma(|\sigma|) \subset H_a = \mathcal{V}(U_a)$. Finally, by ($*$), choose a neighborhood W_a of a in X such that $W_a \cup \mathrm{st}\,(\mathrm{W_a}, \mathfrak{B}) \subset \mathrm{V_a}$.

Let $W = \bigcup_{a\in A} W_a$. Then W is a neighborhood of A and $W \subset A \cup \beta^{-1}(|L|)$. Let $L_W := \{\sigma \prec L \mid \exists\, \tau \prec L$ such that $\sigma \subset \tau$ and $\beta^{-1}(|\tau|) \cap W \neq \emptyset\}$. We easily see that L_W is a subcomplex of L and $W \subset A \cup \beta^{-1}(|L_W|)$.

Suppose that $\sigma \prec L_W$. There are a simplex $\tau \prec L$ and $a(\sigma) \in A$ such that $\sigma \subset \tau$ and $\beta^{-1}(|\tau|) \cap W_{a(\sigma)} \neq \emptyset$. Since $\beta^{-1}(|\tau|)$ is contained in an element of $\mathfrak{B}$, then $\beta^{-1}(|\sigma|) \subset \beta^{-1}(|\tau|) \subset \mathrm{st}\,(\mathrm{W}_{\mathrm{a}(\sigma)}, \mathfrak{B}) \subset \mathrm{V}_{\mathrm{a}(\sigma)}$. Hence

$$\beta^{-1}(|\sigma|) \subset U_{a(\sigma)} \quad \text{and} \quad \gamma(|\sigma|) \subset \mathcal{V}(U_{a(\sigma)}). \tag{1.5}$$

In view of Lemma 1.29, there is a level function λ for K such that $\lambda \leq 2\dim K + 1 = 2n+3$. Now we shall show how to select, for each $\sigma \prec K$, an element $U_\sigma \in \mathfrak{A}_{\lambda(\sigma)}$ so that the hypotheses of Lemma 1.30 are satisfied.

Let $\sigma \prec L_W$. Since $U_{a(\sigma)} \in \mathfrak{A}_{2n+3}$ and $\mathfrak{A}_{2n+3}$ is inscribed to $\mathfrak{A}_{\lambda(\sigma)}$, there is a member $U_\sigma \in \mathfrak{A}_{\lambda(\sigma)}$ such that $U_{a(\sigma)} \subset U_\sigma$. Then, by (1.5), $\beta^{-1}(|\sigma|) \subset U_\sigma$ and $\gamma(|\sigma|) \subset \mathcal{V}(U_\sigma) = \mathcal{U}_{2n+3}(U_\sigma) \subset \mathcal{U}_{\lambda(\sigma)}(U_\sigma) \subset \mathcal{U}_{\lambda(\sigma)-1}(U_\sigma^\varphi)$. If $\sigma \prec K$

but $\sigma \not\prec L_W$, then $\beta^{-1}(|\sigma|)$ is contained in an element of $\mathfrak{B}$ and $\mathfrak{B}$ refines $\mathfrak{A}_{\lambda(\sigma)}$. Hence there is $U_\sigma \in \mathfrak{A}_{\lambda(\sigma)}$ such that $\beta^{-1}(|\sigma|) \subset U_\sigma$.

It is clear that if $\sigma \subset \tau \prec K$, then $U_\sigma \cap U_\tau \neq \emptyset$ for $\emptyset \neq \beta^{-1}(|\sigma|) \subset \beta^{-1}(|\sigma|) \cap \beta^{-1}(|\tau|) \subset U_\sigma \cap U_\tau$.

It follows from Lemma 1.30 that $\gamma||L_W| : |L_W| \to Y$ extends to a map $\delta : |K| \to Y$ such that $\delta(|\sigma|) \subset \mathcal{U}_{\lambda(\sigma)-1}(U_\sigma^\varphi)$ for each $\sigma \prec K$. Therefore $\delta \circ \beta|\beta^{-1}(|L_W|) = \gamma \circ \beta|\beta^{-1}(|L_W|)$.

Let us define $F : X \to Y$ by $F|A := f$ and $F|(X \setminus A) := \delta \circ \beta$. To see that F is continuous recall a continuous map $f^+ : A \cup \beta^{-1}(|L|) \to Y$ given by formula (1.4) and observe that $F|A \cup \beta^{-1}(|L_W|) = f^+|A \cup \beta^{-1}(|L_W|)$.

Clearly $\mathrm{Gr(F|A)} = \mathrm{Gr(f)} \subset \mathcal{V} = \mathcal{U}_{\mathrm{2n+3}} \subset \mathcal{U}_0 \subset \mathcal{U}$. Let $x \in X \setminus A$. Then, for some $\sigma \prec K$, $\beta(x) \in |\sigma|$. So $x \in \beta^{-1}(|\sigma|) \subset U_\sigma \subset U_\sigma^\varphi \in \mathfrak{A}_{\lambda(\sigma)-1}$ and $F(x) = \delta \circ \beta(x) \in \delta(|\sigma|) \subset \mathcal{U}_{\lambda(\sigma)-1}(U_\sigma^\varphi)$. But since $\mathfrak{A}_{\lambda(\sigma)}$ refines $\mathfrak{A}_0$, U_σ^φ is contained in some member U_σ^* of $\mathfrak{A}_0$. Thus $x \in U_\sigma^*$ and $F(x) \in \mathcal{U}_0(U_\sigma^*)$. In view of (1.3) (see Lemma 1.27), it follows now that $F(x) \in \mathcal{U}(x)$, i.e. $\mathrm{Gr(F|X \setminus A)} \subset \mathcal{U}$. Hence $\mathrm{Gr(F)} \subset \mathcal{U}$, i.e $F : X \to Y$ is a $\mathcal{U}$-approximation of φ and $F|A = f$. □

1.33 Corollary *Let $0 \leq n < \infty$ and let $\varphi : X \multimap Y$ be a UV^n-valued map from a space X, $\dim X \leq n+1$, to a space Y. Then φ is approximable.*

This corollary follows from Theorem 1.22 by setting $A = \emptyset$ and observing that in this case the assumptions that X, Y are metrizable and Y is locally n-connected are superfluous.

1.E. Further results

Actually our Theorems 1.22 and 1.23 lie between Theorem 1.21 (iii) and the following theorem which can be deduced from results of Toruńczyk [170]. Namely, a variant of Proposition 6.3 of [170] yields the following result which forms a version of Theorem 1.23 with a weaker conclusion but under more general assumptions.

1.34 Theorem *Let $\varphi : X \multimap Y$ be a UV^ω-valued map from an* ANR *X to a metric space Y. If A is a closed subset of X and N is a neighborhood* *of A, then for every neighborhood $\mathcal{U}$ of* $\mathrm{Gr}(\varphi)$ *and any selection $f : N \to Y$ of φ ([10]), there is a $\mathcal{U}$-approximation $F : X \to Y$ of φ such $F|A = f|A$.*

First let us formulate a version of Proposition 6.3 of Toruǹczyk [170].

1.35 Proposition *Let $f : X \to Y$ be a proper surjection from a metric space X onto a metric space Y and suppose that there is a closed set $S \subset Y$ such that the inclusion $f^{-1}(y) \hookrightarrow X$ has UV^{ω}-property for any $y \in S$ and $f^{-1}(y)$ is a singleton for $y \in Y \setminus S$. Suppose M is an* ANR*, $u : M \to Y$ and $\eta : M \to (0, \infty)$ is a continuous function. If K is a closed subset of M, U is a neighborhood of K in M and $v : U \to X$ is a map such that $f \circ v = u|U$, then there is a map $g : M \to X$ such that $g|K = v|K$ and $d(f \circ g(x), u(x)) < \eta(x)$ for all $x \in M$.*

Proof It is a modification of the proof of Proposition 6.3 from [170]. Let $Z(f)$ denote the cylinder of f and let $p : Z(f) \to Y$ be the natural collapse map. Clearly there is a "natural" projection $p' : Z(f) \setminus S \to X$ such that $p'|X = 1_X$ and $f \circ p' = p|Z(f) \setminus S$. As in [170, Prop. 6.3], there is a map $w : M \to Z(f)$ such that $w|K = v|K$ and $p \circ w = u$. In the present situation, $Z(f) \setminus S$ is LC^n rel.$Z(f)$ (see [170, p. 101] or [65]) at each point of S for all $0 \leq n < \infty$. Hence S is locally homotopically negligible in $Z(f)$ in virtue of [170, Th. 2.8]. Therefore, there is an "η-short" homotopy joining w to a map $w' : M \to Z(f) \setminus S$ such that $w'|K = v|K$. Now, to complete the proof, define $g : M \to X$ putting $g = p' \circ w'$. □

Proof of Theorem 1.34 Let N be a neighborhood of $A, \mathcal{U}$ be a neighborhood of $\mathrm{Gr}(\varphi)$ and assume that $f : N \to Y$ is a selection of φ. Let $\mathfrak{D}$ be the upper semicontinuous decomposition of $X \times Y$ whose elements are single points $\{(x, y)\}$, $y \notin \varphi(x)$, and sets of the form $\{x\} \times \varphi(x)$ for $x \in X$ and let $q : X \times Y \to X \times Y/\mathfrak{D}$ denote the quotient map. Observe that $Z := X \times Y/\mathfrak{D}$ is a metrizable space. Set $S = q(\mathrm{Gr}(\varphi))$. Then $q^{-1}(z)$ is a singleton for $z \in Z \setminus S$ and $q^{-1}(z) \hookrightarrow X \times Y$ has UV^{ω}-property for $z \in S$. Consider maps $u : X \to Z$, $v : N \to X \times Y$ given by $u(x) = q(\{x\} \times \varphi(x))$ and $v(x) = (x, f(x))$ for $x \in X$. Then $q \circ v = u|N$. Finally, let $\varepsilon : X \to (0, \infty)$ be a continuous function corresponding to $\mathcal{U}$ as in Proposition 1.2 (i).

Let $\pi : X \times Y \to Y$ denote the projection. Using methods similar to those presented in the proof of Proposition 1.2 (i) and since the map

[10] i.e., $f(x) \in \varphi(x)$ for all $x \in N$.

$\pi \circ q^{-1} : Z \multimap Y$ is upper semicontinuous, one constructs a continuous function $\eta : X \to (0, \infty)$ such that

$$\begin{aligned} \pi \circ q^{-1}(B(u(x), \eta(x))) &\subset B(\pi \circ q^{-1} \circ u(B(x, \varepsilon(x))), \varepsilon(x)) \\ &= B(\varphi(B(x, \varepsilon(x))), \varepsilon(x)). \end{aligned} \tag{1.6}$$

Proposition 1.35 provides a map $w : X \to X \times Y$ such that $w|A = v|A$ and $d(q \circ w(x), u(x)) < \eta(x)$ for each $x \in X$. We define $F : X \to Y$ putting $F = \pi \circ w$. Hence $F|A = f$. Moreover, for each $x \in X$, $F(x) \in \pi \circ q^{-1}(B(u(x), \eta(x)))$. In view of (1.6), F is an ε-approximation of φ. $\square$

Setting $A = B = \emptyset$ in Theorem 1.34 we obtain the following approximation result.

1.36 Corollary *A UV^ω-valued map $\varphi : X \multimap Y$ from an* ANR *X to a metric space Y is approximable.*

In order to formulate a positive result concerning relative approximability in an even more general case we shall need the following definition.

1.37 Definition Let $\mathfrak{A}$ be an open cover of a space X and let A be a closed subset of X. We say that the space X (resp. the pair (X, A)) is *properly $\mathfrak{A}$-dominated by a space Z* (*resp. by a pair* (Z, C)) if there are a map $p : X \to Z$ (resp. $p : (X, A) \to (Z, C)$), a proper map $r : Z \to X$ (resp. $r : (Z, C) \to (X, A)$) and a homotopy $h : X \times [0, 1] \to X$ (resp. $h : (X, A) \times [0, 1] \to (X, A)$) such that $h_0 = 1_X$, $h_1 = r \circ p$ and $\{h(\{x\} \times [0, 1])\}_{x \in X}$ refines the cover $\mathfrak{A}$.

In the early 1980's, Ancel and Toruńczyk independently asked which ANR's are properly dominated by locally finite dimensional polyhedra? For instance, the standard construction of dominating polyhedra of ANR's yields that every separable or locally compact ANR, being strongly paracompact, is properly $\mathfrak{A}$-dominated by a locally finite, hence locally finite dimensional polyhedron for any open cover $\mathfrak{A}$. The question whether every ANR satisfies this property remains open.

It turns out that for spaces that are properly dominated by locally finite dimensional polyhedra we have the following fact concerning relative approximability over (A, B).

1.38 Theorem *Let X be a space which is properly $\mathfrak{A}$-dominated by a locally finite dimensional polyhedron for every open cover $\mathfrak{A}$ of X. Let $\varphi : X \multimap Y$ be UV^ω-valued map into a space Y. Then, for every neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ and every open cover $\mathfrak{A}$ of X, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ such that if A, B are closed subsets of X and $\mathrm{st}\,(\mathrm{A}, \mathfrak{A}) \subset \mathrm{B}$, $f : B \to Y$ is a $\mathcal{V}$-approximation of φ, then there exists a $\mathcal{U}$-approximation $F : X \to Y$ of φ with $F|A = f|A$. In particular, φ is relatively approximable over (A, B) where $A \subset X$ is closed and B is a closed neighborhood of A.*

Proof The last statement follows from the first one since there is an open cover $\mathfrak{A}$ of X such that $\mathrm{st}\,(\mathrm{A}, \mathfrak{A}) \subset \mathrm{int}\,\mathrm{B}$.

Take a neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ and an open cover $\mathfrak{A}$ of X. Since $\varphi = \varphi \circ 1_X$, then Lemma 1.25 (ii) provides neighborhoods $\mathcal{W}$ of $\mathrm{Gr}(\varphi)$ (in $X \times Y$) and $\mathcal{N}$ of the diagonal Δ_X such that

$$\mathcal{W} \circ \mathcal{N} \subset \mathcal{U}. \tag{1.7}$$

Similarly, there is a neighborhood $\mathcal{M}$ of Δ_X such that $\mathcal{M} \circ \mathcal{M}^{-1} \subset \mathcal{N}$. Let $\mathfrak{B}$ be a star refinement of $\mathfrak{A}$ which also refines the cover $\{\mathcal{M}(x)\}_{x \in X}$. If $V \in \mathfrak{B}$ and $x \in V$, then

$$V \subset \mathcal{N}(x). \tag{1.8}$$

Indeed, if $x \in V \subset \mathcal{M}(x')$ for some $x' \in X$, then $V \subset \mathcal{M} \circ \mathcal{M}^{-1}(x) \subset \mathcal{N}(x)$.

By our assumption, there are a locally finite dimensional polyhedron P, a map $p : X \to P$, a proper map $r : P \to X$ and a homotopy $h : X \times [0, 1] \to X$ such that $\{h(\{x\} \times [0, 1])\}_{x \in X}$ refines $\mathfrak{B}$ and $h_0 = 1_X$, $h_1 = r \circ p$. Hence, by (1.8), for each $x \in X$,

$$x,\, r \circ p(x) \in h(\{x\} \times [0, 1]) \subset \mathcal{N}(x). \tag{1.9}$$

Clearly $\varphi \circ r$ is a UV^ω-valued map. Moreover, $\mathrm{Gr}(\varphi \circ \mathrm{r}) \subset \mathcal{U}' := \{(\mathrm{z}, \mathrm{y}) \in \mathrm{P} \times \mathrm{Y} \mid (\mathrm{r}(\mathrm{z}), \mathrm{y}) \in \mathcal{W}\}$. It is easy to see that $\mathcal{U}'$ is an open subset of $P \times Y$.

By Theorem 1.23, there is neighborhood $\mathcal{V}'$ of $\mathrm{Gr}(\varphi \circ \mathrm{r})$ in $P \times Y$ such that, for any subpolyhedron Q of P and any $\mathcal{V}'$-apprroximation $f' : Q \to Y$ of $\varphi \circ r$, there is a $\mathcal{U}'$-approximation $F' : P \to Y$ of $\varphi \circ r$ with $F'|Q = f'$.

In view of Lemma 1.25 (ii) and the properness of r, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ (in $X \times Y$) such that $\mathcal{W}' := \{(z, y) \in P \times Y \mid (r(z), y) \in \mathcal{V}\} \subset \mathcal{V}'$. Moreover, we may assume that $\mathcal{V} \subset \mathcal{W}$.

We now take closed subsets A and B such that $\mathrm{st}\,(\mathrm{A}, \mathfrak{A}) \subset \mathrm{B}$ and let $f : B \to Y$ be a $\mathcal{V}$-approximation of φ. We are to show that there exists a $\mathcal{U}$-approximation $F : X \to Y$ of φ such that $F|A = f|A$.

Let $C = \mathrm{cl}\,\mathrm{st}\,(\mathrm{A}, \mathfrak{B})$ and $D = \mathrm{cl}\,\mathrm{st}\,(\mathrm{C}, \mathfrak{B})$. Since $r \circ p(C) \subset h(C \times [0,1]) \subset \mathrm{st}\,(\mathrm{C}, \mathfrak{B})$, we obtain that $\mathrm{cl}\, p(C) \subset r^{-1}(D)$. Since $\mathrm{st}\,(\mathrm{A}, \mathfrak{A}) \subset \mathrm{B}$ and $\mathfrak{B}$ star refines $\mathfrak{A}$, then $\mathrm{st}\,(\mathrm{D}, \mathfrak{B}) \subset \mathrm{B}$ and, thus, $D \subset \mathrm{int}\,\mathrm{B}$. Consequently, $\mathrm{cl}\, p(C) \subset r^{-1}(\mathrm{int}\,\mathrm{B})$. Hence there is a subpolyhedron Q of P such that $p(C) \subset Q \subset r^{-1}(B)$. Since $r(Q) \subset B$, then $f \circ (r|Q) : Q \to Y$ is defined and $\mathrm{Gr}(\mathrm{f} \circ (\mathrm{r}|\mathrm{Q})) \subset \mathcal{W}' \subset \mathcal{V}'$, i.e. $f \circ (r|Q)$ is a $\mathcal{V}'$-approximation of $\varphi \circ r$. Therefore it admits an extension $F' : P \to Y$ being a $\mathcal{U}'$-approximation of $\varphi \circ r$.

Obviously $A \subset \mathrm{int}\,\mathrm{C}$. There is an Urysohn's map $\lambda : X \to [0,1]$ such that $\lambda|A \equiv 0$ and $\lambda|(X \setminus \mathrm{int}\,\mathrm{C}) \equiv 1$. Consider a map $F : X \to Y$ given by

$$F(x) = \begin{cases} f(h(x, \lambda(x)) & \text{if } x \in C \\ F'(p(x)) & \text{if } x \in X \setminus \mathrm{int}\,\mathrm{C}. \end{cases}$$

The map F is well-defined because, for $x \in C$, $h(\{x\} \times [0,1]) \subset \mathrm{st}\,(\mathrm{C}, \mathfrak{B}) \subset \mathrm{B}$ (f is defined on B). F is continuous for if $x \in C \setminus \mathrm{int}\,\mathrm{C}$, then $\lambda(x) = 1$ and $p(x) \in p(C) \subset Q$; hence $f(h(x, \lambda(x)) = f(h(x,1)) = f \circ r \circ p = F'(p(x))$. Moreover $F|A = f|A$.

If $x \in C$, then by (1.9), $(x, h(x, \lambda(x))) \in \mathcal{N}$ and $(h(x, \lambda(x)), f(h(x, \lambda(x)))) \in \mathcal{V}$ since $\mathrm{Gr}(\mathrm{f}) \subset \mathcal{V}$. Hence $(x, F(x)) = (x, f(h(x, \lambda(x)))) \in \mathcal{V} \circ \mathcal{N} \subset \mathcal{W} \circ \mathcal{N} \subset \mathcal{U}$. For $x \in X \setminus \mathrm{int}\,\mathrm{C}$, $(p(x), F(x)) = (p(x), F'(p(x))) \in \mathcal{U}'$ hence $(r \circ p(x), F(x)) \in \mathcal{W}$. By (1.9), $(x, r \circ p(x)) \in \mathcal{N}$ so $(x, F(x)) \in \mathcal{W} \circ \mathcal{N} \subset \mathcal{U}$ in view of (1.7). This shows that indeed F is a $\mathcal{U}$-approximation of φ. □

The next result is a direct generalization of Theorem 1.21 (iv).

1.39 Theorem *Let (X, A) be an* ANR*-pair which is properly $\mathfrak{A}$-dominated by a locally finite dimensional polyhedral pair for each open cover $\mathfrak{A}$ of X. If $\varphi : X \multimap Y$ is a UV^ω-valued map into a space Y, then it is relatively approximable over A.*

First we shall need the following lemma (comp. [20, Lemma 2.2]).

1.40 Lemma *Let (X, A) be an* ANR*-pair and $\varphi : X \multimap Y$ be a set-valued map into a space Y. Let $M := X \times \{0\} \cup A \times [0,1]$. For any neighborhood $\mathcal{U}$*

of $\mathrm{Gr}(\varphi)$*, there is a neighborhood* $\mathcal{U}_0$ *of* $\mathrm{Gr}(\varphi)$ *with the following property: for every map* $g : M \to Y$ *such that* $(x, g(x,t)) \in \mathcal{U}_0$ *for each* $(x,t) \in M$*, there is an extension* $G : X \times [0,1] \to Y$ *of* g *such that* $(x, G(x,t)) \in \mathcal{U}$ *for all* $x \in X$ *and* $t \in [0,1]$*.*

Proof Let $\mathcal{U}' := \{(x,t,y) \in X \times [0,1] \times Y \mid (x,y) \in \mathcal{U}\}$; it is clear that $\mathcal{U}'$ is an open neighborhood of $\mathrm{Gr}(\varphi')$ in $X \times [0,1] \times Y$ where $\varphi' = \varphi \circ \pi$ and $\pi : X \times [0,1] \to X$ is the projection. By Lemma 1.25 (ii), there are a neighborhood $\mathcal{U}_0'$ of $\mathrm{Gr}(\varphi')$ and a neighborhood $\mathcal{N}$ of the diagonal in $(X \times [0,1])^2$ such that $\mathcal{U}_0' \circ \mathcal{N} \subset \mathcal{U}'$.

Clearly M is a neighborhood retract of $X \times [0,1]$; hence there is a neighborhood retraction $r : U \to M$. It is easy to show that there exists a neighborhood V of M in $X \times [0,1]$ such that $V \subset U$ and, for each $(x,t) \in V$, $r(x,t) \in \mathcal{N}(x,t)$.

By Lemma 1.25 (ii), and since π is proper, there is a neighborhood $\mathcal{U}_0$ of $\mathrm{Gr}(\varphi)$ such that $\{(x,t,y) \in X \times [0,1] \times Y \mid (x,y) \in \mathcal{U}_0\} \subset \mathcal{U}_0'$. Take $g : M \to Y$ such that $(x, g(x,t)) \in \mathcal{U}_0$ for all $(x,t) \in M$ and define $g' = g \circ (r|V) : V \to Y$. It is obvious that, for all $(x,t) \in V$, $(x,t,g'(x,t)) \in \mathcal{U}_0' \circ \mathcal{N} \subset \mathcal{U}'$; and hence

$$(x, g'(x,t)) \in \mathcal{U}. \tag{1.10}$$

Take a neighborhood N of A in X such that $N \times [0,1] \subset V$, an Urysohn's function $\lambda : X \to [0,1]$ such that $\lambda|A \equiv 1$, $\lambda|X \setminus N \equiv 0$ and define $G : X \times [0,1] \to Y$ putting $G(x,t) = g'(x, \lambda(x)t)$. By (1.10), $(x, G(x,t)) \in \mathcal{U}$ for all $(x,t) \in X \times [0,1]$ and $G|M = g$. □

Proof of Theorem 1.39 Take a neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$. Lemma 1.40 provides a neighborhood $\mathcal{U}_0$ of $\mathrm{Gr}(\varphi)$ according to X, A, φ and $\mathcal{U}$.

As in the proof of Theorem 1.38, there are neighborhoods $\mathcal{W}$ of $\mathrm{Gr}(\varphi)$ and $\mathcal{N}$ of Δ_X such that $\mathcal{W} \circ \mathcal{N} \subset \mathcal{U}_0$. Moreover, let $\mathfrak{A}$ be an open cover of X such that, for each $U \in \mathfrak{A}$ and $x \in U$, $U \subset \mathcal{N}(x)$.

Now let us take a locally finite dimensional polyhedral pair (P,Q), a map $p : (X,A) \to (P,Q)$ and a proper map $r : (P,Q) \to (X,A)$ such that 1_X and $r \circ p$ are joined by a homotopy $h : (X,A) \times [0,1] \to (X,A)$ such that $\{h(\{x\} \times [0,1])\}_{x \in X}$ refines $\mathfrak{A}$. Hence, for each $x \in X$, $t \in [0,1]$, $h(x,t) \in \mathcal{N}(x)$.

The map $\varphi \circ r$ is a UV^{ω}-valued map and $\mathrm{Gr}(\varphi \circ \mathrm{r})$ lies in an open set $\mathcal{U}' := \{(z, y) \in P \times Y \mid (r(z), y) \in \mathcal{W}\}$. By Theorem 1.23, we get a neighborhood $\mathcal{V}'$ of $\mathrm{Gr}(\varphi \circ \mathrm{r})$ such that any $\mathcal{V}'$-approximation $f' : Q \to Y$ of $\varphi \circ r$ admits an extension $F' : P \to Y$ being a $\mathcal{U}'$-approximation $F' : P \to Y$ of $\varphi \circ r$.

In view of the properness of r and Lemma 1.25 (ii), there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ such that $\mathcal{V} \subset \mathcal{W}$ and $\{(z, y) \in P \times Y \mid (r(z), y) \in \mathcal{V}\} \subset \mathcal{V}'$.

Take an arbitary $\mathcal{V}$-approximation $f : A \to Y$ of φ and let $f' = f \circ r$. Clearly f' is a $\mathcal{V}'$-approximation of $\varphi \circ r$. Hence there is a $\mathcal{U}'$-approximation $F' : P \to Y$ of $\varphi \circ r$ extending f'.

Define $g : M = X \times \{0\} \cup A \times [0, 1] \to Y$ be given by

$$g(x, t) = \begin{cases} F'(p(x)) & \text{if } x \in X,\ t = 0 \\ f \circ h(x, 1 - t) & \text{if } x \in A,\ t \in [0, 1]. \end{cases}$$

It is easy to see that g is well-defined and continuous. Moreover, for any $(x, t) \in M$, $(x, g(x, t)) \in \mathcal{U}_0$. Due to the choice of $\mathcal{U}_0$ we gather, by Lemma 1.40, that there is an extension $G : X \times [0, 1] \to Y$ of g such that $(x, G(x, t)) \in \mathcal{U}$.

Finally we define $F = G(\,\cdot\,, 1)$. Then $F|A = f$ and $\mathrm{Gr(F)} \subset \mathcal{U}$. □

Now let us turn to Question 1.6. We easily get the following corollaries.

1.41 Corollary *Let $\varphi : X \multimap Y$ be a UV^n-valued map from a space X to Y, $0 \le n \le \omega$. If X, Y are metrizable space, $n < \infty$ and Y is locally n-connected (resp. X is a locally finite dimensional polyhedron with the Whitehead topology, $n = \omega$), then for any neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ with the following property: if A is a closed subset of X such that $\dim(X \setminus A) \le n$ (resp. A is a closed subpolyhedron of X), $f, g : X \to Y$ are $\mathcal{V}$ approximations of φ, $h : A \times [0, 1] \to Y$ is a homotopy joining $f|A$ to $g|A$ and h_t is a $\mathcal{V}$-approximation of φ for all $t \in [0, 1]$, then there is a homotopy $H : X \times [0, 1] \to Y$ joining f to g such that $H|A \times [0, 1] = h$ and H_t is a $\mathcal{U}$-approximation of φ.*

Proof Let $X' = X \times [0, 1]$, $\pi : X' \to X$ be the projection and let $\varphi' = \varphi \circ \pi : X' \multimap Y$. It is clear that φ' is a UV^n-valued map and X' is a locally finite dimensional polyhedron whenever X is so. Take a neighborhood $\mathcal{U}$

of $\mathrm{Gr}(\varphi)$ in $X \times Y$. Then $\mathcal{U}' := \{(x,t,y) \in X' \times Y \mid (x,y) \in \mathcal{U}\}$ is a neighborhood of $\mathrm{Gr}(\varphi')$ in $X' \times Y$.

Theorem 1.22 (resp. 1.23) provides a neighborhood $\mathcal{V}'$ of $\mathrm{Gr}(\varphi')$ in $X' \times Y$ such that if A' is a closed subset of X', $\dim(X' \setminus A') \leq n+1$ (resp. A' is a closed subpolyhedron of X') and $h' : A' \to Y$ is a $\mathcal{V}'$-approximation of φ', then there is a $\mathcal{U}'$-approximation $H : X' \to Y$ of φ' such that $H|A' = h'$.

Since π is proper, Lemma 1.25 (ii) provides a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ such that $\{(x,t,y) \in X' \times Y \mid (x,y) \in \mathcal{V}\} \subset \mathcal{V}'$.

Now let A be a closed subset of X such that $\dim(X \setminus A) \leq n$ (resp. A be a closed subpolyhedron of X). If we set $A' = X \times \{0,1\} \cup A \times [0,1]$, then clearly $X' \setminus A' \subset (X \setminus A) \times [0,1]$ and thus, $\dim(X' \setminus A') \leq n+1$ (resp. A' is a closed polyhedron of X'). Take $f, g : X \to Y$ and $h : A \times [0,1] \to Y$ as in the formulation of the corollary. If we define $h' : A' \to Y$ by $h'(x,t) = h(x,t)$ for $x \in A$, $t \in [0,1]$ and $h'(\,\cdot\,,0) = f$, $h'(\,\cdot\,,1) = g$, then h' is a $\mathcal{V}'$-approximation of φ' and hence, it admits an extension $H : X' \to Y$ being a $\mathcal{U}'$-approximation of φ'. It is easy to see that H is the required homotopy. □

Evidently if we set above $A = \emptyset$, then we obtain the results concerning homotopy approximability of φ.

In a similar way one gets a corollary concerning homotopy approximability over (A, B) of φ in case a space X is properly $\mathfrak{A}$-dominated by a locally finite dimensional polyhedron for each open cover $\mathfrak{A}$ of X. Namely the following result holds.

1.42 Corollary *Let X be a space properly $\mathfrak{A}$-dominated by a locally finite dimensional polyhedron for any open cover $\mathfrak{A}$ of X and let $\varphi : X \multimap Y$ be a UV^ω-valued map into a space Y. For any neighborghood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ and every open cover $\mathfrak{A}$ of X, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ with the following property. If A, B are closed subsets of X, $\mathrm{st}\,(\mathrm{A}, \mathfrak{A}) \subset \mathrm{B}$, $f, g : X \to Y$ are $\mathcal{V}$-approximations of φ and $h : B \times [0,1] \to Y$ is a homotopy joining $f|B$ to $g|B$ and h_t is a $\mathcal{V}$-approximation of φ for each $t \in [0,1]$, then there is a homotopy $H : X \times [0,1] \to Y$ joining f to g, $H|A \times [0,1] = h|A \times [0,1]$ and H_t is a $\mathcal{U}$-approximation of φ for all $t \in [0,1]$. In particular, φ is homotopy approximable.*

Proof First one proves that $X' := X \times [0,1]$ is also properly $\mathfrak{A}'$-dominated by a locally finite dimensional polyhedron for every open cover $\mathfrak{A}'$ of X'.

Let $\mathcal{U}$ be a neighborhood of $\mathrm{Gr}(\varphi)$ and $\mathfrak{A}$ be an open cover of X. As in the proof of Corollary 1.41, we define the neighborhood $\mathcal{U}'$ of the graph of the map $\varphi' : X' \multimap Y$. We also have to produce an appropriate cover of X'. Namely let $\mathfrak{A}' := \{U \times J \mid A \in \mathfrak{A},\ J \in \mathfrak{I}\}$ where $\mathfrak{I} := \{[0,1/3), (1/4,3/4), (2/3,1]\}$. Theorem 1.38 may now be stated in terms of φ', $X', \mathcal{U}'$ and $\mathfrak{A}'$ and a neighborhood $\mathcal{V}'$ of $\mathrm{Gr}(\varphi')$ with the necessary properties is obtained. We define a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ as before in the proof of Corollary 1.41.

Let A, B be closed sets in X, $\mathrm{st}\,(\mathrm{A}, \mathfrak{A}) \subset \mathrm{B}$. If we define A' as in the proof of Corollary 1.41 and put $B' := X \times ([0,1/3] \cup [2/3,1]) \cup B \times [0,1]$, then $\mathrm{st}\,(\mathrm{A}', \mathfrak{A}') \subset \mathrm{B}'$.

Now take $f, g : X \to Y$ and $h : B \times [0,1]$ as in the formulation of the corollary.

In order to define a map $h' : B' \to Y$ whose extension H onto X' will become the required homotopy let us define a map $h'' : X \times \{0,1\} \cup B \times [0,1] \to Y$ by the formula

$$h''(x,t) = \begin{cases} f(x) & \text{if } x \in X,\ t = 0 \\ g(x) & \text{if } x \in X,\ t = 1 \\ h(x,t) & \text{if } x \in B,\ t \in [0,1] \end{cases}$$

Urysohn's Lemma provides a map $\lambda : X \to [0,1]$ such that $\lambda | A \equiv 0$ and $\lambda | (X \setminus \mathrm{int}\,\mathrm{B}) \equiv 1$. For $(x,t) \in X'$, set

$$r(x,t) = (x, \min\{\max\{1 + 2\lambda(x))t - \lambda(x), 0\}, 1\}).$$

Then $r(B') \subset X \times \{0,1\} \cup B \times [0,1]$, $r|A' = 1_{A'}$ and $\pi \circ r = \pi$. Finally we define $h' : B' \to Y$ by $h' := h'' \circ (r|B')$. It follows easily that $h'(\,\cdot\,,0) = f$ and $h'(\,\cdot\,,1) = g$ on X and $h'|A \times [0,1] = h|A \times [0,1]$. It remains to show that $\mathrm{Gr}(\mathrm{h}') \subset \mathcal{V}'$. For any $(x,t) \in B'$, $h'(x,t) = f(x)$ or $h'(x,t) = g(x)$ or $h'(x,t) = h(x,s)$ where $(x,s) = r(x,t)$. Thus, h' is indeed a $\mathcal{V}'$-approximation of φ'.

By Theorem 1.38, h' has an extension H onto X' providing the required homotopy joining f to g.

The last assertion follows if we set $A = B = \emptyset$ and $\mathfrak{A} = \{X\}$. □

1.F. Additional remarks

Let us observe here that the approximability of a set-valued map is in a sense a sufficient condition for the map to have UV-values. Namely, one shows easily that if $\varphi : X \multimap Y$ is a set-valued map of metric spaces with the following property: for every compact ANR T with $\dim T \leq n+1$, for every closed sub-ANR A of T, every continuous map $j : T \to X$ and every $\varepsilon > 0$, there is $\delta > 0$ such than any δ-approximation $f : A \to Y$ of $\varphi \circ j$ extends to an ε-approximation $F : T \to Y$ of $\varphi \circ j$, then φ is a UV^n-valued map.

Most of facts concerning approximability of set-valued maps extend to compositions of maps. Here is a sample result.

1.43 Proposition *Let $\varphi : X \to Y$ be an approximable set-valued map between spaces and let $g : Y \to Z$. Then $g \circ \varphi$ is approximable.*

Proof Let $\mathcal{W}$ be a neighborhood of $\mathrm{Gr}(g \circ \varphi)$ in $X \times Z$ and consider a map $G : X \times Y \to X \times Z$ given by $G(x, y) = (x, g(y))$ for $(x, y) \in X \times Y$. Any point $p \in \mathrm{Gr}(\varphi)$ has a neighborhood $\mathcal{U}_p$ in $X \times Y$ such that $G(\mathcal{U}_p) \subset \mathcal{W}$.

Let $\mathcal{U} = \bigcup_{p \in \mathrm{Gr}(\varphi)} \mathcal{U}_p$. If $f : X \to Y$ is a $\mathcal{U}$-approximation of φ, then $g \circ f$ is a $\mathcal{W}$-approximation of $g \circ \varphi$. □

Observe that even if φ is a UV^n-valued map, $0 \leq n \leq \omega$, then $g \circ \varphi$ has, in general, no longer "regular", i.e. satisfying any UV-properties, values. Such maps arise quite naturally in applications; for instance, the evaluation at a fixed time of the Poincaré – Andronov operator of translation along trajectories of a differential equation with a continuous right-hand side is of that type (see Proposition 6.7).

At last, let us provide the following example showing that (i) the "locally n-connected" hypothesis cannot be deleted from Theorem 1.22, and (ii) the hypothesis" (X, A) is an ANR-pair properly dominated by a locally finite dimensional polyhedral pair" cannot be deleted from Theorem 1.39.

1.44 Example Let $Z := \{0\} \cup \{1/n \mid n = 1, 2, \ldots\} \subset \mathbf{R}$ and let $X := Z \times [0, 1]$. Moreover, let $Y = \{(\lambda z, 1 - \lambda) \in \mathbf{R}^2 \mid z \in Z, \lambda \in [0, 1]\}$ be the cone over Z with the vertex $v = (0, 1)$. We define a map $\varphi : X \multimap Y$ by

$$\varphi(z,t) = \begin{cases} \{0\} \times [0,1] & \text{if } z = 0,\ t \in [0,1] \\ \{(z,0)\} & \text{if } z \neq 0,\ t \in [0,1]. \end{cases}$$

Then φ is a UV^ω-valued map. Finally let $A := Z \times \{0, 1\} \cup \{0\} \times [0, 1]$; A is closed in X.

We shall show that, for any neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$, there is a $\mathcal{V}$-approximation $f : A \to Y$ of φ admittting no extension onto any neighborhood of A.

Indeed, let $\mathcal{V}$ be a neighborhood of $\mathrm{Gr}(\varphi)$. Since $(\{0\}\times[0, 1])\times\{(0, 0)\} \subset \mathrm{Gr}(\varphi) \subset \mathcal{V}$, then there is an $m \geq 1$ such that $(J \times [0, 1]) \times \{(0, 0)\} \subset \mathcal{V}$ where $J := \{0\} \cup \{1/n \mid n \geq m\}$. Set $B := \mathrm{cl}\,(A \setminus (J \times \{1\})) = (Z \times \{0\}) \cup ((Z \setminus J) \times \{1\}) \cup (\{0\} \times [0, 1])$ and define $f : A \to Y$ by

$$f(z,t) = \begin{cases} (z,0) & \text{if } (z,t) \in B \\ (0,0) & \text{if } (z,t) \in J \times \{1\}. \end{cases}$$

If $(z, t) \in B$, then $f(z, t) \in \varphi(z, t)$; if $z \in J$, then $((z, 1), f(z, 1)) = ((z, 1),(0, 0)) \in \mathcal{V}$. Hence $\mathrm{Gr(f)} \subset \mathcal{V}$.

Assume that f extends to a map $f' : N \to Y$ where N is a neighborhood of A in X. Then there must exist an $m' \geq m$ such that $J' \times [0, 1] \subset N$ where $J' = \{0\} \cup \{1/n \mid n \geq m'\}$. For each $n \geq m'$, $c_n := f'|\{1/n\} \times [0, 1]$ is a path in Y from $c_n(0) = f(1/n, 0)$ to $c_n(1) = f(1/n, 1)$. Hence, for each $n \geq m'$, there is a point $t_n \in [0, 1]$ such that $c_n(t_n)$ is the vertex v of Y. The sequence (t_n) converges (possibly after passing to a subsequence) to a point $t \in [0, 1]$. Then $(1/n, t_n) \to (0, t) \in A$, so $f'(1/n, t_n) \to f(0, t)$. But this implies that $f(0, t) = v$. This is a contradiction because $f(0, t) = (0, 0)$.

Chapter 2.

THE VIETORIS THEOREM

2.A. The Sklyarenko result

In 1927 L. Vietoris [171] proved his famous theorem stating that a continuous surjective map whose fibres are acyclic with respect to the constructed therein homology theory (based on the notion of true cycles and applicable to metric spaces) induces an isomorphism of the homology groups of spaces. Vietoris's homology theory developed into the Čech homology groups of compact spaces and Čech cohomology groups for arbitrary spaces. The Vietoris theorem has evolved as well and the following remarkable result has been established – see [24, 25].

2.1 Theorem [Begle (1950)] *Let X, Y be paracompact spaces, G be an abelian group and let $f : X \to Y$ be a continuous closed surjection. If there is an integer $N \geq 0$ such that, for each $0 \leq k < N$ and $y \in Y$, $\check{H}^k(f^{-1}(y); G) = \check{H}^k(*; G)$, then the homomorphism $f^* = \check{H}^k(f) : \check{H}^k(Y; G) \to \check{H}^k(X; G)$, induced by f, is*

(i) *an isomorphism if $0 \leq k \leq N-1$;*

(ii) *a monomorphism if $k = N$.*

Above $*$ stands for a one-point space.

Perhaps the most elementary proof of this result was given by Lawson [129] in 1973. The author has shown it as a simple consequence of his main theorem stating that any two taut cohomology theories on a paracompact space coinciding on points are isomorphic.

In 1963 Sklyarenko [161] obtained a significant generalization of the Vietoris-Begle theorem. Let again $f : X \to Y$ be a closed surjection and let $\mathcal{A}$ be a sheaf of abelian groups over Y ([1]). For an integer $k \geq 1$, define

$$\mathrm{s}^0(\mathrm{f};\mathcal{A}) := \{\mathrm{y} \in \mathrm{Y} \mid \mathrm{H}^0(\mathrm{f}^{-1}(\mathrm{y});\mathcal{A}^*) \neq \mathcal{A}_\mathrm{y}\}$$

$$\mathrm{s}^\mathrm{k}(\mathrm{f};\mathcal{A}) := \{\mathrm{y} \in \mathrm{Y} \mid \mathrm{H}^\mathrm{k}(\mathrm{f}^{-1}(\mathrm{y});\mathcal{A}^*) \neq 0\}$$

($H^*(\,\cdot\,;\mathcal{A})$ denotes the cohomology theory with coefficients in the sheaf $\mathcal{A}$) and, for an integer $N \geq 1$, let us define the *Vietoris indices of* f by

$$\mathrm{i}^\mathrm{N}(\mathrm{f};\mathcal{A}) := \inf\{\mathrm{n} \geq 0 \mid \max_{0\leq \mathrm{k}\leq \mathrm{N}-1}\{\mathrm{rd}_\mathrm{Y}(\mathrm{s}^\mathrm{k}(\mathrm{f};\mathcal{A})) + \mathrm{k}\} + 1 < \mathrm{n}\}\ ([2]).$$

Note that if there is no $n \geq 0$ such that $\mathrm{rd}_\mathrm{Y}(\mathrm{s}^\mathrm{k}(\mathrm{f};\mathcal{A})) + \mathrm{k} + 1 < \mathrm{n}$ for $0 \leq k \leq N-1$, then, by the definition, we have $\mathrm{i}^\mathrm{N}(\mathrm{f};\mathcal{A}) = \infty$; moreover, $\mathrm{i}^\mathrm{N}(\mathrm{f};\mathcal{A})$ never takes value 1.

Additionally, we let

$$\begin{aligned} \mathrm{i}^\mathrm{N}(\mathrm{f};\mathcal{A}) &= -\infty \qquad \text{for } N = -1, 0;\\ \mathrm{i}(\mathrm{f};\mathcal{A}) &= \sup_{N\geq 0} \mathrm{i}^\mathrm{N}(\mathrm{f};\mathcal{A}). \end{aligned}$$

In the sequel if a sheaf $\mathcal{A}$ is constant and equal to G (resp. $G = \mathbf{Z}$), then in the above notation we write $\mathrm{s}^*(\,\cdot\,;\mathrm{G}), \mathrm{i}^*(\,\cdot\,;\mathrm{G})$ and $\mathrm{i}(\,\cdot\,;\mathrm{G})$ (resp. $\mathrm{s}^*(\,\cdot\,)$, $\mathrm{i}^*(\,\cdot\,)$ and $\mathrm{i}(\,\cdot\,)$) unless it leads to an ambiguity.

In particular,

$$\mathrm{i}(\mathrm{f}) = \mathrm{i}(\mathrm{f};\mathbf{Z}). \tag{2.1}$$

[1] Let us recall notation concerning sheaves [39]: $\mathcal{A}_y$ denotes the fibre of a sheaf $\mathcal{A}$ over $y \in Y$; a sheaf over the whole space and those induced by it over subspaces are denoted by the same letter; by $\mathcal{A}^*$ we denote the inverse image of a sheaf $\mathcal{A}$ under a map f; if $B \subset Y$ is closed, then $\mathcal{A}_B$ denotes the sheaf induced by $\mathcal{A}$ equal to $\mathcal{A}$ over $Y \setminus B$ and 0 elsewhere; if a sheaf $\mathcal{A}$ is constant and equal to an abelian group G, then we write G in place of $\mathcal{A}$: in particular G_B denotes the sheaf equal to G over $Y \setminus B$ and 0 elsewhere.

[2] for $A \subset Y$, $\mathrm{rd}_\mathrm{Y}(\mathrm{A}) := \sup\{\dim \mathrm{C} \mid \mathrm{C} \text{ is closed in } \mathrm{Y}, \mathrm{C} \subset \mathrm{A}\}$. Moreover, we let $\mathrm{rd}_\mathrm{Y}(\emptyset) = -\infty$.

2.2 Theorem [Sklyarenko (1963)] *If there is* $N \geq 0$ *such that* $\mathrm{i^N(f;\mathcal{A}) \leq N}$, *then, for* $q \geq 0$,

$$f^* = H^q(f) : H^q(Y;\mathcal{A}) \to H^q(X;\mathcal{A}^*) \ (^3)$$

is an epimorphism if $q = \mathrm{i^N(f;\mathcal{A})} - 1$, *an isomorphism if* $\mathrm{i^N(f;\mathcal{A}) \leq q \leq N - 1}$ *and a monomorphism if* $q = N$. *If* $\mathrm{i(f;\mathcal{A})} < \infty$, *then* f^* *is an epimorphism for* $q = \mathrm{i(f;\mathcal{A})} - 1$ *and an isomorphism for* $q \geq \mathrm{i(f;\mathcal{A})}$.

2.3 Remark Let $B \subset Y$ be closed.

(i) Suppose that $A = f^{-1}(B)$ and G is an abelian group; if we take a sheaf $\mathcal{A} = \mathcal{G}_{\mathcal{B}}$, then $H^q(X;\mathcal{A}^*) = \check{H}^q(X,A;G)$ and $H^q(Y;\mathcal{A}) = \check{H}^q(Y,B;G)$ for any $q \geq 0$. Thus if, for some $N \geq 0$,

$$\begin{aligned} m &= \inf\{n \mid \max_{0\leq k\leq N-1}\{\mathrm{rd_Y(\{y \in Y \setminus B \mid \check{H}^k(f^{-1}(y);G)} \\ &\qquad \neq \check{H}^k(*;G)\}) + k\} + 1 < n\} \end{aligned}$$

and $m \leq N$, then $f^* : \check{H}^q(Y,B;G) \to \check{H}^q(X,A;G)$ is an epimorphism for $m - 1 \leq q \leq N - 1$ and a monomorphism for $m \leq q \leq N$. Evidently $m = \mathrm{i^N(f;G_B)}$.

If, for $0 \leq k \leq N - 1$, sets $\mathrm{s^k(f;G_B) = \{y \in Y \setminus B \mid \check{H}^k(f^{-1}(y);G) \neq \check{H}^k(*;G)\}}$ are empty, then above $m = 0$ and we obtain a relative version of the Vietoris-Begle Theorem 2.1.

(ii) From the universal coefficients theorem it follows that if $f : X \to Y$ is a perfect map (⁴), then for any $k \geq 0$, we have $\mathrm{s^k(f;\mathcal{A}_B) \subset s^k(f;\mathbf{Z}_B) \cup s^{k+1}(f;\mathbf{Z}_B)}$. Therefore if, for some $N \geq 0$, $\mathrm{i^N(f;\mathbf{Z}_B)} \leq 0$, then $\mathrm{i^{N-1}(f;\mathcal{A}_B)} \leq 0$ for any sheaf $\mathcal{A}$; this means that, for $y \in Y \setminus B$, fibres $f^{-1}(y)$ of f are acyclic in dimensions k, $0 \leq k \leq N - 1$ and $\mathcal{A}_y$-acyclic in dimensins $0 \leq k \leq N - 2$. In particular, if $\mathrm{i(f;\mathbf{Z}_B)} = 0$, then $\mathrm{i(f;\mathcal{A}_B)} = 0$.

(iii) Observe that a standard way to obtain a relative version of the Vietoris-Begle theorem for groups $\check{H}^*(\cdot\,,\cdot;G)$ via exact cohomological sequences and the five-lemma would give a worse result under stronger assumptions. However, recalling that $\check{H}^*(X,A;G) = \check{H}^*(X/A,a_0;G)$, $\check{H}^*(Y,B;G) = \check{H}^*(Y/B,b_0;G)$ and $f^* = (f_A : (X/A,a_0) \to (Y/B,b_0))^*$, we shall also get the result stated in (i) directly applying Theorem 2.2 and taking a constant and equal to G sheaf over Y/B.

(iv) Białynicki-Birula in [28] has generalized the Vietoris-Begle theorem

[3] note that f^* is a monomorphism for $q = 0$.

[4] i.e. closed with compact fibres.

in another direction. He considers three spaces X, Y and T and closed surjections $f : X \to Y$, $g : Y \to T$ and $h = g \circ f$ and shows that the assertion of this theorem holds provided, for each $t \in T$, f induces an isomorphism $\check{H}^q(g^{-1}(t); G) \to \check{H}^q(h^{-1}(t); G)$ for $0 \le q \le N-1$ and a monomorphism for $q = N$. Sklyarenko [161] extends this result in a similar manner as before.

Let us now collect several properties of the above defined number $\mathrm{i}^{\mathrm{N}}(\mathrm{f}; \mathcal{A})$ where $N \ge 0$ and $\mathcal{A}$ is an arbitrary sheaf.

2.4 Proposition

(i) *The sequence* $N \mapsto \mathrm{i}^{\mathrm{N}}(\mathrm{f}; \mathcal{A})$ *is nondecreasing. If* $\dim X < \infty$*, then for each* $N \ge \dim X + 1$, $\mathrm{i}^{\mathrm{N}}(\mathrm{f}, \mathcal{A}) = \mathrm{i}^{\dim \mathrm{X}+1}(\mathrm{f}, \mathcal{A}) = \mathrm{i}(\mathrm{f}, \mathcal{A})$*. If* $\dim X, \dim Y < \infty$*, then for each* $N \ge 0$, $\mathrm{i}^{\mathrm{N}}(\mathrm{f}, \mathcal{A}) \le \dim \mathrm{Y} + \min\{\dim \mathrm{X}, \mathrm{N} - 1\} + 2$.

(ii) *If* $f_B = f|A : A \to B$ *(recall that* $A = f^{-1}(B)$*), then* $\mathrm{i}^{\mathrm{N}}(\mathrm{f_B}; \mathcal{A}) \le \mathrm{i}^{\mathrm{N}}(\mathrm{f}; \mathcal{A})$ *for any* $N \ge 0$.

(iii) *Let* $f_1 : X \to Y$ *and* $f_2 : Y \to Z$ *be closed surjections. If, for some* $N \ge 1$, $\mathrm{i}^{\mathrm{N}}(\mathrm{f_1}; \mathcal{A}) = 0$*, then* $\mathrm{i}^{\mathrm{N}}(\mathrm{f_2} \circ \mathrm{f_1}; \mathcal{A}) = \mathrm{i}^{\mathrm{N}}(\mathrm{f_2}; \mathcal{A})$.

(iv) $\mathrm{i}^{\mathrm{N}}(\mathrm{f}; \mathcal{A}_{\mathrm{B}}) \le \mathrm{i}^{\mathrm{N}}(\mathrm{f}; \mathcal{A})$ *for any* $N \ge 0$.

Let us derive a simple corollary from the Sklyarenko Theorem 2.2. Let G be an arbitrary abelian group. As before by $\mathbf{Z}_B$ we denote a sheaf being constant and equal $\mathbf{Z}$ over an open set $Y \setminus B$ and 0 on B.

2.5 Corollary *Let* $f : X \to Y$ *be a perfect surjection such that* $A = f^{-1}(B)$*, where* $B \subset Y$ *is closed, and suppose that, for some* $N \ge 0$, $\mathrm{i}^{\mathrm{N}}(\mathrm{f}; \mathbf{Z}_{\mathrm{B}}) < \mathrm{N}$*. If the space* Y *is compact or the group* G *is finitely generated, then for each* $q \ge 0$*, the induced homomorphism*

$$f^* : \check{H}^q(Y, B; G) \to \check{H}^q(X, A; G)$$

is an epimorphism for $q = \mathrm{i}^{\mathrm{N}}(\mathrm{f}; \mathbf{Z}_{\mathrm{B}}) - 1$*, an isomorphism for* $\mathrm{i}^{\mathrm{N}}(\mathrm{f}; \mathbf{Z}_{\mathrm{B}}) \le \mathrm{q} \le \mathrm{N} - 2$ *and a monomorphism for* $q = N - 1$*. Consequently, if* $\mathrm{i}(\mathrm{f}; \mathbf{Z}_{\mathrm{B}}) < \infty$*, then* f^* *is an epimorphism for* $q = \mathrm{i}(\mathrm{f}; \mathbf{Z}_{\mathrm{B}}) - 1$ *and an isomorphism for* $q \ge \mathrm{i}(\mathrm{f}; \mathbf{Z}_{\mathrm{B}})$.

In particular, if for some $N \ge 0$, $\mathrm{i}^{\mathrm{N}}(\mathrm{f}; \mathbf{Z}_{\mathrm{B}}) \le 0$ *(resp.* $\mathrm{i}(\mathrm{f}; \mathbf{Z}_{\mathrm{B}}) = 0$*), then* f^* *is an isomorphism for all* $0 \le q \le N-2$*, a monomorphism for* $q = N-1$ *(resp. an isomorphism for all* $q \ge 0$*) and no additional assumptions* $(*)$ *concerning* Y *or* G *are necessary.*

Proof The assertion holds for $G = \mathbf{Z}$. Since the universal coefficients sequence for the Čech cohomology is exact whenever Y is compact (hence X is also compact because f being perfect is proper) or the group G is finitely generated (see [166]), we complete the proof of the first and second part of the assertion.

If $\mathrm{i}^{\mathrm{N}}(\mathrm{f}; \mathbf{Z}_{\mathrm{B}}) \leq 0$ (resp. $\mathrm{i}(\mathrm{f}; \mathbf{Z}_{\mathrm{B}}) = 0$), then $\mathrm{i}^{\mathrm{N}-1}(\mathrm{f}; \mathrm{G}_{\mathrm{B}}) \leq 0$ (resp. $\mathrm{i}(\mathrm{f}; \mathrm{G}_{\mathrm{B}}) = 0$) and the third assertion follows from the Sklyarenko theorem and Remark 2.3 (ii). □

2.B. The Vietoris theorem and spectra

It is well-known that the Čech cohomology $\check{H}^*(\,\cdot\,; G)$ corresponds to the Eilenberg-MacLane spectrum $\mathbf{K}(G) = \{K(G, n)\}_{n=-\infty}^{\infty}$ ([5]) (in the sense that $\check{H}^n(X; G) = \mathbf{K}(G)^n(X)$). Hence the following question arises:

2.6 Question *Is there a version of the Vietoris-Begle theorem for spectral cohomologies?*

2.7 Spectra Recall that if $\mathbf{E} = \{E_n, e_n\}_{n=-\infty}^{\infty}$ is a CW-*spectrum* (i.e. for each $n \in \mathbf{Z}$, E_n and $e_n : SE_n \to E_{n+1}$ belong to the category of pointed CW-complexes), then for a pointed CW-complex X and an integer n, one defines an abelian group $\mathbf{E}^n(X)$, a *suspension* isomorphism $\sigma_n : \mathbf{E}^{n+1}(SX) \to \mathbf{E}^n(X)$ ([6]) and shows that $\mathbf{E}^*(\,\cdot\,) = \{\mathbf{E}^n(\,\cdot\,), \sigma^n(\,\cdot\,)\}_{n=-\infty}^{\infty}$ is an (extraordinary reduced) cohomology theory on the category of pointed CW-complexes – see [169]. As usual the cofunctor $\mathbf{E}^*(\,\cdot\,)$ may be extended to the cofunctor of the (reduced) cohomology $\mathbf{E}^*(\,\cdot\,)$ on the category of all CW-complexes.

Moreover, the groups $\mathbf{E}^n(X)$, X being any (pointed) topological space, are defined as $\lim\limits_{\overrightarrow{\alpha \in \Lambda}} \{\mathbf{E}^n(X_\alpha), p^*_{\alpha\beta}, \Lambda\}$, where $\{X_\alpha, p_{\alpha\beta}, \alpha \in \Lambda\}$ is the Čech system of the space X (see [150]), i.e. one considers the Čech extension of the cofunctor $\mathbf{E}^*(\,\cdot\,)$ from the category of (pointed) CW-complexes onto the category of all (pointed) topological spaces.

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[5]For $n \geq 1$, $K(G, n)$ is an Eilenberg-MacLane space of (G, n)-type and, for $n \leq 0$, $K(G, n) = \{*\}$.

[6]for $k \geq 1$, $S^k X$ denotes the k-th (reduced) suspension of X; $SX = S^1 X$.

A positive answer to Question 2.6 is given by Dydak and Kozlowski in the following result – see [58, Th. A].

2.8 Theorem [Dydak-Kozlowski (1991)] *Let* $\mathbf{E}$ *be an arbitrary spectrum and let* $f : X \to Y$ *be a closed surjection of paracompact spaces. If the Brouwer-Čech dimension* $\mathrm{Ind}\,\mathrm{Y} = \mathrm{N_Y} < \infty$ *and, for some integer* m*,*

$$f^* : \mathbf{E}^n(\{y\}) \to \mathbf{E}^n(f^{-1}(y)) \tag{2.2}$$

is an isomorphism for each $m \leq n \leq m + N_Y$ *and* $y \in Y$*, then*

$$f^* : \mathbf{E}^q(Y) \to \mathbf{E}^q(X)$$

is an isomorphism and for $q = m + N_Y$ *a monomorphism for* $q = m + N_Y + 1$*.*

The authors observed and provided an example showing that if instead of (2.2) one assumes that

$$\mathbf{E}^n(f^{-1}(y)) \cong \mathbf{E}^n(\{y\}), \tag{2.3}$$

then the assertion does not hold.

However, if $\mathbf{E} = \mathbf{K}(G)$, then clearly assumptions (2.2) and (2.3) are equivalent. Hence in this case the Dydak-Kozlowski theorem constitutes a version of the classical Vietoris-Begle theorem. Indeed, suppose that $m \geq 1$ and put $N = m + N_Y + 1$. Clearly, if $0 \leq k \leq m - 1$, then $\mathrm{rd_Y(s^k(f;G))} \leq \mathrm{N_Y}$ and, for $m \leq k \leq N - 1$, $\mathrm{s^k(f;G)} = \emptyset$. Hence $\mathrm{i^N(f;G)} \leq \mathrm{N}$. The Sklyarenko Theorem 2.2 states that $f^* : \check{H}^q(Y;G) \to \check{H}^q(X;G)$ is (at least) an epimorphism for $q = N - 1$ and a monomorphism for $q = N$. The Dydak-Kozlowski result additionally implies that if $q = N - 1$ it is also an isomorphism. If $m \leq 0$, then again putting $N = m + N_Y + 1$, we see that $\mathrm{i^N(f;G)} = 0$. Looking carefully at the Dydak-Kozlowski result, we observe that in this case its assertion gives the same information as the Sklyarenko theorem or the Vietoris-Begle one does:

$f^* : \check{H}^q(Y;G) \to \check{H}^q(X;G)$ *is an isomorphism for any* $0 \leq q \leq N - 1$ *and a monomorphism for* $q = N$*.*

2.9 Let $\mathbf{E}$ be an Ω-*spectrum.* It means that a map $e_n' : E_n \to \Omega E_{n+1}$ dual to e_n (i.e. given by the formula $e_n'(x)(s) = e_n[s,x]$ for $x \in E_n$ and $s \in S^1$)

is a weak homotopy equivalence. Hence, in particular, for any $k \geq 0$ and $n \in \mathbf{Z}$, we have $\pi_k(E_n) \cong \pi_k(\Omega E_{n+1}) \cong \pi_{k+1}(E_{n+1})$. For instance $\mathbf{K}(G)$ is an Ω-spectrum. A basic result is that in this case $\mathbf{E}^n(X) \cong [X; E_n]$; hence Theorem 2.8 has a particularly nice form: it states that f induces a bijective correspondence

$$f^{\#} : [Y; E_n] \to [X; E_n], \qquad n = m + N_Y,$$

between respective sets of homotopy classes. It therefore might be viewed as a *homotopy* version of the Vietoris theorem.

However, in the next sections we shall be interested in homotopy properties of f stated in terms of the behaviour of $f^{\#} : [Y; S^n] \to [X; S^n]$. Unfortunately, the spherical spectrum $\mathbf{S} = \{S^n\}_{n=-\infty}^{\infty}$ ([7]) is not an Ω-spectrum. Such spectra give rise to sort of stable cohomology theories; hence the result of Dydak-Kozlowski should be viewed rather as a result of a *stable* character.

2.10 To see that better let $\mathbf{E} = \mathbf{S}$. Clearly, for any paracompact space X and $n \in \mathbf{Z}$,

$$\mathbf{S}^n(X) = \varinjlim_{k \geq 0} [S^k X; S^{n+k}]$$

is, by definition, the n-th *stable cohomotopy* group $\pi_s^n(X)$ of X.

If, for instance, X and Y are paracompact spaces, $\dim Y \leq \operatorname{Ind} Y = N_Y < \infty$, $\dim X \leq N_X \leq \infty$ and $f : X \to Y$ is a closed surjection such that, for an integer m,

$$\forall m \leq q \leq m + N_Y \ \forall y \in Y \ \ \pi_s^q(f^{-1}(y)) \overset{f^*}{\cong} \pi_s^q(\{y\}),$$

then

$$f^* : \pi_s^n(Y) \cong \pi_s^n(X)$$

for $n = m + N_Y$. By the suspension theorem (see [166, Th. 8.5.11]), $\pi_s^n(X) \cong [S^k X; S^{n+k}]$ and $\pi_s^n(Y) \cong [S^l Y; S^{n+l}]$ where k, l are such that $2n + k \geq N_X + 2$ and $2n + l \geq N_Y + 2$. In particular, if

$$n \geq \frac{1}{2}(\max\{N_X, N_Y\} - k) + 1 \tag{2.4}$$

[7] for $n < 0$, we put $S^n = \{*\}$.

for some $k \geq 0$, then

$$(S^k f)^{\#} : [S^k Y; S^{n+k}] \to [S^k X; S^{n+k}]$$

($n = m + N_Y$) is a bijection. This is perhaps the best we can achieve with regard to the homotopy behaviour of f under the above assumptions.

A natural question arises what is going on when estimate (2.4) does not hold.

2.11 Question *What assumptions give better information concerning the transformation $f^{\#} : [Y; S^n] \to [X; S^n]$, induced by f, when n is in the unstable area, i.e. $n < \frac{1}{2}\max\{N_X, N_Y\} + 1$ or, more generally, the transformation $(S^k f)^{\#} : [S^k Y; S^{n+k}] \to [S^k X; S^{n+k}]$ when $\max\{N_X, N_Y\} > 2n + k - 2$?*

Some light onto these questions is shed by the following result being a generalization of a theorem proved by the author [112].

2.12 Theorem *Let X, Y be paracompact spaces $\dim X = N_X$, $\dim Y = N_Y$, and let P be a compact (metric) k-connected ANR, $k \geq 1$. If $f : X \to Y$ is a closed surjection with compact fibres such that, for $N \geq N_X + N_Y + 2$, $\mathrm{i^N(f) < N}$, then the transformation*

$$f^{\#} : [Y; P] \to [X; P],$$

induced by f, is a surjection if $\mathrm{i^N(f) - 2 \leq k}$ and a bijection if $\mathrm{i^N(f) - 1 \leq k}$.

The proof of a further generalization (see Theorem 2.17) will be given in the next section.

Let us note the following

2.13 Corollary *Assume that $f : X \to Y$ is as above and let $n \geq 2$. For any $k \geq 0$, the transformation*

$$(S^k f)^{\#} : [S^k Y; S^{n+k}] \to [S^k X; S^{n+k}]$$

is a surjection if $\mathrm{i^N(f) - 1 \leq n}$ and a bijection if $\mathrm{i^N(f) \leq n}$.

Proof If $k = 0$, then it is enough to observe that S^n is $(n-1)$-connected and invoke the Theorem 2.12 putting $P = S^n$. The case $k \geq 1$ will be treated in Section 3.A.. □

2.C. (Co)homotopy version of the Vietoris-Sklyarenko theorem

As above all *spaces* are supposed to be paracompact and a *pair* (X, A) consists of a space X and its *closed* subset A; a pair (X, A) is a (complete) ANR-pair if X and A are (complete) metric absolute neighborhood retracts.

2.14 Definition We say that a topological space X is of (n, m)-*type*, $1 \leq n, m \leq \infty$ if it is path connected and $\pi_i(X) = 0$ for each $i \leq n-1$ or $i \geq m+1$. More generally, a pair (X, A) is of (n, m)-type if it is 1-connected and $\pi_i(X, A) = 0$ if $2 \leq i \leq n-1$ or $i \geq m+1$.

Clearly, from the definition it follows that if X (resp. (X, A)) is of (n, m)-type and is not ∞-connected, then $m \geq n$ and $n < \infty$. Obviously X (resp. (X, A)) is of (n, ∞)-type if and only if X (resp. (X, A)) is $(n-1)$-connected.

In the rest of this section we assume that pairs (X, X'), (Y, Y') and a *perfect surjection*

$$f : (X, X') \to (Y, Y') \qquad \text{such that } X' = f^{-1}(Y')$$

are given. Moreover, let us pose

2.15 Assumption

(i) *(P, P') is a paracompact pair having the homotopy type of an* ANR-*pair.*

(ii) *P is of (n, m)-type, $1 \leq n, m \leq \infty$;*

(iii) *(P, P') is of $(n+1, m+1)$-type;*

(iv) *P (resp. (P, P')) is homotopically simple, that is i-simple for any $i \geq 1$ (resp. $i \geq 2$);*

(v) *if Y (resp. Y') is not compact, we assume that $\pi_i(P)$ (resp. $\pi_i(P')$) is finitely generated for any $i \geq 1$.*

2.16 Remark

(i) Condition 2.15 (i) is fulfilled if (P, P') is a CW-pair. Indeed, each CW-pair has the homotopy type of a polyhedral pair being homotopy equivalent to itself endowed with the metric topology – see [134]; a simplicial pair with the metric topology is homotopy equivalent to its telescope being a complete ANR-pair – see [57].

Observe, moreover, that an ANR-pair, being homotopy dominated by a CW-pair and hence having the homotopy type of a CW-pair (see [134]), is homotopy equivalent to a complete ANR-pair. Therefore, Assumption 2.15 implies that (P, P') has the homotopy type of a complete ANR-pair.

Recall also here that any complete ANR is an ANE – see Preliminaries.

(ii) Observe that if conditions 2.15 (i),(ii) and (iii) hold, the space P (resp. P') is simply connected and has the homotopy type of a compact ANR whenever Y (resp. Y') is not compact, then all above assumptions are satisfied. Indeed, if Y (resp. Y') is not compact, then P (resp. P') has the homotopy type of a simply connected compact polyhedron (see [173]). Hence, by the generalized Hurewicz theorem [166, Ch. 9.6, Cor. 16], $\pi_i(P)$ (resp. $\pi_i(P')$) is a finitely generated abelian group for any $i \geq 2$.

Let us now state the main results of this section.

2.17 Theorem (Generalized Vietoris Theorem I) *Suppose that, for some* $N \geq 1$*, we have* $\mathrm{i}^{\mathrm{N}}(\mathrm{f}) < \mathrm{N}$ *and Assumptions* 2.15 (i)–(v) *are satisfied.*

(i) (Case $m = \infty$**)** *If* $N \geq \dim X + \dim Y + 2$*, then the transformations* $f^{\#} : [Y, Y'; P, P'] \to [X, X'; P, P']$ *and* $f^{\#} : [Y, P] \to [X, P]$ *induced by* f *are:*

1. *surjective if* $n \geq \mathrm{i}^{\mathrm{N}}(\mathrm{f}) - 1$*;*
2. *bijective if* $n \geq i^N(f)$ *– see also Remark* 2.24 *below.*

(ii) (Case $m < \infty$**)** *The transformations* $f^{\#} : [Y, Y'; P, P'] \to [X, X'; P, P']$ *and* $f^{\#} : [Y, P] \to [X, P]$ *are:*

1. *surjective if* $\mathrm{i}^{\mathrm{N}}(\mathrm{f}) - 1 \leq \mathrm{n}$ *and* $m \leq N - 2$*;*
2. *injective if* $\mathrm{i}^{\mathrm{N}}(\mathrm{f}) \leq \mathrm{n}$ *and* $m \leq N - 1$*.*

Consequently $f^{\#}$ *is a bijection if* $i^N(f) \leq n$ *and* $m \leq N - 2$*.*

2.18 Theorem (Generalized Vietoris Theorem II) *Suppose that there is* $N \geq 1$ *such that* $\mathrm{i}^{\mathrm{N}}(\mathrm{f}; \mathcal{A}) < \mathrm{N}$ *where* $\mathcal{A} = \mathbf{Z}_{Y'}$ (8) *and let* p *be an arbitrary point in* P*. Let Assumptions* 2.15 (i), (ii) *and* (iv), (v) *be satisfied.*

(i) (Case $m = \infty$**)** *If* $N \geq \dim X + \dim Y + 2$*, then* $f^{\#} : [Y, Y'; P, p] \to [X, X'; P, p]$ *is:*

1. *a surjection for* $n \geq \mathrm{i}^{\mathrm{N}}(\mathrm{f}) - 1$*;*
2. *a bijection for* $n \geq \mathrm{i}^{\mathrm{N}}(\mathrm{f})$*.*

(ii) (Case $m < \infty$**)** *The transformation* $f^{\#}$ *is:*

1. *a surjection if* $\mathrm{i}^{\mathrm{N}}(\mathrm{f}) - 1 \leq \mathrm{n}$ *and* $m \leq N - 2$*;*
2. *an injection if* $\mathrm{i}^{\mathrm{N}}(\mathrm{f}) \leq \mathrm{n}$ *and* $m \leq N - 1$*.*

Consequently if $\mathrm{i}^{\mathrm{N}}(\mathrm{f}) \leq \mathrm{n}$ *and* $m \leq N - 2$*, then* $f^{\#}$ *is a bijection* (9)*.*

The reader will easily see the analogies of Theorems 2.17, 2.18 and Corollary 2.5. On the other hand: if Y is either compact or an abelian group G is finitely generated, $n \geq 1$ and P is the Eilenberg-MacLane complex $\mathbf{K}(G, n)$ (being a space of (n, n)-type and, as a CW-complex – homotopy equivalent to some complete (metric) ANR), then $\check{H}^n(X, X'; G) = [X, X'; P, *]$, $\check{H}^n(Y, Y'; G) = [Y, Y'; P, *]$ where $* \in P$, and, by Theorem 2.18, $f^* : \check{H}^n(Y, Y'; G) \to \check{H}^n(X, X'; G)$ is an epimorphism for $\mathrm{i}^{\mathrm{N}}(\mathrm{f}) - 1 \leq \mathrm{n} \leq \mathrm{N} - 2$ and a monomorphism for $\mathrm{i}^{\mathrm{N}}(\mathrm{f}) \leq \mathrm{n} \leq \mathrm{N} - 1$. Therefore Theorems 2.17 and 2.18 constitute a sort of generalizations of Corollary 2.5.

Moreover, the above results to some extent generalize and correspond to the following version of an unpublished result of Kozlowski (see [172, App. B, p. 117], [141, Th. 3.10]) and results stated in the other paper by Dydak and Kozlowski [57, Cor. 1] (see also [59, Th. 10.4.4] and compare also [108]) where conditions concerning $\mathrm{i}^*(\mathrm{f})$ are replaced by the cell-likeness of f. Recall that a perfect surjection $f : X \to Y$ is a *cell-like map* if, for each $y \in Y$, $f^{-1}(y) \in UV^{\infty}$ – in other words $f^{-1}(y)$ is a cell-like set (recall Definition 1.10 and Remark 1.11 (iii)). In view of Proposition 1.15, the following result holds true.

2.19 Theorem (Cell-like Vietoris Theorem I) *Suppose that* f *is a cell-like map and* P *satisfies Assumption* 2.15 (i)*. If either*

[8] Recall that $\mathbf{Z}_{Y'}$ denotes a sheaf constant (and equal to $\mathbf{Z}$) over $Y \setminus Y'$ and trivial over Y'.

[9] Observe that Theorem 2.18 *does not follow directly* from Theorem 2.17. If we put $P' = \{p\}$, then (P, P') is *not* of $(n + 1, m + 1)$-type.

(i) $\operatorname{Ind} Y < \infty$,
or
(ii) *there is a positive integer* m *such that* $\pi_i(P) = 0$ *for* $i \geq m+1$,
then the transformation $f^{\#} : [Y; P] \to [X; P]$ *induced by* f *is a bijection.*

In case X, Y are compact metric spaces, the proof of part (i) is given in [59] and of part (ii) in [172]. The general case (ii) is treated in [57] while (i) follows implicitly from [58, Th. B].

The above result admits a relative version, too.

2.20 Theorem (Cell-like Vietoris Theorem II) *Suppose that* f *is a cell-like map and* (P, P') *satisfies Assumption* 2.15 (i). *If either*
(i) $\operatorname{Ind} Y < \infty$,
or
(ii) *there exists a positive integer* m *such that* $\pi_i(P) = \pi_i(P') = 0$ *for* $i \geq m+1$,
then f *induces a bijective transformation* $f^{\#} : [Y, Y'; P, P'] \to [X, X'; P, P']$.

Proof We may actually assume that (P, P') is a complete ANR-pair. The argument is now rather standard. Let Z (resp. Z') be a cylinder of $f : X \to Y$ (resp $f' : X' \to Y'$). We are to prove that the transformation $i^{\#} : [Z, Z'; P, P'] \to [X, X'; P, P']$, induced by the inclusion $i : (X, X') \to (Z, Z')$, is bijective.

Consider a diagram

$$\begin{array}{ccccc} (Z', X') & \xrightarrow{i_2} & (Z, X') & \xrightarrow{j} & (Z, Z') \\ & & \uparrow{\scriptstyle i_1} & \nearrow{\scriptstyle i} & \\ & & (X, X') & & \end{array}$$

where all maps are the inclusions.

I. By Theorem 2.19, the transformation $[Z; P] \to [X; P]$, induced by the inclusion $X \to Z$, is bijective. Hence and since the pair (Z, X) has the homotopy extension property with respect to P, we gather that $i_1^{\#}$ is surjective.

To see that $i_1^{\#}$ is also injective consider a diagram

$$\begin{array}{ccc} Z \times \{0,1\} \cup X \times I & \xrightarrow{\kappa} & Z \times I \\ & {}_{r_1}\searrow \quad \swarrow_{r_2} & \\ & Y \times I & \end{array}$$

where κ is the inclusion and $r_i([x,t],\lambda) = (f(x),\lambda)$ for $\lambda \in I$ and $[x,t] \in Z$, $i = 1,2$. It is clear that r_2 is a deformation retraction and r_1 is a cell-like map (for $y \in Y$, $\lambda \in (0,1)$, $r_1^{-1}(y,\lambda) = f^{-1}(y) \times \{\lambda\}$ and, for $\lambda = 0,1$, $r_1^{-1}(y,\lambda) = C(f^{-1}(y)) \times \{\lambda\}$ where $C(f^{-1}(y))$ is the cone over $f^{-1}(y)$); therefore, again by theorem 2.19, $\kappa^{\#} : [Z' \times I, P] \to [Z' \times \{0,1\} \cup X' \times I, P]$ is a bijection because, in view of results from [152] (see also [153]), $\mathrm{Ind}\,\mathrm{Y} \times [0,1] < \infty$. This shows that any homotopy joining $g_0 \circ i_1$ and $g_1 \circ i_1$, where $g_0, g_1 : (Z, X') \to (P, P')$, extends to a homotopy joining g_0 to g_1.

II. We shall show that $j^{\#}$ is a bijection, too. Observe that, for each $g : (Z', X') \to (P, P')$, there exists $\overline{g} : (Z', X') \to (P, P')$ such that $\overline{g}(Z') \subset P'$ and $g \simeq \overline{g}\,[\mathrm{rel}\,\mathrm{X}']$. Indeed, let $g' : X' \to P'$ be induced by g. Since $i'^{\#} : [Z', P'] \to [X', P']$, where $i' = i|X'$, is bijective in view of Theorem 2.19, there is $g'' : Z' \to P'$ such that $g'' \circ i' \simeq g'$. Let $k : P' \to P$ be the inclusion. Then $k \circ g'' \circ i' \simeq k \circ g' = g \circ i'$. Clearly the transformation $[Z', P] \to [X', P]$, induced by i', is bijective (again by Theorem 2.19). Hence $g \simeq k \circ g''$ and $k \circ g''(Z') \subset P'$. Because (Z', X') (resp. $(Z' \times I, Z' \times \{0,1\} \cup X' \times I)$) has homotopy extension property (HEP) with respect to P' (resp. P), we get the desired $\overline{g}$ (comp. [166, Chapter 1, Ex. D.4]).

Now let $g : (Z, X') \to (P, P')$; by the above argument, there is $\overline{g} : (Z', X') \to (P, P')$ such that $\overline{g}(Z') \subset P'$ and $h : \overline{g} \simeq g \circ i_2\,[\mathrm{rel}\,\mathrm{X}']$. It is obvious that h has an extension $H : Z \times I \to P$ such that $H(\,\cdot\,,1) = g$ and $H(\,\cdot\,,0)|Z' = \overline{g}$. Hence $H(Z' \times \{0\}) \subset P'$, $H(\,\cdot\,,0) \circ j \simeq g$. Hence $j^{\#}[H(\,\cdot\,,0)] = [g]$ and $j^{\#} : [Z, Z'; P, P'] \to [Z, X'; P, P']$ is surjective.

Suppose that $h : g_0 \circ j \simeq g_1 \circ j$ where $g_i : (Z, Z') \to (P, P')$, $i = 0,1$. Obviously $h(Z' \times \{0,1\} \cup X' \times I) \subset P'$ i.e. $h : (Z \times I, Z' \times \{0,1\} \cup X' \times I) \to (P, P')$.

Arguing as in part I, we see that the transformation $[Z' \times I, P] \to [Z' \times \{0,1\} \cup X' \times I, P]$, induced by the inclusion $Z' \times \{0,1\} \cup X' \times I \to Z' \times I$, is bijective.

Reasoning exactly as in the first paragraph of this part of the proof we gather that there exists $\overline{h} : (Z' \times I, Z' \times \{0,1\} \cup X' \times I) \to (P, P')$ such that $\overline{h}(Z' \times I) \subset P'$ and $H : \overline{h} \simeq (h|Z' \times I)\,[\mathrm{rel}\, Z' \times \{0,1\} \cup X' \times I]$. Let $\overline{H} : (Z \times \{0,1\} \cup Z' \times I, Z' \times \{0,1\} \cup X' \times I) \times I \to (P, P')$ be given by the formula

$$\overline{H}(z,t,\lambda) = \begin{cases} g_t(z) & \text{if } t = 0,1,\ \lambda \in I,\ z \in Z \\ H(z,t,\lambda) & \text{if } t,\lambda \in I,\ z \in Z'. \end{cases}$$

Because $\overline{H}(\,\cdot\,,\,\cdot\,,1) = h|Z \times \{0,1\} \cup Z' \times I$ is extendable onto $Z \times I$, so is $\overline{H}(\,\cdot\,,\,\cdot\,,0)$ giving rise to a map $G : (Z \times I, Z' \times I) \to (P, P')$ such that $G : g_0 \simeq g_1$. □

2.21 Remark One sees easily the differences between Theorems 2.17 and 2.20. The advantage of 2.20 (i) is that we do not have to assume that dimension of X is finite; instead a relatively strong assumption on fibres of f is required. In view of Corollary 1.16 and Remark 1.18, if f is a cell-like map, then $\mathrm{i(f)} = 0$. Therefore 2.17 (ii) implies 2.20 (ii).

Finally, let us remark that Theorem 2.19 (i) (resp. Theorem 2.20 (i)) holds if P (resp. (P, P')) has the homotopy type of an ANE (resp. ANE-pair).

Before we give the proofs of Theorems 2.17 and 2.18, let us consider the following examples showing the nature of Assumptions 2.15 and those of 2.17 or 2.18 and 2.19, 2.20.

2.22 Example

(i) Consider the Kahn acyclic infinite-dimensional compactum X (recalled in Example 1.19 (i)) admitting an essential map $g : X \to P = S^3$. Taking $f : X \to Y = *$ we see that $f^\# : [Y, P] \to [X, P]$ is not surjective.

(ii) Dranishnikov [53] gives an example of a compact metric space Y' such that $\dim Y' = \infty$ but the (integral) *cohomological dimension* c-dim Y' = 3. By the theorem of Edwards (see [172, Sec. 6, p. 113]), there is a compact metric space X', $\dim X' = 3$, and a cell-like map $f' : X' \to Y'$. Clearly, for any $N \geq 1$, $\mathrm{i^N(f')} = 0$. There is a closed subspace $Y \subset Y'$, $\dim Y = \infty$ with $[Y, S^4] \neq 0$ but $[X, S^4] = 0$ where $X = f'^{-1}(Y)$. Hence $f^\#$, where $f = f'|X : X \to Y$, is not injective (recall that, by Proposition 1.4, $\mathrm{i(f)} = 0$).

(iii) In Example 1.19 (ii) we gave an example of a compact acyclic ANR such that $\pi_1(X)$ is not abelian. Let $P = X$ and $f : X \to Y = *$. Clearly $[X, P] \neq 0$ and, hence, $f^\#$ is not surjective (P is not homotopically simple since it has a non-abelian fundamental group).

(iv) Let $f : S^3 \to S^2$ be the Hopf fibration. Evidently, for any $N \geq 1$, $\mathrm{i}^{\mathrm{N}}(\mathrm{f}) = 5$, but for $P = S^3$ no assertion of Theorem 2.17 holds.

(v) Taylor gives an example of a metric space X and a cell-like map $f : X \to Y = Q$ onto the Hilbert cube Q which is not a shape equivalence, (i.e. there are polyhedra P such that $f^{\#} : [Q, P] \to [X, P]$ is not bijective) showing that assumptions concerning Y in Theorem 2.19 are essential.

Our approach to the proof of our main results is classical and essentially based on the obstruction theory (see e.g. [102, Ch. VI]) and the Sklyarenko Theorem 2.2. In order to proceed further we recall some well-known notions and introduce a necessary notation.

2.23 Bridges The family of all locally finite open coverings of a space X is denoted by $\Omega(X)$. For a pair (X, A) and a covering $\mathfrak{A} \in \Omega(X)$, let $(X_{\mathfrak{A}}, A_{\mathfrak{A}})$ be a polyhedral pair where $X_{\mathfrak{A}}$ (resp. $A_{\mathfrak{A}}$) is the space of the nerve of $\mathfrak{A}$ (resp. of the covering $\mathfrak{A}|A = \{U \cap A \mid U \in \mathfrak{A}\}$) endowed with the weak topology. (Observe that the family $\{(\mathfrak{A}, \mathfrak{A}|A) \mid \mathfrak{A} \in \Omega(X)\}$ is cofinal in the family of all open coverings of the pair (X, A) directed by the usual relation "$\preceq$" of refinement).

Let $f : (X, A) \to (Y, B)$ (resp. $g : A \to Y$). After [102, Ch. II, Ex. B] we say that a covering $\mathfrak{A} \in \Omega(X)$ is a *bridge for* f (resp. for g) if there is a *bridge map* $f_{\mathfrak{A}} : (X_{\mathfrak{A}}, A_{\mathfrak{A}}) \to (Y, B)$ (resp. $g_{\mathfrak{A}} : A_{\mathfrak{A}} \to Y$) such that, for any canonical map $p_{\mathfrak{A}} : (X, A) \to (X_{\mathfrak{A}}, A_{\mathfrak{A}})$, $f_{\mathfrak{A}} \circ p_{\mathfrak{A}} \simeq f$ (resp. $g_{\mathfrak{A}} \circ (p_{\mathfrak{A}}|A) \simeq g$).

It is well-known that if (Y, B) is an ANR-pair, then:

(i) there exists a bridge $\mathfrak{A} \in \Omega(X)$ of f (resp. of g);

(ii) a refinement of a bridge is again a bridge;

(iii) if $\mathfrak{A}, \mathfrak{B} \in \Omega(X)$ are bridges of f (resp. g) with bridge maps $f_{\mathfrak{A}}, f_{\mathfrak{B}}$ (resp. $g_{\mathfrak{A}}, g_{\mathfrak{B}}$), then there is a common refinement $\mathfrak{D} \in \Omega(X)$ of $\mathfrak{A}, \mathfrak{B}$ such that, for any canonical projections $p_{\mathfrak{A}\mathfrak{D}} : (X_{\mathfrak{D}}, A_{\mathfrak{D}}) \to (X_{\mathfrak{A}}, A_{\mathfrak{A}})$ and $p_{\mathfrak{B}\mathfrak{D}} : (X_{\mathfrak{D}}, A_{\mathfrak{D}}) \to (X_{\mathfrak{A}}, A_{\mathfrak{A}})$, $f_{\mathfrak{A}} \circ p_{\mathfrak{A}\mathfrak{D}} \simeq f_{\mathfrak{B}} \circ p_{\mathfrak{B}\mathfrak{D}}$ (resp. $g_{\mathfrak{A}} \circ (p_{\mathfrak{A}\mathfrak{D}}|A_{\mathfrak{D}}) \simeq g_{\mathfrak{B}} \circ (p_{\mathfrak{B}\mathfrak{D}}|A_{\mathfrak{D}})$).

Proof of Theorem 2.17 Clearly if $n = \infty$ or $n > m$, then P and (P, P') are ∞-connected and the assertions follow trivially. Therefore in the sequel we assume that $n < \infty$ and $n \leq m$. Moreover we may assume that actually (P, P') is a complete ANR-pair. Recall also that any complete ANR is an abslute neighborhood extensor for the class of paracompact spaces; hence each paracompact pair has the homotopy extension property (HEP) with respect to it.

Let Z (resp. Z') be the cylinder of $f : X \to Y$ (resp. of $f' = f|X' : X' \to Y'$). Therefore X', X and Z' are closed subspaces of Z, $X' \subset Z'$ and $\dim Z \leq \dim X + \dim Y + 1$ (obviously Z is paracompact). There is a (strong) deformation retraction $r : (Z, Z') \to (Y, Y')$ and $f = r \circ i$ where $i : (X, X') \to (Z, Z')$ is the inclusion. Evidently:

(1) $r^{\#} : [Y, Y'; P, P'] \to [Z, Z'; P, P']$ is a bijection.

(2) Let G be an abelian group being finitely generated whenever Y is not compact. For any $q \geq 0$, $\check{H}^q(Z; G) = \check{H}^q(Y; G)$, $\check{H}^q(Z'; G) = \check{H}^q(Y'; G)$ and hence, by Corollary 2.5 and the cohomology exact sequence, for any $\mathrm{i^N(f) \leq q \leq N - 1}$, $\check{H}^q(Z, X; G) = 0 = \check{H}^q(Z', X'; G)$.

Let $i_1 : (X, X') \to (Z, X')$ be the inclusion.

(3) To prove surjectivity of $i_1^{\#}$ and $i^{\#} : [Z; P] \to [X; P]$ let $i^N(f) - 1 \leq n$ (and $m \leq N - 2$ in case (ii)) and let $g : (X, X') \to (P, P')$. We claim that there is an extension $\overline{g} : Z \to P$ of g.

Indeed,there is a bridge $\mathfrak{A} \in \Omega(Z)$ of g with a bridge map $g_{\mathfrak{A}} : X_{\mathfrak{A}} \to P$. Since P is $(n-1)$-connected, $g_{\mathfrak{A}}$ has an extension $\overline{g}_{\mathfrak{A}} : Z^n_{\mathfrak{A}} \cup X_{\mathfrak{A}} \to P$ (here and below $Z^n_{\mathfrak{A}}$ denotes the n-dimensional skeleton of $Z_{\mathfrak{A}}$). Assume that $g_{\mathfrak{A}}$ has an extension $\overline{g}_{\mathfrak{A}} : Z^k_{\mathfrak{A}} \cup X_{\mathfrak{A}} \to P$ for some $n \leq k$. Observe that under our assumptions, $\pi_k(P)$ is an abelian group being finitely generated when Y is not compact. The $(k+1)$-dimensional obstruction set $\mathcal{O}^{k+1}(g_{\mathfrak{A}}) \subset \check{H}^{k+1}(Z_{\mathfrak{A}}, X_{\mathfrak{A}}; \pi_k(P))$ is nonempty. Let $c \in \mathcal{O}^{k+1}(g_{\mathfrak{A}})$. Since $\mathrm{i^N(f) \leq k + 1}$, $\check{H}^{k+1}(Z, X; \pi_k(P)) = 0$, and there is a bridge $\mathfrak{B} \in \Omega(Z)$, $\mathfrak{A} \preceq \mathfrak{B}$, of g such that, for any canonical projection $p_{\mathfrak{A}\mathfrak{B}} : (Z_{\mathfrak{B}}, X_{\mathfrak{B}}) \to (Z_{\mathfrak{A}}, X_{\mathfrak{A}})$, $0 = p^*_{\mathfrak{A}\mathfrak{B}}(c) \in \mathcal{O}^{k+1}(g_{\mathfrak{A}} \circ (p_{\mathfrak{A}\mathfrak{B}}|X_{\mathfrak{B}})) \subset \check{H}^{k+1}(Z_{\mathfrak{B}}, X_{\mathfrak{B}}; \pi_k(P))$. Hence a map $g_{\mathfrak{B}} = g_{\mathfrak{A}} \circ (p_{\mathfrak{A}\mathfrak{B}}|X_{\mathfrak{B}})$, being a $\mathfrak{B}$-bridge map for g, has an extension $\overline{g}_{\mathfrak{B}} : Z^{k+1}_{\mathfrak{B}} \cup X_{\mathfrak{B}} \to P$.

After a finite number of steps, we get a bridge $\mathfrak{D} \in \Omega(Z)$ of g with a bridge map $g_{\mathfrak{D}} : X_{\mathfrak{D}} \to P$ having an extension $\overline{g}_{\mathfrak{D}} : Z^l_{\mathfrak{D}} \to P$ where $l = \min\{N - 1, \dim Z\}$. If $\dim Z \leq \dim X + \dim Y + 1 \leq N - 1$, then $Z^l_{\mathfrak{D}} = Z_{\mathfrak{D}}$. Otherwise, if $m \leq N - 2$, then, since $\pi_i(P) = 0$ for $i \geq m + 1$, we may further extend $\overline{g}_{\mathfrak{D}}$ to get a map $Z_{\mathfrak{D}} \to P$ denoted, as before, by $\overline{g}_{\mathfrak{D}}$.

By HEP of (Z, X) with respect to P, g has a desired extension onto Z.

Therefore $i_1^{\#} : [Z, X'; P, P'] \to [X, X'; P, P']$ and $i^{\#} : [Z; P] \to [X; P]$ are surjections.

(4) To prove the injectivity of $i_1^{\#}$ and $i^{\#}$, let $\mathrm{i^N(f)} \leq \mathrm{n}$ (and $m \leq N-1$ in case (ii)) and consider maps $g_j : (Z, X') \to (P, P')$, $j = 0, 1$, such that $h : g_0 \circ i_1 \simeq g_1 \circ i_1$. Define $\overline{h} : Z \times \{0,1\} \cup X \times I \to P$ by the formula

$$\overline{h}(z,t) = \begin{cases} g_t(z) & \text{if } t = 0,1,\ z \in Z \\ h(z,t) & \text{if } t \in I,\ z \in X. \end{cases}$$

Since $\check{H}^q(Z \times I, Z \times \{0,1\} \cup X \times I; \pi_{q-1}(P)) = \check{H}^{q-1}(Z, X; \pi_{q-1}(P)) = 0$ for $\mathrm{i^N(f)} + 1 \leq \mathrm{q} \leq \mathrm{N}$, arguing as in (3), we get an extension $H : Z \times I \to P$ of $\overline{h}$, $H(X' \times I) \subset P'$, thus a homotopy $H : g_0 \simeq g_1$.

This already completes the proof of the part concerning $f^{\#} : [Y; P] \to [X; P]$.

In order to proceed with the relative case consider the inclusions

$$(Z', X') \xrightarrow{i_2} (Z, X') \xrightarrow{j} (Z, Z').$$

(5) Assume that $\mathrm{i^N(f)} - 1 \leq \mathrm{n}$ (and $m \leq N-2$ in case (ii)). First we shall prove that, given a map $g : (Z', X') \to (P, P')$, there is a map $\overline{g} : (Z', X') \to (P, P')$ such that $\overline{g}(Z') \subset P'$ and $\overline{g} \simeq g\,[\mathrm{rel\,X'}]$.

There is a bridge $\mathfrak{A} \in \Omega(Z')$ of g with a bridge map $g_{\mathfrak{A}} : (Z'_{\mathfrak{A}}, X'_{\mathfrak{a}}) \to (P, P')$.

Since (P, P') is n-connected (and, at least, 1-connected) there exists a map $\overline{g}_{\mathfrak{A}} : (Z'_{\mathfrak{A}}, X'_{\mathfrak{A}}) \to (P, P')$ such that $\overline{g}_{\mathfrak{A}} \simeq g_{\mathfrak{A}}$ and $\overline{g}_{\mathfrak{A}}(Z'^k_{\mathfrak{A}}) \subset P'$ for some $k \geq \max\{n, 1\}$.

Reasoning similarly as in (3) but applying the theory of obstructions to the deformation we get a bridge $\mathfrak{B} \in \Omega(Z')$ and a bridge map $g_{\mathfrak{B}} : (Z'_{\mathfrak{B}}, X'_{\mathfrak{B}}) \to (P, P')$ such that $g_{\mathfrak{B}}(Z') \subset P'$. Let $g' = g_{\mathfrak{B}} \circ p_{\mathfrak{B}}$ where $p_{\mathfrak{B}} : (Z', X') \to (Z'_{\mathfrak{B}}, X'_{\mathfrak{B}})$ is a canonical map. Then $g' \simeq g$ and $g'(Z') \subset P'$. Since (Z', X') (resp. $(Z' \times I, Z' \times \{0,1\} \cup X' \times I)$) has HEP with respect to P' (resp. to P), we get the required $\overline{g}$.

The rest of the proof is similar to the last part of the argument used to show Theorem 2.20. For the sake of completeness we give details.

(6) Let $g : (Z, X') \to (P, P')$. By (5), there is a map $\overline{g} : (Z', X') \to (P, P')$ such that $\overline{g}(Z') \subset P'$ and $h : \overline{g} \simeq g \circ i_2\,[\mathrm{rel\,X'}]$. Clearly h can

be extended to a homotopy $H : Z \times I \to P$ such that $H(\cdot, 1) = g$ and $H(\cdot, 0)|Z' = \overline{g}$. Therefore $H(Z' \times \{0\}) \subset P'$ and $H(\cdot, 0) \circ j \simeq g$. Hence $j^{\#}[H(\cdot, 0)] = [g]$ and $j^{\#} : [Z, Z'; P, P'] \to [Z, X'; P, P']$ is surjective.

(7) To prove injectivity of $j^{\#}$ suppose that $\mathrm{i^N(f)} \leq \mathrm{n}$ (and $m \leq N-1$ in case (ii)), $g_i : (Z, Z') \to (P, P')$, $i = 0, 1$, and $h : g_0 \circ j \simeq g_1 \circ j$. Evidently $h(Z' \times \{0,1\} \cup X' \times I) \subset P'$. Since $\check{H}^q(Z' \times I, Z' \times \{0,1\} \cup X' \times I; \pi_q(P, P')) = \check{H}^{q-1}(Z', X'; \pi_q(P, P')) = 0$ for $\mathrm{i^N(f) + 1 \leq k + 1 \leq q \leq N}$, arguing as in (5), there is $\overline{h} : (Z' \times I, Z' \times \{0,1\} \cup X' \times I) \to (P, P')$ such that $\overline{h}(Z' \times I) \subset P'$ and $H : \overline{h} \simeq (h|Z' \times I)\, [\mathrm{rel}\, \mathrm{Z' \times \{0,1\} \cup X' \times I}$. Let $\overline{H} : (Z \times \{0,1\} \cup Z' \times I, Z' \times \{0,1\} \cup X' \times I) \times I \to (P, P')$ be given by the formula

$$\overline{H}(z, t, \lambda) = \begin{cases} g_t(z) & \text{if } t = 0, 1,\ \lambda \in I,\ z \in Z \\ H(z, t, \lambda) & \text{if } t, \lambda \in I,\ z \in Z'. \end{cases}$$

Since $\overline{H}(\cdot, \cdot, 1) = h|Z \times \{0,1\} \cup Z' \times I$ has the extension h onto $Z \times I$, the map $\overline{H}(\cdot, \cdot, 0)$ has an extension $G : (Z \times I, Z' \times I) \to (P, P')$ and $G : g_0 \simeq g_1$.

Summing up: $i^{\#} : [Z, Z'; P, P'] \to [X, X'; P, P']$ is the composition $i_1^{\#} \circ j^{\#}$; hence, in case (i) it is a surjection for $n \geq \mathrm{i^N(f)} - 1$, an injection if $\mathrm{i^N(f) \leq n}$. In case (ii) it is a surjection if $\mathrm{i^N(f) - 1 \leq n \leq m \leq N - 2}$ and an injection for $\mathrm{i^N(f) \leq n \leq m \leq N - 1}$. Since $f^{\#} = i^{\#} \circ r^{\#}$, in view of (1), we complete the proof. □

2.24 Remark

(i) Observe that it is enough to assume that $N \geq \dim Z + 1$. Moreover, note that $\mathrm{i^N(f) < N}$ for $N \geq \dim Z + 1$ if and only if $\mathrm{i(f) < N}$ since, for such N, $\mathrm{i^N(f) = i^{\dim X + 1}(f) = i(f)}$ – see Proposition 2.4. Recall that if $\dim X, \dim Y < \infty$, then $\mathrm{i(f)} < \infty$. Therefore $f^{\#}$ is surjective (bijective) whenever $n \geq \mathrm{i(f)} - 1$ ($n \geq \mathrm{i(f)}$).

(ii) The main tool of the above proof is that $\check{H}^q(Z, X; G) = 0 = \check{H}^q(Z', X'; G)$ if G is an abelian group (finitely generated whenever Y is not compact) and $\mathrm{i^N(f) \leq q \leq N - 1}$. Suppose that f is a Vietoris map, i.e. $\mathrm{i(f) = 0}$. Then, in view of the second part of Corollary 2.5, Assumption 2.15 (v) is not necessary in order to get the assertions of Theorems 2.17 and 2.18. Recall also that in this case $\mathrm{i^N(f) = i^N(f; \mathbf{Z}_{Y'}) = 0}$ for any N.

Proof of Theorem 2.18: Let $\overline{X} = X/X'$, $\overline{Y} = Y/Y'$ and let $\varphi_X : X \to \overline{X}$, $\varphi_Y : Y \to \overline{Y}$ be the canonical projections. If $\overline{x} = \varphi_X(X')$,

$\overline{y} = \varphi_Y(Y')$, then $\varphi_X^{\#} : [\overline{X}, \overline{x}; P, p] \to [X, X'; P, p]$ and $\varphi_Y^{\#} : [\overline{Y}, \overline{y}; P, p] \to [Y, Y'; P, p]$ are bijections. Define $\overline{f} : (\overline{X}, \overline{x}) \to (\overline{Y}, \overline{y})$ by the formula $\overline{f} \circ \varphi_X = \varphi_Y \circ f$. Clearly $\overline{f}$ is a continuous closed surjection and spaces $\overline{X}$, $\overline{Y}$ are paracompact.

Observe now that $\mathrm{i^N(\bar{f})} = \mathrm{i^N(f;}\mathcal{A})$. Since $\overline{f}^{-1}(\overline{y}) = \{\overline{x}\}$ we may argue as in the proof of Theorem 2.17 to get the desired result. □

If in Theorem 2.17 $m = \infty$, we can also get a different result, but one still needs some restrictions concerning the dimension. We have seen that assumptions concerning the cohomological dimension will not do (recall Example 2.22 (ii)). However the deformation dimension (see Definition 1.13) seems to be the right choice.

We give an absolute version of the result here, leaving the relative case to the reader.

2.25 Proposition *Let $f : X \to Y$ be a perfect surjection between paracompact spaces.*

(i) *If, for some $N \geq 2$, the deformation dimensions* $\mathrm{def\,dim\,X}, \mathrm{def\,dim\,Y} \leq N - 2$ *and* $\mathrm{i^N(f) < N}$ *and P satisfies Assumptions* 2.15 (i), (iv) *and* (v) *and is $(n-1)$-connected, $n \geq 1$, then $f^{\#} : [Y; P] \to [X; P]$ is a surjection for* $\mathrm{i^N(f) - 1 \leq n}$ *and an injection for* $\mathrm{i^N(f) \leq n}$.

(ii) [57, Cor. 2] *If* $\mathrm{def\,dim\,X}, \mathrm{def\,dim\,Y} < \infty$*, f is a cell-like map and P satisfies the conditions from* 2.15*, then $f^{\#} : [Y; P] \to [X; P]$ is a bijection.*

Proof First observe that we may replace P by a homotopy equivalent CW-complex (still denoted by P). By attaching k-cells, $k \geq N$ to P, we can obtain a CW-complex Q such that $\pi_i(Q) = 0$ for each $i \geq N - 1$. Hence Q is of $(n, N-2)$-type. Since $(N-1)$-dimensional skeletons of P and Q coincide and $\mathrm{def\,dim\,X \leq N - 2}$, $\mathrm{def\,dim\,Y \leq N - 2}$, the inclusion $i : P \to Q$ induces bijections $i_{\#} : [X; P] \to [X; Q]$ and $i_{\#} : [Y; P] \to [Y; Q]$. On the other hand, since Q has the homotopy type of an absolute neighborhood retract for paracompact spaces, in view of Theorem 2.17 (ii), we have that $f^{\#} : [Y; Q] \to [X; Q]$ is a surjection for $\mathrm{i^N(f) - 1 \leq n}$ and an injection for $\mathrm{i^N(f) \leq n}$. This completes the proof of (i).

As stated above, part (ii) is [57, Cor. 2] where the proof similar to that of part (i) but based on a version of 2.19 (ii) instead of 2.17 (ii) was given. It also follows from part (i) in view of Remark 2.21. □

Since evidently $\mathrm{def\,dim\,X} \leq \mathrm{dim\,X}$ and $\mathrm{def\,dim\,Y} \leq \mathrm{dim\,Y}$, 2.25 (i) is an obvious augumentation of Theorem 2.17 (i). However it does not constitute an extension of 2.17 (i). Indeed, if $\max\{\dim X, \dim Y\} + 2 = M$ and $\max\{\mathrm{def\,dim\,X}, \mathrm{def\,dim\,Y}\} + 2 = \mathrm{N}$, then clearly $M \geq N$ and $\mathrm{i^M(f)} \geq \mathrm{i^N(f)}$, so there is no relation between numbers $N - \mathrm{i^N(f)}$ and $M - \mathrm{i^M(f)}$.

Let us note the following corollary

2.26 Corollary *Let X, Y be homotopically simple complete* ANR*s such that $\pi_i(X)$ and $\pi_i(Y)$ are finitely generated groups for any $i \geq 1$. If X is of (n,m)-type, Y is of $(n,m+1)$-type, $1 \leq n,m \leq \infty$, $f : X \to Y$ is a perfect surjection such that, for some $N \geq 0$, $\mathrm{i^N(f)} < \mathrm{N}$, $\mathrm{i^N(f)} \leq \mathrm{n}$ and*

(i) $m \leq N - 2$*; or*

(ii) $\dim X + \dim Y + 2 \leq N$*; or*

(iii) $\max\{\mathrm{def\,dim\,X}, \mathrm{def\,dim\,Y}\} + 2 \leq \mathrm{N}$,

then f is a homotopy equivalence.

Proof: We easily gather that in each case $f^{\#} : [Y;X] \to [X,X]$ is a surjection; hence there is $g : Y \to X$ such that $g \circ f \simeq id_X$. For finitely generated abelian group G, $g^* : \check{H}^q(X;G) \to \check{H}^q(Y;G)$ is a monomorphism for any $q \geq 0$ and an isomorphism for $\mathrm{i^N(f)} \leq \mathrm{q} \leq \mathrm{N} - 1$. Therefore $\check{H}^q(Z_g, Y; G) = 0$ (Z_g is the cylinder of g) if $\mathrm{i^N(f)} + 1 \leq \mathrm{q} \leq \mathrm{N}$. Arguing as in the proof of Theorem 2.17, we gather that $g^{\#} : [X;Y] \to [Y,Y]$ is a surjection; hence there is $f' : X \to Y$ such that $f' \circ g \simeq id_Y$. We see therefore that $g_{\#} : \pi_i(Y) \to \pi_i(X)$ and $f_{\#} : \pi_i(X) \to \pi_i(Y)$ are isomorphisms for any $i \geq 1$. This completes the proof. □

Chapter 3.

COHOMOTOPY GROUPS

3.A. Cohomotopy groups and the Vietoris theorem

3.1 Homotopy and Vietoris' theorem S. Smale [164] was perhaps the first to observe that a Vietoris type theorem holds in terms of homotopy groups. Let X be an arbitrary topological space, let Y be paracompact and locally n-connected ($n \geq 1$); if instead acyclicity one assumes that each fibre of a given perfect (or merely closed) surjection $f : X \to Y$ is n-connected and locally $(n-1)$-connected, then $f_\# : \pi_q(X, x) \to \pi_q(Y, f(x))$ is an isomorphism for any $0 \leq q \leq n$. If, additionally, Y is dominated by a polyhedron (i.e. has the homotopy type of a CW-complex), then $f_\#$ is an epimorphism for $q = n + 1$ as well. It was shown independently by many authors that the asumption concerning fibres of f may be still relaxed. Namely, one has to suppose that, for each $y \in Y$ and each neighborhood U of $f^{-1}(y)$ there is a neigborhood $V \subset U$ of $f^{-1}(y)$ such that any singular k-sphere in V is inessential in U $(0 \leq k \leq n)$ – see [54, 13, 122] and others. In other words fibres of f should have UV^n-property. Compare also the papers [56, 159].

In this section we present some Vietoris type theorems stated in terms of cohomotopy groups. However again our hypotheses are not "categorical": the assumption concerning fibres is stated in the language of cohomology.

3.2 In Theorem 2.17 one may take, for instance, $(P, P') = (D^{n+1}, S^n)$ (where D^{n+1} is the $(n+1)$-dimensional closed unit ball), $n \geq 1$. If $n = 0$, then, although not all of the assumptions are satisfied, the assertion also holds if $\mathrm{i(f)} = 0$.

Taking $(P, P') = (S^n, s_0)$, where s_0 is a base point, Theorems 2.18, 2.20 and 2.25 imply that

3.3 Corollary *Consider a perfect surjection $f : (X, X') \to (Y, Y')$ such that $f^{-1}(Y') = X'$.*

(i) *If* $\dim X, \dim Y < \infty$ *and* $n \geq 1$*, then* f *induces a map*

$$f^{\#} : [Y, Y'; S^n, s_0] \to [X, X'; S^n, s_0]$$

being a surjection if $n = \mathrm{i(f)} - 1$ *and a bijection if* $\mathrm{i(f)} \leq \mathrm{n}$ (1).

(ii) *If* $\mathrm{Ind\,Y} < \infty$ *and* f *is a cell-like map, then* $f^{\#}$ *is always a bijection.*

(iii) *If* $\max\{\mathrm{def\,dim\,X}, \mathrm{def\,dim\,Y}\} < \infty$*, then* $f^{\#} : [Y; S^n] \to [X; S^n]$ *is a surjection for* $n = \mathrm{i(f)} - 1$ *and a bijection for* $n \geq \mathrm{i(f)}$.

Let us note also the following corollary – comp. Corollary 2.13. As usual, by $\Sigma^k X$ $(S^k X)$ we denote the k-th unreduced (reduced) suspension of a (pointed) space X.

3.4 Corollary *If* $f : X \to Y$ *is a perfect surjection of finite-dimensional (pointed) paracompact spaces (resp.* $\mathrm{Ind\,Y} < \infty$, X *is arbitrary and* f *is a cell-like map), then, for each* $k \geq 0$, $(\Sigma^k f)^{\#} : [\Sigma^k Y; S^{n+k}] \to [\Sigma^k X; S^{n+k}]$ $((S^k f)^{\#} : [S^k Y; S^{n+k}] \to [S^k X; S^{n+k}])$ *is a surjection for* $n = \mathrm{i(f)} - 1$ *and a bijection for* $n \geq \mathrm{i(f)}$ *(resp. a bijection for all* n*). If* $k \geq 2$*, then* $(S^k f)^{\#}$ *is an epimorphism and an isomorphism, respectively.*

Similiar results hold when $\mathrm{def\,dim\,X}, \mathrm{def\,dim\,Y} < \infty$ *(we leave the precise formulation to the reader).*

For it is enough to argue as in the proof of Theorem 2.17 but instead of the cylinder of $\Sigma^k f$ (resp. $S^k f$) one should consider the space $\Sigma^k Z$ (resp. $S^k Z$) where Z is the cylinder of f.

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[1] This assertion holds also when $\mathrm{i(f)} = 0$ and $n = 0$.

3.5 Remark More generally, the above assertions 3.3 (i) and 3.4 (resp. 3.3 (iii)) stay true if we replace $\mathrm{i(f)}$ with $\mathrm{i^N(f)}$ where $\dim X + \dim Y + 2 \leq N$ in 3.3 (i) or 3.4 (resp. $\max\{\mathrm{def\,dim\,X}, \mathrm{def\,dim\,Y}\} + 2 \leq \mathrm{N}$ in 3.3 (iii)) and assume that $i^N(f) < N$.

3.6 Cohomotopy groups Something more can be said with regard to the above statements. In [138], the author proves that given a compact pair (X, A), $\dim X < \infty$ and $\check{H}^q(X, A) = 0$ for $q \geq 2m - 1$, $m \geq 1$, the set $[X, A; S^n, s_0]$ admits the structure of an abelian group by the usual Borsuk method (see [165, 102] and comp. [142]) provided $n \geq m$ and is usually denoted by $\pi^n(X, A)$ and called the *cohomotopy group* of (X, A).

Essentially by the same methods one can introduce the group structure in the set $\pi^n(X, A) := [X, A; S^n, s_0]$, $n \geq m$, where (X, A) is a pair such that $\dim X < \infty$ and $\check{H}^q(X, A) = 0$ for $q \geq 2m - 1$ (2). Moreover, if $f : (X, A) \to (Y, B)$, $\dim Y < \infty$ and $\check{H}^q(Y, B) = 0$ for $q \geq 2m - 1$, then $f^\# : \pi^n(Y, B) \to \pi^n(X, A)$, $n \geq m$, is a homomorphism.

If $\mathrm{def\,dim\,X} < 2\mathrm{m} - 1$ for some $m \geq 1$, then as proved by [150, (4.2)], $\pi^n(X) := [X; S^n]$ again admits the structure of an abelian group.

Combinig these remarks with Corollary 3.3 we get the whole variety of conditions implying that a perfect surjection of paracompact spaces induces an isomorphisms of cohomotopy groups.

3.7 Corollary *Let $f : (X, X') \to (Y, Y')$ be a perfect surjection.*

(i) *Let $\check{H}^q(Y, Y') = 0$ for $q \geq 2m - 1$ where $m \geq 1$. If X, Y are finite-dimensional, then, for $n = \max\{m, \mathrm{i(f)} - 1\}$, $f^\# : \pi^n(Y, Y') \to \pi^n(X, X')$ is an epimorphism and an isomorphism when $\max\{m, \mathrm{i(f)}\} \leq \mathrm{n}$.*

If $\mathrm{Ind\,Y} < \infty$ and f is cell-like, then $f^\# : \pi^n(Y, Y') \to \pi^n(X, X')$ is an isomorphism for any $n \geq m$.

(ii) *If $\max\{\mathrm{def\,dim\,X}, \mathrm{def\,dim\,Y}\} < 2\mathrm{m} - 1$, $m \geq 1$, then $f^\# : \pi^n(Y) \to \pi^n(X)$ is an epimorphism for $n = \max\{\mathrm{i(f)} - 1, \mathrm{m}\}$ and an isomorphism for $n \geq \max\{\mathrm{i(f)}, \mathrm{m}\}$.*

The above facts have straightforward implications in the coincidence (fixed-point) theory. Some other more concrete applications will be studied in Chapters 5, 6, 7 and 8.

[2] If $\dim X \leq 2m - 1$, then this follows from the suspension theorem.

Let $\dim X < \infty$, $f : (X, X') \to (B^{m+1}, S^m)$ be a perfect surjection and let $g : X \to \mathbf{R}^{m+1}$ be a map such that $(f - g)(X) \subset \mathbf{R}^{n+1}$ (that means that $f = (f_1, f_2)$, $g = (g_1, g_2)$ where $f_1, g_1 : X \to \mathbf{R}^{n+1}$ and $f_2 = g_2 : X \to \mathbf{R}^{m-n}$) Let us assume that $m \geq n \geq \mathrm{i(f)}$. Next suppose that $(f - g)(X') \subset \mathbf{R}^{n+1} \setminus \{0\}$. Therefore, without loss of generality we may assume that $f - g : (X, X') \to (B^{n+1}, S^n)$.

3.8 Proposition *If the element* $(f^{\#})^{-1}([f - g]) \in [B^{m+1}, S^m; B^{n+1}, S^n] \cong \pi^n(S^m)$ *is nontrivial, then* f *and* g *have a coincidence, i.e. there is* $x_0 \in X$ *such that* $f(x_0) = g(x_0)$*.*

Proof Assume to the contrary that $(f - g)(X) \subset B^{n+1} \setminus \{0\}$. Hence there is $F : (X, X') \to (B^{n+1}, S^n)$ such that $f - g \simeq F[\mathrm{rel\, X'}]$ and $F(X) \subset S^n$. By Corollary 3.3, there is $F' : B^{m+1} \to S^n$ such that $F' \circ f \simeq F : X \to S^n$. Obviously $F' \circ f \simeq F : (X, X') \to (B^{n+1}, S^n)$, as well. Since F' is inessential and $[f - g] = f^{\#}[F']$, we get a contradiction. □

In particular we obtain

3.9 Corollary *Let* $f : (X, X') \to (B^{n+1}, S^n)$*,* $n \geq 0$ *be a perfect surjection and let* $\dim X < \infty$*. If* $g : X \to \mathbf{R}^{n+1}$ *is such that*

$$\|g(x)\| \leq \|f(x)\| \quad \text{for } x \in X' \tag{3.1}$$

and $\mathrm{i(f)} \leq \mathrm{n}$*, then maps* f *and* g *have a coincidence.*

Proof Assumption (3.1) immediately implies that $f - g \simeq f$, i.e. the element $(f^{\#})^{-1}([f - g])$ is nontrivial. □

3.10 Remark

(i) Clearly Assumption (3.1) (called the Rothe type condition) is necessary to show that $f - g \simeq f$. Any other condition (the Altman or Krasnoselskii type one) of this type would do – see [55, II.(5.1)] for the fixed point theoretical context.

(ii) One easily sees that the element $(f^{\#})^{-1}([f - g]) \in \pi^n(S^m)$ in Proposition 3.8 constitutes a natural generalization of the *fixed-point index*. For if $(X, X') = (B^{m+1}, S^m)$, $m = n$ and $f = id$, then $(f^{\#})^{-1}([f - g]) = \mathrm{ind\, g}$ – see [52].

(iii) The results stated above are valid when the domain X of a perfect surjection f is a finite-dimensional paracompactum. However these results still hold true with obvious modifications if we assume that $\mathrm{def\,dim\,X} \leq \mathrm{N} - 2$ and $\mathrm{i^N(f)} < \mathrm{N}$ for some $N \geq 2$ or that X is arbitrary but f is a cell-like map.

A concept underlying Proposition 3.8 suggests a way to extend Corollaries 3.3 and 3.7. The idea is that restrictions concerning the dimension should be replaced by restrictions concerning the "admissible" category of maps.

First we shall start our considerations with description of this category and cohomotopy functors on it.

3.B. Infinite dimensional cohomotopy sets and groups

In this Section we shall study a version of the cohomotopy theory of Gęba [76, 79]. We have to extend this theory a bit since, originally, the functor of Gęba is defined for pairs of subsets of a Banach space while we need it for arbitrary pairs.

3.11 Definition Let $(E, \|\cdot\|)$ be a Banach space, $\dim E \leq \infty$, and consider the following generalized Leray-Schauder category $\mathcal{LS}(E)$ (comp. [76, 79, 146]):

- objects of this category are triples $(X, A; u)$ where X is a paracompact space, A is a closed subset of X and $u : X \to E$ is a *proper* ([3]) map such that the set $u(X)$ is *bounded* and $u^{-1}(L)$ is a compact subset of X for each finite-dimensional (linear) subspace $L \subset E$.

- morphisms between objects $(X, A; u)$ and $(Y, B; v)$ are maps $f : (X, A) \to (Y, B)$ such that $v \circ f = u$.

Obviously a pair $(X; u)$, where X and u are as above, is identified with the triple $(X, \emptyset; u)$ and, therefore, considered also as an object in $\mathcal{LS}(E)$.

[3] i.e. such that $u^{-1}(K)$ is compact for a compact set $K \subset E$ (e.g. a perfect map is proper).

In most cases, we consider X as a closed bounded subset of a larger ambient normed space E' and assume that u is actually defined on E', is proper and bounded when restricted to any closed bounded subset of E'. In such a case, by abuse of notation, we write $(X, A; u)$ to denote the triple $(X, A; u|X)$.

3.12 Suppose that a *base point* s_0 in the unit sphere $S^E = \{x \in E \mid \|x\| = 1\}$ of the space E is given. By Z we denote the straight line spanned by s_0, i.e. $Z = \{x \in E \mid x = \lambda s_0, \lambda \in \mathbf{R}\}$ and let $Z_- = \{\lambda s_0 \mid \lambda \leq 0\}$ be a negative (closed) half-line determined by Z.

3.13 u-fields Let $(X, A; u) \in \mathcal{LS}(E)$. By a *$u$-field* on (X, A) we understand a *compact* ([4]) map $g : X \to E$ such that $(u - g) : (X, A) \to (E \setminus \{0\}, E \setminus Z_-)$.

We say that two u-fields $g_i : X \to E$ on $(X, A; u)$, $i = 0, 1$, are *u-homotopic* – written $g_0 \simeq_u g_1$ – if there is a compact map $h : X \times [0, 1] \to E$, called a *$u$-homotopy*, such that $h(\,\cdot\,, i) = g_i$, $i = 0, 1$, and $(u - h(\,\cdot\,, t)) : (X, A) \to (E \setminus \{0\}, E \setminus Z_-)$ for all $t \in [0, 1]$.

Clearly the relation "$\simeq_u$" of u-homotopy is an equivalence relation; the homotopy class of a u-field g is denoted by $[g]_u$; the collection of all u-homotopy classes is denoted by $\pi^E(X, A; u)$.

It is clear that π^E is an h-cofunctor (the notion of a homotopy of morphisms in $\mathcal{LS}(E)$ is obvious) from $\mathcal{LS}(E)$ to the category of sets and, given a morphism $f : (X, A; u) \to (Y, B; v)$, we have $\pi^E(f) : \pi^E(Y, B; v) \to \pi^E(X, A; u)$ defined by the formula $\pi^E(f)([g]_v) = [g \circ f]_u$ for any v-field $g : Y \to E$ on $(Y, B; v)$.

3.14 For any finite-dimensional linear subspace $L \subset E$ such that $Z \subset L$, let

$$X_L := u^{-1}(L)\ ({}^5), \quad A_L := A \cap X_L$$

and

$$\Sigma^L(X_L, A_L) = [X_L, A_L; L \setminus \{0\}, L \setminus Z_-].$$

Observe that there is a (canonical) bijection

$$\Sigma^L(X_L, A_L) \cong [X_L, A_L; S^L, s_0]$$

[4] i.e. $\mathrm{cl}\, g(X)$ is compact in E.

[5] we should rather write $X_L = X \cap u^{-1}(L)$ since as stated above u might be defined on a larger ambient space.

where S^L is the unit sphere in L. In the sequel we identify $\Sigma^L(X_L, A_L)$ with $[X_L, A_L; S^L, s_0]$ via this bijection.

3.15 By Λ we shall denote the family of all finite-dimensional linear subspaces $L \subset E$ such that $Z \subset L$ directed by inclusion.

After [79, p. 194] we introduce an orientation in E. Namely, by the *orientation* of E we mean the family $\mathcal{O} = \{\mathcal{O}_L\}_{L\in\Lambda}$ where, for any $L \in \Lambda$, $\mathcal{O}_L$ is a fixed orientation on L.

Let $L, N \in \Lambda$, $L \subset N$ and $\dim L = \dim N - 1$. In this case L cuts N into two *closed* halfspaces denoted, according to the orientations $\mathcal{O}_N$ and $\mathcal{O}_L$, by N^+ and N^-. We treat S^L as the equator of the unit sphere S^N in N; hence N^+ (resp. N^-) determines the north (resp. south) hemisphere S^N_+ (resp. S^N_-) in S^N.

If, as above, $X_N = u^{-1}(N)$, $A_N = A \cap X_N$ and $X^{\pm}_N := u^{-1}(N^{\pm})$, $A^{\pm}_N := A \cap X^{\pm}_N$ then $X_N = X^+_N \cup X^-_N$, $A_N = A^+_N \cup A^-_N$ and $X_L = X^+_N \cap X^-_N$, $A_L = A^+_N \cap A^-_N$.

As usual (comp. [76, p. 182]), one defines the coboundary (Mayer-Vietoris) set-transformation of the relative triad $(X_N, A_N; X^+_N, A^+_N; X^-_N, A^-_N)$

$$\Delta_{LN} : [X_L, A_L; S^L, s_0] \to [X_N, A_N; S^N, s_0].$$

Here is a brief description of the definition of Δ_{LN}. Let Y be a space obtained from X_N by identification of A_N to a single point $y_0 \in Y$, let $\eta : (X_N, A_N) \to (Y, y_0)$ be the quotient projection and let $Y^{\pm} = \eta(X^{\pm}_N)$, $Y_0 = Y^+ \cap Y^-$. Then the space Y_0 may be identified with the one obtained from X_L by collapsing A_L to the point y_0 and the map $\eta_0 : (X_L, A_L) \to (Y_0, y_0)$, given by $\eta_0(x) = \eta(x)$ for $x \in X_L$, is the quotient map. Clearly $\eta^{\#}_0 : [Y_0, y_0; S^L, s_0] \to [X_L, A_L; S^L, s_0]$ and $\eta^{\#} : [Y, y_0; S^N, s_0] \to [X_N, A_N, S^N, s_0]$ are bijections. Let $i_0 : Y_0 \to (Y_0, y_0)$ and $i : Y \to (Y, y_0)$ are inclusions, then $i^{\#}_0 : [Y_0, y_0; S^L, s_0] \to [Y_0, S^L]$ and $i^{\#} : [Y, y_0; S^N, s_0] \to [Y, S^N]$ are bijections, as well. Now let $\overline{\delta} : [Y_0, S^L] \to [Y^+, Y_0; S^N, s_0]$ be a transformation defined as follows: given an element $[f] \in [Y_0, S^L]$ represented by a map $f : Y_0 \to S^L$ extend it to a map $f' : (Y^+, Y_0) \to (S^N_+, S^L)$ (it is possible since this hemisphere is contractible); take a homotopy $h_t : (S^N_+, S^L) \to (S^N, S^N_-)$, $0 \le t \le 1$, such that h_0 is the inclusion and h_1 maps S^L onto s_0 and maps $S^N_+ \setminus S^L$ homeomorphically onto $S^N \setminus s_0$; at last put

$\overline{\delta}([f]) = [h_1 \circ f']$ [6]. Next let $k : (Y^+, Y_0) \to (Y, Y^-)$ and $j : Y \to (Y, Y^-)$ be the inclusions. Clearly $k^\# : [Y, Y^-; S^N, s_0]) \to [Y^+, Y_0; S^N, s_0]$ is a bijective transformation (excision – see [165, Th. 7.6]). At last we define

$$\Delta_{LN} = \eta^\# \circ (i^\#)^{-1} \circ j^\# \circ (k^\#)^{-1} \circ \overline{\delta} \circ i_0^\# \circ (\eta_0^\#)^{-1}.$$

The transformation Δ_{LN} also admits the following geometric description. Having an element $[G] \in [X_L, A_L; S^L, s_0]$ represented by a map $G : (X_L, A_L) \to (S^L, s_0)$ and an arbitrary extension $G' : (X_N, A_N) \to (S^N, s_0)$ of G such that $G'(X_N^\pm) \subset N^\pm$, then $[G'] = \Delta_{LN}([G])$.

Therefore it is easy to see that Δ_{LN} does not depend on the choice N^+ and N^-, i.e. if we change the orientations and according to them N^+ and N^- change position, then it does not affect the transformation Δ_{LN}.

Clearly Δ_{LN} induces a transformation (denoted by the same symbol)

$$\Delta_{LN} : \Sigma^L(X_L, A_L) \to \Sigma^N(X_N, A_N).$$

If $L, L_1, L_2 \in \Lambda$, $L_1 \neq L_2$, $L \subset L_i$ and $\dim L_i = \dim L + 1$, $i = 1, 2$, then there is a unique $N \in \Lambda$ such that $\dim N = \dim L_i + 1$. It is easy to see that $\Delta_{L_1 N} \circ \Delta_{LL_1} = \Delta_{L_2 N} \circ \Delta_{LL_2}$.

Now if $L \in \Lambda$ and $L \subset N \in \Lambda$, then there is a chain of subspaces $L = L_0 \subset L_1 \subset \ldots \subset L_{n+1} = N$ such that $\dim L_{i+1} = \dim L_i + 1$ for $i = 0, \ldots, n$. Hence we may define a transformation $\Delta_{LN} = \Delta_{L_n L_{n+1}} \circ \ldots \circ \Delta_{L_0 L_1} : \Sigma^L(X_L, A_L) \to \Sigma^N(X_N, A_N)$. Taking into account the above statement, it can be easily shown that Δ_{LN} is well-defined, i.e. it does not depend on the choice of the above chain of subspaces.

Given the third subspace $M \in \Lambda$, $M \supset N$, we see that $\Delta_{LM} = \Delta_{NM} \circ \Delta_{LN}$. Additionally let us put $\Delta_{LL} = id$.

3.16 Let us note the following simple property. Assume that $L \in \Lambda$ and $G : (X_L, A_L) \to (L \setminus \{0\}, L \setminus Z_-)$ and consider a compact extension $g : X \to L$ of the map $X_L \ni x \mapsto u(x) - G(x) \in L$. Evidently g is a u-field on $(X, A; u)$. Let $L \subset N \in \Lambda$ and let $Q = (u - g)|X_N$. Then $\Delta_{LN}([G]) = [Q]$.

[6] we see that $\overline{\delta}$ is nothing else but the coboundary homomorphism defined with respect to the special – described above – choice of the equator and the north (south) hemispheres in S^N.

Indeed, first suppose that $\dim N = \dim L + 1$. Observe that $Q|X_L = G$; moreover $Q(X_N^{\pm}) \subset N^{\pm}$. Then, by the very definition of Δ_{LN} and its geometric description, $[Q] = \Delta_{LN}([G])$ (see also [76, Lemma 2.4]). If $\dim N > \dim L + 1$, then in order to get the assertion one can iterate the above argument.

3.17 Hence we may define a direct system $\Sigma = \{\Sigma^L(X_L, A_L), \Delta_{LN} \mid L, N \in \Lambda, L \subset N\}$ of sets. Let $(\Sigma^E(X, A; u), \{\sigma^L\}_{L\in\Lambda})$ be *the direct limit* (in the category of sets) of the system Σ, i.e.

$$\Sigma^E(X, A; u) := \lim_{\overrightarrow{L\in\Lambda}} \Sigma^L(X_L, A_L)$$

and $\sigma^L : \Sigma^L(X_L, A_L) \to \Sigma^E(X, A; u)$, $L \in \Lambda$, is the canonical transformation.

3.18 For each $L \in \Lambda$, consider a transformation

$$\xi^L : \Sigma^L(X_L, A_L) \to \pi^E(X, A; u)$$

which assigns to the homotopy class $[G]$ of a map $G : (X_L, A_L) \to (L \setminus \{0\}, L \setminus Z_-)$ the homotopy class $[g]_u$ of a u-field $g : X \to L$ on (X, A) being an arbitrary compact extension onto X of the map $X_L \ni x \mapsto G^L(x) = u(x) - G(x)$. The transformation $\xi^L : [G] \mapsto [g]_u$ is well-defined since if $\overline{G} \simeq G : (X_L, A_L) \to (L \setminus \{0\}, L \setminus Z_-)$ and $\overline{g}$ is a compact extension onto X of the map $X_L \ni x \mapsto u(x) - \overline{G}(x) \in L$, then clearly $\overline{g} \simeq_u g$. Moreover ξ^L is even injective if we restrict "admissible" u-homotopies h to those that map $X \times [0,1]$ into L (i.e such that $(u - h(\,\cdot\,, t)) : (X, A) \to (L \setminus \{0\}, L \setminus Z_-))$ because if such h joins some u-fields g and $\overline{g}$, then $G = (u - g)|X_L \simeq \overline{G} = (u - \overline{g})|X_L : (X_L, A_L) \to (L \setminus \{0\}, L \setminus Z_-)$.

3.19 Lemma *The family $\{\xi^L \mid L \in \Lambda\}$ of set-transformations is* compatible *with Σ, i.e. for any $L, N \in \Lambda$, $L \subset N$, $\xi^N \circ \Delta_{LN} = \xi^L$.*

Proof It is enough to consider the case $\dim N = \dim L + 1$. Let $G : (X_L, A_L) \to (L \setminus \{0\}, L \setminus Z_-)$ and let $G' : (X_N, A_N) \to (N \setminus \{0\}, N \setminus Z_-)$ be an extension of G such that $G'(X_N^{\pm}) \subset N^{\pm}$. Next let $g : X \to L$, $g' : X \to N$ be compact extensions of the maps $X_L \ni x \mapsto u(x) - G(x) \in L$

and $X_N \ni x \mapsto u(x) - G'(x) \in N$, respectively. Consider a map $X_N \times [0,1] \ni (x,t) \mapsto H(x,t) = u(x) - [(1-t)g(x) + tg'(x)] \in N$. Observe that $H(\,\cdot\,,0) = (u-g)|X_N$, $H(\,\cdot\,,1) = (u-g')|X_N = G'$ and $H(x,t) = G(x)$ for $x \in X_L$, $t \in [0,1]$.

Suppose that $0 = H(x,t)$, i.e $(t-1)[u(x) - g(x)] = tG'(x)$, for some $x \in X_N$, $t \in [0,1]$. Then clearly $(x,t) \in (X_N \setminus X_L) \times (0,1)$. If $x \in X_N^+ \setminus X_L$, then $(t-1)[u(x) - g(x)] \in N^- \setminus L$ and $tG'(x) \in N^+$; while if $x \in X_N^- \setminus X_L$, then $(t-1)[u(x) - g(x)] \in N^+ \setminus L$ and $tG'(x) \in N^-$. In both cases we get a contradiction.

Similarly one shows that $H(x,t) \notin Z_-$ for $x \in A_N$ and $t \in [0,1]$.

Therefore $H : (X_N, A_N) \times [0,1] \to (N \setminus \{0\}, N \setminus Z_-)$. Now we see that any compact extension $h : X \times [0,1] \to N$ of the map

$$X \times \{0,1\} \cup X_N \times [0,1] \mapsto \begin{cases} g(x) & \text{if } x \in X,\ t = 0 \\ g'(x) & \text{if } x \in X,\ t = 1 \\ u(x) - H(x,t) & \text{if } x \in X_N,\ t \in [0,1] \end{cases}$$

provides a u-homotopy joining g to g'. □

Hence there is a unique (limit) set-transformation

$$\xi : \Sigma^E(X, A; u) \to \pi^E(X, A; u)$$

such that

$$\xi \circ \sigma^L = \xi^L.$$

3.20 Proposition *The transformation ξ is bijective.*

Proof Indeed take $[g]_u \in \pi^E(X, A; u)$ where $g : X \to E$ is a u-field on (X, A). Since g is a compact map the set $K = \operatorname{cl} g(X)$ is compact. For any $\varepsilon > 0$, there is a finite-dimensional linear subspace E_ε of E and a Schauder projection (see e.g. [55]) $p_\varepsilon : K \to E_\varepsilon$ such that $\|p_\varepsilon(x) - x\| < \varepsilon$ for $x \in K$.

Since u is proper and $u(X)$ is bounded, there exists $\varepsilon > 0$ such that, for all $x \in X$, $\|u(x) - p \circ g(x)\| \geq \varepsilon$ and, for all $x \in A$, $d(u(x) - p \circ g(x), Z_-) \geq \varepsilon$ where we have put $p := p_\varepsilon$. Let $L := E_\varepsilon$ and put $g' := p \circ g : X \to L$, $g_L = g'|X_L$ and $u_L := u|X_L$. It is obvious that $(u_L - g_L) : (X_L, A_L) \to (L \setminus \{0\}, L \setminus Z_-)$; moreover $g \simeq_u g'$. Therefore $[g]_u = \xi^L([G])$ where $G = u_L - g_L$. In other words $\pi^E(X, A; u) = \bigcup_{L \in \Lambda} \xi^L(\Sigma^L(X_L, A_L))$.

Suppose now that there are subspaces $L, N \in \Lambda$ and maps $G_0 : (X_L, A_L) \to (L \setminus \{0\}, L \setminus Z_-)$, $G_1 : (X_N, A_N) \to (N \setminus \{0\}, N \setminus Z_-)$ such that $\xi^L([G_0]) =$

$\xi^N([G_1])$; thus if u-field $g_i : X \to L$ represents $\xi^L([G_i])$, $i = 0, 1$, then there is a u-homotopy $h : g_0 \simeq_u g_1 : X \to E$.

If h itself is finite-dimensional, say $h(X \times [0,1]) \subset M \in \Lambda$ (clearly $L, N \subset M$), and $(X_M, A_M) \ni x \mapsto Q_i(x) = u(x) - g_i(x) \in (M \setminus \{0\}, M \setminus Z_-)$, $i = 0, 1$, then $Q_0 \simeq Q_1 : (X_M, A_M) \to (M \setminus \{0\}, M \setminus Z_-)$ and, in view of 3.16, $\Delta_{LM}([G_0]) = [Q_0] = [Q_1] = \Delta_{NM}([G_1])$.

If h is not finite-dimensional, then take $\varepsilon' > 0$, $\varepsilon' < \varepsilon$, such that if $t \in [0,1]$, then $\|p' \circ h(x,t) - u(x)\| \geq \varepsilon'$ for $x \in X$ and $d(u(x) - p' \circ h(x,t), Z_-) \geq \varepsilon'$ for $x \in A$ where $p' : \mathrm{cl}\, h(X \times [0,1]) \to M \supset L, N$, $\dim M < \infty$, is a Schauder projection with $\|p'(x) - x\| < \varepsilon'$ for $x \in \mathrm{cl}\, h(X \times [0,1])$. Now $p' \circ h : p' \circ g_0 \simeq_u p' \circ g_1$ and $Q_0 \simeq Q'_0 \simeq Q'_1 \simeq Q_1 : (X_M, A_M) \to (M \setminus \{0\}, M \setminus Z_-)$ where $Q'_i(x) = u(x) - p' \circ g_i(x)$ for $x \in X_M$, $i = 0, 1$. Thus again $\Delta_{LM}([G_0]) = [Q_0] = [Q_1] = \Delta_{NM}([G_1])$.

This completes the proof of our proposition. $\square$

3.21 Remark

(i) It is clear that defining the direct system Σ, instead of Λ, one may take an arbitrary *cofinal* subfamily Λ' of Λ and it does not affect the structure of $\pi^E(X, A; u)$.

(ii) In view of the above considerations $\pi^E(X, A; u)$ may be treated as *a* direct limit of Σ with respect to the compatible family $\{\xi^L \mid L \in \Lambda\}$.

3.22 Definition We say that an object $(X, A; u)$ is *regular* if there exists a positive integer $m_0 \leq \dim E$ such that, for all linear subspaces L of E with $\dim L \geq m_0$ and $q \geq 2\dim L - 3$, $\check{H}^q(u^{-1}(L), A \cap u^{-1}(L)) = 0$.

We say that $(X, A; u)$ is of *finite type* if $\dim X_L < \infty$ for all $L \in \Lambda$.

If $(X, A; u)$ is a regular object of finite type, $L \in \Lambda$ and $\dim L = m+1 \geq m_0$, then $\check{H}^q(X_L, A_L) = 0$ for $q \geq 2\dim L - 3 = 2m - 1$ and $\dim X_L < \infty$. Hence $\pi^m(X_L, A_L)$ has the structure of an abelian group – see 3.6.

In order to provide some necessary examples we need to recall the notion of a Fredholm operator.

3.23 Fredholm operators Let E', E be Banach spaces. The set of all bounded linear operators $E' \to E$ is denoted by $\mathcal{L}(E', E)$. We say that $F \in \mathcal{L}(E', E)$ is a *Fredholm operator* if $\dim \mathrm{Ker}(F) < \infty$ and $\dim \mathrm{Coker(F)} <$

∞ (recall that $\mathrm{Ker}(F) := \{x \in E' \mid F(x) = 0\}$ is the *null-space* of F, $\mathrm{Coker(F)} := \mathrm{Y/R(F)}$ where $\mathrm{R}(F) := F(E')$ is the *range* of F, and the quotient space is understood in the algebraic sense, i.e. regardless the topology).

By the *index* of a Fredholm operator F we mean the number

$$\mathrm{ind(F)} := \dim \mathrm{Ker(F)} - \dim \mathrm{Coker(F)}.$$

The set of all Fredholm operators is denoted by $\mathcal{F}(E', E)$ and

$$\mathcal{F}_m(E', E) = \{F \in \mathcal{F}(E', E) \mid \mathrm{ind(F)} = \mathrm{m}\}.$$

If $F \in \mathcal{F}(E', E)$, then both $\mathrm{Ker}(F)$ and $\mathrm{R}(F)$ are closed subspaces and direct summands of E' and E, respectively; i.e. there are finite dimensional projectors $P : E' \to E'$, $Q : E \to E$ such that $\mathrm{R}(P) = \mathrm{Ker}(F)$, $\mathrm{Ker}(P) \oplus \mathrm{Ker}(F) = E'$ and $\mathrm{Ker}(Q) = \mathrm{R}(F)$, $\mathrm{R}(Q) \oplus \mathrm{R}(F) = E$. Hence there is $G \in \mathcal{L}(E, E')$ such that

$$G \circ F = 1_{E'} - P, \quad F \circ G = 1_E - Q.$$

If $F \in \mathcal{L}(E', E)$ and $\dim \mathrm{Coker(F)} < \infty$, then $\dim \mathrm{Coker(F)} = \dim \mathrm{Ker(F^*)}$ where $F^* : E^* \to E'^*$ is the adjoint operator. Hence if $F \in \mathcal{L}(E', E)$ and $\dim \mathrm{Ker}(F)$, $\dim \mathrm{Ker}(F^*) < \infty$, then F is a Fredholm operator; in particular a self-adjoint $F \in \mathcal{L}(E')$, where E' is Hilbert space, with the finite-dimensional null-space is a Fredholm operator of index 0.

One shows that $\mathcal{F}(E', E)$ is an open subset of $\mathcal{L}(E', E)$ (in the operator-norm topology) and the function $\mathrm{ind} : \mathcal{F}(\mathrm{E}', \mathrm{E}) \to \mathbf{Z}$ is continuous. Moreover, if $F \in \mathcal{F}(E', E)$ and $C : E' \to E$ is a compact linear operator, then $F + C$ is a Fredholm operator with index ind(F).

3.24 Example

(i) The simplest example of a regular object of finite type in $\mathcal{LS}(E)$, $\dim E \geq 4$, is as follows: take a closed bounded $X \subset E$, A a closed subset of X and $u = i : X \hookrightarrow E$.

(ii) More generally: if E' is a Banach space, $F \in \mathcal{F}_k(E', E)$, $k \geq 0$, $A \subset X \subset E'$ are closed bounded and $u = F|X$, then $(X, A; u)$ is of finite type. It is a regular object provided $\dim E \geq 4 + k$.

(iii) Let (X, A) be a paracompact pair (A is closed in X) such that $\dim X < \infty$. If $\dim X < 2 \dim E - 3$, then for any proper bounded $u : X \to E$, $(X, A; u)$ is a regular object of finite type in $\mathcal{LS}(E)$.

3.25 Proposition *For any regular object $(X, A; u) \in \mathcal{LS}(E)$ of finite type, the set $\pi^E(X, A; u)$ admits the structure of an abelian group.*

Proof According to Remark 3.21, when defining Σ we may take Λ to be the collection of all finite-dimensional subspaces linear subspaces of E of dimension $\geq m_0$ containing Z. In this case, for each $L \in \Lambda$, $\dim L = m+1$, the set $\Sigma^L(X_L, A_L) = [X_L, A_L; S^L, s_0]$ has the structure of an abelian group. Precisely: for any $L \in \Lambda$, there is an orthogonal linear isomorphism $o_L : L \to \mathbf{R}^{m+1}$ which preserves orientations (on L we have the orientation $\mathcal{O}_L$ and on $\mathbf{R}^{m+1}$ we consider the canonical orientation); now we pull-back the group structure of $\pi^m(X_L, A_L)$ onto $\Sigma^L(X_L, A_L)$ through o_L. Observe that if $o'_L : L \to \mathbf{R}^{m+1}$ is another orientation preserving orthogonal isomorphism, then it induces the identical group structure in $\Sigma^L(X_L, A_L)$ as o_L does.

Hence if $L, N \in \Lambda$, $L \subset N$, then

$$\Delta_{LN} : \Sigma^L(X_L, A_L) \to \Sigma^N(X_N, A_N)$$

is a homomorphism. Therefore Σ is the direct system of abelian groups; so $\Sigma^E(X, A; u)$ and $\pi^E(X, A; u)$ have the group structure (observe that proving the bijectivity of ξ one may always assume that a Schauder projection p_ε takes values in a space $E_\varepsilon \in \Lambda$). □

3.26 Example For some instances, the group $\pi^E(X, \varphi)$ might be easily computed. If E' is a Banach space, $F : E' \to E$ is a Fredholm operator of index $k \geq 0$, $\dim E \geq k+3$, $X = S$ is a unit sphere in E' and $u = F|S$, then (S, u) is a regular object and $\pi^E(S, u) = \pi_s^{-k}(S^0) \cong \pi_k^s(S^0)$ is the k-th stable homotopy group of spheres.

To see that, let Λ' be a subfamily in Λ consisting of linear subspaces of sufficiently large dimension and having a "general position" with respect to the range $\mathrm{R}(F)$, that is $L \in \Lambda'$ if and only if $L = L' \oplus T$ where $L', T \in \Lambda$, $T \oplus R(F) = E$ and $L' \subset R(F)$ (e.g. $T = \mathrm{R}(Q)$ – see 3.23). Clearly Λ' is cofinal in Λ and thus $\pi^E(S, u) = \lim\limits_{\substack{\longrightarrow \\ L \in \Lambda'}} \pi^{\dim L-1}(S^{\dim L-1+k})$ because $u^{-1}(L) = S \cap F^{-1}(L')$ and $\dim F^{-1}(L') = \dim L' + \dim \mathrm{Ker}(F) = \dim L + k$. If $\dim L$ is sufficiently large ($\dim L \geq k+3$), then by the Freundenthal suspension theorem, $\pi^{\dim L-1}(S^{\dim L-1+k}) \cong \pi_k^s(S^0)$. Moreover, since the hemispheres are contractible, the homomorphisms Δ_{LN}, $L, N \in \Lambda'$, $L \subset N$, are isomorphisms in this case (see also e.g. [165, Th. 9.1]); hence the assertion.

Let us remark that an analogous result concerning the homotopy classification of compact perturbations of a Fredholm operator of nonnegative index has been established first by Svarc [168] (see also [76]).

3.27 Let $(X, A; u)$ and $(Y, B; v)$ be (resp. regular of finite type) objects from $\mathcal{LS}(E)$ and let $f : (X, A; u) \to (Y, B; v)$ be a morphism. In case the objects are regular, let m_0 be such that $\check{H}^q(X_L, A_L) = 0 = \check{H}^q(Y_L, B_L)$ for all $q \geq 2 \dim L - 3$ where L is a linear subspace of E with $\dim L \geq m_0$ and $(Y_L, B_L) = (v^{-1}(L), B \cap v^{-1}(L))$. As above Λ is a family of finite-dimensional linear subspaces containing Z (resp. with dimension at least m_0). It is clear that, for each $L \in \Lambda$, f induces the map $f_L : (X_L, A_L) \to (Y_L, B_L)$; hence the set-transformation (resp. homomorphism)

$$f_L^{\#} : \Sigma^L(Y_L, B_L) \to \Sigma^L(X_L, A_L)$$

such that a diagram

$$\begin{array}{ccc}
\Sigma^L(Y_L, B_L) & \xrightarrow{f_L^{\#}} & \Sigma^L(X_L, A_L) \\
\downarrow \Delta_{LN} & & \downarrow \Delta_{LN} \\
\Sigma^N(Y_N, B_N) & \xrightarrow{f_N^{\#}} & \Sigma^N(X_N, A_N)
\end{array}$$

is commutative. Therefore a (unique) transformation (resp. homomorphism) $\Sigma^E(f) : \Sigma^E(Y, B; v) \to \Sigma^E(X, A; u)$ is well-defined. It is also easy to check the commutativity of diagrams

$$\begin{array}{ccc}
\Sigma^L(Y_L, B_L) & \xrightarrow{f_L^{\#}} & \Sigma^L(X_L, A_L) \\
\downarrow \xi^L & & \downarrow \xi^L \\
\pi^E(Y, B; v) & \xrightarrow{\pi^E(f)} & \pi^E(X, A; u)
\end{array}$$

and

$$\begin{array}{ccc}
\Sigma^E(Y, B; v) & \xrightarrow{\Sigma^E(f)} & \Sigma^E(X, A; u) \\
\downarrow \xi & & \downarrow \xi \\
\pi^E(Y, B; v) & \xrightarrow{\pi^E(f)} & \pi^E(X, A; u)
\end{array}$$

which shows that π^E is cofunctor from $\mathcal{LS}(E)$ to the category of sets (resp. abelian groups).

From Corollaries 3.3 and 3.7 and 3.27 we get immediately the following version of the Vietoris theorem.

3.28 Theorem *Let $(X, A; u), (Y, B; v) \in \mathcal{LS}(E)$ and a morphism $f : (X, A) \to (Y, B)$ be such that $f : (X, A) \to (Y, B)$ is a perfect surjection, $A = f^{-1}(B)$.*

(i) *If $\mathrm{i(f)} < \dim \mathrm{E} \le \infty$ and objects $(X, A; u)$, $(Y, B; v)$ are of finite type, then*

$$\pi^E(f) : \pi^E(Y, B; v) \to \pi^E(X, A; u)$$

is a bijection. If objects $(X, A; u)$, $(Y, B; v)$ are also regular, then $\pi^E(f)$ is an isomorphism.

If $\dim E < \infty$ and $\mathrm{i(f)} = \dim \mathrm{E}$, then $\pi^E(f)$ is a surjection (an epimorphism provided the objects are additionally regular).

(ii) *If $(Y, B; v)$ is of finite type, Y is a metric space and f is a cell-like map, then $\pi^E(f)$ is a bijection.*

3.29 The introduced cofunctor π^E admits a generalization in analogy to the *infinite-dimensional stable cohomotopy* theory of Gęba – see [76].

Namely assume that $\dim E = \infty$, a filtration $\{E^n\}_{n=0}^{\infty}$ of linear subspaces such that $\dim E^n = n$, $E^n \subset E^{n+1}$ and a family of complementing closed subspaces $\{E_n\}$ i.e. such that $E^n \oplus E_n = E$ and $E_n \subset E_{n-1}$ for each $n \ge 1$ are given. Additionally assume that $Z \subset \bigcap_{n=0}^{\infty} E_n$.

For each object $(X, A; u)$ in $\mathcal{LS}(E)$ and $n \ge 0$ let $\pi^{\infty-n}(X, A; u)$ be the set of all u-homotopy classes of compact maps $g : X \to E$ such that $(u - g) : (X, A) \to (E_n \setminus \{0\}, E_n \setminus Z_-)$ (u-homotopies are compact maps $h : X \times [0, 1] \to E$ such that $(u - h(\,\cdot\,, t)) : (X, A) \to (E_n \setminus \{0\}, E_n \setminus Z_-)$).

It is easy to see that $\pi^{\infty-n}(X, A; u) = \pi^{E_n}(X, A; u_n)$ where $u_n = p_n \circ u$ and $p_n : E \to E_n$ is a linear projection parallel to E^n. Moreover, for a linear subspace L being in "general position" with respect to E_n (i.e. such that $L = E^n \oplus L'$ where $L' \subset E_n$) and such that $Z \subset L'$ we have $u^{-1}(L) = u_n^{-1}(L')$. The family of such subspaces is cofinal in the family of all subspaces.

If $(X, A; u)$ is a regular object of finite type, then so is $(X, A; u_n)$. Therefore in this case one may pull-back the group structure from $\pi^{E_n}(X, A; u_n)$ onto $\pi^{\infty-n}(X, A; u)$, $n \ge 0$.

In this way one obtains a family $\{\pi^{\infty-*}\}_{n=0}^{\infty}$ of cofunctors – the so-called *infinite-dimensional stable cohomotopy theory.* Arguing as in [76] one may show that the family $\{\pi^{\infty-n}\}_{n=0}^{\infty}$ gives rise to the (extraordinary) cohomology theory (i.e satisfying the Eilenberg-Steenrod axioms save the dimension one).

Now we shall study some other properties of the functor π^E necessary in the sequel.

3.30 Let $(X, A; u)$ be an object in $\mathcal{LS}(E)$. Put $\overline{E} := E \oplus \mathbf{R}$. It is clear that we may treat E as a linear subspace in $\overline{E}$, therefore Z is also a subspace in $\overline{E}$. Let $\overline{u} : X \to \overline{E}$, $\overline{u}(x) = (u(x), 0)$ for $x \in X$. Clearly $\overline{u}$ is proper and bounded and, for each finite-dimensional linear subspace L in $\overline{E}$, $\overline{u}^{-1}(L) = u^{-1}(L \cap E)$ is a compact subset of X. Hence $(X, A; \overline{u})$ is an object in $\mathcal{LS}(\overline{E})$; it is regular (resp. of finite type) if $(X, A; u)$ is so.

According to 3.13 one may consider the set $\pi^{E+1}(X, A; u) := \pi^{\overline{E}}(X, A; \overline{u})$. Recall that a compact map $\overline{g} : X \to \overline{E}$ is a $\overline{u}$-field on $(X, A; \overline{u})$ if $(\overline{u} - g) : (X, A) \to (\overline{E} \setminus \{0\}, \overline{E} \setminus Z_-)$ and two $\overline{u}$-fields $\overline{g}_0, \overline{g}_1$ are $\overline{u}$-homotopic provided there is a compact map $\overline{h} : X \times [0, 1] \to \overline{E}$ such that $\overline{h}(\,\cdot\,, i) = \overline{g}_i$, $i = 0, 1$ and $(\overline{u} - \overline{h}(\,\cdot\,, t)) : (X, A) \to (\overline{E} \setminus \{0\}, \overline{E} \setminus Z_-)$.

Analogously as in 3.17 one may define a direct system $\overline{\Sigma}$ of sets with respect to the object $(X, A; \overline{u})$. Observe that when defining this system instead of considering the collection of all finite-dimensional subspaces of $\overline{E}$ containing Z we may take the family $\overline{\Lambda}$ of subspaces of the form $\overline{L} = L \oplus \mathbf{R}$ where $L \in \Lambda$ because such a family is cofinal in the latter one. Clearly $\overline{\Lambda}$ can be identified with Λ; for each $\overline{L} \in \overline{\Lambda}$, $\overline{u}^{-1}(\overline{L}) = u^{-1}(L) = X_L$.

However we shall study a different system. For each $L \in \Lambda$, let

$$\Sigma^{L+1}(X_L, A_L) = [X_L, A_L; \overline{L} \setminus \{0\}, \overline{L} \setminus Z_-] \cong [X_L, A_L; S^{\overline{L}}, s_0].$$

If $L, N \in \Lambda$, $\dim N = \dim L + 1$, and $[\overline{G}] \in \Sigma^{L+1}(X_L, A_L)$ is represented by a map $\overline{G} : (X_L, A_L) \to (S^{\overline{L}}, s_0)$, then $\Delta_{LN}^{+1}([\overline{G}])$ denotes the homotopy class of an arbitrary extension $\overline{G}' : (X_N, A_N) \to (S^{\overline{N}}, s_0)$ of $\overline{G}$ such that $\overline{G}'(X_N^{\pm}) \subset N^{\pm} \oplus \mathbf{R}$. The transformation

$$\Delta_{LN}^{+1} : \Sigma^{L+1}(X_L, A_L) \to \Sigma^{N+1}(X_N, A_N)$$

is well-defined since clearly it admits an algebraic description in analogy with the one given in 3.15.

Let

$$\Sigma^{+1} = \{\Sigma^{L+1}(X_L, A_L), \Delta^{+1}_{LN} \mid L, N \in \Lambda, L \subset N\}.$$

Let, for $L \in \Lambda$,

$$\xi^{L+1} : \Sigma^{L+1}(X_L, A_L) \to \pi^{E+1}(X, A; u)$$

be a transformation assigning to any homotopy class $[\overline{G}]$ of a map $\overline{G} : (X_L, A_L) \to (\overline{L} \setminus \{0\}, \overline{L} \setminus Z_-)$ the homotopy class $[\overline{g}]_{\overline{u}}$ of an arbitrary compact extension onto X of the map $X_L \ni x \mapsto \overline{u}(x) - \overline{G}(x) \in \overline{L}$. As in Lemma 3.19, we show easily that the family $\{\xi^{L+1}\}_{L\in\Lambda}$ is compatible with the system Σ^{+1}.

If the direct limit of Σ^{+1} is denoted by $\Sigma^{E+1}(X, A; u)$, then, as in Proposition 3.20, we show that a (unique) limit set-transformation $\xi^{+1} : \Sigma^{E+1}(X, A; u) \to \pi^{E+1}(X, A; u)$ is a bijection. Moreover, if the object $(X, A; u) \in \mathcal{LS}(E)$ is regular, then $\pi^{E+1}(X, A; u)$ admits the structure of an abelian group and ξ^{+1} is an isomorphism.

3.31 Example If $\dim E = n < \infty$, then for any object $(X, A; u) \in \mathcal{LS}(E)$, $\pi^E(X, A; u) = \pi^{n-1}(X, A)$ and $\pi^{E+1}(X, A; u) = \pi^n(X, A)$. In this case $(X, A; u)$ is regular if and only if $\check{H}^q(X, A) = 0$ for each $q \geq 2n - 3$.

3.32 Coboundary transformation Let $L \in \Lambda$, $\overline{L} = L \oplus \mathbf{R}$ and let $S^{\overline{L}}$ be the unit sphere in $\overline{L}$. Clearly S^L may be treated as the equator of $S^{\overline{L}}$ which in an obvious way determines the north (south) hemisphere $S^{\overline{L}}_+$ ($S^{\overline{L}}_-$). Consider a diagram

$$\begin{array}{ccc} [A_L, S^L] & & [X_L, A_L; S^{\overline{L}}, s_0] \\ \uparrow \alpha & & \downarrow \gamma \\ [X_L, A_L; S^{\overline{L}}_+, S^L] & \xrightarrow{\beta} & [X_L, A_L; S^{\overline{L}}, S^{\overline{L}}_-] \end{array}$$

in which the transformation α is induced by the operation of restriction and the transformations β and γ are induced by inclusions. Since the hemispheres $S^{\overline{L}}_{\pm}$ are contractible, we easily see that the transformations α and γ are bijective. Therefore the transformation

$$\delta^L = \gamma^{-1} \circ \beta \circ \alpha^{-1} : [A_L, S^L] \to [X_L, A_L; S^{\overline{L}}, s_0]$$

is well-defined.

In this way we have also defined a transformation (denoted by the same symbol)

$$\delta^L : \Sigma^L(A_L) \to \Sigma^{L+1}(X_L, A_L).$$

It is easy to check that if $L, N \in \Lambda$, $L \subset N$, then due to our definition of Δ^{+1}_{LN}, the following diagram

$$\begin{array}{ccc} \Sigma^L(A_L) & \xrightarrow{\delta^L} & \Sigma^{L+1}(X_L, A_L) \\ \downarrow{\scriptstyle \Delta_{LN}} & & \downarrow{\scriptstyle \Delta^{+1}_{LN}} \\ \Sigma^N(A_N) & \xrightarrow{\delta^N} & \Sigma^{N+1}(X_N, A_N) \end{array}$$

is commutative and the limit transformation $\Delta : \Sigma^E(A; u) \to \Sigma^{E+1}(X, A; u)$ is defined. Hence we have a transformation

$$\delta = \xi^{+1} \circ \Delta \circ \xi^{-1} : \pi^E(A; u) \to \pi^{E+1}(X, A; u)$$

being a homomorphism whenever objects $(A; u)$ and $(X, A; u)$ are regular of finite type. Observe that from the very definition of δ it follows that the following diagram

$$\begin{array}{ccc} \Sigma^L(A_L) & \xrightarrow{\delta^L} & \Sigma^{L+1}(X_L, A_L) \\ \downarrow{\scriptstyle \xi^L} & & \downarrow{\scriptstyle \xi^{L+1}} \\ \pi^E(A; u) & \xrightarrow{\delta} & \pi^{E+1}(X, A; u) \end{array}$$

is commutative.

It follows that if $f : (X, A; u) \to (Y, B; v)$ is a morphism in $\mathcal{LS}(E)$, then there is a commutative diagram

$$\begin{array}{ccc} \pi^E(A;u) & \xrightarrow{\delta} & \pi^{E+1}(X,A;u) \\ {\scriptstyle \pi^E(f|A)}\uparrow & & \uparrow{\scriptstyle \pi^{E+1}(f)} \\ \pi^E(B;v) & \xrightarrow{\delta} & \pi^{E+1}(Y,B;v). \end{array}$$

Let $(X,A;u) \in \mathcal{LS}(E)$, consider the inclusions $i : A \to X$ and $j : X \to (X,A)$. Evidently i,j are morphisms from $\mathcal{LS}(E)$.

3.33 Proposition *If objects $(X,A;u)$, $(A;u)$ in $\mathcal{LS}(E)$ are regular* ([7]) *and of finite type, then the following sequence*

$$\pi^E(X,A;u) \xrightarrow{\pi^E(j)} \pi^E(X;u) \xrightarrow{\pi^E(i)} \pi^E(A;u) \xrightarrow{\delta} \pi^{E+1}(X,A,u)$$
$$\xrightarrow{\pi^{E+1}(j)} \pi^{E+1}(X;u) \xrightarrow{\pi^{E+1}(i)} \pi^{E+1}(A;u)$$

is exact.

Proof It follows from the exactness of the cohomotopy sequence [102]

$$\Sigma^L(X_L,A_L) \xrightarrow{j^\#} \Sigma^L(X_L) \xrightarrow{i^\#} \Sigma(A_L) \xrightarrow{\delta^L} \Sigma^{L+1}(X_L,A_L)$$
$$\xrightarrow{j^\#} \Sigma^{L+1}(X_L) \xrightarrow{i^\#} \Sigma^{L+1}(A_L)$$

and the definitions of the transformations (recall that the direct limit of a direct system of exact sequences is again exact). □

3.34 Proposition (Strong excision) *If $(X,A;u)$ is an object (resp. a regular object of finite type) in $\mathcal{LS}(E)$, B is a closed subset of X such that*

[7] This assumption is a bit too strong: if $\check{H}^q(A_L) = 0 = \check{H}^q(X_L,A_L)$ for $q \geq 2\dim L-2$ and all linear subspaces $L \supset Z$ of E of sufficiently large dimension, then the exactness also takes place.

$X = A \cup B$, then the inclusion $i : (B, A \cap B) \to (X, A)$ induces a bijective transformation (resp. an isomorphism) $\pi^E(X, A; u) \to \pi^E(B, A \cap B; u)$.

Proof It is evident that $(B, A \cap B; u)$ is an object in $\mathcal{LS}(E)$ and i is a morphism. Moreover $(B, A \cap B; u)$ is, in view of the strong excision property of the (Čech) cohomology, regular if $(X, A; u)$ is so (and of finite type whenever $(X, A; u)$ is so). By [165, Th. 7.6], for each linear subspace L (resp. $L \in \Lambda$), $i^\# : [X_L, A_L; S^L, s_0] \to [B_L, (A \cap B)_L; S^L, s_0]$ is bijective (resp. isomorphic), hence so is $i^\# : \Sigma^L(X_L, A_L) \to \Sigma^L(B_L, (A \cap B)_L)$. Therefore the assertion. □

Let us recapitulate our results.

3.35 Theorem *π^E defines a generalized cohomotopy cofunctor on the category $\mathcal{LS}(E)$*

$$\mathcal{LS}(E) \ni (X, A; u) \mapsto \pi^E(X, A; u);$$

to the category of sets and to the category of abelian groups when restricted to regular objects of finite type. Any morphism $f : (X, A; u) \to (Y, B; v)$ induces a transformation $\pi^E(f) : \pi^E(Y, B; v) \to \pi^E(X, A; u)$, being a homomorphism provided $(X, A; u)$, $(Y, B; v)$ are regular of finite type, by the formula $\pi^E(f)([g]_v) = [g \circ f]_u$ for any v-field $g : Y \to E$ on $(Y, B; v)$. Moreover:

(i) *$\pi^E(Id_{(X,A)}) = Id_{\pi^E(X,A;u)}$;*

(ii) *$\pi^E(f' \circ f) = \pi^E(f) \circ \pi^E(f')$ where $f' : (Y, B; v) \to (Z, C; s)$ is a morphism from $\mathcal{LS}(E)$;*

(iii) *there is a transformation $\delta : \pi^E(A; u) \to \pi^{E+1}(X, A; u)$, being a homomorphism if objects $(X, A; u)$ and $(A; u)$ are regular of finite type, such that $\pi^{E+1}(f) \circ \delta = \delta \circ \pi^E(f|A)$;*

(iv) *if objects $(X, A; u)$ and $(A; u)$ are regular of finite type, then the following sequence*

$$\pi^E(X, A; u) \xrightarrow{\pi^E(j)} \pi^E(X; u) \xrightarrow{\pi^E(i)} \pi^E(A; u) \xrightarrow{\delta} \pi^{E+1}(X, A, u)$$

$$\xrightarrow{\pi^{E+1}(j)} \pi^{E+1}(X; u) \xrightarrow{\pi^{E+1}(i)} \pi^{E+1}(A; u)$$

is exact (transformations different from δ are induced by the inclusions);

(v) *if morphisms $f, f' : (X, A; u) \to (Y, B; v)$ are homotopic, then*

$\pi^E(f) = \pi^E(f')$*;*

(vi) *if* B *is closed in* X*,* $A \cup B = X$*, then* $\pi^E(X, A; u) \cong \pi^E(B, A \cap B; u)$*;*

(vii) $\pi^E(pt; u) = 0$ *for any map* $u : pt \to E$ *(*pt *is a one-point space).*

Chapter 4.

TOPOLOGY OF MORPHISMS

Let X, Y be spaces and let $m \geq 0$ be an integer.

4.1 Definition Let $m \geq 0$ be an integer. A set-valued map $\varphi : X \multimap Y$ between spaces is called

(i) *m-acyclic* if, for each integer $k \geq 0$,

$$\mathrm{rd_X\{x \in X \mid \check{H}^k(\varphi(x)) \neq \check{H}^k(*)\} < m - k - 1}$$

where $*$, as usual, stands for a one-point space;

(ii) *$\mathcal{CE}$-valued map* if $\varphi(x) \in UV^\infty$ (a cell-like set) for each $x \in X$ ([1]).

By $\mathcal{A}_m$ (resp. $\mathcal{CE}$) we denote the class of all m-acyclic (resp. $\mathcal{CE}$) set-valued maps. Thus, for instance, for pairs (X, X'), (Y, Y'),

$$\mathcal{A}_m(X, X'; Y, Y') = \{\varphi : (X, X') \multimap (Y, Y') \mid \varphi \text{ is } m\text{-acyclic}\}.$$

Observe that $\mathcal{A}_0 = \mathcal{A}_1$ is the class of set-valued maps with acyclic values; $\mathcal{CE} \subset \mathcal{A}_{\backslash} \subset \mathcal{A}_{\Updownarrow}$ for $0 \leq n \leq m$.

[1] Recall that a map φ is a UV^∞-valued map (see 1.20) if, for each $x \in X$, the inclusion $\varphi(x) \hookrightarrow Y$ has UV^∞-property; hence a $\mathcal{CE}$-valued map is UV^∞-valued provided Y is an ANE.

We say that a map $\varphi : X \multimap Y$ belongs to $\widetilde{\mathcal{A}}_m$, $m \geq 0$, if $\varphi \in \mathcal{A}_m$ and

$$\sup_{x \in X} \dim \varphi(x) < \infty.$$

Evidently, if $\dim Y < \infty$, then $\widetilde{\mathcal{A}}_m(X,Y) = \mathcal{A}_m(X,Y)$.

It is clear that the composition $\varphi_2 \circ \varphi_1$ of maps $\varphi_1 \in \mathcal{A}_m(X,Y)$ and $\varphi_2 \in \mathcal{A}_m(Y,Z)$ (or $\varphi_1, \varphi_2 \in \mathcal{CE}$) is, in general, neither m-acyclic nor $\mathcal{CE}$.

As stated in the introduction set-valued maps being finite compositions of members of $\mathcal{CE}$ or $\mathcal{A}_0$ are of our main interest. Unfortunately these maps cannot be considered without a deeper insight into their structure reflected by particular factorizations. This leads us to the notion of a *morphism*.

4.A. The class of morphisms

4.2 Definition Let $p : W \to X$. We say (comp. [31, 43, 111]) that p belongs to the class $\mathcal{V}$ (p is a $\mathcal{V}$*-map*) – written $p \in \mathcal{V}(W;X)$ – if

(i) p is a perfect surjection;

(ii) $\mathrm{i(p)} < \infty$ (see Section 2.A., (2.1)).

We say that $p : W \to X$ is a $\widetilde{\mathcal{V}}$*-map* ($p \in \widetilde{\mathcal{V}}(W;X)$) if p is a $\mathcal{V}$-map and

(iii) $\dim p := \sup_{x \in X} \dim p^{-1}(x) < \infty$.

Additionally, for closed $W' \subset W$, $X' \subset X$, a map $p : (W,W') \to (X,X')$ is a $\mathcal{V}$*-map* – written $p \in \mathcal{V}(W,W';X,X')$ – if $p \in \mathcal{V}(W;X)$ and $p^{-1}(X') = W'$; for an integer $m \geq 0$, $\mathcal{V}_m(W,W';X,X') = \{p \in \mathcal{V}(W,W';X,X') \mid \mathrm{i(p)} \leq \mathrm{m}\}$ and, similarly, $\widetilde{\mathcal{V}}_m = \widetilde{\mathcal{V}} \cap \mathcal{V}_m$.

Clearly $\mathcal{V}_0 = \mathcal{V}_1$ (resp. $\widetilde{\mathcal{V}}_0 = \widetilde{\mathcal{V}}_1$) is the class of *Vietoris maps* (resp. finite-dimensional Vietoris maps), i.e. satisfying (i) with **Z**-acyclic fibres; $\mathcal{V}_n \subset \mathcal{V}_m$ (resp. $\widetilde{\mathcal{V}}_n \subset \widetilde{\mathcal{V}}_m$) for $0 \leq n \leq m$; and $\mathcal{V} = \bigcup_{m \geq 0} \mathcal{V}_m$, $\widetilde{\mathcal{V}} = \bigcup_{m \geq 0} \widetilde{\mathcal{V}}_m$.

4.3 Remark
(i) Recall that the preimage of a (paracompact) space under a perfect

map is again a space (i.e. is paracompact).

(ii) If $p \in \mathcal{V}(W,X)$, $\dim W < \infty$, then $p \in \widetilde{\mathcal{V}}(W,X)$; conversely, by a theorem on dimension-lowering maps (see [162]), if $p \in \widetilde{\mathcal{V}}(W,X)$ and $\dim X < \infty$, then $\dim W < \infty$.

(iii) Observe that $p : (W,W') \to (X,X')$ is a $\mathcal{V}_m$-map (resp. $\widetilde{\mathcal{V}}_m$-map), $m \geq 0$, if and only if $p^{-1}(X') = W'$ and a transformation $X \ni x \mapsto p^{-1}(x) \in 2^W$ is a set-valued map belonging to $\mathcal{A}_m$ (resp. $\widetilde{\mathcal{A}}_m$).

The following result is just a restatement of Corollary 2.5.

4.4 Theorem *Let $p : (W,W') \to (X,X')$ be a $\mathcal{V}_m$-map, $m \geq 0$, and G be an abelian group. If the space X is compact or the group G is finitely generated, then the induced homomorphism*

$$p^* : \check{H}^k(X,X';G) \to \check{H}^k(W,W';G)$$

is an isomorphism for each $k \geq m$ and an epimorphism for $k = m-1$.

If p is a $\mathcal{V}_I$-map, then p^* is an isomorphism for all $k \geq 0$ under no additional conditions imposed onto X or G.

Below we collect some other basic properties of $\mathcal{V}$-maps (for easy proofs – see [111, p. 15–17]).

4.5 Proposition

(i) *Let $p \in \mathcal{V}_m(W,X)$ (resp. $p \in \widetilde{\mathcal{V}}_m$), $m \geq 0$. If B is an F_σ-subset of X, then $p_B := p|p^{-1}(B) : p^{-1}(B) \to B$ is a $\mathcal{V}_m$-map (resp. $\widetilde{\mathcal{V}}_m$-map). The same holds for an arbitrary set $B \subset X$ provided $m = 0$.*

(ii) *Consider a triad*

$$(W_1,W_1') \xrightarrow{q_1} (Y,Y') \xleftarrow{p_2} (W_2,W_2')$$

where p_2 is a $\mathcal{V}_0$-map (resp. $\widetilde{\mathcal{V}}_0$-map). In the pull-back *of this triad, i.e. the cotriad*

$$(W_1,W_1') \xleftarrow{\overline{p}_2} (W,W') \xrightarrow{\overline{q}_1} (W_2,W_2')$$

where $W = W_1 \boxtimes W_2 := \{(w_1,w_2) \in W_1 \times W_2 \mid q_1(w_1) = p_2(w_2)\}$ is the fibre-product *of W_1 and W_2, $\overline{p}_2(w_1,w_2) = w_1$, $\overline{q}_1(w_1,w_2) = w_2$ and $W' = \overline{p}_2^{-1}(W_1')$, the map $\overline{p}_2$ is a $\mathcal{V}_0$-map (resp. $\widetilde{\mathcal{V}}_0$-map)* ([2]).

[2] observe that $W = W_1 \boxtimes W_2$ is closed in $W_1 \times W_2$ since p_2 is a perfect map.

(iii) *Let* $p_1 : W \to X$ *be a* $\mathcal{V}_0$*-map and* $p_2 : X \to Y$. *Then* $P := p_2 \circ p_1$ *is a* $\mathcal{V}_m$*-map,* $m \geq 0$*, if and only if* p_2 *is such a map; if* p_2 *is a* $\widetilde{\mathcal{V}}$*-map, then* p_1 *is a* $\widetilde{\mathcal{V}}$*-map if and only if* P *is a* $\widetilde{\mathcal{V}}$*-map.*

Let (X, X'), (Y, Y') be pairs of spaces and $m \geq 0$. By $D_m(X, X'; Y, Y')$ (resp. $\widetilde{D}_m(X, X'; Y, Y')$) we denote the class of all cotriads

$$(X, X') \xleftarrow{p} (W, W') \xrightarrow{q} (Y, Y')$$

where p is a $\mathcal{V}_m$-map (resp. $\widetilde{\mathcal{V}}_m$-map). Obviously $D_0 = D_1$ and $D_n \subset D_m$ (resp. $\widetilde{D}_0 = \widetilde{D}_1$ and $\widetilde{D}_n \subset \widetilde{D}_m$) and $\widetilde{D}_m \subset D_m$ for $0 \leq n \leq m$. Additionally, we put

$$D(X, X'; Y, Y') := \bigcup_{m \geq 0} D_m(X, X'; Y, Y')$$

(resp. $\widetilde{D} = \bigcup_{m \geq 0} \widetilde{D}_m$); hence $\widetilde{D} \subset D$.

4.6 Definition We say that cotriads

$$(X, X') \xleftarrow{p_i} (W_i, W_i') \xrightarrow{q_i} (Y, Y'), \qquad i = 1, 2$$

from $D(X, X'; Y, Y')$ (resp. $\widetilde{D}$) are *equivalent* – written $(p_1, q_1) \approx (p_2, q_2)$ – if there exists a cotriad

$$(X, X') \xleftarrow{p} (W, W') \xrightarrow{q} (Y, Y')$$

(resp. with finite-dimensional map p) and $\mathcal{V}_0$-maps $f_i : (W, W') \to (W_i, W_i')$ such that the following diagram is commutative

$$\begin{array}{ccccc}
 & & (W_1, W_1') & & \\
 & \swarrow_{p_1} & \uparrow f_1 & {}_{q_1}\searrow & \\
(X, X') & \xrightarrow{p} & (W, W') & \xrightarrow{q} & (Y, Y') \\
 & \nwarrow^{p_2} & \downarrow f_2 & {}_{q_2}\nearrow & \\
 & & (W_2, W_2') & &
\end{array}$$

i.e. $p_i \circ f_i = p$ and $q_i \circ f_i = q$, $i = 1, 2$.

4.7 Remark

(i) Observe that if $(p_i, q_i) \in D(X, X'; Y, Y')$ (resp. $\widetilde{D}$), $i = 1, 2$, $(p_1, q_1) \approx (p_2, q_2)$ and p_1 (or p_2) is a $\mathcal{V}_m$-map (resp. a $\widetilde{\mathcal{V}}_m$-map), $m \geq 0$, then by Proposition 4.5 (iii), p_2 (or p_1) and the existing map (resp. finite-dimensional map) p are in fact $\mathcal{V}_m$-maps (resp. $\widetilde{\mathcal{V}}_m$-maps). Moreover, if $p_i \in \widetilde{\mathcal{V}}$, then the existing $\mathcal{V}_0$-maps f_i are finite-dimensional, i.e. $f_i \in \widetilde{\mathcal{V}}_0$, $i = 1, 2$.

(ii) The relation "$\approx$" is reflexive, symmetric and, in view of Proposition 4.5 (ii), (iii), transitive.

4.8 Morphisms Elements of the quotient

$$M(X, X'; Y, Y') = D(X, X'; Y, Y')/ \approx$$

or

$$\widetilde{M}(X, X'; Y, Y') = \widetilde{D}(X, X'; Y, Y')/ \approx$$

are called *morphisms* (resp. *finite-dimensional morphisms*) and are denoted by Greek letters $\Phi, \Psi, \ldots$.

As usual, if $(p, q) \in D(X, X'; Y, Y')$ (resp. $\widetilde{D}$), then the morphism in M (resp. $\widetilde{M}$) represented by (p, q) is denoted by $[(p, q)]_{\approx}$.

By $M_m(X, X'; Y, Y')$ (resp. $\widetilde{M}_m(X, X'; Y, Y')$), $m \geq 0$, we denote the set of all morphisms from $M(X, X'; Y, Y')$ (resp. $\widetilde{M}$) which are represented by cotriads $(p, q) \in D_m(X, X'; Y, Y')$ (resp. $\widetilde{D}_m$). Elements of M_m (resp. $\widetilde{M}_m$) are called *m-morphisms* (resp. *finite-dimensional m-morphisms*).

Clearly $M_0 = M_1$, $M_n \subset M_m$ (resp. $\widetilde{M}_0 = \widetilde{M}_1$, $\widetilde{M}_n \subset \widetilde{M}_m \subset M_m$) for $0 \leq n \leq m$ and $M = \bigcup_{m \geq 0} M_m$ (resp. $\widetilde{M} = \bigcup_{m \geq 0} \widetilde{M}_m$).

Additionally, for a further convenience, if $\Phi = [(p, q)]_{\approx}$, then we let

$$\mathrm{i}(\Phi) := \mathrm{i}(\mathrm{p}).$$

This definition is obviously correct since it does not depend on the choice of a representing cotriad (p, q).

4.9 Morphisms vs. set-valued maps Any morphism Φ from the class $M_m(X, X'; Y, Y')$, $m \geq 0$, *determines* a multivalued transformation: for each $x \in X$, let $\varphi_\Phi(x) = q(p^{-1}(x))$ where $(p, q) \in \Phi$. This map is well-defined (the definition does not depend on the choice of (p, q) in Φ) and the transformation $X \ni x \mapsto \varphi_\Phi(x)$ in fact defines a set-valued map $\varphi_\Phi : (X, X') \multimap (Y, Y')$ in $\mathcal{A}^c_m$ (and in $\widetilde{\mathcal{A}}^c_m$ if $\Phi \in \widetilde{M}_m$), where $\mathcal{A}^c_m$ (resp. $\widetilde{\mathcal{A}}^c_m$) denotes the class of (finite) compositions of maps from $\mathcal{A}_m$ (resp. $\widetilde{\mathcal{A}}_m$). Observe that different morphisms may determine the same set-valued map $\varphi \in \mathcal{A}^c_m$ (resp. $\widetilde{\mathcal{A}}^c_m$).

Conversely, we say that a map $\varphi : (X, X') \multimap (Y, Y')$ is *determined by a morphism* if there exists $\Phi \in M(X, X'; Y, Y')$ such that $\varphi(x) = \varphi_\Phi(x)$ for any $x \in X$.

4.10 Example

(i) Any map $\varphi \in \mathcal{A}_m(X, Y)$ (resp. $\widetilde{\mathcal{A}}_m$), $m \geq 0$, is determined by a morphism $\Phi \in M_m(X, Y)$ (resp. $\widetilde{M}_m$). Indeed, one can represent Φ by the so-called *canonical* cotriad $X \xleftarrow{p_\varphi} W_\varphi \xrightarrow{q_\varphi} Y$ where $W_\varphi = \mathrm{Gr}(\varphi)$ is the graph of φ and p_φ, q_φ are the projections.

(ii) Any map $g : X \to Y$ is determined by a unique 0-morphism belonging to $\widetilde{M}_0(X, Y)$. Indeed, if $(p, q) \in D_0(X, Y)$ and $g(x) = q(p^{-1}(x))$ for any $x \in X$, then $(p, q) \approx (id_X, g) \in \widetilde{D}_0$. In this sense we have the inclusion

$$\mathcal{C}(X, X'; Y, Y') \subset \widetilde{M}_0(X, X'; Y, Y').$$

4.11 Composition of morphisms Let (X, X'), (Y, Y') and (Z, Z') be pairs of spaces, $\Phi_1 \in M_m(X, X'; Y, Y')$, $m \geq 0$, and $\Phi_2 \in M_0(Y, Y'; Z, Z')$. If $(p_1, q_1) \in D_m$ and $(p_2, q_2) \in D_0$ represent Φ_1, Φ_2, respectively, then we define the composition $\Phi_2 \circ \Phi_1$ of the morphisms Φ_1 and Φ_2 as the morphism represented by the cotriad $(p_1 \circ \overline{p}_2, q_2 \circ \overline{q}_1) \in D_m(X, X'; Z, Z')$ where $(\overline{p}_2, \overline{q}_1)$ is the *pull-back* (see 4.5 (ii)) of the triad

$$(p_1^{-1}(X), p_1^{-1}(X')) \xrightarrow{q_1} (Y, Y') \xleftarrow{p_2} (p_2^{-1}(Y), p_2^{-1}(Y')).$$

It is easy to verify that, in view of Proposition 4.5 (ii), (iii), the composition is well-defined, i.e. does not depend on the choice of (p_i, q_i), $i = 1, 2$, $\Phi_2 \circ \Phi_1 \in M_m(X, X'; Z, Z')$ and, for any $x \in X$, $\varphi_{\Phi_2 \circ \Phi_1}(x) = \varphi_{\Phi_2} \circ \varphi_{\Phi_1}(x)$.

Analogously we define the composition $\Phi_2 \circ \Phi_1$ of $\Phi_1 \in \widetilde{M}_m$, $m \geq 0$, and $\Phi_2 \in \widetilde{M}_0$. Clearly $\Phi_2 \circ \Phi_1 \in \widetilde{M}_m$.

4.12 Remark

(i) Our notion of morphisms originates from [86] where a different relation of equivalence between elements of $D_0(X, X'; Y, Y')$ was considered.

Namely: cotriads

$$(X, X') \xleftarrow{p_i} (W_i, W_i') \xrightarrow{q_i} (Y, Y'),$$

$i = 1, 2$, from D_0 are said to be equivalent (in the sense of [86]) if there exist maps $f : (W_1, W_1') \to (W_2, W_2')$ and $g : (W_2, W_2') \to (W_1, W_1')$ such that $p_1 = p_2 \circ f$, $q_1 = q_2 \circ f$ and $p_2 = p_1 \circ g$, $q_2 = q_1 \circ g$. In [111], the above relation was modified in order to get some additional properties: cotriads $(p_i, q_i) \in D_m(X, X'; Y, Y')$, $m \geq 0$, are said to be equivalent (in the sense of [111]) if there exists a homeomorphism $h : (W_1, W_1') \to (W_2, W_2')$ such that $p_1 = p_2 \circ h$ and $q_1 = q_2 \circ h$.

The relation introduced in 4.6 is more general than the one from [111] and, when restricted to D_0, clearly different than the one from [86]. It seems that now our relation suits the study of homotopical properties of morphisms better. We shall discuss this problem below (see Remark 4.27 (iii)).

(ii) If $B \subset X$ is an F_σ-set, then the *restriction* $\Phi|B \in M_m(B, Y)$ (resp. $\widetilde{M}_m$) for $\Phi \in M_m(X, Y)$ (resp. $\widetilde{M}_m$), $m \geq 0$, is defined. For if $\Phi = [(p, q)]_\approx$, then $\Phi|B = \Phi \circ i_B = [(p_B, q|p^{-1}(B))]_\approx$ where $i_B : B \to X$ is the inclusion – see Proposition 4.5. If $m = 0$, then it holds for an arbitrary B. More generally, if $f : X \to Y$ is a closed map such that $\dim f \leq n$ and $\Phi \in M_m(Y, Z)$ (resp. $\Phi \in \widetilde{M}_m$), $m \geq 0$, then identifying f with a morphism from $M_0(X, Y)$ represented by (id_X, f) (recall 4.10), one can consider the composition $\Phi \circ f$ defined in the same manner as in 4.11 (although the assumptions of 4.11 are not fulfilled) and then $\Phi \circ f \in M_{m+n}(X, Z)$ (resp. $\Phi \circ f \in \widetilde{M}_{m+n}$) – see [111, Prop. (2.7), (3.7)] for details. If Φ is a 0-morphism, then so is $\Phi \circ f$ for an arbitrary f.

(iii) By 4.11, we have defined a category $\mathbf{M}_0$ (resp. $\widetilde{\mathbf{M}}_0$) with pairs of spaces as objects and $M_0(X, X'; Y, Y')$ (resp. $\widetilde{M}_0(X, X'; Y, Y')$) as the set of arrows from (X, X') to (Y, Y') (comp. [86]). Evidently, the category of topological spaces is a subcategory of both $\mathbf{M}_0$ and $\widetilde{\mathbf{M}}_0$.

(iv) The category $\mathbf{M}_0$ reminds the category of right fractions – see [75]. In [44] Calvert studies categories of fractions $\mathbf{T}(\Sigma(n)^{-1})$ and $\mathbf{T}(\Sigma^{-1})$, where $\Sigma(n)$, $n > 0$, is the class of perfect surjections $p : X \to Y$ such that $\check{H}^k(p^{-1}(y)) = \check{H}^k(*)$ for $y \in Y$, $0 \leq k < n$, and $\Sigma = \bigcap_{n \geq 1} \Sigma(n)$. It is clear (see [75]) that arrows in these categories can be regarded as diagrams $X \xleftarrow{p} W \xrightarrow{q} Y$ where $p \in \Sigma(n)$ (resp. $p \in \Sigma$). Since $\Sigma = \mathcal{V}_0$, the category $\mathbf{T}(\Sigma^{-1})$ may be identified (up to the relation $\approx$) with our category $\mathbf{M}_0$.

4.13 (i) As a straightforward consequence of Theorem 4.4 we have the following, already extensively used, observation (comp. [64, 96, 84, 43, 111]):

If $\Phi \in M_m(X, X'; Y, Y')$ *(resp.* $\Phi \in \widetilde{M}_m(X, X'; Y, Y')$*),* $m \geq 0$*, then for any* $k \geq m$*, one can define the* induced homomorphism

$$\Phi^* = \check{H}^k(\Phi) : \check{H}^k(Y, Y'; G) \to \check{H}^k(X, X'; G)$$

where G *is an abelian group being finitely generated whenever* X *is not compact, by putting*

$$\Phi^* := (p^*)^{-1} \circ q^*$$

where $(X, X') \xleftarrow{p} (W, W') \xrightarrow{q} (Y, Y') \in D_m$ *represents* Φ*.*

It is clear that this definition does not depend on the choice of (p, q) in Φ and if $m = 0$, then no restrictions concerning G or X are needed.

(ii) If $\Phi \in M(X, X'; Y, Y')$ and $\Phi' \in M(X', Y')$ is induced by Φ, then $\check{H}^{k+1}(\Phi) \circ \delta_X = \delta_Y \circ \check{H}^k(\Phi')$ where $k \geq \mathrm{i}(\Phi)$ and δ_X , δ_Y are the connecting homomorphisms of the pairs (X, X') and (Y, Y'), respectively.

(iii) Let $\varphi \in \mathcal{A}_m(X, Y)$ (resp. $\varphi \in \widetilde{\mathcal{A}}_m(X, Y)$), $m \geq 0$, be determined by a morphism $\Phi = [X \xleftarrow{p} W \xrightarrow{q} Y]_{\approx} \in M_m(X, Y)$ (resp. $\widetilde{M}_m$), then $p = p_\varphi \circ f$ and $q = q_\varphi \circ f$ where $f : W \to W_\varphi$, $f(w) = (p(w), q(w)), w \in W$ (see Example 4.10 (i)). Therefore, we may define $\varphi^* = \check{H}^k(\varphi) = \check{H}^k(\Phi)$, $k \geq m$, where $\Phi \in M_m(X, Y)$ is an arbitrary morphism determining φ.

4.14 Example In the case of set-valued maps determined by morphisms one cannot define the induced homomorphism regardless the underlying morphism. For, it is easy to see that if $W = \{(z, u) \in S^1 \times S^1 \mid |z - u| \leq \sqrt{2}\}$, $p : W \to S^1$, $q_i : W \to S^1$ are given by $p(z, u) = z$ and $q_i(z, u) = u^{i+1}$, $i = 1, 2$, then $\Phi_i = [(p, q_i)]_{\approx}$, $i = 1, 2$, determine the same map, but $\check{H}^1(\Phi_1) \neq \check{H}^1(\Phi_2)$.

4.15 Cartesian product of morphisms Let X, Y_1, Y_2 be spaces, let $\Phi_1 \in M_m(X, Y_1)$, $m \geq 0$, and $\Phi_2 \in M_0(X, Y_2)$. There exists a morphism $(\Phi_1, \Phi_2) \in M_m(X, Y_1 \times Y_2)$ such that $(\Phi_1, \Phi_2)(x) = \Phi_1(x) \times \Phi_2(x)$ for any $x \in X$. Indeed, let $X \xleftarrow{p_j} W_j \xrightarrow{q_j} Y_j$ represent Φ_j, $j = 1, 2$. Consider a diagram

$$\begin{array}{cccccc} & & & W_1\boxtimes W_2 \xrightarrow{q} \; Y_1\times Y_2 & & \\ & & \swarrow^{\overline{p}_2} & & \searrow^{\overline{p}_1} & \\ X \xleftarrow{p_1} & W_1 & & & & W_2 \\ & & \nwarrow_{p_1} & & \nearrow_{p_2} & \\ & & & X & & \end{array}$$

with the cotriad $W_1 \xleftarrow{\overline{p}_2} W_1\boxtimes W_2 \xrightarrow{\overline{p}_1} W_2$ being the pull-back of the triad $W_1 \xrightarrow{p_1} X \xleftarrow{p_2} W_1$ and $q(w_1, w_2) = (q_1(w_1), q_2(w_2))$. The morphism (Φ_1, Φ_2) is represented by the pair (p, q) where $p = p_1 \circ \overline{p}_2$.

In particular, if $f : X \to Y_2$ is a map, then we may consider the morphism $(\Phi, f) \in M_m(X, Y_1 \times Y_2)$ for any $\Phi \in M_m(X, Y_1)$, $m \geq 0$. With obvious changes one also defines the morphism $(f, \Phi) \in M_m(X, Y_2 \times Y_1)$.

If $\Phi \in M_m(X, E)$, $m \geq 0$, $f : X \to E$, where E is a normed space, then we may consider a morphism $f + \Phi \in M_m(X, E)$ such that, for each $x \in X$, $(f + \Phi)(x) = f(x) + \Phi(x) = \{f(x) + y \mid y \in \Phi(x)\}$. For this purpose it is enough to define $f + \Phi = + \circ (\Phi, f)$ where $+ : E \times E \to E$ is given by $+(y', y'') = y'' + y'$. Clearly this definition is correct in view of the above definition and 4.11.

In particular, one considers *fields* of the form $i - \Phi$ where $\Phi \in M_m(X, E)$, $m \geq 0$, $X \subset E$ and $i : X \to E$ is the inclusion.

In a similar way, given morphisms $\Phi_1 \in M_0(X_1, Y_1)$, $\Phi_2 \in M_m(X_2, Y_2)$, $m \geq 0$, we may define the morphism $\Phi_1 \times \Phi_2 \in M_{n+m}(X_1 \times X_2, Y_1 \times Y_2)$, where $n \geq \dim X_1$, such that $\Phi_1 \times \Phi_2(x_1, x_2) = \Phi_1(x_1) \times \Phi_2(x_2)$.

4.16 Remark It is easy to see that (Φ, f) (resp. (f, Φ)) admits a simpler description. Namely, if Φ is represented by the pair $X \xleftarrow{p} W \xrightarrow{q} Y_1$, then (Φ, f) (resp. (f, Φ)) is represented by $X \xleftarrow{p} W \xrightarrow{\overline{q}} Y_1 \times Y_2$ where $\overline{q}(w) = (q(w), f \circ p(w))$ (resp. $\widetilde{q}(w) = (f \circ p(w), q(w))$) for $w \in W$. This cotriad is obviously equivalent to the one defined above.

4.B. Homotopy properties of morphisms

4.17 Definition Let $\Phi_j \in M(X, X'; Y, Y')$ (resp. $\Phi_j \in \widetilde{M}$), $j = 0, 1$, and let $n \geq 0$. We say that the morphisms Φ_0, Φ_1 are *n-homotopic in M* (resp. in $\widetilde{M}$) – written $\Phi_0 \simeq_n \Phi_1$ – if there exists a *homotopy* $\Phi : \Phi_0 \simeq_n \Phi_1$ i.e. a morphism $\Phi \in M_n(X \times I, X' \times I; Y, Y')$ (resp. $\Phi \in \widetilde{M}_n$) such that $\Phi \circ i_j = \Phi_j$ (where $i_j : X \to X \times I$, $i_j(x) = (x, j)$, $x \in X$), $j = 0, 1$.

4.18 Remark Let $\Phi : \Phi_0 \simeq_n \Phi_1$ in M (resp. in $\widetilde{M}$), $n \geq 0$.

(i) If $m \geq n$, then $\Phi_0 \simeq_m \Phi_1$. If $n = 1$, then $\Phi_0 \simeq_0 \Phi_1$, i.e. relations "$\simeq_0$" and "$\simeq_1$" are identical.

(ii) By Remark 4.12 (ii), $\Phi_j \in M_n$ (resp. $\Phi_j \in \widetilde{M}_n$) for $j = 0, 1$.

(iii) Assume that $(X, X') \xleftarrow{p_j} (W_j, W_j') \xrightarrow{q_j} (Y, Y') \in D$ (resp. $\widetilde{D}$) represents Φ_j, $j = 0, 1$, and $(X \times I, X' \times I) \xleftarrow{p} (W, W') \xrightarrow{q} (Y, Y') \in D_n$ (resp. $\widetilde{D}_n$) represents Φ. There are cotriads $(X, X') \xleftarrow{P_j} (\overline{W}_j, \overline{W}_j') \xrightarrow{Q_j} (Y, Y') \in D$ and $\mathcal{V}_0$-maps $k_j : (\overline{W}_j, \overline{W}_j') \to (W_j, W_j')$, $g_j : (\overline{W}_j, \overline{W}_j') \to (p^{-1}(X \times \{j\}), p^{-1}(X' \times \{j\}))$ such that $p_j \circ k_j = P_j$, $q_j \circ k_j = Q_j$ and $p \circ g_j = i_j \circ P_j$, $q \circ g_j = Q_j$, $j = 0, 1$.

4.19 Proposition *The relations*

$$\begin{aligned} \simeq_n |M_m(X, X'; Y, Y') &= \{(\Phi_0, \Phi_1) \mid \Phi_j \in M_m(X, X'; Y, Y'), \\ &\qquad j = 0, 1;\ \Phi_0 \simeq_n \Phi_1 \text{ in } M\} \\ \simeq_n |\widetilde{M}_m(X, X'; Y, Y') &= \{(\Phi_0, \Phi_1) \mid \Phi_j \in \widetilde{M}_m(X, X'; Y, Y'), \\ &\qquad j = 0, 1;\ \Phi_0 \simeq_n \Phi_1 \text{ in } \widetilde{M}\} \end{aligned}$$

are equivalence relations provided $n \geq m + 1$.

Proof It is easy to see that the relation $\simeq_n |M_m$ (resp. $\simeq_n |\widetilde{M}_m$) is symmetric and, by Lemma 4.20 below, transitive already for $n \geq m$. However, for a morphism $\Phi_0 \in M_m(X, Y)$ (resp. $\widetilde{M}_m$), $m > 1$, we can build its self-homotopy only in $M_{m+1}(X \times I, Y)$ (resp. $\widetilde{M}_{m+1}$). Namely if $X \xleftarrow{p} W \xrightarrow{q} Y \in D_m$ (resp. $\widetilde{D}_m$) represents Φ_0, then representing a morphism Φ by $X \times I \xleftarrow{P} W \times I \xrightarrow{Q} Y$, where $P(w, t) = (p(w), t)$, $Q(w, t) = q(w)$ for $w \in W$, $t \in I$, we see that P is a $\mathcal{V}_{m+1}$-map (resp. $\widetilde{\mathcal{V}}_{m+1}$-map) and $\Phi : \Phi_0 \simeq_{m+1} \Phi_0$. □

4.20 Lemma *Assume that* $X_i = X \times [\frac{1}{2}i, \frac{1}{2}(i + 1)]$, $i = 0, 1$, *and let* $j_i : X_0 \cap X_1 \to X_i$, $r_i : X_i \to X \times I$ *be the inclusions. If, for* $i = 0, 1$, $\Phi_i \in M_m(X_i, Y)$ *(resp.* $\Phi_i \in \widetilde{M}_m$*),* $m \geq 0$, *and* $\Phi_0 \circ j_0 = \Phi_1 \circ j_1$, *then there is* $\Phi \in M_m(X \times I, Y)$ *(resp.* $\Phi \in \widetilde{M}_m$*) such that* $\Phi \circ r_i = \Phi_i$, $i = 0, 1$.

Proof Let $X_i \xleftarrow{p_i} W_i \xrightarrow{q_i} Y \in D_m$ represent Φ_i, $i = 0, 1$. There are a cotriad $X_0 \cap X_1 \xleftarrow{p} W \xrightarrow{q} Y \in D_m$ and $\mathcal{V}_0$-maps $f_i : W \to p_i^{-1}(X_0 \cap X_1) \subset W_i$ such that $p = p_i \circ f_i$, $q = q_i \circ f_i$, $i = 0, 1$.

Let $W_I = W \times I$ and define $g : W \times \{0, 1\} \to W_0 \oplus W_1$ by $g(w, i) = f_i(w)$, $w \in W$, $i = 0, 1$; next put $\overline{W} = W_I \cup_g W_0 \oplus W_1$. Moreover, let $h_i : W_i \to \overline{W}$, $i = 0, 1$, and $h : W_I \to \overline{W}$ be the respective quotient maps.

We shall define a cotriad $X \times I \xleftarrow{\overline{p}} \overline{W} \xrightarrow{\overline{q}} Y \in D_m$ and $\mathcal{V}_0$-maps $\overline{f}_i : \overline{W}_i \to W_i$, where $\overline{W}_i = \overline{p}^{-1}(X_i)$, such that $p_i \circ \overline{f}_i = \overline{p}|\overline{W}_i$ and $q_i \circ \overline{f}_i = \overline{q}|\overline{W}_i$ for $i = 0, 1$.

To this end consider a map $\overline{p} : \overline{W} \to X \times I$ given by the formulae

$$\begin{aligned} \overline{p}(h_i(w_i)) &= p_i(w_i) && \text{for } w_i \in W_i,\ i = 0, 1, \\ \overline{p}(h(w, t)) &= p(w) && \text{for } w \in W,\ t \in I. \end{aligned}$$

One can easily see that $\overline{p}$ is a well-defined continuous surjection. For any closed subset $C \subset \overline{W}$, $\overline{p}(C) = p_0(h_0^{-1}(C)) \cup p \circ \pi(h^{-1}(C)) \cup p_1(h_1^{-1}(C))$ where $\pi : W_I \to W$ is the projection. For any $x \in X$,

$$\overline{p}^{-1}(x) = \begin{cases} h_i(p_i^{-1}(x)) & \text{if } x \in X_i \setminus (X_0 \cap X_1),\ i = 0, 1; \\ h(p^{-1}(x) \times I) & \text{if } x \in X_0 \cap X_1. \end{cases}$$

Hence $\overline{p}$ is a perfect surjection.

Observe that, for $x \in X_0 \cap X_1$, $h : p^{-1}(x) \times I \to \overline{p}^{-1}(x)$ is a perfect surjection and, for each $\overline{w} \in \overline{p}^{-1}(x)$ (i.e. $\overline{w} = h(w, t)$ where $w \in p^{-1}(x)$, $t \in I$),

$$h^{-1}(\overline{w}) = \begin{cases} (w, t) & \text{if } t \in (0, 1) \\ f_i^{-1}(f_i(w)) \times \{i\} & \text{if } t = i = 0, 1. \end{cases}$$

hence $h : p^{-1}(x) \times I \to \overline{p}^{-1}(x)$ is a $\mathcal{V}_0$-map.

Altogether, for any integer $k \geq 0$,

$$\begin{aligned} \mathrm{s^k(\overline{p}) \cap X_0 \cap X_1} &= \mathrm{s^k(p)}, \\ \mathrm{s^k(\overline{p}) \cap [X_i \setminus (X_0 \cap X_1)]} &\subset \mathrm{s^k(p_i)} \end{aligned}$$

for $i = 0, 1$, and thus, the simple properties of dim imply that $\mathrm{rd_{X \times I}(s^k(\overline{p})) < m - k - 1}$, i.e. $\overline{p}$ is a $\mathcal{V}_m$-map.

Define $\overline{q} : \overline{W} \to Y$ similarly as $\overline{p}$ above.

Since $\overline{W}_i = \overline{p}^{-1}(X_i) = h_i(W_i) \cup h(W_I)$, we define $\overline{f}_i : W_i \to W_i$ by the formulae

$$\begin{aligned} \overline{f}_i(h_i(w_i)) &= w_i && \text{for } w_i \in W_i, \\ \overline{f}_i(h(w, t)) &= f_i(w) && \text{for } w \in W,\ t \in I;\ i = 0, 1. \end{aligned}$$

Analogously as before we check that $\overline{f}_i$ is a $\mathcal{V}_0$-map and $p_i \circ \overline{f}_i = \overline{p}|\overline{W}_i$, $q_i \circ \overline{f}_i = \overline{q}|\overline{W}_i$, $i = 0, 1$.

At last it is enough to represent $\Phi \in M_m(X \times I, Y)$ by $(\overline{p}, \overline{q})$.

It is clear that $\Phi \in \widetilde{M}_m(X \times I, Y)$ provided $\Phi_i \in \widetilde{M}_m(X_i, Y)$, $i = 0, 1$.□

4.21 Remark
(i) If $\Phi_i \in M(X, X'; Y, Y')$ (or $\Phi_i \in \widetilde{M}$), $i = 0, 1$, and $\Phi_0 \simeq_n \Phi_1$ in M (or in $\widetilde{M}$), $n \geq 0$, then for $k \geq n$, $\check{H}^k(\Phi_i)$ are defined and $\check{H}^k(\Phi_0) = \check{H}^k(\Phi_1)$.
(ii) The cofunctor $\check{H}^*$ of (integral) Čech cohomology may be extended from the topological category $\mathbf{T}$ to the category $\mathbf{M}_0$ (or $\widetilde{\mathbf{M}}_0$) (comp. [86]).

4.22 Definition By $M_m[X, X'; Y, Y']_n$ (resp. $\widetilde{M}_m[X, X'; Y, Y']_n$), $m \geq 0$, $n \geq m + 1$, we denote the set of all n-homotopy classes $[\Phi]_n = \{\Psi \in M_m(X, X'; Y, Y') \mid \Phi \simeq_n \Psi \,\text{in}\, M\}$ of $\Phi \in M_m(X, X'; Y, Y')$ (resp. $[\Phi]_n = \{\Psi \in \widetilde{M}_m \mid \Phi \simeq_n \Psi \,\text{in}\, \widetilde{M}\}$ of $\Phi \in \widetilde{M}_m$).

In some of the following paragraphs we shall try to answer the question under what conditions a given morphism $\Phi \in M(X, X'; Y, Y')$ (resp. $\Phi \in \widetilde{M}$) is homotopic to a map $f : (X, X') \to (Y, Y')$.
Let us first start with the following simple observation.

4.23 Proposition *If $(X, X') \xleftarrow{p} (W, W') \xrightarrow{q} (Y, Y') \in D_m$ (resp. $\widetilde{D}_m$), $m \geq 0$, represents a morphism $\Phi \in M_m$ (resp. $\widetilde{M}_m$) and there exists a map $f : (X, X') \to (Y, Y')$ such that a diagram*

$$\begin{array}{ccc} (W, W') & \xrightarrow{q} & (Y, Y') \\ \downarrow{\scriptstyle p} & \nearrow{\scriptstyle f} & \\ (X, X') & & \end{array}$$

is homotopically commutative (i.e. there is $h : (W, W') \times I \to (Y, Y')$ such that $h(\,\cdot\,, 0) = f \circ p$, $h(\,\cdot\,, 1) = q$), then $\Phi \simeq_{m+1} f$ in M (resp. $\widetilde{M}$).

Proof Let (Z, Z') be the cylinder of p (i.e. Z is the cylinder of $p : W \to X$ and Z' is the cylinder $p' = p|W' : W' \to X'$).

Define $P : (Z, Z') \to (X \times I, X' \times I)$ and $F : (Z, Z') \to (Y, Y')$ by $P[w, t] = (p(w), t)$, $F[w, t] = f \circ p(w)$ for $w \in W$, $t \in I$ and $P[x] = (x, 1)$, $F[x] = f(x)$ for $x \in X$. Evidently $F[\,\cdot\,, 0] = h(\,\cdot\,, 0)$. Since the inclusion $i : (W, W') \to (Z, Z')$ is a cofibration, there is $H : (Z, Z') \times I \to (Y, Y')$ such that $H(\,\cdot\,, 0) = F$ and $H|W \times I = h$. Let, for $[w, t] \in Z$, $Q[w, t] = H([w, t], 1 - t)$ and $Q([x]) = H([x], 0)$. Then $Q : (Z, Z') \to (Y, Y')$ and the cotriad (P, Q) represents a homotopy $\Psi \in M_{m+1}(X \times I, X' \times I; Y, Y')$ (resp. $\widetilde{M}_{m+1}$) such that $\Psi : \Phi \simeq_{m+1} f$. □

The above result and its context were the main motivation for the (co)homotopy version of the Vietoris theorem presented in a general framework in Sections 2.C and 3.A. We shall now apply it to get one of the main theorems of the present dissertation.

In analogy with 2.15 let us now pose the following

4.24 Assumption

(i) (X, X') *is a paracompact pair and* $\dim X < \infty$*;*

(ii) (Y, Y') *is a paracompact pair having the homotopy type of an* ANR-*pair;*

(iii) Y *is* $(k-1)$*-connected,* $k \geq 1$*;*

(iv) (Y, Y') *is* k*-connected;*

(v) Y *(resp.* (Y, Y')*) is homotopically simple, that is* i*-simple for any* $i \geq 1$ *(resp.* $i \geq 2$*);*

(vi) *if* X *(resp.* X'*) is not compact, we assume that* $\pi_i(Y)$ *(resp.* $\pi_i(Y')$*) is finitely generated for any* $i \geq 1$*,*

Recall that under these assumption a $\widetilde{\mathcal{V}}_m$-map $p : (W, W') \to (X, X')$ induces a set-transformation $p^{\#} : [X, X'; Y, Y'] \to [W, W'; Y, Y']$ being a surjection if $k \geq m - 1$ and a bijection if $k \geq m$ (see 2.17 or 3.3). See also Remark 2.24 (ii).

4.25 Theorem (Identification Theorem I) *Let Assumptions* 4.24 *be satisfied. If* $\Phi \in \widetilde{M}_m(X, X'; Y, Y')$ *(resp.* $\Phi \in \widetilde{M}_m(X, Y)$*),* $m \geq 0$*, and* $m - 1 \leq k$*, then there exists a map* $f : (X, X') \to (Y, Y')$ *(resp.* $f : X \to Y$*) such that* $\Phi \simeq_{m+1} f$*. If* $k \geq m$*, then a map* f *is determined uniquely up to homotopy.*

Therefore, there are 1–1 *correspondences*

$$\widetilde{M}_m[X, X'; Y, Y']_n \leftrightarrow [X, X'; Y, Y];$$

$$\widetilde{M}_m[X,Y]_n \leftrightarrow [X,Y];$$
$$\widetilde{M}_m[X,X';Y,y]_n \leftrightarrow [X,X';Y,y],$$

where y is an arbitrary point of Y, provided $m \geq 0$ and $m+1 \leq n \leq k+1$.

In particular, under Assumption 4.24 (i) *and the same restrictions concerning m, n and k, there is a* 1–1 *correpondence*

$$\widetilde{M}_m[X,X';S^k,s_0]_n \leftrightarrow \pi^k(X,X').$$

In the above statements if $m = 0$, then Assumption 4.24 (vi) *is not necessary.*

First, we shall need the following lemma.

4.26 Lemma *Let $p_j : W_j \to X$, $j = 0,1$, be a $\mathcal{V}_m$-map (resp. $\widetilde{\mathcal{V}}_m$-map), $m \geq 0$. There is a $\mathcal{V}_{m+1}$-map (resp. $\widetilde{\mathcal{V}}_{m+1}$-map) $p : W \to X \times I$ and homeomorphisms $h_j : p^{-1}(X \times \{j\}) \to W_j$ such that $i_j \circ p_j \circ h_j = p|p^{-1}(X \times \{j\})$, $j = 0,1$.*

Proof Let $W = W_0 \times [0, \frac{1}{2}] \oplus W_1 \times [\frac{1}{2}, 1] \cup_g X$ where $g : W \times \{\frac{1}{2}\} \oplus W_1 \times \{\frac{1}{2}\} \to X$ is given by $g(w_j, \frac{1}{2}) = p_j(w_j)$ for $w_j \in W_j$, $j = 0,1$.

Let $t_0 : W_0 \times [0, \frac{1}{2}] \oplus W_1 \times [\frac{1}{2}, 1] \to W$ and $t_1 : X \to W$ be the quotient maps. We define $p : W \to X \times I$ by the formulae

$$\begin{aligned} p(t_0(w,\lambda)) &= (p_j(w), \lambda), & w \in W_j,\ j \leq 2\lambda \leq j+1,\ j = 0,1; \\ p(t_1(x)) &= \left(x, \frac{1}{2}\right), & x \in X. \end{aligned}$$

It is easy to see that p is a well-defined perfect surjection.

For $(x,\lambda) \in X \times I$,

$$p^{-1}(x,\lambda) = \begin{cases} t_1(x) & \text{if } \lambda = \frac{1}{2}; \\ t_0(p_j^{-1}(x) \times \{\lambda\}) & \text{if } j \leq 2\lambda \leq j+1,\ \lambda \neq \frac{1}{2},\ j = 0,1. \end{cases}$$

Hence, for any integer $r \geq 0$,

$$\mathrm{s^r(p) = s^r(p_0)} \times \left[0, \frac{1}{2}\right) \cup \mathrm{s^r(p_1)} \times \left(\frac{1}{2}, 1\right];$$

one easily checks that p is a $\mathcal{V}_{m+1}$-map (resp. $\widetilde{\mathcal{V}}_{m+1}$-map).
Putting $h_j(t_0(w_j, j)) = w_j$ for $j = 0,1$ and $w_j \in W_j$ we complete the proof.
□

Proof of 4.25 Let $\Phi \in \widetilde{M}_m(X,X';Y,Y')$ be represented by $(X,X') \xleftarrow{p} (W,W') \xrightarrow{q} (Y,Y') \in \widetilde{D}_m$. If $k \geq m-1$, then by Theorem 2.17, there exists a a homotopy class $[f] \in [X,X';Y,Y']$ such that $p^{\#}([f]) = [q]$. If $k \geq m$, then $[f]$ is determined uniquely. In view of Proposition 4.23, $\Phi \simeq_{m+1} f$.

Assume that $m+1 \leq n \leq k+1$ and let us consider the correspondence

$$A : [\Phi]_n \mapsto [f] = (p^{\#})^{-1}[q]. \tag{4.1}$$

To show that the above definition is correct, take $\Phi_j \in \widetilde{M}_m(X,X';Y,Y')$, $j = 0,1$, such that $\Phi : \Phi_0 \simeq_n \Phi_1$ and assume that $(X,X') \xleftarrow{p_j} (W_j,W_j') \xrightarrow{q_j} (Y,Y') \in \widetilde{D}_m$ represents Φ_j, $j = 0,1$, and $(X \times I, X' \times I) \xleftarrow{p} (W,W') \xrightarrow{q} (Y,Y') \in \widetilde{D}_n$ represents Φ. There are cotriads $(X,X') \xleftarrow{P_j} (\overline{W}_j,\overline{W}_j') \xrightarrow{Q_j} (Y,Y') \in \widetilde{D}$ and $\mathcal{V}_0$-maps $k_j : (\overline{W}_j,\overline{W}_j') \to (W_j,W_j')$, $g_j : (\overline{W}_j,\overline{W}_j') \to (p^{-1}(X \times \{j\}), p^{-1}(X' \times \{j\}))$ such that $p_j \circ k_j = P_j$, $q_j \circ k_j = Q_j$ and $p \circ g_j = i_j \circ P_j$, $q \circ g_j = Q_j$, $j = 0,1$.

Since $n \leq k+1$, by Theorem 2.17, there is a map $F : (X,X') \times I \to (Y,Y')$ such that $F \circ p \simeq q$. Assume that $f_j : (X,X') \to (Y,Y')$ is a unique (up to homotopy) map such that $f_j \circ p_j \simeq q_j$, $j = 0,1$. Hence $f_j \circ P_j \simeq Q_j$ and, letting $F_j = F(\,\cdot\,,j)$, $F_j \circ P_j \simeq Q_j$. Since P_j is clearly a $\mathcal{V}_m$-map (see Proposition 4.5 (iii)), $f_j \simeq F_j$, $j = 0,1$. Thus $f_0 \simeq f_1$ ([3]).

Clearly A is surjective: for any $[f] \in [X,X';Y,Y']$, we have

$$A([[(id_X,f)]_{\approx}]_n) = [f].$$

To show the injectivity of A, let $\Phi_j \in \widetilde{M}_m(X,X';Y,Y')$ be represented by a cotriad $(X,X') \xleftarrow{p_j} (W_j,W_j') \xrightarrow{q_j} (Y,Y') \in \widetilde{D}_m$ and suppose that $A([\Phi_j]_n) = [f_j]$, $j = 0,1$, where $f : f_0 \simeq f_1$. In view of Lemma 4.26, there is a $\mathcal{V}_{m+1}$-map $p : (W,W') \to (X \times I, X' \times I)$ and homeomorphisms $h_j : (p^{-1}(X \times \{j\}), p^{-1}(X' \times \{j\})) \to (W_j,W_j')$ such that $i_j \circ p_j \circ h_j = p|p^{-1}(X \times \{j\})$, $j = 0,1$. It is easy to see that $f \circ p \circ h_j^{-1} = f_j \circ p_j \simeq q_j$; define $\overline{q} : (p^{-1}(X \times \{0,1\}), p^{-1}(X' \times \{0,1\})) \to (Y,Y')$ by the formula $\overline{q}(w) = q_j \circ h_j(w)$ for $w \in p^{-1}(X \times \{j\})$, $j = 0,1$. Since $f \circ (p|p^{-1}(X \times \{0,1\})) \simeq \overline{q}$, then by HEP ([4]), $\overline{q}$ has an extension $q : (W,W') \to (Y,Y')$.

A morphism $\Phi \in \widetilde{M}_{m+1}(X \times I, X' \times I; Y,Y')$ represented by (p,q) furnishes an n-homotopy $(n \geq m+1)$ $\Phi : \Phi_0 \simeq_n \Phi_1$. □

[3] In particular it follows that if $\Phi \in \widetilde{M}_m(X,X';Y'Y')$, $m \geq 0$, and $g \simeq_{m+1} \Phi$, where $g : (X,X') \to (Y,Y')$, then $g \simeq f$ where $[f] = A([\Phi]_n)$ and $m+1 \leq n \leq k+1$.

[4] clearly, without loss of generality we may assume here that (Y,Y') is a complete ANR-pair.

4.27 Remark Assume that (X, X'), (Y, Y') are as above.

(i) Theorem 4.25 implies, quite unexpectedly, that if $\Phi_j \in \widetilde{M}_m(X, X'; Y, Y')$, $m \geq 0$, $j = 0, 1$, and $\Phi_0 \simeq_n \Phi_1$, $m+1 \leq n \leq k+1$, then $\Phi_0 \simeq_{m+1} \Phi_1$. Moreover, $\check{H}^s(\Phi_0) = \check{H}^s(\Phi_1)$ for $s \geq m$ (comp. Remark 4.21 (i)). For, then $A[\Phi_0]_n = [f] = A[\Phi_1]_n$ and, evidently, $\check{H}^s(\Phi_j) = \check{H}^s(f)$ for $s \geq m$ ($j = 0, 1$).

(ii) Suppose that $(p_j, q_j) \in \widetilde{D}_0(X, X'; Y, Y')$ (resp. $(p_j, q_j) \in \widetilde{D}_m$, $m \geq 0$), $j = 1, 2$, and that (p_1, q_1) and (p_2, q_2) are equivalent in the sense of [86] (resp. in the sense of [111]). Then, carefully studying the proof of Theorem 4.25, we see that $[(p_1, q_1)]_\approx \simeq_0 [(p_2, q_2)]_\approx$ (resp. $[(p_1, q_1)]_\approx \simeq_{m+1} [(p_2, q_2)]_\approx$). In other words, the different relations of equivalence of pairs considered in [86] and [111] lead to the same homotopy theory of morphisms (see Remark 4.12).

4.28 If (Z, Z') is a paracompact pair such that $\dim Z < \infty$ and $\check{H}^s(X, X') = 0 = \check{H}^s(Z, Z')$ for $s \geq 2n - 1$, $n \geq 1$, $\Phi \in \widetilde{M}_m(X, X'; Z, Z')$, then one can define a homomorphism

$$\Phi^\# : \pi^k(Z, Z') \to \pi^k(X, X')$$

for any $k \geq \max\{n, m\}$. Indeed, in view of Corollary 3.7, for any $f : (Z, Z') \to (S^k, s_0)$, we may put

$$\Phi^\#([f]) = (p^\#)^{-1}([f \circ q])$$

where $(p, q) \in \widetilde{D}_m$ represents Φ. To see that $\Phi^\#$ is well-defined it is enough to note that $\Phi^\#([f]) = A([f \circ \Phi]_{m+1}$ (see Theorem 4.25 and (4.1)).

4.C. Classification and extension of morphisms

Having the equivalence Theorem 4.25 we are prepared for one of the next main results of this chapter.

4.29 Theorem (Classification Theorem) *Let Y be a $(k-1)$-connected CW-complex, $k \geq 1$ (and if $k = 1$, suppose that Y is homotopically simple), Y' be a closed contractible subset of Y, $y_0 \in Y'$ and let $i : (Y, y_0) \to (Y, Y')$ be the inclusion. Assume that $u = (i^*)^{-1}(\kappa)$ where $\kappa \in \check{H}^k(Y, y_0; \pi_k(Y))$ is a k-characteristic element of Y. If (X, X') is a pair,* $\dim X < \infty$*, X*

is compact or the group $\pi_i(Y)$ is finitely generated for any $i \geq 1$, then the transformation

$$B : \widetilde{M}_m[X, X'; Y, Y']_n \to \check{H}^k(X, X'; \pi_k(Y)),$$

where $m \geq 0$, $m+1 \leq n \leq k+1$, given by the formula

$$B([\Phi]_n) = \check{H}^k(\Phi)(u), \qquad \Phi \in \widetilde{M}_m(X, X'; Y, Y'),$$

is:

(i) *bijective if $\check{H}^s(X, X'; \pi_s(Y)) = 0 = \check{H}^{s+1}(X, X'; \pi_s(Y))$ for any $s > k$;*

(ii) *surjective if $\check{H}^{s+1}(X, X'; \pi_s(Y)) = 0$ for $s > k$.*

Proof Evidently B is well-defined (see Remark 4.21 (i)). In view of Theorem 4.25, we have a bijective correspondence $A : \widetilde{M}_m[X, X'; Y, Y']_n \to [X, X'; Y, Y']$; next $i_\# : [X, X'; Y, y_0] \to [X, X'; Y, Y']$ is clearly bijective. The Hopf classification map $C : [X, X'; Y, y_0] \to \check{H}^k(X, X'; \pi_k(Y))$ given by $C([f]) = f^*(\kappa)$ for $f : (X, X') \to (Y, y_0)$, is bijective (resp. surjective) if (i) (resp. (ii)) is satisfied (see [102, 166, 143] and [149]). Now observe that we may put $B = C \circ i_\#^{-1} \circ A$. □

In particular, there is the following classification.

4.30 Corollary *If $\dim X < \infty$, $\check{H}^s(X) = 0$ for $s \geq k+1$, $k \geq 1$, then a correspondence*

$$B : \widetilde{M}_m[X, S^k]_{m+1} \to \check{H}^k(X),$$

where $m \geq 0$, $m \leq k$, given by $B[\Phi]_{m+1} = \Phi^*(\kappa)$, where κ is a generator of $\check{H}^k(S^k)$, is bijective.

4.31 For any $\Phi \in M_m(S^k, S^k)$, $0 \leq m \leq k$, $k \geq 1$, one can define the *topological degree* $\deg(\Phi)$ *of* Φ by the formula

$$\Phi^*(\kappa) = \deg(\Phi) \cdot \kappa$$

(κ is as in Corollary 4.30) – comp. [84]. Similarly one can define $\deg(\varphi)$ for $\varphi \in \mathcal{A}_m(S^k, S^k)$ (see 4.13 (iii)). It is clear that if $\Phi_j \in M(S^k, S^k)$, $j = 0, 1$, and $\Phi_0 \simeq_n \Phi_1$, $0 \leq n \leq k$, then $\deg(\Phi_0) = \deg(\Phi_1)$ (resp. if $\varphi_j \in \mathcal{A}(S^k, S^k)$, $j = 0, 1$, are homotopic within $\mathcal{A}_n$, $0 \leq n \leq k$, then $\deg(\varphi_0) = \deg(\varphi_1)$). However it is not known whether the converse fact holds. However, as a simple corollary of Corollary 4.30, we have:

4.32 Theorem (Hopf Degree Theorem) *If* $\Phi_j \in \widetilde{M}_m(S^k, S^k)$, $0 \le m \le k$, *and* $\deg(\Phi_0) = \deg(\Phi_1)$, *then* $\Phi_0 \simeq_{m+1} \Phi_1$.

In particular, if $\varphi \in \mathcal{A}_m(S^k, S^k)$, $0 \le m \le k$, $j = 0, 1$, *and* $\deg(\varphi_0) = \deg(\varphi_1)$, *then* φ_0 *and* φ_1 *are homotopic within the class* $\widetilde{\mathcal{M}}_{m+1}$.

The last statement constitutes a partial answer to the long standing open question concerning the validity of the Hopf theorem for acyclic set-valued maps (see [31, § 2.3]) and shows that in order to study acyclic maps appropriately it is natural to consider them in the framework of maps determined by morphisms.

It seems obvious that the problem of the extension for set-valued maps is quite complex. Once again (as in the case of homotopies) this problem has to be related to different classes of maps. We shall now give a sufficient condition for the existence of extensions of morphisms from $\widetilde{M}$ (or maps from $\widetilde{\mathcal{M}}$).

4.33 Lemma *Let a space* X *have the topology compatible with its closed covering* $\{X_j\}_{j\in J}$ *and suppose that, for each* $j \in J$, *there is a* $\mathcal{V}_m$*-map (resp.* $\widetilde{\mathcal{V}}_m$*-map)* $p_j : W_j \to X_j$, $m \ge 0$; *for any* $i, j \in J$, *there is a homeomorphism* $h_{ij} : p_i^{-1}(X_i \cap X_j) \to p_j^{-1}(X_i \cap X_j)$ *such that* $p_j \circ h_{ij} = p_i|p_i^{-1}(X_i \cap X_j)$ *and* $h_{ii} = id$, $h_{ik} = h_{jk} \circ h_{ij}$ *on* $p_i^{-1}(X_i \cap X_j \cap X_k)$ *for* $k \in J$. *If* $m = 0, 1$ *(or* $m > 1$ *and the covering* $\{X_j\}$ *is* σ*-locally finite), then there exists a* $\mathcal{V}_m$*-map (resp.* $\widetilde{\mathcal{V}}_m$*-map)* $p : W \to X$ *and homeomorphisms* $h_j : W_j \to p^{-1}(X_j)$ *such that* $p \circ h_j = p_j$, $j \in J$.

Proof In the free union $\overline{W} = \bigvee_{j\in J} W_j$ consider a relation R: if $w_i \in W_i$, $w_j \in W_j$, $i, j \in J$, then $(w_i, i)\, R\, (w_j, j)$ if and only if $w_i \in p_i^{-1}(X_i \cap X_j)$, $w_j \in p_j^{-1}(X_i \cap X_j)$ and $h_{ij}(w_i) = w_j$. Clearly R is a closing equivalence relation.

Let $t_j : W_j \to W := \overline{W}/R$ be the quotient map, $j \in J$. Define $p : W \to X$ by the formulae

$$p(t_j(w)) = p_j(w)$$

for $w \in W_j$, $j \in J$.One checks easily that, under our assumptions, p is a $\mathcal{V}_m$-map (resp. $\widetilde{\mathcal{V}}_m$-map) and since $p^{-1}(X_j) = t_j(W_j)$, a map $h_j := t_j : W_j \to p^{-1}(X_j)$ is a homeomorphism. □

Let us assume now that (X, A) is a finite-dimensional polyhedral pair with a fixed triangulation, $\dim(X \setminus A) \le N$ and let $m \ge 0$. If $m > 1$, assume additionally that the family $\{A, \Delta_j\}_{j\in J}$, where $\{\Delta_j\}_{j\in J}$ is the set of all simplices of X not in A, is a σ-locally finite covering of X.

4.34 Lemma *Let $p : W \to A$ be a $\mathcal{V}_m$-map (resp. $\widetilde{\mathcal{V}}_m$-map), $m \geq 0$. There exists a $\mathcal{V}_n$-map (resp. $\widetilde{\mathcal{V}}_n$-map) $\overline{p} : \overline{W} \to X$ and a homeomorphism $h : W \to \overline{p}^{-1}(A)$ such that $\overline{p}\circ h = p$ where $n = 0$ if $m = 0, 1$ and $n = m+N$ if $m > 1$.*

Proof We shall construct a family $\{p_k : W_k \to \overline{X}^{k-1}\}_{k=0}^{N+1}$, where, for $s \geq -1$, $\overline{X}^s = X^s \cup A$, and X^s is the s-dimensional skeleton of X ($X^{-1} = \emptyset$), of maps such that:

(i) p_k is a $\mathcal{V}_0$-map (resp. $\widetilde{\mathcal{V}}_0$-map) if $m = 0, 1$ and a $\mathcal{V}_{m+k-1}$-map (resp. $\widetilde{\mathcal{V}}_{m+k-1}$-map) if $m > 1$;

(ii) for each $1 \leq k \leq N + 1$, there is a homeomorphism $g_k : W_{k-1} \to p_k^{-1}(\overline{X}^{k-2})$ such that $p_k \circ g_k = p_{k-1}$;

(iii) for each $0 \leq k \leq N+1$, there is a homeomorphism $h_k : W \to p_k^{-1}(A)$ such that $p_k \circ h_k = p$.

Let us put $p_0 := p$ and $h_0 := id$. Assume that, for any $0 \leq i \leq k-1$, where $1 \leq k \leq N + 1$, p_i, g_i and h_i satisfying (i), (ii) and (iii) has already been constructed. We shall build a space W_k, $p_k : W_k \to \overline{X}^{k-1}$ and homeomorphisms g_k, h_k.

Let $\{\Delta_j\}_{j\in J_k}$ be the family of all $(k-1)$-dimensional simplices in $X \setminus A$. Denote by W_k^j, $j \in J_k$, the (unreduced) cone $C\, p_{k-1}^{-1}(\partial\Delta_j)$ ($\partial\Delta_j$ denotes the boundary of Δ_j, $\partial\Delta_j \subset \overline{X}^{k-2}$), i.e. $W_k^j = (p_{k-1}^{-1}(\partial\Delta_j) \times I)/(p_{k-1}^{-1}(\partial\Delta_j) \times \{0\})$. Fix a homeomorphism $f_j : B^{k-1} \to \Delta_j$ (B^{k-1} is the closed unit disc in $\mathbf{R}^{k-1}$) such that $f_j(S^{k-2}) = \partial\Delta_j$, $j \in J_k$. For $[w, \lambda] \in W_k^j$, we put

$$p_k^j[w, \lambda] = f_j(\lambda f_j^{-1}(p_{k-1}(w))).$$

It is easy to see that $p_k^j : W_k^j \to \Delta_j$ is a $\mathcal{V}_0$-map (resp. $\widetilde{\mathcal{V}}_0$-map) if $m = 0, 1$ and a $\mathcal{V}_{m+k-1}$-map (resp. $\widetilde{\mathcal{V}}_{m+k-1}$-map) if $m > 1$.

Let $W_k^0 := W_{k-1}$ and let $p_k^0 := p_{k-1} : W_k^0 \to \overline{X}^{k-2}$. In view of Lemma 4.33, there exists a space W_k, a map $p_k : W_k \to \overline{X}^{k-1}$ being a $\mathcal{V}_0$-map (resp. $\widetilde{\mathcal{V}}_0$-map) if $m = 0, 1$ and a $\mathcal{V}_{m+k-1}$-map (resp. $\widetilde{\mathcal{V}}_{m+k-1}$-map) if $m > 1$ and a homeomorphism $g_k : W_{k-1} \to p_k^{-1}(\overline{X}^{k-2})$ such that $p_k \circ g_k = p_{k-1}$. Now it is enough to put $h_k = (g_k | p_{k-1}^{-1}(A)) \circ h_{k-1}$.

By induction we complete the construction. At last, we put $\overline{p} = p_{N+1}$ and $h = h_{N+1}$. □

4.35 Theorem (Extension Theorem) *Let Y, y_0 and κ satisfy the assumptions of Theorem 4.29 and (X, A) be as above. If X is compact or the group $\pi_s(Y)$ is finitely generated for any $s \geq 1$), $\check{H}^{s+1}(X, A; \pi_s(Y)) = 0$ for $s > k$, $\Phi \in \widetilde{M}_m(A, Y)$, $0 \leq m \leq k+1$, then there exists a morphism $\overline{\Phi} \in \widetilde{M}_n(X, Y)$ such that $\overline{\Phi} \circ i = \Phi$, where $i : A \to X$ is the inclusion and $n = 0$ if $m = 0, 1$ and $n = m + N$ if $m > 1$, provided*

$$\delta \circ \check{H}^k(\Phi)(\kappa) = 0 \qquad in\ \check{H}^{k+1}(X, A; \pi_k(Y)). \tag{4.2}$$

If $m = 0, 1$, then an extension $\overline{\Phi} \in \widetilde{M}_0(X, Y)$ exists if and only if condition (4.2) is satisfied.

Proof Let $A \xleftarrow{p} W \xrightarrow{q} Y \in \widetilde{D}_m$ represent Φ. In view of Theorem 2.17, there is $f : A \to Y$ such that $f \circ p \simeq q$. By the classical Hopf extension theorem (see [102, 166, 143]), there is an extension $\overline{f} : X \to Y$ of f if and only if $\delta \circ \check{H}^k(f)(\kappa) = \delta \circ \check{H}^k(\Phi)(\kappa) = 0$.

In view of Lemma 4.34, there is a $\widetilde{\mathcal{V}}_n$-map $\overline{p} : \overline{W} \to X$ and a homeomorphism $h : W \to \overline{p}^{-1}(A)$ such that $\overline{p} \circ h = p$. If (4.2) holds, then $\overline{f} \circ \overline{p}|\overline{p}^{-1}(A) \simeq q \circ h^{-1}$. By HEP, $q \circ h^{-1}$ has an extension $\overline{q} : W \to Y$. Letting $\overline{\Phi} \in \widetilde{M}_n(X, Y)$ be represented by $(\overline{p}, \overline{q}) \in \widetilde{D}_n$ we complete the proof. □

The reader will easily reformulate the above theorem for set-valued maps from the class $\widetilde{\mathcal{M}}$.

4.D. The class of CE-morphisms

It seems that the notion of a morphism is sufficiently general with regard to set-valued maps acting between finite-dimensional spaces. However in order to obtain the Identification Theorem 4.25 we have to restrict ourselves to morphisms from $\widetilde{M}$ i.e. represented by cotriads (p, q) where p is a $\widetilde{\mathcal{V}}$-map, so its domain W is finite-dimensional, as well. It is a serious restriction, especially when it comes to applications. In the present section we shall show how one can overcome this restriction. Let us recall the following definition.

4.36 Definition We say that $p : W \to X$ is a *cell-like map* (p is a *CE-map*) – written $p \in CE(W; X)$ – provided it is perfect and, for each $x \in X$, $p^{-1}(x)$ is a cell-like set. As before $p : (W, W') \to (X, X')$ is a *CE-map* – written $p \in CE(W, W'; X, X')$ if $p \in CE(W; X)$ and $p^{-1}(X') = W'$.

It is clear that $CE \subset \mathcal{V}_m$ for each $m \geq 0$.

Again, by Theorem 2.2 or Corollary 2.5 we have

4.37 Theorem *Let $p : (W, W') \to (X, X')$ be a CE-map and G be an abelian group. The induced homomorphism*

$$p^* : \check{H}^k(X, X'; G) \to \check{H}^k(W, W'; G)$$

is an isomorphism for each $k \geq 0$.

4.38 Proposition
(i) *Let $p \in CE(W, X)$. If $B \subset X$, then $p_B := p|p{-}1(B) : p^{-1}(B) \to B$ is a CE-map.*

(ii) *If in a triad*

$$(W_1, W_1') \xrightarrow{q_1} (Y, Y') \xleftarrow{p_2} (W_2, W_2')$$

p_2 is a CE-map, then in the pull-back of this triad,

$$(W_1, W_1') \xleftarrow{\overline{p}_2} (W, W') \xrightarrow{\overline{q}_1} (W_2, W_2')$$

where $W = W_1 \boxtimes W_2 := \{(w_1, w_2) \in W_1 \times W_2 \mid q_1(w_1) = p_2(w_2)\}$, $\overline{p}_2(w_1, w_2) = w_1$, $\overline{q}_1(w_1, w_2) = w_2$ and $W' = \overline{p}_2^{-1}(W_1')$, the map $\overline{p}_2$ is a CE-map.

Observe that the composition of CE-maps between ANRs is again a CE-map – see [3, Th. 5.14] and [48, Cor. 40.2 (2)] (for separable ANRs – see [97]). Otherwise there are no results and this is a serious obstacle.

As before let (X, X'), (Y, Y') be pairs of spaces. By $D_{CE}(X, X'; Y, Y')$ we denote the class of all cotriads

$$(X, X') \xleftarrow{p} (W, W') \xrightarrow{q} (Y, Y')$$

where p is a CE-map.

Obviously $D_{CE} \subset D_0$. Therefore in order to introduce an equivalence relation into D_{CE} we may consider the relation "$\approx$" from 4.6. However it would involve some superfluous restrictions. Hence we have the following simpler, but sufficient for our needs, defintion given in analogy to [111].

4.39 Definition We say that cotriads

$$(X, X') \xleftarrow{p_i} (W_i, W_i') \xrightarrow{q_i} (Y, Y'), \quad i = 1, 2$$

from $D_{CE}(X, X'; Y, Y')$ are *equivalent* – written $(p_1, q_1) \approx (p_2, q_2)$ – if there exists a homeomorphism

$$f : (W_1, W_1') \to (W_2, W_2')$$

such that the following diagram

$$\begin{array}{ccccc} & & (W_1, W_1') & & \\ & \swarrow^{p_1} & & {}_{q_1}\searrow & \\ (X, X') & & \downarrow f & & (Y, Y') \\ & \nwarrow^{p_2} & & {}_{q_2}\nearrow & \\ & & (W_2, W_2') & & \end{array}$$

is commuative. The relation "$\approx$" is clearly an equivalence.

4.40 CE-morphisms Elements of the quotient

$$M_{CE}(X, X'; Y, Y') = D_{CE}(X, X'; Y, Y')/\approx$$

are called *CE-morphisms* and again are denoted by Greek letters $\Phi, \Psi, \ldots$.

If $(p, q) \in D_{CE}(X, X'; Y, Y')$, then the morphism in M_{CE} represented by $(p, q) \in D_{CE}$ is denoted by $[(p, q)]_{\approx}$.

As before any morphism $\Phi \in M_{CE}(X, X'; Y, Y')$ *determines* a multivalued transformation $X \ni x \mapsto \varphi_\Phi(x) = q(p^{-1}(x))$ where $(p, q) \in \Phi$. This map is well-defined and the transformation φ_Φ defines a set-valued map $\varphi_\Phi : (X, X') \multimap (Y, Y')$ being a composition of a CE-set-valued map and a single-valued one. Clearly different CE-morphisms may determine the same set-valued map φ. We say that a map $\varphi : (X, X') \multimap (Y, Y')$ is *determined by a CE-morphism* if there exists $\Phi \in M_{CE}(X, X'; Y, Y')$ such that $\varphi(x) = \varphi_\Phi(x)$ for any $x \in X$.

4.41 Example

(i) Any map $\varphi \in \mathcal{CE}(\mathcal{X}, \mathcal{Y})$ is determined by a morphism $\Phi \in M_{CE}(X, Y)$. Indeed, one can represent Φ by a cotriad $X \xleftarrow{p_\varphi} W_\varphi \xrightarrow{q_\varphi} Y$ where $W_\varphi = \mathrm{Gr}(\varphi)$ is the graph of φ and p_φ, q_φ are the projections.

(ii) In particular, any map $g : X \to Y$ is determined by a CE-morphism. In this case it is convenient to identify g with the CE-morphism represented by the cotriad (id_X, g) (being equivalent to the cotriad $X \xleftarrow{p_g} \mathrm{Gr(g)} \xrightarrow{\mathrm{q_g}} \mathrm{Y}$ where p_g, q_g are the respective projections).

(iii) CE-morphisms arise quite naturally in the context of differential inclusions – Section 6.B.

4.42 Remark

(i) Contrary to Remark 4.12 (iii), where we have defined categories $\mathbf{M}_0$ and $\widetilde{\mathbf{M}}_0$, there is perhaps no way to define a category $\mathbf{M}_{CE}$ with CE-morphisms as the set of arrows in a similar way because there is no satisfactory notion of the composition of CE-morphisms (the one defined in 4.11 does not apply in general since, as we mentioned above, the composition of CE-maps is, in general, no longer a CE-map).

However, if $f : X \to Y$ and $\Phi \in M_{CE}(Y, Z)$ is represented by a cotriad $Y \xleftarrow{p} W \xrightarrow{q} Z$, then identifying f with a morphism from $M_{CE}(X, Y)$ represented by (id_X, f), one can consider the composition $\Phi \circ f$ as the CE-morphism represented by the cotriad $X \xleftarrow{\overline{p}} X \boxtimes W \xrightarrow{q \circ \overline{f}} Z$ where the cotriad $X \xleftarrow{\overline{p}} X \boxtimes W \xrightarrow{\overline{f}} W$ is the fibre-product of the triad $X \xrightarrow{f} Y \xleftarrow{p} W$ and then $\Phi \circ f \in M_{CE}(X, Z)$. In this way, we may define the restriction $\Phi|B \in M_{CE}(B, Y)$ for *any* $B \subset X$ and $\Phi \in M_{CE}(X, Y)$ – comp. Remark 4.12.

(ii) As in 4.15 and Remark 4.16, if $\Phi \in M_{CE}(X, Y_1)$ and $f : X \to Y_2$, then the morphism $(\Phi, f) \in M_{CE}(X, Y_1 \times Y_2)$ is defined. It is represented by the cotriad $(p, (q, f \circ p))$ where $\Phi = [(p, q)]_\approx$. Similarly, under appropriate assumptions one may consider fields of the form $i - \Phi$ for any CE-morphism Φ.

(iii) As above if $\Phi \in M_{CE}(X, X'; Y, Y')$, then for any $k \geq 0$, one can define the *induced homomorphism*

$$\Phi^* = \check{H}^k(\Phi) : \check{H}^k(Y, Y'; G) \to \check{H}^k(X, X'; G)$$

where G is an arbitrary abelian group – see Theorem 4.37 and 4.13.

4.43 Definition Let $\Phi_j \in M_{CE}(X,X';Y,Y')$, $j=0,1$. We say that the morphisms Φ_0, Φ_1 are *homotopic in* M_{CE} – written $\Phi_0 \simeq_{CE} \Phi_1$ – if there exists a *homotopy* $\Phi : \Phi_0 \simeq_{CE} \Phi_1$, i.e. a morphism $\Phi \in M_{CE}(X \times I, X' \times I; Y, Y')$ such that $\Phi \circ i_j = \Phi_j$.

4.44 Proposition *The relation*

$$\simeq_{CE} := \{(\Phi_0,\Phi_1) \mid \Phi_j \in M_{CE}(X,X';Y,Y'),\ j=0,1;\ \Phi_0 \simeq_{CE} \Phi_1\}$$

is an equivalence relation.

Proof It is clear that $\simeq_{CE}$ is reflexive and symmetric. Now let us proceed with the proof of its transitivity. We shall deal with the absolute case (the relative one is proven analogously).

Suppose that $\Phi_{-1}, \Phi_0, \Phi_1 \in M_{CE}(X;Y)$ and $\Psi_{-1} : \Phi_{-1} \simeq_{CE} \Phi_0$, $\Psi_1 : \Phi_0 \simeq_{CE} \Phi_1$ are the respective CE-homotopies. Rescaling the time we may suppose that $\Psi_{-1} \in M_{CE}(X \times [-1,0];Y)$ and $\Psi_1 \in M_{CE}(X \times [0,1];Y)$. Let cotriads $X \times [-1,0] \stackrel{\overline{p}_{-1}}{\longleftarrow} \overline{W}_{-1} \stackrel{\overline{q}_{-1}}{\longrightarrow} Y$ and $X \times [0,1] \stackrel{\overline{p}_1}{\longleftarrow} \overline{W}_1 \stackrel{\overline{q}_1}{\longrightarrow} Y$ represent Ψ_{-1} and Ψ_1, respectively, and let $X \stackrel{p_0}{\longleftarrow} W_0 \stackrel{q_0}{\longrightarrow} Y$ represent Φ_0. Let us put $W_{\pm 1} = p_{\pm 1}^{-1}(X \times \{0\})$, and $p_{\pm 1} = \overline{p}_{\pm 1}|W_{\pm 1}$, $q_{\pm 1} = \overline{q}_{\pm 1}|W_{\pm 1}$. Under our assumptions $(p_{-1}, q_{-1}) \approx (p_0, q_0) \approx (p_1, q_1)$ hence there is a homeomorphism $h : W_{-1} \to W_1$ such that $p_1 \circ h = p_{-1}$ and $q_1 \circ h = q_{-1}$.

Let $\overline{W} = \overline{W}_{-1} \cup_h \overline{W}_1$ and let $h_{\pm 1} : \overline{W}_{\pm 1} \to \overline{W}$ be the natural quotient projection. Evidently $h_{\pm 1}$ is a homeomorphic embedding and we may define $\overline{p} : \overline{W} \to X \times [-1,1]$ by the formula

$$\overline{p}(h_{\pm 1}(w_{\pm 1})) := \overline{p}_{\pm 1}(w_{\pm 1})$$

for $w_{\pm 1} \in \overline{W}_{\pm 1}$. It is easy to see that $\overline{p}$ is a well-defined CE-mapping. If we define $\overline{q} : \overline{W} \to Y$ similarly, then the cotriad $(\overline{p}, \overline{q})$ represents a morphism Ψ such that $\Psi : \Phi_{-1} \simeq_{CE} \Phi_1$. $\square$

4.45 Definition By $M_{CE}[X,X';Y,Y']$ we denote the set of all CE-homotopy classes $[\Phi]_{CE} = \{\Psi \in M_{CE}(X,X';Y,Y') \mid \Phi \simeq_{CE} \Psi\}$ of $\Phi \in M_{CE}(X,X';Y,Y')$.

It is easy to see that Proposition 4.23 also holds for CE-morphisms:

4.46 Proposition *Given $\Phi \in M_{CE}(X, X'; Y, Y')$ represented by a cotriad $(X, X') \xleftarrow{p} (W, W') \xrightarrow{q} (Y, Y') \in D_{CE}$ and a map $f : (X, X') \to (Y, Y')$, if $f \circ p \simeq q$, then $\Phi \simeq_{CE} f$.*

Proof may be obtained exactly as that of Proposition 4.23. and (ii) follows from Theorem 2.20 and part (i). □

Similarly as before we get a result concerning the homotopy classification of CE-morphisms. Namely in view of Theorems 2.20, 2.19 we have

4.47 Theorem (Identification Theorem II) *Assume that $(X, X'), (Y, Y')$ are paracompact pairs, (Y, Y') has the homotopy type of an* ANR*-pair and* $\operatorname{Ind} X < \infty$. *If $\Phi \in M_{CE}(X, X'; Y, Y')$ (resp. $\Phi \in M_{CE}(X, Y)$), then there exists a unique (up to homotopy) map $f : (X, X') \to (Y, Y')$ (resp. $f : X \to Y$) such that $\Phi \simeq_{CE} f$.*

There is a 1-1 *correspondence*

$$\begin{array}{rcl} M_{CE}[X, X'; Y, Y'] & \longleftrightarrow & [X, X'; Y, Y'] \\ M_{CE}[X; Y] & \longleftrightarrow & [X; Y] \\ [[(p, q)]_{\approx}]_{CE} & \mapsto & [f] = (p^{\#})^{-1}[q]. \end{array}$$

Observe that, in view of Remark 2.21, the above result is also true if we replace (Y, Y') by an ANE-pair.

To get the proof of Theorem 4.47 we use the same argument as in the proof of Theorem 4.25 but instead Proposition 4.23 we have to use Proposition 4.46 and instead Lemma 4.26 we must employ the following lemma.

4.48 Lemma *Let $p_j : W_j \to X$, $j = 0, 1$, be a CE-map. There is a space W and a CE-map $p : W \to X \times I$ such that $W_j = p^{-1}(X \times \{j\})$ and $p|p^{-1}(X \times \{j\}) = i_j \circ p_j$, $j = 0, 1$.*

This is nothing else but a version of Lemma 4.26; in order to prove it we may proceed exactly as we did there and observe that one may treat W_j as the closed subspace of the constructed space W.

Let us end this section with the following example.

4.49 Example Let (X, X') be a space and let (Y, Y') be a pair of subsets of a normed space. Assume that $\Phi \in M_m(X, X'; Y, Y')$, $m \geq 0$ (resp. $\widetilde{M}_m$ or $\Phi \in M_{CE}(X, X'; Y, Y')$) and a map $f : (X, X') \to (Y, Y')$ such that, for each $x \in X$ (resp. $x \in X'$) and $t \in I$, $(1-t)\Phi(x)+tf(x) = \{(1-t)y+tf(x) \mid y \in \Phi(x)\} \subset Y$ (resp. $(1-t)\Phi(x) + tf(x) \subset Y'$) is given. Then, we easily see that $f \circ p \simeq q : (W, W') \to (Y, Y')$ (take the linear homotopy between $f \circ p$ and q) where $(X, X') \xleftarrow{p} (W, W') \xrightarrow{q} (Y, Y')$ represents Φ. Hence, in view of Propositions 4.23 and 4.46, $\Phi \simeq_{m+1} f$ in M (resp. $\widetilde{M}$) or $\Phi \simeq_{CE} f$.

4.E. Infinite dimensional case

In this short section we shall establish possibly infinite-dimensional counterparts of Identification Theorems 4.25 and 4.47.

Let $(E, \|\cdot\|)$ be a Banach space, $\dim E \leq \infty$ and let $(X, A; u) \in \mathcal{LS}(E)$ – see 3.11 and recall the notation from Section 3.B. and from paragraphs 3.12, 3.13.

4.50 By $\widetilde{M}(X, A; u)$ (resp. $M_{CE}(X, A; u)$) we denote the class of all morphisms $\Phi \in \widetilde{M}(X, E)$ (resp. $M_{CE}(X, E)$) such that:

(i) $u - \Phi \in \widetilde{M}(X, A; E \setminus \{0\}, E \setminus Z_-)$ (resp. $M_{CE}(X, A; E \setminus \{0\}, E \setminus Z_-)$);

(ii) Φ determines a compact set-valued map ([5]);

(iii) if $\dim E = m + 1 < \infty$, $m \geq 0$, then $\mathrm{i}(\Phi) \leq m$ (see 4.8).

If $A = \emptyset$, then we write $\widetilde{M}(X; u)$ (or $M_{CE}(X; u)$) instead of $\widetilde{M}(X, \emptyset; u)$ (or $M_{CE}(X, \emptyset; u)$).

Within $\widetilde{M}(X, A; u)$ (resp. $M_{CE}(X, A; u)$) we consider the following notion of homotopy: if $\Phi_j \in \widetilde{M}(X, A; u)$ (resp. $M_{CE}(X, A; u)$), $j = 0, 1$, then we say that Φ_0, Φ_1 are *u-homotopic* – written $\Phi_0 \simeq_u \Phi_1$ – whenever there is Φ such that:

(j) $\Phi \circ i_j = \Phi_j$ (where $i_j : X \to X \times I$, $i_j(x) = (x, j)$ for $x \in X$), $j = 0, 1$;

(jj) $u - \Phi \in \widetilde{M}(X \times I, A \times I; E \setminus \{0\}, E \setminus Z_-)$ (resp. $u - \Phi \in M_{CE}(X \times I, A \times I; E \setminus \{0\}, E \setminus Z_-)$);

(jjj) Φ determines a compact map;

(jv) if $\dim E = m + 1 < \infty$, then $\mathrm{i}(\Phi) \leq m + 1$.

[5] i.e. $\mathrm{cl}\,\Phi(X)$ is compact in E.

Arguing as in Propositions 4.19 and 4.44, we show that the relation of u-homotopy is an equivalence. By $M[X, A; u]$ (resp. $\widetilde{M}[X, A; u]$ or $M_{CE}[X, A; u]$) we denote the set of all homotopy classes $[\varPhi]$ of $\varPhi \in M(X, A; u)$ (resp. $\widetilde{M}(X, A; u)$ or $M_{CE}(X, A; u)$).

4.51 Theorem (Identification Theorem III) *There is a* 1-1 *correspondence*

$$\widetilde{M}[X, A; u] \longleftrightarrow \pi^E(X, A; u)$$

$$(\textit{resp. } M_{CE}[X, A; u] \longleftrightarrow \pi^E(X, A; u))$$

provided $(X, A; u)$ *is of finite type* (*resp. and* X *is a metric space*).

Proof Let $\alpha \in \widetilde{M}[X, A; u]$ (resp. $M_{CE}[X, A; u]$) and let $\varPhi \in \alpha$. Suppose that a cotriad

$$X \xleftarrow{p} W \xrightarrow{q} E \in \widetilde{D}$$

(resp. D_{CE}) represents $\varPhi$.

It is evident that $(W, p^{-1}(A); v)$, where $v = u \circ p$, is an object in $\mathcal{LS}(E)$. Moreover if $\varPhi \in \widetilde{M}$, then $(W, p^{-1}(A); v)$ is of finite-type. Similarly the map $q : W \to E$ is a v-field on $(W, p^{-1}(A))$ and p is a morphism between the objects $(W, p^{-1}(A); v)$ and $(X, A; u)$. Therefore, in view of Theorem 3.28,

$$\pi^E(p) : \pi^E(X, A; u) \to \pi^E(W, p^{-1}(A); v)$$

is a bijection.

Let

$$A(\alpha) = \beta := \pi^E(p)^{-1}([q]_v) \in \pi^E(X, A; u). \tag{4.3}$$

The verification that $A : \widetilde{M}[X, A; u] \to \pi^E(X, A; u)$ is well-defined (i.e. does not depend on the choice of $\varPhi \in \alpha$ and $(p, q) \in \varPhi$) and bijective goes as in the proof of Theorem 4.25. □

4.52 Remark

(i) Theorem 4.51 shows, in particular, that for any $\varPhi \in \widetilde{M}(X, A; u)$ (resp. $M_{CE}(X, A; u)$) and any (p, q) that represents $\varPhi$, there is a unique (up to u-homotopy) u-field $f : X \to E$ such that $f \circ p \simeq_v q$. This implies that $\varPhi \simeq_u f$ within $\widetilde{M}(X, A; u)$ (resp. $M_{CE}(X, A; u)$) (evidently f may be considered an element of $\widetilde{M}(X, A; u)$ (resp. $M_{CE}(X, A; u)$).

(ii) If the object $(X, A; u)$ is of finite type and regular, then $\widetilde{M}(X, A; u)$ and $M_{CE}(X, A; u)$ admit the structure of an abelian group (inherited from $\pi^E(X, A; u)$).

4.F. Concluding remarks

4.53 As the basic classes of set-valued maps of this dissertation we consider the collections

- $\mathcal{M}_m$ (resp. $\widetilde{\mathcal{M}}_m$), $m \geq 0$, of maps determined by morphisms from M_m (resp. $\widetilde{M}_m$);
- $\mathcal{M}_{CE}$ of maps determined by CE-morphisms (that is compositions of $\mathcal{CE}$-valued maps with single-value ones).

Additionally, let $\mathcal{M} = \bigcup_{m\geq 0} \mathcal{M}_m$ (resp. $\widetilde{\mathcal{M}} = \bigcup_{m\geq 0} \widetilde{\mathcal{M}}_m$).

In view of Example 4.10,

$$\mathcal{CE} \subset \mathcal{A}_{\Updownarrow} \subset \mathcal{M}_{\Updownarrow} \subset \mathcal{A}^{\rfloor}_{\Updownarrow}\ (^6), \qquad \widetilde{\mathcal{A}}_{\Updownarrow} \subset \widetilde{\mathcal{M}}_{\Updownarrow} \subset \widetilde{\mathcal{A}}^{\rfloor}_{\Updownarrow},$$

for all $m \geq 0$, and, by (1.11),

$$\mathcal{CE} \subset \mathcal{M}_{\mathcal{CE}} \subset \mathcal{A}^{\rfloor}_{\prime} = \mathcal{M}_{\prime}, \qquad \widetilde{\mathcal{A}}^{\rfloor}_{\prime} = \widetilde{\mathcal{M}}_{\prime}.$$

Moreover, if $\varphi_1 \in \mathcal{A}_m(X,Y)$ (resp. $\varphi_1 \in \widetilde{\mathcal{A}}_m$), $m \geq 0$, $\varphi_2 \in \mathcal{A}_0(Y,Z)$ (resp. $\varphi_2 \in \widetilde{\mathcal{A}}_0$), then $\varphi_2 \circ \varphi_1 \in \mathcal{M}_m(X,Z)$ (resp. $\widetilde{\mathcal{M}}_m$).

In [84] (see also [43]) maps from $\mathcal{M}_0$ are called *strongly admissible* (*admissible* are maps having strongly admissible selections).

Taking into account that different (also from the homological point of view) morphisms may determine the same set-valued map, in the sequel we prefer to speak about morphisms, instead set-valued maps, in order to study them together with their (equivalent) factorizations. In our opinion it leads to a stronger theory. In this way, results about set-valued maps are stated as corollaries.

However, having in mind these subtleties, sometimes we identify maps from $\mathcal{M}_m$ (or $\mathcal{M}_{CE}$) with underlying morphisms.

The equivalence Theorems 4.25, 4.47 and their corollaries have the following consequences in the theory of set-valued maps.

---

[6] recall that by $\mathcal{F}^c$ we denote the class of finite compositions of maps from the class $\mathcal{F}$.

4.54 Corollary

(i) *Let Assumptions* 4.24 *be satisfied. If* $\varphi : (X, X') \multimap (Y, Y')$ *belongs to the class* $\widetilde{\mathcal{M}}_m$, $m \geq 0$ (*for instance* $\varphi \in \tilde{\mathcal{A}}_m$; *or* $\varphi \in \mathcal{A}_m$ *and* $\dim Y < \infty$), *then* φ *is homotopic to a single-valued map within the class* $\widetilde{\mathcal{M}}_{m+1}$ *provided* $m - 1 \leq k$. *This map is unique up to homotopy if* $m \leq k$.

In particular, if $\varphi \in \widetilde{\mathcal{A}}_0^c$, *then* φ *is homotopic to a single-valued map within* $\widetilde{\mathcal{A}}_0^c$ (*in this case Assumption* 4.24 (vi) *is not necessary*).

(ii) *Suppose that* Ind $< \infty$ *and* (Y, Y') *has the homotopy type of an* ANR-*pair* (*or an* ANE-*pair*). *Any map* $\varphi : (X, X') \multimap (Y, Y')$ (*resp.* $\varphi : X \multimap Y$) *from the class* $\mathcal{M}_{CE}$, *for instance* $\varphi \in \mathcal{CE}$, *is homotopic to a uniquely* (*up to homotopy*) *determined single-valued map.*

The above corollary gives an answer to an old question of A. Granas and generalizes results of [38] and [29] where cellular and so-called strongly acyclic maps, respectively, with values in S^n, $n \geq 1$, were considered. Sometimes, the existing single-valued continuous map homotopic to a set-valued one in an appropriate sense is called a *homotopy approximation*.

Corollary 4.54 may be still generalized. Namely, suppose that $\varphi_i : X_{i-1} \multimap X_i$, $i = 1, \ldots, k$, $X_0 = X$, $X_k = Y$ and let $\varphi = \varphi_k \circ \ldots \circ \varphi_1 : X \multimap Y$. If $\varphi_i \in \mathcal{A}_0$ (resp. $\widetilde{\mathcal{A}}_0$), then $\varphi \in \mathcal{A}_0^c = \mathcal{M}_0$ (resp. $\widetilde{\mathcal{M}}_0$) and φ has a homotopy approximation provided X and Y satisfy appropriate conditions.

4.55 Proposition

(i) *If* $\dim X_i < \infty$ $(i = 0, 1, \ldots, k-1)$, Y *has the homotopy type of an* ANR *and is* $(k-1)$-*connected,* $\varphi_i \in \widetilde{\mathcal{A}}_m$ $(i = 1, \ldots, k)$, $m - 1 \leq k$, *then* φ *is homotopic within the class* $\widetilde{\mathcal{M}}_{m+1}^c$ *to a single-valued map* $f : X \to Y$ *provided* Y *is homotopically simple and* $\pi_i(Y)$ *is finitely generated* $(i \geq 1)$ *whenever* X *is not compact.*

(ii) *If* $\mathrm{Ind}\,\mathrm{X_i} < \infty$ $(i = 0, 1, \ldots, k-1)$, Y *has the homotopy type of an* ANR (*or an* ANE), $\varphi_i \in \mathcal{CE}$ $(i = 1, \ldots, k)$, *then* φ *is homotopic within the class* $\mathcal{M}_{CE}^c$ *to a single valued map* $f : X \to Y$.

Proof It is sufficient to consider $k = 2$. Suppose that φ_i is determined by an m-morphism (resp. CE-morphism) represented by a cotriad $X_{i-1} \xleftarrow{p_i} W_i \xrightarrow{q_i} X_i$, $i = 1, 2$. Clearly, in view of Theorem 2.17 (resp. 2.19), there is a map $f' : X_1 \to Y$ such that $f' \circ p_2 \simeq q_2$ and a map $f : X \to Y$ such that $f \circ p_1 \simeq f' \circ q_1$. Invoking Proposition 4.23 (resp. 4.46) we complete the proof. □

4.56 Graph-approximations vs. homotopy approximations It is natural to pose the following question: given an approximable set-valued map (see Definition 1.1) admitting homotopy approximations, what is the relation between them and graph approximations?

Moreover, in Chapter 1, we have studied conditions implying the graph-approximability of maps whose values satisfy some UV-properties. But we studied there no homotopy relations between a set-valued map and its graph approximations. It seems that on the level of Chapter 1 such problem has no satisfactory solution.

We are now in a position to get an answer to these problems.

4.57 Theorem *Let X be a metric space and Y be an* ANR. *Suppose that an approximable set-valued map $\varphi : X \multimap Y$ has the following property: there is a map $f : X \to Y$ such that $f \circ p \simeq q$ where $p : \mathrm{Gr}(\varphi) \to \mathrm{X}$ and $q : \mathrm{Gr}(\varphi) \to \mathrm{Y}$ are the projections. Then there is a neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi) \subset \mathrm{X} \times \mathrm{Y}$ such that $f \simeq g$ where $g : X \to Y$ is an arbitrary $\mathcal{U}$-approximation of φ.*

Proof Let $\{\mathcal{U}_j\}_{j \in J}$ stand for the family of all neighborhoods of $G := \mathrm{Gr}(\varphi)$ in $X \times Y$. Clearly the set J is directed by the relation: $i \le j \Leftrightarrow \mathcal{U}_i \supset \mathcal{U}_j$ for $i, j \in J$. For any $i, j \in J$, $i \le j$, let $\xi_{ij} : [\mathcal{U}_i; Y] \to [\mathcal{U}_j; Y]$ and $\xi_i : [\mathcal{U}_i; Y] \to [G, Y]$ be maps induced by the inclusions $\mathcal{U}_j \hookrightarrow \mathcal{U}_i$ and $G \hookrightarrow \mathcal{U}_i$, respectively; clearly $\xi_i = \xi_j \circ \xi_{ij}$. Hence $\{[\mathcal{U}_i; Y]; \xi_{ij} \mid i, j \in J, i \le j\}$ is a direct system. It is not difficult (using the fact that Y is an ANR) to check that $([G; Y], \{\xi_i\}_{i \in J})$ is a direct limit of this system.

Consider projections $X \xleftarrow{p_i} \mathcal{U}_i \xrightarrow{q_i} Y$, $i \in J$. It is obvious that $p^\# = \xi_i \circ p_i^\#$ and $p_j^\# = \xi_{ij} \circ p_i^\#$ for any $i \le j$ where $p^\# : [X; Y] \to [G; Y]$ and $p_i^\# : [X; Y] \to [\mathcal{U}_i; Y]$ are induced by p nad p_i, respectively.

Fix $i \in J$, then $\xi_i[q_i] = [q] = p^\#[f] = \xi_i \circ p_i^\#[f]$. Therefore there is $j \ge i$ in J such that

$$\xi_{ij}[q_i] = \xi_{ij} \circ p_i^\#[f]$$

hence

$$[q_j] = \xi_{ij}[q_i] = \xi_{ij} \circ p_i^\#[f] = p_j^\#[f].$$

Let $g : X \to Y$ be an arbitrary $\mathcal{U}_j$-approximation of φ; i.e. $W := \mathrm{Gr(g)} \subset \mathcal{U}_\mathrm{j}$. It is clear that since $f \circ p_j \simeq q_j$, then $f \circ (p_j|W) \simeq q_j|W$. Now we easily see that $f \simeq g$. □

4.58 Corollary *Let X be a metric space,* $\dim X < \infty$*, Y be an* ANR *and $\varphi : X \multimap Y$ be a $\mathcal{CE}$-valued map (see Definition* 4.1 (ii)*). If A denotes the bijection from Theorem* 4.47*, then*

$$A[\varphi]_{CE} = [g] \tag{4.4}$$

where $g : X \to Y$ is a sufficiently close graph approximation of φ.

Moreover, $\varphi \simeq_{CE} g$.

Proof In view of Corollary 1.33, φ is approximable (recall that the inclusion $\varphi(x) \hookrightarrow Y$ has UV^∞-property or each $x \in X$) and, by Theorem 2.19, φ has the property from Theorem 4.57. The assertion follows immediately from Theorems 4.57 and 4.47. □

Let us now carefully study possibilities to extend the above results. Is it true that having a map φ from $\mathcal{M}_{CE}$ and its arbitrary but sufficiently close graph-approximation g, $\varphi \simeq_{CE} g$? It seems that in general the answer to this question is negative (or at least the author cannot prove it is true). However it appears that we may always find a graph-approximation which will do.

4.59 Theorem *Let the assumptions of Corollary* 4.58 *concerning X and Y be satisfied and let $\varphi : X \multimap Y$ belong to $\mathcal{M}_{CE}(X, Y)$. Suppose that φ is determined by a morphism represented by a cotriad $X \xleftarrow{p} W \xrightarrow{q} Y$ from D_{CE}, where W is a subset of an* ANR *Z, and that q admits an extension onto a neighborhood of W in Z (for instance it holds whenever W is a closed subset of Z). Then, for any neighborhood $\mathcal{W}$ of* $\mathrm{Gr}(\varphi)$ *in $X \times Y$, there is a $\mathcal{W}$-approximation $g : X \to Y$ of φ such that $\varphi \simeq_{CE} g$.*

Proof We consider a set-valued map $\varphi_p : X \multimap W$, for each $x \in X$, $\varphi_p(x) = p^{-1}(x)$. Let $G = \mathrm{Gr}(\varphi_\mathrm{p})$ and let $X \xleftarrow{\widetilde{p}} G \xrightarrow{\widetilde{q}} W$ be the projections. Obviously $p \circ \widetilde{q} = \widetilde{p}$. Let $q' : W' \to Y$ be an extension of q onto a neighborhood W' of W in Z and let $i : W \to W'$ be the inclusion. By Theorem 2.19 and since W' is an ANR, there is a unique (up to homotopy) map $f : X \to W'$ such that $f \circ p \simeq i$. Hence $f \circ \widetilde{p} \simeq i \circ \widetilde{q}$. Evidently $i \circ \widetilde{q}$ is the projection from the graph $\mathrm{Gr}(\mathrm{i} \circ \varphi_\mathrm{p}) = \mathrm{G}$. Therefore, in virtue of Theorem 4.57, there is a neighborhood $\mathcal{U}'$ of $\mathrm{Gr}(\mathrm{i} \circ \varphi_\mathrm{p})$ in $X \times W'$ such that $g' \simeq f$ where $g' : X \to W'$ is a $\mathcal{U}'$-approximation of $i \circ \varphi_p$. Hence $g' \circ p \simeq i$ and $q' \circ g' \circ p \simeq q' \circ i = q$. By Proposition 4.46, $\varphi \simeq_{CE} g = q' \circ g'$.

On the other hand, in view of Proposition 1.43, if $\mathcal{U}'$ is sufficiently small, then g is a $\mathcal{W}$-approximation of φ since clearly $\varphi = q' \circ (i \circ \varphi_p)$. □

Finally, let us observe that the results stated and proved above in this section admit infinite dimensional versions in the spirit of Section 4.E. We leave the precise formulation to the reader.

PART II: APPLICATIONS

Chapter 5.

THE COINCIDENCE INDICES FOR CLASSES OF MAPS

Above we have started the study of the general class $\widetilde{\mathcal{M}}$ of set-valued maps determined by morphisms. This class contains, for instance, the class $\widetilde{\mathcal{A}}_0^c$ of compositions of acyclic (with respect to Čech integral cohomology) maps having finite covering dimension of values. This study was based on the homotopy properties of maps from $\widetilde{\mathcal{M}}$. It appears (see Theorems 4.25, 4.47, 4.51 and Corollary 4.54) that under some mild assumptions on spaces, each map from this class is homotopy equivalent to a single-valued (continuous) map. It enabled us to get a full homotopy classification (in the spirit of the Hopf theorems) of the class $\widetilde{\mathcal{M}}$.

In the present chapter, developing ideas from the author's paper [113], we shall apply these results in the context of coincidence index (in particular, fixed-point index) theory of morphisms and set-valued maps.

Since 1959 and the early paper of A. Granas [93], several successful approaches to the fixed-point index theory were presented (recall, for instance, the attitude of Cellina and Lasota [47] or Ma [135]). The authors usually study the fixed-point index maps from the class $\mathcal{A}_0^c$ defined on open subsets of a Euclidean space ([43, 44, 84])) or a Banach space ([37, 42, 72, 44, 119, 86]). See also [31] or [163] for extensive review and references.

Those papers rely heavily on homological methods based on the Vietoris-Begle theorem and the work of Dold [52] (see also [94]). Indices defined there are usually rational-valued (except for [42]) and are either multivalued or depend on the particular decomposition of a map from $\mathcal{A}_0^c$ (see [86] or [119] and [37, 84, 42] where the most general situation is considered). The definitions provided there are simple and elegant; they allow to obtain the Lefschetz theory in its full generality; however, the common lack (except for some special cases – see [44]) of the commutativity and, consequently, of the (mod p) properties and even the topological invariance of these indices makes it impossible to generalize them for maps on ANR-s.

In 1981, Siegberg and Skordev (see [163], [69] and [60]) overcame this shortcoming and gave another description of an index satisfying all the axioms (including the commutativity and the (mod p) properties) for the class of maps admitting the so-called A-systems (i.e. basically maps from $\mathcal{A}_0^c$). Their definition also involves a homological approach based on chain approximations (having roots in [64, 144]) and, hence, is rather complicated and difficult; the index is again $\mathbf{Q}$-valued and its relation to the indices of Calvert and Bryszewski-Górniewicz is not clear.

In [29, 90] and [21] the authors present different, the so-called homotopical, approaches to the index theory based on single-valued graph-approximations; results of [29] however apply only to strongly acyclic and acyclic maps on ENR-s (euclidean neighbourhood retracts) and it is not clear how to extend this method to ANR-s or maps from $\mathcal{A}_0^c$, while the index in [90, 21] applies only to a special subclass of $\mathcal{A}_0^c$ (consisting of compositions of UV-maps).

Paper [113] presents an attitude in the spirit of [31, 80, 90] where the homotopy equivalence, implied by the existence of single-valued graph approximations, was used. This attitude is actually based on identification theorems (given in Chapter 4) coming from [112]. It joins the homological approach with the homotopical one and allows to develop and prove some new properties of the Bryszewski-Górniewicz fixed-point index (see [42, 43] and also [86]) for maps from $\widetilde{\mathcal{M}}$ defined on open subsets of a Banach space. It also yields an integer-valued fixed-point index satisfying all the axioms for locally compact maps in $\widetilde{\mathcal{M}}_0$ (recall that $\widetilde{\mathcal{M}}_0 = \widetilde{\mathcal{A}}_0^c$) defined on arbitrary ANR-s. Our fixed-point index is the index in the sense of uniqueness: the Calvert and the Skordev-Siegberg indices when applied to $\widetilde{\mathcal{M}}_0$ are equal to our index (see (1.8) (iii), (2.8)). Therefore, slightly restricting the class of maps (basically we study the class $\widetilde{\mathcal{A}}_0^c$), a simple and elegant fixed-point index introduced and developed in [37, 43, 42, 84, 86] and [119] is *de facto* extended yielding the unique and consequent with the previous research index theory of locally compact set-valued maps of open subsets of arbitrary ANR-s.

Here we adopt the approach from [113] to coincidence problems under the presence of the dimension defect. It appears that our methods combined with those from Chapter 3 provide a convenient framework for such problems. Namely, using a purely homotopic approach we present a coincidence index theory for maps (or morphisms) acting between subsets of (possibly infinite dimensional) Banach spaces E', E. This index provides sort of algbraic count of coincidence points of maps F and φ where $F : E' \to E$ is a Fredholm operator (see – 3.23), $\varphi : X \multimap E$ is a compact set-valued map and $X \subset E'$ is bounded and closed. The introduced theory enables us to put into a unified framework some other earlier studied degrees or indices or other homotopy invariants (comp. [139], [77, 81], [30], [106]) studied in the single valued context together with a set-valued case. Set-valued maps are studied here for the first time (save vague announcements from [33]).

5.1 Remark Throughout the rest of this work we shall use the following notation. By Π_k, $k \geq 0$, we denote the k-th *stable homotopy group of spheres*, i.e.

$$\Pi_k := \varinjlim_{n \geq 0} \pi_{n+k}(S^n).$$

It is well-known that if $k + 2 \leq n$, then there is the *suspension* (or the Freudenthal) isomorphism $\pi_{n+k}(S^n) \cong \pi_{n+k+1}(S^{n+1})$; hence $\Pi_k \cong \pi_{n+k}(S^n)$. In the sequel we ignore this isomorphism and actually *identify* these groups. See [102] for some computation of Π_k for various k.

5.A. The generalized degree – finite-dimensional case

By $\mathcal{C}(m, n)$, $m \geq n$, we denote the class of all pairs (f, U) where U is an open bounded subset of $\mathbf{R}^m$ and $f : (\mathrm{cl}\, U, \mathrm{bd\, U}) \to (\mathbf{R}^{\mathrm{n}}, \mathbf{R}^{\mathrm{n}} \setminus \{0\})$ is a continuous map.

We say that (f_0, U), (f_1, U) from $\mathcal{C}(m, n)$ are *homotopic* if there is a *homotopy* $h : (\mathrm{cl}\, U \times [0, 1], \mathrm{bd} \times [0, 1]) \to (\mathbf{R}^{\mathrm{n}}, \mathbf{R}^{\mathrm{n}} \setminus \{0\})$ such that $h(\,\cdot\,, i) = f_i$, $i = 0, 1$.

In [81], the authors defined the *generalized degree* $d : \mathcal{C}(m,n) \to \pi_m(S^n)$. However the argument used there is rather different and a bit more complex, one can easily see that it may be summarized as follows.

Let $(f,U) \in \mathcal{C}(m,n)$. Since $f(\mathrm{bd\,U}) \subset \mathbf{R}^{\mathrm{n}} \setminus \{0\} \subset \mathrm{S^n} \setminus \{\mathrm{s_{-1}}\}$, where we identify S^n with the one-point compactification of $\mathbf{R}^n$, i.e. $S^n = \mathbf{R}^n \cup \{\infty\}$, and $0 \in \mathbf{R}^n$ is identified with the south pole s_{-1} of S^n, and $S^n \setminus \{s_{-1}\}$ is an absolute retract, there is a continuous map $f^* : S^m \setminus U \to S^n \setminus \{s_{-1}\}$ such that $f^*|\mathrm{bd\,U} = \mathrm{f}|\mathrm{bd\,U}$. Define $f' : S^m \to S^n$ by the formula

$$f'(x) = \begin{cases} f^*(x) & \text{for } x \in S^m \setminus U \\ f(x) & \text{for } x \in U \end{cases}$$

and put $d(f,U) = [f'] \in \pi_m(S^n)$, where, as usual, $[f']$ denotes the homotopy class of f'. It is easy to see that this definition does not depend on the choice of the auxuliary map f^*.

The defined degree d enjoys the properties similar to those of the ordinary (Brouwer) degree $\deg_B$ and in case $m = n$ it coincides with $\deg_B$.

The advantage is that one may define the degree d on $\mathcal{C}(m,n)$ without restrictions concerning m and n (save the obvious one: $m \geq n$ ([1])), however, if for instance $m = 10$ and $n = 6$, then $d(f,U) = 0$ for any $(f,U) \in \mathcal{C}(10,6)$ because $\pi_{10}(S^6) = 0$, i.e. d does not distinguish essential and inessential maps in this case.

Let us now formalize the above construction of the generalized topological degree for the case $m < 2n-2$ (observe that the additivity property of d in [81] was obtained when $m < 2n-3$). Note that, in view of the suspension isomorphism, $\pi^r(S^{m-n+r}) \cong \pi_{m-n+r}(S^r) \cong \Pi_{m-n}$ provided $r \geq m-n+2$; so, in particular, this isomorphism holds for any $r \geq n-1$).

Let again $(f,U) \in \mathcal{C}(m,n)$ where $n \leq m < 2n-2$. Assume, without loss of generality, that $\mathrm{cl}\, U \subset D^m$ (it is possible since U is bounded) and that $\dim \mathrm{bd\,U} \leq \mathrm{m} - 1$ (if not, then the localization property below – see Proposition 5.4 (ii) – shows that such an assumption does not cause loss of generality). Consider the following sequence of abelian groups and homomorphisms:

$$\begin{aligned} \pi^{n-1}(\mathrm{bd\,U}) &\xrightarrow{\delta_1} \pi^n(\mathrm{cl}\,U, \mathrm{bd\,U}) \xleftarrow{\mathrm{j}^\#} \pi^{\mathrm{n}}(\mathrm{D^m}, \mathrm{D^m} \setminus \mathrm{U}) \\ &\xrightarrow{i^\#} \pi^n(D^m, S^{m-1}) \xleftarrow{\delta_2} \pi^{n-1}(S^{m-1}) \end{aligned} \tag{5.1}$$

[1]if $m < n$, then of course $d(f,U)$ is defined and always equal to 0.

in which δ_1 denotes the coboundary homomorphism of the pair $(\mathrm{cl}\, U, \mathrm{bd\, U})$, $j : (\mathrm{cl}\, U, \mathrm{bd\, U}) \to (\mathrm{D^m}, \mathrm{D^m} \setminus \mathrm{U})$, $i : (D^m, S^{m-1}) \to (D^m, D^m \setminus U)$ are the inclusions and δ_2 is the coboundary homomorphism of the pair (D^m, S^{m-1}). Clearly $j^{\#}$ is the excision isomorphism and δ_2 is an isomorphism in view of the contarctibility of D^m and the exactness of the cohomotopy sequence of the pair (D^m, S^{m-1}).

Let

$$\kappa = \delta_2^{-1} \circ i^{\#} \circ (j^{\#})^{-1} \circ \delta_1. \tag{5.2}$$

and let $\alpha := [f|\mathrm{bd\, U}] \in \pi^{\mathrm{n}-1}(\mathrm{bd\, U})$; without loss of generality we have identified here $[\mathrm{bd\, U}; \mathbf{R}^{\mathrm{n}} \setminus \{0\}]$ with $\pi^{n-1}(\mathrm{bd\, U})$.

5.2 Definition By the *generalized degree of f on U* we understand the element

$$\deg(f, U) := \kappa(\alpha) \in \pi^{n-1}(S^{m-1}) \cong \Pi_{m-n}.$$

5.3 Remark

(i) If $U = B^m$, then we see that $\kappa = id_{\pi^{n-1}(S^{m-1})}$ and $\deg(f, U) = \alpha \in \pi^{n-1}(S^{m-1})$.

(ii) We can easily see that $\deg(f, U) = d(f, U)$ up to the suspension isomorphism.

(iii) Now study the case $m = n \geq 3$. We shall show that our degree $\deg : \mathcal{C}(n, n) \to \pi^{n-1}(S^{n-1}) \cong \mathbf{Z}$ is identical with the ordinary Brouwer degree $\deg_B$. Let $(f, U) \in \mathcal{C}(n, n)$ and let $K = f^{-1}(0)$; then K is a compact subset of U. Consider the following sequence

$$\begin{aligned} \check{H}^n(\mathbf{R}^n, \mathbf{R}^n \setminus \{0\}) &\xrightarrow{f^*} \check{H}^n(U, U \setminus K) \xleftarrow{\cong} \check{H}^n(\mathbf{R}^n, \mathbf{R}^n \setminus K) \\ &\longrightarrow \check{H}^n(\mathbf{R}^n, \mathbf{R}^n \setminus B^n) \xleftarrow{\cong} \check{H}^n(\mathbf{R}^n, \mathbf{R}^n \setminus \{0\}) \end{aligned}$$

where all unmarked arrows are induced by the respective inclusions, the first isomorphism comes from excision and the second one is induced by the strong deformation retraction of $\mathbf{R}^n \setminus \{0\}$ onto $\mathbf{R}^n \setminus B^n$. As is well-known if we denote the composed homomorphism by η and choose a generator 1 of $\check{H}^n(\mathbf{R}^n, \mathbf{R}^n \setminus \{0\})$, then the Brouwer degree $\deg_B(f, U)$ is given by the equality $\eta(1) = \deg_B(f, U) \cdot 1$. Using rather obvious appropriate excisions and commutative diagrams we see that the homomorphism η may be replaced by the one arising from the following sequence

$$\begin{aligned} &\check{H}^{n-1}(\mathbf{R}^n \setminus \{0\}) \xrightarrow{f^*} \check{H}^{n-1}(\mathrm{bd\, U}) \xrightarrow{\delta} \check{\mathrm{H}}^{\mathrm{n}}(\mathrm{cl\, U}, \mathrm{bd\, U}) \\ &\xleftarrow{\cong} \check{H}^n(D^n, D^n \setminus U) \longrightarrow \check{H}^n(D^n, S^{n-1}) \underset{\cong}{\xleftarrow{\delta}} \check{H}^{n-1}(S^{n-1}). \end{aligned}$$

This sequence is, by the Hopf theorem – see [102, Th.11.5], isomorphic to the sequence

$$\begin{aligned}\pi^{n-1}(\mathbf{R}^n\setminus\{0\}) &\xrightarrow{f\#} \pi^{n-1}(\mathrm{bd\,U}) \xrightarrow{\delta} \pi^{\mathrm{n}}(\mathrm{cl\,U},\mathrm{bd\,U})\\ &\xleftarrow{\cong} \pi^n(D^n, D^n\setminus U) \longrightarrow \pi^n(D^n, S^{n-1}) \underset{\cong}{\xleftarrow{\delta}} \pi^{n-1}(S^{n-1})\end{aligned}$$

showing that indeed

$$\deg_B(f,U)=\deg(f,U).$$

(iv) Of course $\mathcal{C}(m,n)\subset\mathcal{C}(m,m)$, $0\le n\le m$. Therefore, given $(f,U)\in\mathcal{C}(m,n)$, one may define $\deg_B(f,U)\in\mathbf{Z}$. However, in case $n<m$, we easily see that $\deg_B(f,U)=0$. Hence, under the presence of dimension defect, i.e. $n<m$, the usual Brouwer degree is useless.

Let us now collect (to some extent after [81]) some basic properties of the generalized degree.

5.4 Proposition *Let $(f,U)\in\mathcal{C}(m,n)$, $n\le m<2n-2$.*
(i) (Existence) *If* $\deg(f,U)\neq 0\in\pi^{n-1}(S^{m-1})$*, then* $0\in f(U)$.
(ii) (Localization) *If* $\Gamma\subset U$ *is open and* $0\notin f(\mathrm{cl}\,U\setminus\Gamma)$*, then* $(f|\mathrm{cl}\,\Gamma,\Gamma)\in\mathcal{C}(m,n)$ *and* $\deg(f,\Gamma)=\deg(f,U)$.
(iii) (Additivity) *If* U_1, U_2 *are open disjoint subsets of* U *such that* $0\notin f(\mathrm{cl}\,U\setminus(U_1\cup U_2))$*, then* $(f|\mathrm{cl}\,U_i,U_i)\in\mathcal{C}(m,n)$*,* $i=1,2$*, and* $\deg(f,U)=\deg(f,U_1)+\deg(f,U_2)$.
(iv) (Homotopy) *If* (f,U) *is homotopic to* (g,U) *in* $\mathcal{C}(m,n)$*, then* $\deg(f,U)=\deg(g,U)$.
(v) (Boundary Dependence) *If* $(g,U)\in\mathcal{C}(m,n)$ *and* $f|\mathrm{bd\,U}=\mathrm{g}|\mathrm{bd\,U}$*, then* $\deg(f,U)=\deg(g,U)$.

Proof The proofs (in a different setting) may be found in [81]. For the sake of completeness we provide independent proofs.

(i) If $0\notin f(U)$, then $f:\mathrm{cl}\,U\to\mathbf{R}^n\setminus\{0\}$ and, hence, $\delta_1([f|\mathrm{bd\,U}])=0\in\pi^{\mathrm{n}}(\mathrm{cl\,U},\mathrm{bd\,U})$ (see (5.1)); so $\deg(f,U)=0$.

(ii) Let us consider the following diagram

$$\begin{array}{ccccc}
\pi^{n-1}(\mathrm{bd\,U}) & \xleftarrow{i_U} & \pi^{n-1}(\mathrm{cl}\,U \setminus \Gamma) & \xrightarrow{i_\Gamma} & \pi^{n-1}(\mathrm{bd}\,\Gamma) \\
\downarrow \delta_1 & & \downarrow \delta_1 & & \downarrow \delta_1 \\
\pi^n(\mathrm{cl}\,U, \mathrm{bd\,U}) & \longleftarrow & \pi^n(\mathrm{cl}\,U, \mathrm{cl}\,U \setminus \Gamma) & \longrightarrow & \pi^n(\mathrm{cl}\,\Gamma, \mathrm{bd}\,\Gamma) \\
\uparrow & & \uparrow & & \uparrow \\
\pi^n(D^m, D^m \setminus U) & \longleftarrow & \pi^n(D^m, D^m \setminus \Gamma) & \longrightarrow & \pi^n(D^m, D^m \setminus \Gamma) \\
& \searrow & \downarrow & \swarrow & \\
& & \pi^n(D^m, S^{m-1}) & &
\end{array}$$

where all unmarked arrows, i_Γ and i_U are induced by inclusions. It is clear that this diagram is commutative; moreover $i_\Gamma([f|\mathrm{cl}\,U \setminus \Gamma]) = [f|\mathrm{bd}\,\Gamma]$ and $i_U([f|\mathrm{cl}\,U \setminus \Gamma]) = [f|\mathrm{bd\,U}]$. This shows that indeed $\deg(f, U) = \deg(f, \Gamma)$.

(iii) Without loss of generality we may assume that actually $f : \mathrm{bd\,U} \to \mathrm{S^{n-1}}$ and, in view of (ii) above, that $\mathrm{cl}\,U_1 \cap \mathrm{cl}\,U_2 = \emptyset$ and $U_1 \cup U_2 = U$. For $i = 1, 2$, let $g_i : \mathrm{bd\,U} \to \mathrm{S^{n-1}}$ be given by the formula

$$g_i(x) = \begin{cases} f(x) & \text{for } x \in \mathrm{bd\,U_i} \\ s_0 & \text{for } x \in \mathrm{bd\,U} \setminus \mathrm{bd\,U_i} \end{cases}$$

where $s_0 = (1, 0, \ldots, 0) \in S^{n-1}$.

It is a routine to show that $\kappa([g_i]) = \deg(f, U_i)$ (see (5.2)), $i = 1, 2$.

Next recall the definition of the addition in $\pi^{n-1}(\mathrm{bd\,U})$. It is clear that $\alpha := [f|\mathrm{bd\,U}] = [\mathrm{g_1}] + [\mathrm{g_2}]$. Therefore

$$\deg(f, U) = \kappa(\alpha) = \kappa([g_1]) + \kappa([g_2]) = \deg(f, U_1) + \deg(f, U_2).$$

(iv), (v) are trivial in view of our definition. □

5.5 Remark Observe that the existence property (see Proposition 5.4) may be extended a bit. In fact, given $(f, U) \in \mathcal{C}(m, n)$, if a point $y \in \mathbf{R}^n$ lies in the same connected component V of $\mathbf{R}^n \setminus f(\mathrm{bd\,U})$ as 0 does, then (f, U) is homotopic to $(g, U) \in \mathcal{C}(m, n)$ where $g(x) = f(x) - y$ for $x \in \mathrm{cl}\,U$. Hence if $\deg(f, U) \neq 0$, then $y \in f(U)$, i.e. $V \subset f(U)$.

Suppose now that $(f,U) \in \mathcal{C}(m+1,n+1)$, $m < 2n-2$, and f is such that

$$f(\operatorname{cl} U \cap \mathbf{R}^{m+1}_{\pm}) \subset \mathbf{R}^{n+1}_{\pm}$$

where $\mathbf{R}^{n+1}_{\pm} := \{x = (x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \mid \pm x_{n+1} \geq 0\}$.

Let $U_0 = U \cap \mathbf{R}^m$ and let $f_0 : (\operatorname{cl} U_0, \mathrm{bd\,U_0}) \to (\mathbf{R}^{\mathrm{n}}, \mathbf{R}^{\mathrm{n}} \setminus \{0\})$ be the restriction of f to $\operatorname{cl} U_0$.

5.6 Proposition (Suspension) *Under the above assumptions*

$$\deg(f,U) = \deg(f_0,U_0)$$

(*up to the suspension isomorphism*).

Proof For $x \in \mathbf{R}^{m+1}$, set $\overline{x} = (x_1, ..., x_m)$ and consider the homotopy $h : \operatorname{cl} U \times [0,1] \to \mathbf{R}^{n+1}$ given, for $(\overline{x}, x_{m+1}) \in \operatorname{cl} U$, by the formula

$$h(\overline{x}, x_{m+1}, t) = (1-t) f(\overline{x}, x_{m+1}) + t f'(\overline{x}, x_{m+1})$$

where $f'(\overline{x}, x_{m+1}) = (\overline{f}_0(\overline{x}), x_{m+1})$ and $\overline{f}_0 : \mathbf{R}^m \to \mathbf{R}^n$ is an extension of f_0. It is clear that h has no zeros on $\mathrm{bd\,U}$. Hence, by the homotopy invariance of deg, $\deg(f,U) = \deg(f',U)$. But since zeros of f' are exactly those of f_0, in view of the localization property, $\deg(f',U) = \deg(f',V')$ where $V' = V \times (-\varepsilon, \varepsilon)$, $V \subset U_0$ is an open set such that $\deg(f_0,U_0) = \deg(f_0,V)$, $\varepsilon > 0$ is sufficiently small and $V \times (-\varepsilon, \varepsilon) \subset U$. Observe that, for $\overline{x} \in V$, $\overline{f}_0(\overline{x}) = f_0(\overline{x})$.

Consider now the sequences (of the form (5.1)) defining $\deg(f',V')$ and

$\deg(f_0, V)$, respectively, and the following commutative diagram

$$\begin{array}{ccc}
\pi^n(\mathrm{bd\,V}) & \xrightarrow{\Delta} & \pi^n(\mathrm{bd\,V'}) \\
\downarrow & & \downarrow \\
\pi^n(\mathrm{cl}\,V, \mathrm{bd\,V}) & \longrightarrow & \pi^{n+1}(\mathrm{cl}\,V', \mathrm{bd\,V'}) \\
\uparrow & & \uparrow \\
\pi^n(D^m, D^m \setminus V) & \longrightarrow & \pi^{n+1}(D^{m+1}, D^{m+1} \setminus V') \\
\downarrow & & \downarrow \\
\pi^n(D^m, S^{m-1}) & \longrightarrow & \pi^{n+1}(D^{m+1}, S^m) \\
\uparrow & & \uparrow \\
\pi^{n-1}(S^{m-1}) & \xrightarrow{\Delta_1} & \pi^n(S^m)
\end{array}$$

where Δ, Δ_1 and all other horizontal arrows are the corresponding Mayer-Vietoris homomorphisms. It is obvious that $\Delta[f_0|\mathrm{bd\,V}] = [\mathrm{f'}|\mathrm{bd\,V'}]$ and, since Δ_1 is nothing else but the suspension isomorphism, we get that $\deg(f', V') = \deg(f_0, V)$. This completes the proof. □

5.7 The degree deg may be easily generalized. Assume that $U \subset \mathbf{R}^m$ is open bounded and $f : \mathrm{cl}\,U \to \mathbf{R}^n$, $n \le m < 2n - 2$. Suppose that A is a connected subset of $\mathbf{R}^n \setminus f(\mathrm{bd\,U})$. We define

$$\deg(f, U; A) := \deg(f_y, U)$$

where $f_y : \mathrm{cl}\,U \to \mathbf{R}^n$ is given by $f_y(x) = f(x) - y$ and y is an arbitrary point in A. Since clearly $(f_y, U) \in \mathcal{C}(m, n)$, $\deg(f_y, U)$ is defined and our definition is correct because it does not depend on the choice of y in view of the homotopy invariance of deg. Indeed A is contained in a component of $\mathbf{R}^n \setminus f(\mathrm{bd\,U})$; hence any two points of A can be joined by a path disjoint from $f(\mathrm{bd\,U})$ and the argument used in Remark 5.5 may be applied.

It is obvious that if $\deg(f, U; A) \neq 0 \in \Pi_{m-n}$, then $A \subset f(U)$.

Having this we are in a position to state and prove the following important result. Let f be as above, $n \geq 3$, and suppose that V is an open bounded subset of $\mathbf{R}^n$ such that $f(\mathrm{cl}\, U) \subset \mathrm{cl}\, V$. By $\{V_j\}_{j\in J}$ we denote the family of all components of $V \setminus f(\mathrm{bd\,U})$. Evidently J is at most a countable set. Finally, suppose that $g : (\mathrm{cl}\, V, \mathrm{bd\,V}) \to (\mathbf{R}^{\mathrm{n}}, \mathbf{R}^{\mathrm{n}} \setminus \{0\})$ and that $0 \notin g \circ f(\mathrm{bd\,U})$. In other words $(g \circ f, U) \in \mathcal{C}(m, n)$ and $(g, V) \in \mathcal{C}(n, n)$.

5.8 Theorem (Multiplication) *Under the above assumption, the following formula holds*

$$\deg(g \circ f, U) = \sum_{j\in J} \deg_B(g, V_j) \cdot \deg(f, U; V_j).$$

Proof First observe that the right hand side of the above formula correctly defines an element of Π_{m-n} since, for each $j \in J$, $\deg_B(g, V_j)$ is a well-defined integer (observe that, for any $j \in J$, $\mathrm{bd\,V_j} \subset \mathrm{f(bd\,U)} \cup \mathrm{bd\,V}$; hence $(g, V_j) \in \mathcal{C}(n, n)$). Moreover, the summation in this formula is finite for $g^{-1}(0)$ is compact, so that, being covered by the disjoint open sets V_j, it meets only a finite number of them.

Let $U_j := f^{-1}(V_j)$, $j \in J$. The sets U_j are disjoint, if $x \in \mathrm{cl}\, U \setminus \bigcup_{j\in J} U_j$, then $g \circ f(x) \neq 0$, hence by the additivity property

$$\deg(g \circ f, U) = \sum_{j\in J} \deg(g \circ f, U_j).$$

We are now to prove that, for all $j \in J$,

$$\deg(g \circ f, U_j) = \deg_B(g, V_j) \cdot \deg(f_y, U_j)$$

where $f_y(x) = f(x) - y$, $x \in \mathrm{cl}\, U$, and y is a point from V_j. To this end fix $j \in J$, a point $y \in V_j$ and suppose that $\deg(g, V_j) = d \in \mathbf{Z}$.

Observe that $f : (\mathrm{cl}\, U_j, \mathrm{bd\,U_j}) \to (\mathrm{cl\,V_j}, \mathrm{bd\,V_j})$. Moreover, in view of the Lefschetz duality (see [166, Th. 6.2.19]), $\check{H}^n(\mathrm{cl}\, V_j, \mathrm{bd\,V_j}) = \mathrm{H_0(V_j)} = \mathbf{Z}$ because V_j is connected and open. Let s_y denotes the shift map $\mathrm{bd\,V_j} \ni \mathrm{x} \overset{\mathrm{s_y}}{\mapsto} \mathrm{x} - \mathrm{y} \in \mathbf{R}^{\mathrm{n}} \setminus \{0\}$. In fact one shows easily that the composition

$$\check{H}^{n-1}(\mathbf{R}^n \setminus \{0\}) \xrightarrow{s_y^*} \check{H}^{n-1}(\mathrm{bd\,V_j}) \xrightarrow{\delta} \check{\mathrm{H}}^{\mathrm{n}}(\mathrm{cl\,V_j}, \mathrm{bd\,V_j})$$

is an isomorphism. By the already mentioned Hopf theorem [102, Th. 11.5], we obtain that also the composition

$$\eta : \pi^{n-1}(\mathbf{R}^n \setminus \{0\}) \xrightarrow{s_y^\#} \pi^{n-1}(\mathrm{bd\,V_j}) \xrightarrow{\delta} \pi^{\mathrm{n}}(\mathrm{cl\,V_j}, \mathrm{bd\,V_j})$$

is an isomorphism. If $\nu \in \pi^{n-1}(\mathbf{R}^n \setminus \{0\})$ is a generator, then let $\mu = \eta(\nu) \in \pi^n(\mathrm{cl}\, V_j, \mathrm{bd\,V_j})$ be the generator obtained via this isomorphism.

It is also easy to see that the image of ν under the composition

$$\pi^{n-1}(\mathbf{R}^n \setminus \{0\}) \xrightarrow{g^\#} \pi^{n-1}(\mathrm{bd\,V_j}) \xrightarrow{\delta} \pi^{\mathrm{n}}(\mathrm{cl\,V_j}, \mathrm{bd\,V_j})$$

is exactly $d \cdot \mu$.

Now recall the homomorphism κ defined by the sequence (5.1) (where it is necessary to replace U by U_j) and observe that $\deg(f_y, U_j)$ is the image of ν under the homomorphism

$$\pi^{n-1}(\mathbf{R}^n \setminus \{0\}) \xrightarrow{s_y^\#} \pi^{n-1}(\mathrm{bd\,V_j}) \xrightarrow{\mathrm{f}^\#} \pi^{\mathrm{n-1}}(\mathrm{bd\,U_j}) \xrightarrow{\kappa} \pi^{\mathrm{n-1}}(\mathrm{S^{m-1}}).$$

However, in view of the commutativity of the diagram

$$\begin{array}{ccc} \pi^n(\mathrm{cl}\, U_j, \mathrm{bd\,U_j}) & \xrightarrow{f^\#} & \pi^n(\mathrm{cl}\, V_j, \mathrm{bd\,V_j}) \\ \uparrow\delta & & \uparrow\delta \\ \pi^{n-1}(\mathrm{bd\,U_j}) & \xrightarrow{f^\#} & \pi^{n-1}(\mathrm{bd\,V_j}) \end{array}$$

we gather that $\deg(f_y, U_j)$ is the image of μ under the composition

$$\begin{array}{rl} \pi^n(\mathrm{cl}\, V_j, \mathrm{bd\,V_j}) & \xrightarrow{f^\#} \pi^n(\mathrm{cl}\, U_j, \mathrm{bd\,U_j}) \xleftarrow{\mathrm{j}^\#} \pi^{\mathrm{n}}(\mathrm{D^m}, \mathrm{D^m} \setminus \mathrm{U}) \\ & \xrightarrow{i^\#} \pi^n(D^m, S^{m-1}) \xleftarrow{\delta_2} \pi^{n-1}(S^{m-1}) \end{array}$$

Now it is easy to see that $\deg(g \circ f, U_j)$ is the image of ν under the composition

$$\pi^{n-1}(\mathbf{R}^n \setminus \{0\}) \xrightarrow{g^\#} \pi^{n-1}(\mathrm{bd\,V_j}) \xrightarrow{\mathrm{f}^\#} \pi^{\mathrm{n-1}}(\mathrm{bd\,U_j}) \xrightarrow{\delta} \pi^{\mathrm{n}}(\mathrm{cl\,U_j}, \mathrm{bd\,U_j}) \xleftarrow{\mathrm{j}^\#}$$
$$\pi^n(D^m, D^m \setminus U) \xrightarrow{i^\#} \pi^n(D^m, S^{m-1}) \xleftarrow{\delta_2} \pi^{n-1}(S^{m-1}).$$

By the above arguments we see therefore that

$$\deg(g \circ f, U_j) = d \cdot \deg(f_y, U_j).$$

□

5.9 Remark

(i) It is well to emphasize that V can be chosen to be *any* convenient bounded open subset of the domain of g containing $f(\mathrm{cl}\, U)$. It must, however, be chosen so that $g^{-1}(0)$ does not meet $\mathrm{bd}\, \mathrm{V}$.

(ii) Let $(f, U) \in \mathcal{C}(m, n)$, $n \leq m < 2n - 2$. Then $(-f, U) \in \mathcal{C}(m, n)$ and $\deg(-f, U) = (-1)^n \deg(f, U)$.

5.10 Corollary (Product) *Let $(f_1, U_1) \in \mathcal{C}(m_1, n_1)$, $n_1 \leq m_1 < 2n_1 - 2$, $(f_2, U_2) \in \mathcal{C}(n_2, n_2)$. Then, $(f, U_1 \times U_2) \in \mathcal{C}(m_1 + n_2, n_1 + n_2)$, where for $f = f_1 \times f_2 : \mathrm{cl}\,(U_1 \times U_2) \to \mathbf{R}^{n_1 + n_2}$, and if $n_1 + n_2 \geq 3$, then*

$$\deg(f, U_1 \times U_2) = \deg_B(f_2, U_2) \cdot \deg(f_1, U_1).$$

Proof Clearly $\deg(f, U_1 \times U_2) = \deg(g \circ h, U_1 \times U_2)$ where

$$\mathrm{cl}\,(U_1 \times U_2) \ni (x_1, x_2) \mapsto h(x_1, x_2) = (f_1(x_1), x_2) \in \mathbf{R}^{n_1} \times \mathrm{cl}\, U_2,$$

$$\mathbf{R}^{n_1} \times \mathrm{cl}\, U_2 \ni (y_1, y_2) \mapsto g(y_1, y_2) = (y_1, f_2(y_2)) \in \mathbf{R}^{n_1 + n_2}.$$

By the multiplication property

$$\deg(g \circ h, U_1 \times U_2) = \sum_{j \in J} \deg_B(g, V_j) \cdot \deg(h, U_1 \times U_2; V_j),$$

where $\{V_j\}_{j \in J}$ are the connected components of the set $B \times U_2 \setminus g(\mathrm{bd}\,(\mathrm{U}_1 \times \mathrm{U}_2))$ and B is a sufficiently large open ball in $\mathbf{R}^{n_1}$ such that $f_1(\mathrm{cl}\, U_1) \subset \mathrm{cl}\, B$.

It is easy to see that, for each $j \in J$, $\deg(h, U_1 \times U_2; V_j) = \deg(f_1, U_1)$ by the suspension property. Moreover, by the additivity property,

$$\sum_{j \in J} \deg(g, V_j) = \deg(g, B \times U_2).$$

Since, in view of the suspension property again, $\deg(g, B \times U_2) = \deg(f_2, U_2)$, we complete the proof. □

Note that the suspension property is, in fact, equivalent to the product property.

5.11 Remark Proposition 5.6 gives an idea how to remove the annoying assumption concerning dimensions: $n \leq m < 2n - 2$. Namely consider $(f, U) \in \mathcal{C}(m, n)$ where $m \geq n$ and $m \geq 2n - 2$. Let

$$\widetilde{U} = \{(x, \lambda) \in \mathbf{R}^{m+1} \mid x \in U, \lambda \in (-1, 1)\}$$

and let $\widetilde{f} : \operatorname{cl} \widetilde{U} \to \mathbf{R}^{n+1}$ be given by the formula

$$\widetilde{f}(x, \lambda) = (f(x), \lambda) \ (^2).$$

Clearly $(\widetilde{f}, \widetilde{U}) \in \mathcal{C}(m', n')$ where $m' = m + 1$ and $n' = n + 1$. If already $m' < 2n' - 1$ (i.e. $m = 2n - 2$), we may define

$$\deg(f, U) := \deg(\widetilde{f}, \widetilde{U}).$$

If still $m' \geq 2n' - 1$, then we may continue such constructions and after $k = m - 2n + 3$ steps we encounter the situation which enable us to define the degree as in Definition 5.1. In what follows by $\deg(f, U)$ we understand the member of Π_{m-n} constructed as indicated here.

It is clear that the degree thus defined satisfies all the properties enlisted above in Propositions 5.4, 5.6, Theorem 5.8 and Corollary 5.10.

5.B. The generalized degree – general situation

In this section we shall use results of Section 3.B. Therefore we shall maintain the notation used there.

Suppose that E', E are Banach spaces and $F : E' \to E$ is a Fredholm linear operator of nonnegative index $\mathrm{ind(F) = k} \geq 0$. Clearly this implies that $\dim E' = \dim E + k$. Further suppose that E is oriented (the meaning of the orientation in the case $\dim E = \infty$ is explained in 3.15).

5.12 Example

(i) If we suppose that $\dim E', \dim E < \infty$, then the operator $F : E' \to E$ given by $F(x) \equiv 0$ is a Fredholm map of index $\dim E' - \dim E$.

(ii) Let $E' = E \oplus \mathbf{R}^k$ where $n \geq 0$. Define $F : E' \to E$ by the formula $F(x, \lambda) = x$ for any $x \in E$ and $\lambda \in \mathbf{R}^k$. Then F is a Fredholm operator of index k. In other words F is a projection of a Banach space onto its closed subspace of finite codimension. In particular ($k = 0$), F is the identity map.

---

[2] i.e. $\widetilde{f}$ is, topologically, nothing else but the suspension of f. Indeed, if $h : \operatorname{cl} \widetilde{U} \times [0, 1] \to \mathbf{R}^{n+1}$ is given by $h(x, \lambda, t) = ((1 - t|\lambda|)f(x), \lambda)$, then h joins $\widetilde{f}$ to the suspension of f.

5.13 We are now going to study the general coincidence index constituting a homotopy invariant algebraic count of coincidence points of F and a compact map.

Let us denote by $\mathcal{C}^F(E', E)$ the collection of all pairs (g, U) where U is an open bounded subset of E', $g : \mathrm{cl}\, U \to E$ is a compact map such that $g(x) \neq F(x)$ for $x \in \mathrm{bd\,U}$, i.e. $g|\mathrm{bd\,U}$ is a F-field on $\mathrm{bd\,U}$ (recall 3.13).

We say that (g_0, U), (g_1, U) from $\mathcal{C}^F(E', E)$ are *homotopic* if there exists a compact map $h : \mathrm{cl}\, U \times I \to E$ such that $h(\,\cdot\,, i) = g_i$, $i = 0, 1$, and $h(x, t) \neq F(x)$ for any $x \in \mathrm{bd\,U}$, $t \in I$. This means that F-fields g_0 and g_1 on $\mathrm{bd\,U}$ are F-homotopic: $g_0|\mathrm{bd\,U} \simeq_{\mathrm{F}} \mathrm{g_1}|\mathrm{bd\,U}$.

Observe here that homotopy considerations are of little value in the case of Fredholm operators F of negative index. The Sard-Smale theorem asserts that the range of $F - g$, where g is of C^1-class, is nowhere dense. Hence the simple idea of perturbing g with a point $y \in E$ of sufficiently small norm to a map $g' = g - y$ without affecting the solvability of $F - g'$ breaks down in total agreement with Proposition 5.17 (i) below (see also Remark 5.5).

The desired homotopy invariant may be constructed by a suitable application of Gęba's results [77]; it was done by Borisovich *et al.* in [30] (see also [33]) however without a direct reference to [77]. Our approach, implicitly contained in [77, 76], is based on results from Section 3.B. It is also close to the attitude of [106].

5.14 Remark If $\dim E' = m$, $\dim E = n$ and $n \leq m < \infty$, then $k = m - n$ and $(g, U) \in \mathcal{C}^F(E', E)$ if and only if $(F - g, U) \in \mathcal{C}(m, n)$. We may define

$$\mathrm{ind_F(g, U) := deg(F - g, U)}$$

(see Remark 5.11).

Hence in what follows we therefore restrict ourselves to the infinite-dimensional Banach spaces E', E. In this case, if X is a closed bounded subset of E' and $A \subset X$ is closed, then the triple $(X, A; u)$, where $u = F|X$, is a regular object of finite-type in the generalized Leray-Schauder category $\mathcal{LS}(E)$ (see Definitions 3.11 and 3.22). Indeed, it is enough to take $m_0 = k + 4$ in Definition 3.22 to see that $\dim X_L \leq \dim L + k < 2 \dim L - 3$ where $X_L = u^{-1}(L)$ and L is a finite-dimensional linear subspace in E, $\dim L \geq m_0$.

Below we use the notation $(X, A; F)$ to denote the triple $(X, A; u)$.

With no loss of generality assume that $\mathrm{cl}\,U \subset D^{E'}$ ([3]). Since $(\mathrm{cl}\,U, \mathrm{bd}\,\mathrm{U}; F)$ and $(\mathrm{bd}\,\mathrm{U}; \mathrm{F})$ are regular objects of finite type we may consider the following sequence of abelian groups and homomorphisms (recall 3.35)

$$\begin{array}{ccccc} \pi^E(\mathrm{bd}\,\mathrm{U};\mathrm{F}) & \xrightarrow{\delta_1} & \pi^{E+1}(\mathrm{cl}\,U, \mathrm{bd}\,\mathrm{U}; \mathrm{F}) & \xleftarrow{\pi^{E+1}(j)} & \pi^{\mathrm{n}}(\mathrm{D}, \mathrm{D}\setminus \mathrm{U}; \mathrm{F}) \\ & \xrightarrow{\pi^{E+1}(i)} & \pi^n(D, S; F) & \xleftarrow{\delta_2} & \pi^E(S; F) \end{array}$$

where $D := D^{E'}$, $S := S^{E'}$, δ_1 denotes the coboundary homomorphism of the pair $(\mathrm{cl}\,U, \mathrm{bd}\,\mathrm{U}; \mathrm{F})$, $j : (\mathrm{cl}\,U, \mathrm{bd}\,\mathrm{U}) \to (\mathrm{D}, \mathrm{D}\setminus\mathrm{U})$, $i : (D, S) \to (D, D\setminus U)$ are the inclusions (obviously being morphisms of $\mathcal{LS}(E)$) and δ_2 is the coboundary homomorphism of the pair $(D, S; F)$. Clearly $\pi^{E+1}(j)$ is an isomorphism (excision!) and δ_2 is an isomorphism in view of the contractibility of $D_L = F^{-1}(L) \cap D$, for each finite-dimensional linear subspace L of E, and the exactness of the cohomotopy sequence of the pair (D, S) (see Proposition 3.33).

Let $\alpha := [g|\mathrm{bd}\,\mathrm{U}]_{\mathrm{F}} \in \pi^{\mathrm{E}}(\mathrm{bd}\,\mathrm{U}; \mathrm{F})$.

5.15 Definition By the *generalized coincidence index of g on U with respect to F* we understand the element

$$\mathrm{ind}_{\mathrm{F}}(\mathrm{g}, \mathrm{U}) := \kappa(\alpha) \in \pi^{\mathrm{E}}(\mathrm{S}; \mathrm{F}) \cong \Pi_{\mathrm{k}}$$

where $\kappa = \delta_2^{-1} \circ \pi^{E+1}(i) \circ (\pi^{E+1}(j))^{-1} \circ \delta_1$ (see Example 3.26).

5.16 Remark

(i) If $U = B := B^{E'}$, then $\kappa = id_{\pi^E(S;F)}$ and $\mathrm{ind}_{\mathrm{F}}(\mathrm{g}, \mathrm{U}) = [\mathrm{g}|\mathrm{S}]_{\mathrm{F}}$.

(ii) Suppose that $(g, U) \in \mathcal{C}^F(E', E)$ and g is a finite-dimensional map, i.e. $g(\mathrm{cl}\,U)$ is contained in a finite-dimensional linear subspace T of E. Now let $Q : E \to E$ be a linear bounded projection with $\mathrm{Ker}(Q) = \mathrm{R}(F)$ and consider a space $\mathrm{R}(Q) \oplus (I - Q)(T) = \mathrm{R}(Q) + T$. "Adding dimensions" to $(I - Q)(T)$ in $\mathrm{R}(F)$ (according to our assumption $\dim E = \infty$) we construct the smallest linear subspace $L \supset T + \mathrm{R}(Q)$ with $m + 1 = \dim L \geq k + 3$ having a "general position" with respect to the range $\mathrm{R}(F)$, i.e. $L = L' \oplus \mathrm{R}(Q)$ where $L' = \mathrm{R}(F) \cap L$, and such that $T \subset L$. It is clear that $\dim F^{-1}(L) = m + 1 + k$.

To simplify the notation let us put $A := \mathrm{bd}\,\mathrm{U}$, $X := \mathrm{cl}\,U$. Hence $X_L = X \cap F^{-1}(L)$ and $A_L = A \cap F^{-1}(L)$. Let $F_L = F|F^{-1}(L)$, $g_L = g|X_L$

[3]recall that $D^{E'} = \{x \in E' \mid ||x|| \leq 1\}$, $S^{E'} = \mathrm{bd}\,\mathrm{D}^{\mathrm{E}'} = \{\mathrm{x} \in \mathrm{E}' \mid ||\mathrm{x}|| = 1\}$ and $B^{E'} = D^{E'} \setminus S^{E'} = \{x \in E' \mid ||x|| < 1\}$.

and $G = F_L - g_L : (X_L, A_L) \to (L, L \setminus \{0\})$.

We claim that

$$\mathrm{ind_F(g,U) = deg(F_L - g_L, U \cap F^{-1}(L))}.$$

First it is clear that $\xi^L([G|A_L]) = [g|A]_F$ where $\xi^L : \Sigma^L(A_L) \to \pi^E(A;F)$ (see 3.18). Next $\overline{\xi}^L : \pi^m(S^{m+k}) \cong \Sigma^L(F^{-1}(L) \cap S) \to \pi^E(S;F) \cong \Pi_k$ is an isomorphism since $m \geq k+2$. Set $S_L = F^{-1}(L) \cap S$ and $D_L = F^{-1}(L) \cap D$.

Consider a diagram

$$\begin{array}{ccc}
\pi^E(A;F) & \xleftarrow{\xi^L} & \Sigma^L(A_L) \\
\downarrow \delta & & \downarrow \delta^L \\
\pi^{E+1}(X,A;F) & \xleftarrow{\xi^{L+1}} & \Sigma^{L+1}(X_L,A_L) \\
\uparrow & & \uparrow \\
\pi^{E+1}(D,D\setminus U;F) & \xleftarrow{\xi^{L+1}} & \Sigma^{L+1}(D_L,D_L\setminus U_L) \\
\downarrow & & \downarrow \\
\pi^{E+1}(D,S;F) & \xleftarrow{\xi^{L+1}} & \Sigma^{L+1}(D_L,S_L) \\
\uparrow \delta & & \uparrow \delta^L \\
\pi^E(S;F) & \xleftarrow{\overline{\xi}^L} & \Sigma^L(S_L)
\end{array}$$

Now we see that if χ denotes the homomorphism induced by the right column, then $\overline{\xi}^L \circ \chi([G]) = \mathrm{ind_F(g,U)}$. But the right column in this diagram is isomorphic to the sequence

$$\begin{array}{rl}
\pi^m(A_L) \xrightarrow{\delta} & \pi^{m+1}(X_L,A_L) \longleftarrow \pi^{m+1}(D^{m+k+1}, D^{m+k+1} \setminus U_L) \\
\longrightarrow & \pi^{m+1}(D^{m+k+1}, S^{m+k}) \xleftarrow{\delta} \pi^m(S^{m+k})
\end{array}$$

and hence $\chi([G]) = \deg(G, U \cap F^{-1}(L))$.

Something more can be said with regard to the above result. Namely consider $T' = F^{-1}(\mathrm{R}(Q) + T)$, $F' = F|T'$, $g' = g|X \cap T'$ and let $G' =$

$F' - g'$ ([4]). Evidently $\dim T', \dim T < \infty$, $\dim T' - \dim T = k$ but not necessarily $\dim T' < 2\dim T - 2$. According to Remark 5.14, we may still define (through suspensions) $\deg(G', U \cap T')$. As one may expect

$$\deg(G', U \cap T') = \mathrm{ind_F(g, U)}.$$

To see that it is enough to show that $\deg(G', U \cap T') = \deg(G, U \cap F^{-1}(L))$. For simplicity assume that the "dimension defect" here is 1, i.e. $L = \mathrm{R}(Q) \oplus (I - Q)(T) \oplus \mathbf{R}y$ where $y \in \mathrm{R}(F)$. Hence $F^{-1}(L) = T' \oplus \mathbf{R}x$ where $F(x) = y$. It is now clear that G defined above is homotopic to the suspension of G' and this completes the proof. If the "dimension defect" is greater than one, we may iterate this argument.

(iii) In particular, let $E' = \mathbf{R}^m$, $E = \mathbf{R}^n$, $m \geq n$, and $F \equiv 0$. Then F is a Fredholm operator of index $k = m - n$. It is clear that $\mathrm{ind_0(g, U)} = \mathrm{deg(-g, U)}$ for any $(g, U) \in \mathcal{C}(m, n)$.

The defined index $\mathrm{ind_F}$ satisfies all expected properties analogous to those enlisted in Proposition 5.4.

5.17 Proposition *Let $(g, U) \in \mathcal{C}^F(E', E)$.*

(i) (Existence) *If $\mathrm{ind_F(g, U)} \neq 0 \in \Pi_\mathrm{k}$, then g and F have a coincidence point, i.e there is $x_0 \in U$ such that $F(x_0) = g(x_0)$. Moreover, if V is a component of $E \setminus (F - g)(\mathrm{bd}\, U)$ containing 0, then $V \subset (F - g)(U)$.*

(ii) (Localization) *If $V \subset U$ is open and $F(x) \neq g(x)$ for $x \in \mathrm{cl}\, U \setminus V$, then $(g|\mathrm{cl}\, V, V) \in \mathcal{C}^F(E', E)$ and $\mathrm{ind_F(g, U)} = \mathrm{ind_F(g, V)}$.*

(iii) (Additivity) *If U_1, U_2 are open disjoint subsets of U such that $F(x) \neq g(x)$ for $x \in \mathrm{cl}\, U \setminus (U_1 \cup U_2)$, then $(g|\mathrm{cl}\, U_j) \in \mathcal{C}^F(E', E)$, $j = 1, 2$, and $\mathrm{ind_F(g, U)} = \mathrm{ind_F(g, U_1)} + \mathrm{ind_F(g, U_2)}$.*

(iv) (Homotopy) *If $(f, U) \in \mathcal{C}^F(E', E)$ is homotopic to (g, U), then $\mathrm{ind_F(g, U)} = \mathrm{ind_F(g, U)}$.*

(v) (Boundary Dependence) *If $(f, U) \in \mathcal{C}^F(E', E)$ and $f|\mathrm{bd\, U} = \mathrm{g|bd\, U}$, then $\mathrm{ind_F(g, U)} = \mathrm{ind_F(f, U)}$.*

(vi) (Restriction) *Suppose that $g(\mathrm{cl}\, U) \subset T$ where T is a closed subspace of E. Then $\mathrm{ind_F(g, U)} = \mathrm{ind_G(g', U \cap T')}$ where $T' = F^{-1}(T + \mathrm{R}(Q))$, $g' = g|\mathrm{cl}\, U \cap T'$ and $G = F|T'$.*

Proof Properties (iv) and (v) follow from the very definition.

Since F is proper and g is compact, we easily gather that there exists a

[4] recall that $\mathrm{R}(Q) + T = \mathrm{R}(Q) \oplus (I - Q)(T)$.

finite-dimensional approximation $f : \mathrm{cl}\, U \to E$ (i.e. $f(\mathrm{cl}\, U) \subset L_0$, $\dim L_0 < \infty$) such that $f(x) \neq F(x)$ for all x outside a neighborhood of the set $\{x \in \mathrm{cl}\, U \mid g(x) = F(x)\}$ and such that (f, U) and (g, U) being close to each other are homotopic in $\mathcal{C}^F(E', E)$. For this reason it is enough to take a sufficiently fine Schauder projection $p : \mathrm{cl}\, g(\mathrm{cl}\, U) \to L_0 \subset E$, put $f := p \circ g$ and consider the linear homotopy $h(\,\cdot\,, t) = (1-t)g + tf$.

By (iv), $\mathrm{ind_F(g, U) = ind_F(f, U)}$ and by Remark 5.16 (ii), $\mathrm{ind_F(f, U)} = \mathrm{deg(F_L - f_L, U \cap L)}$ where $L = L_0 + \mathrm{R}(Q)$ (see Remark 5.16 (ii)).

Having this, all properties (i)–(iii) and (vi) follow from Proposition 5.4 and Remark 5.5. □

5.18 Remark

(i) Let us consider the case $E' = E$, $F = I := id$ and let $(g, U) \in \mathcal{C}^I(E, E)$. We shall show that our index $\mathrm{ind_I(g, U)} \in \Pi_0 = \mathbf{Z}$ is the Leray-Schauder fixed point index of g on U (see [94]).

In view of Proposition 5.17 (iv), we may assume that g is a finite-dimensional map. By Remarks 5.16 (ii) and 5.3 (iii),

$$\mathrm{ind_I(g, U) = deg(I - g_L, U \cap L) = deg_B(I - g_L, U_L)}$$

and this is nothing else but the definition of the classical Leray-Schauder fixed-point index (see also [94]).

Similarly one shows that if $\mathrm{ind(F) = k = 0}$, then $\mathrm{ind_F(g, U)}$ is the Mawhin degree $\deg_M(F - g, U)$ defined e.g. in [139] or in [140].

(ii) Let E', E and F be as in Example 5.12, let $k > 0$ and let $(g, U) \in \mathcal{C}^F(E', E)$. It is clear that coincidences of g with F are just fixed points of a compact map $g' : \mathrm{cl}\, U \to E'$ given by $g'(x, \lambda) = (g(x, \lambda), \lambda)$ for $(x, \lambda) \in \mathrm{cl}\, U$ ($x \in E$, $\lambda \in \mathbf{R}^k$) and $(g', U) \in \mathcal{C}^I(E', E')$. However, we easily see that the Leray-Schauder fixed point index of g' is 0. Hence this index is useless under the presence of the dimension defect.

As in Section 5.B, one may state the *multiplication property* of $\mathrm{ind_F}$. Suppose that $g : \mathrm{cl}\, U \to E$, where $U \subset E'$ is open bounded, is a compact map. First note that if A is a connected subset of $E \setminus (F - g)(\mathrm{bd\, U})$, then it makes sense to define

$$\mathrm{ind_F(g, U; A) := ind_F(g - y, U)}$$

where y is a point from A. This definition is correct.

Let $(h, V) \in \mathcal{C}^G(E, E'')$ where $G : E \to E''$ is a Fredholm operator of index 0 and $(F-g)(\mathrm{cl}\,U) \subset \mathrm{cl}\,V$. Observe that the family $\{V_j\}_{j\in J}$ of all components of $V \setminus (F-g)(\mathrm{bd\,U})$ is at most countable (for g is compact). Assume that the composition $(G-h)\circ(F-g)$ (being of the form $G\circ F - f$ where $f = G\circ g + h\circ(F-g)$ is a compact map on $\mathrm{cl}\,U$) has no zeros on $\mathrm{bd\,U}$. Under these assumption

$$\mathrm{ind_{G\circ F}(f,U)} = \sum_{\mathrm{j}\in \mathrm{J}} \mathrm{deg_M(G-h,V_j)\cdot ind_F(g,U;V_j)}, \tag{5.3}$$

where $\deg_M(G-h,V_j)$ stands for Mawhin's degree (see [140]). Note that $G\circ F$ is a Fredholm operator of index k and the summation above is finite for $(G-h)^{-1}(0)$ is compact. The easy proof uses Theorem 5.8 – we leave details to the reader.

As a simple corrolary we get the following useful fact. As above, let $P : E' \to E'$ (resp. $Q : E \to E$) be a linear bounded projector onto $\mathrm{Ker}(F)$ (resp. with $\mathrm{Ker}(Q) = \mathrm{R}(F)$).

5.19 Corollary *Assume that, for $(g,U) \in \mathcal{C}^F(E',E)$, there are maps $g_1 : \mathrm{Ker}(P) \to \mathrm{Ker}(Q)$ and $g_2 : \mathrm{R}(P) \to \mathrm{R}(Q)$ and such that $g(x) = g_1(x-Px) + g_2(Px)$. Then*

$$\mathrm{ind_F(g,U) = deg_M(F_1 - g_1, U\cap Ker(P))\cdot deg(g_2, U\cap R(P))}$$

where $F_1 : \mathrm{Ker}(P) \to \mathrm{R}(F)$ is the restriction to $\mathrm{Ker}(P)$ of F and, hence, a Fredholm operator of index 0.

5.20 Example For instance, if $(g, B^{E'}) \in \mathcal{C}^F(E',E)$ is such that $\mathrm{cl}\,U \times [0,1] \ni (x,t) \mapsto h(x,t) = g(t(x-Px)+Px)$ provides a homotopy (i.e. $F(x) \neq h(x,t)$ when $x \in \mathrm{bd\,U}$, $t\in[0,1]$), then

$$\mathrm{ind_F(g,U) = deg(g\circ P, B^{E'}\cap Ker(F))}.$$

Observe that this result may also be easily derived as a consequence of the restriction property of $\mathrm{ind_F}$ – see Proposition 5.17.

5.C. The coincidence index – set-valued case

Now we shall deal with a situation even more general than one from Section 5.B. Namely we are going to define the coincidence index for morphisms (set-valued maps). The homotopy invariant we are going to introduce embraces the invariants discussed above and earlier studied ones e.g. from [47, 135, 43, 37, 42, 44, 119] and [31]. Let again $F : E' \to E$ be a Fredholm operator of index $\mathrm{ind(F) = k}$ and assume that E is oriented.

5.21 In analogy with 4.50, let $\widetilde{M}^F(E', E)$ (resp. $M^F_{CE}(E', E)$) denote the collection of all pairs (Φ, U) where U is an open bounded subset of E', $\Phi \in \widetilde{M}(\mathrm{cl}\, U, E)$ (resp. $\Phi \in M_{CE}(\mathrm{cl}\, U, E)$) such that:

(i) for $x \in \mathrm{bd\, U}$, $F(x) \notin \Phi(x)$;

(ii) Φ determines a compact map;

(iii) if $\dim E = m + 1 < \infty$, $m \geq 0$, then $\mathrm{i}(\Phi) \leq m$ (recall 4.8).

We say that (Φ_0, U), (Φ_1, U) from $\widetilde{M}^F(E', E)$ (resp. $M^F_{CE}(E', E)$) are *homotopic* if there is a *homotopy* $\Phi \in \widetilde{M}(\mathrm{cl}\, U \times I, E)$ (resp. $M_{CE}(\mathrm{cl}\, U \times I, E)$) such that

(j) $\Phi \circ i_j = \Phi_j$, $j = 0, 1$;

(jj) $F(x) \notin \Phi(x, t)$ for $x \in \mathrm{bd\, U}$ and $t \in I$;

(jjj) Φ determines a compact map;

(jv) if $\dim E = m + 1 < \infty$, then $\mathrm{i}(\Phi) \leq m + 1$.

Let $(\Phi, U) \in \widetilde{M}^F(E', E)$ (resp. $M^F_{CE}(E', E)$). Again, without loss of generality we assume that $\mathrm{cl}\, U \subset D$, where D is the unit closed ball in E'. Let $i : \mathrm{bd\, U} \to \mathrm{cl\, U}$ be the inclusion. Evidently $\Psi := \Phi \circ i \in \widetilde{M}(\mathrm{bd\, U; F})$ (resp. $M_{CE}(\mathrm{bd\, U; F})$) – see 4.50. Therefore, by Theorem 4.51 (see also Remark 4.52 (i)), there is a unique (up to F-homotopy) F-field $g : \mathrm{bd\, U} \to \mathrm{E}$ such that $\Phi \simeq_F g$. Precisely, $[g]_F = (\pi^E(p))^{-1}[q]_v$ where a cotriad (p, q) represents Ψ and $v = F \circ p$.

5.22 Definition By the *generalized coincidence index of Φ on U with respect to F* we mean

$$\mathrm{ind_F}(\Phi, U) := \mathrm{ind_F(\overline{g}, U)}$$

where $\overline{g} : \mathrm{cl}\, U \to E$ is an arbitrary compact extension of g.

We easily see that this definition is correct because $(\overline{g}, U) \in \mathcal{C}^F(E', E)$ and $\mathrm{ind_F(\overline{g}, U)}$ does not depend on the choice of g and its extension $\overline{g}$.

5.23 Remark

(i) If Φ is a 0-morphism (resp. CE-morphism) determining a single-valued map g, then $\mathrm{ind_F}(\Phi, U) = \mathrm{ind_F(g, U)}$ (see Example 4.10 (ii) or 4.41).

(ii) As in Remark 5.18 (i), if $E' = E$ and $F = I = id$, then our index $\mathrm{ind_I}(\Phi, U)$ is the Leray-Schauder fixed point index studied, for instance, in [37, 42] or [111] and [119].

Now we shall collect the important properties of the defined index $\mathrm{ind_F}$.

5.24 Theorem *Let* $(\Phi, U) \in \widetilde{M}^F(E', E)$ *(resp.* $M^F_{CE}(E', E)$*).*

(i) (Existence) *If* $\mathrm{ind_F}(\Phi, U) \neq \mathrm{o} \in \Pi_k$*, then* Φ *and* F *have a coincidence point, i.e there is* $x_0 \in U$ *such that* $F(x_0) \in \Phi(x_o)$*.*

(ii) (Localization) *If* $V \subset U$ *is open and* $F(x) \notin \Phi(x)$ *for* $x \in \mathrm{cl}\, U \setminus V$*, then* $(\Phi|\mathrm{cl}\, V, V) \in \widetilde{M}^F(E', E)$ *(resp.* $M^F_{CE}(E', E)$*) and* $\mathrm{ind_F}(\Phi, U) = \mathrm{ind_F}(\Phi, V)$*.*

(iii) (Additivity) *If* U_1*,* U_2 *are open disjoint subsets of* U *such that* $F(x) \notin \Phi(x)$ *for* $x \in \mathrm{cl}\, U \setminus (U_1 \cup U_2)$*, then* $(\Phi|\mathrm{cl}\, U_j \in \widetilde{M}^F(E', E)$ *(resp.* $M^F_{CE}(E', E)$*),* $j = 1, 2$*, and* $\mathrm{ind_F}(\Phi, U) = \mathrm{ind_F}(\Phi, U_1) + \mathrm{ind_F}(\Phi, U_2)$*.*

(iv) (Homotopy) *If* $(\Psi, U) \in \widetilde{M}^F(E', E)$ *is homotopic to* (Φ, U)*, then* $\mathrm{ind_F}(\Phi, U) = \mathrm{ind_F}(\Psi, U)$*.*

(v) (Boundary Dependence) *If* $(\Psi, U) \in \widetilde{M}^F(E', E)$ *(resp.* $M^F_{CE}(E', E)$*) and* $\Psi|\mathrm{bd\, U} = \Phi|\mathrm{bd\, U}$*, then* $\mathrm{ind_F}(\Phi, U) = \mathrm{ind_F}(\Psi, U)$*.*

(vi) (Restriction)*Suppose that* $\Phi(\mathrm{cl}\, U) \subset T$ *where* T *is a closed subspace of* E*. Then* $\mathrm{ind_F}(\Phi, U) = \mathrm{ind_G}(\Phi', U \cap T')$ *where* $T' = F^{-1}(T + \mathrm{R}(Q))$ ([5])*,* $\Phi' = \Phi|\mathrm{cl}\, U \cap T'$ *and* $G = F|T'$*.*

Proof All these properties are simple consequences of Definition 5.22 and Proposition 5.17. Perhaps only (i) requires a comment. If $F(x) \notin \Phi(x)$ for $x \in \mathrm{cl}\, U$, then $\Phi \in \widetilde{M}(\mathrm{cl}\, U; F)$ and $\mathrm{ind_F}(\Phi, U) = \mathrm{ind_F(g, U)}$ where g is an F-field on $\mathrm{cl}\, U$. Hence $F(x) \neq g(x)$ on $\mathrm{cl}\, U$ and $\mathrm{ind_F(g, U)} = 0$. □

5.25 Let us consider a special case: namely let $E' = \mathbf{R}^m$, $E = \mathbf{R}^n$, $1 \leq n \leq m < \infty$ and $F \equiv 0$. All that follows is implicitly contained in the above paragraphs; we enter some details more carefully in order to provide a better illustration of these ideas and for further reference.

In analogy to the notation from Section 5.A it makes sense to write in

[5] Q has the same meaning as in Remark 5.16 (ii).

this case $\widetilde{M}(m,n)$ (or $M_{CE}(m,n)$) instead of $\widetilde{M}^F(E',E)$ (or $M_{CE}^F(E',E)$) to denote the collection of all pairs (Φ,U) where U is an open bounded subset of $E' = \mathbf{R}^m$, $\Phi \in \widetilde{M}_{n-1}(\mathrm{cl}\, U, \mathbf{R}^n)$ (or $\Phi \in M_{CE}(\mathrm{cl}\, U, \mathbf{R}^n)$) such that $0 \notin \Phi(x)$ for $x \in \mathrm{bd\,U}$. Pairs (Φ_0, U), (Φ_1, U) from $\widetilde{M}(m,n)$ (resp. $M_{CE}(m,n)$) are homotopic if there is a homotopy $\Phi \in \widetilde{M}_n(\mathrm{cl}\, U \times I, \mathbf{R}^n)$ (resp. $M_{CE}(m,n)$) such that $\Phi \circ i_j = \Phi_j$, $j = 0,1$, and $0 \notin \Phi(x,t)$ for $x \in \mathrm{bd\,U}$ and $t \in I$.

Let $(\Phi,U) \in \widetilde{M}(m,n)$ (resp. $M_{CE}(m,n)$). Clearly $\Psi = \Phi \circ i \in \widetilde{M}_{m-1}(\mathrm{bd\,U}, \mathbf{R}^{\mathrm{n}})$ (resp. $M_{CE}(\mathrm{bd\,U}, \mathbf{R}^{\mathrm{n}})$) where $i : \mathrm{bd\,U} \to \mathrm{cl\,U}$ is the inclusion. Now according to Definition 5.22, we define $\deg(\Phi,U)$ (instead of $\mathrm{ind_F}$) by

$$\deg(\Phi,U) := \deg(\overline{g},U)$$

where $\overline{g} : \mathrm{cl}\, U \to \mathbf{R}^n$ is an arbitrary extension of a map $g : \mathrm{bd\,U} \to \mathbf{R}^{\mathrm{n}} \setminus \{0\}$ such that $\Psi \simeq_n g$ (resp. $\Psi \simeq_{CE} g$). Such a map g exists in view of Theorem 4.25 (resp. Theorem 4.47).

Equivalently, in case $\Phi \in M_{CE}(\mathrm{cl}\, U, \mathbf{R}^n)$, in order to get an appropriate $g : \mathrm{bd\,U} \to \mathbf{R}^{\mathrm{n}} \setminus \{0\}$ we may use graph approximations. Namely one may employ Theorem 4.59 provided its assumptions hold. In applications to follow we shall deal with maps (morphisms) arising in the study of differential inclusion (see Examples 6.7 and 6.10). In this case such a construction is valid.

Evidently the degree deg thus defined satisfies all the properties enlisted in Proposition 5.4 (or Theorem 5.24).

Finally observe that in order to define $\deg(\Phi,U)$ we need to have $\Phi|\mathrm{bd\,U}$. Hence even if Φ is defined merely on $\mathrm{bd\,U}$, then we may consider its degree $\deg(\Phi,U)$. For instance, in such a setting, the nontriviality of $\deg(\Phi,U)$ will imply the existence of zeros of any extension of Φ onto $\mathrm{cl}\, U$.

Next, observe that given a morphism $\Phi \in M_{n-1}(D^n, \mathbf{R}^n)$ such that $\Phi(S^{n-1}) \subset S^{n-1}$,

$$\deg(\Phi, B^n) = \deg(\Phi)$$

where $\deg(\Phi)$ has been defined in 4.31.

Now we shall present some results providing means to compute $\deg(\Phi,U)$ starting with the following example.

5.26 Example If $(\Phi, U) \in \widetilde{M}(m,n)$ (or $M_{CE}(m,n)$ and $(f,U) \in \mathcal{C}(m,n)$ is such that, for any $x \in \mathrm{bd\,U}$ and $y \in \Phi(x)$,

$$|y - f(x)| < |y| + |f(x)|$$

or, what is equivalent,

$$y \cdot f(x) > -|y||f(x)|(^6),$$

then (Φ, U), (f, U) are homotopic (within $\widetilde{M}(m,n)$ or $M_{CE}(m,n)$) and, consequently, $\deg(\Phi, U) = \deg(f, U)$.

Indeed, the above conditions show that, for any $x \in \mathrm{bd\,U}$ and $t \in I$, $(1-t)\Phi(x) + tf(x) \in \mathbf{R}^n \setminus \{o\}$. Hence, by Example 4.49, $\Phi \simeq_n f$ in $\widetilde{M}(\mathrm{cl}\,U, \mathrm{bd\,U}, \mathbf{R}^\mathrm{n}; \mathbf{R}^\mathrm{n} \setminus \{0\})$ (or $\Phi \simeq_{CE} f$ in $M_{CE}(\mathrm{cl}\,u, \mathrm{bd\,U}; \mathbf{R}^\mathrm{n}, \mathbf{R}^\mathrm{n} \setminus \{0\})$).

As in 5.7, we are in a position, to define $\deg(\Phi, U; A)$, where $\Phi \in \widetilde{M}_{n-1}(\mathrm{cl}\,U, \mathbf{R}^n)$ (or $\Phi \in M_{CE}(\mathrm{cl}\,U, \mathbf{R}^n)$), $U \in \mathbf{R}^m$, $m \geq n$, is open bounded and A is a connected subset of $\mathbf{R}^n \setminus \Phi(\mathrm{bd\,U})$ (note that the set $\Phi(\mathrm{bd\,U})$ is compact), setting

$$\deg(\Phi, U; A) := \deg(\Phi_y, U)$$

where $\Phi_y = \Phi - y$, $y \in A$ is an arbitrary point (see 4.15 or 4.42 (ii)).

5.27 Proposition *Assume that $\Phi \in \widetilde{M}_{n-1}(\mathrm{cl}\,U, \mathbf{R}^n)$, U is as above, V is an open subset of $\mathbf{R}^n$ such that $\Phi(\mathrm{cl}\,U) \subset \mathrm{cl}\,V$ and $(\Psi, V) \in \widetilde{M}(n,n)$, $\Psi \in \widetilde{M}_0$, is such that $0 \notin \Psi \circ \Phi(\mathrm{bd\,U})$. Then $(\Psi \circ \Phi, U) \in \widetilde{M}(m,n)$. If $\{V_j\}_{j \in J}$ is the family of all connected components of the set $V \setminus \Phi(\mathrm{bd\,U})$, then*

$$\deg(\Psi \circ \Phi, U) = \sum_{j \in J} \deg(\Psi, V_j) \cdot \deg(\Phi, U; V_j).$$

Proof As in the proof of Theorem 5.8 we show that all expresions in the above formula are well-defined. In order to show the assertion it is sufficient to show that, for each $j \in J$,

$$\deg(\Psi \circ \Phi, U_j) = \deg_B(\Psi, V_j) \cdot \deg(\Phi, U_j; V_j)$$

where $U_j = \{x \in U \mid \Phi(x) \subset V_j\}$.

Indeed, fix $j \in J$, denote $Z := \Phi(\mathrm{bd\,U_j})$ and suppose that a map g :

[6] recall that "$\cdot$" denotes the scalar product in $\mathbf{R}^n$

$Z \to \mathbf{R}^n \setminus \{0\}$ is such that $\Psi|Z \simeq_0 g$ as morphisms from $\widetilde{M}_0(Z, \mathbf{R}^n \setminus \{0\})$. Using [109, Th. 1.3] we take an extension $\overline{g} : Z \cup \mathrm{cl}\, V_j \to \mathbf{R}^n$ of g having a *unique* zero in $y \in V_j$. By the definition

$$\deg(\Psi, V_j) = \deg(\overline{g}, V_j). \tag{5.4}$$

Observe that $\Phi(\mathrm{cl}\, U_j) \subset Z \cup \mathrm{cl}\, V_j$. Moreover, it is clear that $\Psi \circ (\Phi|\mathrm{bd}\, U_j) \simeq_n \overline{g} \circ (\Phi|\mathrm{bd}\, U_j)$ as morphisms from $\widetilde{M}_{n-1}(\mathrm{bd}\, U_j, \mathbf{R}^n \setminus \{0\})$.

Next let $f : \mathrm{bd}\, U_j \to S^{n-1}(y, r)$, where $r > 0$ is such that $D^n(y, r) \subset V_j$, be a map such that $\Phi|\mathrm{bd}\, U_j \simeq_n f$ as morphisms from $\widetilde{M}_{n-1}(\mathrm{bd}\, U_j, \mathbf{R}^n \setminus \{y\})$. Now it is matter of a simple computation to show that $(\Psi \circ \Phi)|\mathrm{bd}\, U_j$ is homotopic to $\overline{g} \circ f$. If $\overline{f} : \mathrm{cl}\, U_j \to D^n(y, r)$ is an extension of f, then

$$\deg(\Phi, U_j; V_j) = \deg(\overline{f}, U_j; y) \tag{5.5}$$

and $\overline{g} \circ \overline{f} : \mathrm{cl}\, U_j \to \mathbf{R}^n$ is an extension of $\overline{g} \circ f$; hence

$$\deg(\Psi \circ \Phi, U_j) = \deg(\overline{g} \circ \overline{f}, U_j). \tag{5.6}$$

In view of Remark 5.9 and the multiplication property from Theorem 5.8,

$$\deg(\overline{g} \circ \overline{f}, U_j) = \deg(\overline{g}, B^n(y, r)) \cdot \deg(\overline{f}, U_j; y).$$

But, by the localization property, $\deg(\overline{g}, B^n(y, r)) = \deg(\overline{g}, V_j)$. Hence, by (5.4), (5.5) and (5.6), we get the assertion. □

As above, Propsition 5.27 implies the following corollary (recall 4.15).

5.28 Corollary *Let* $(\Phi_1, U_1) \in \widetilde{M}(n_1, n_1)$ *and* $(\Phi_2, U_2) \in \widetilde{M}(m_2, n_2)$, $m_2 \geq n_2$. *If* $\Phi_1 \in \widetilde{M}_0$ *and* $n_1 + n_2 \geq 3$, *then* $(\Phi_1 \times \Phi_2, U_1 \times U_2) \in \widetilde{M}(n_1 + m_2, n_1 + n_2)$ *and*

$$\deg(\Phi_1 \times \Phi_2, U_1 \times U_2) = \deg(\Phi_1, U_1) \cdot \deg(\Phi_2, U_2).$$

It is not difficult to obtain results analogous to those from Proposition 5.27 and Corollary 5.28 concerning $\mathrm{ind}_F : \widetilde{M}^F(E', E) \to \Pi_{\mathrm{ind}(F)}$ (comp. (5.3 and Corollary 5.19).

5.D. Continuation results

In this short section we shall obtain several existence results of the Leray-Schauder type (cf. eg. [140, 95]).

Let again E', E be Banach spaces and let $F : E' \to E$ be a Fredholm operator of index $k \geq 0$. Suppose that U is an open subset of $F' \times \mathbf{R}$ and that $\Phi \in \widetilde{M}(\mathrm{cl}\, U, E)$ (or $\Phi \in M_{CE}(\mathrm{cl}\, U, E)$). If $\dim E = m + 1 < \infty$, then we assume that $\mathrm{i}(\Phi) \leq m$. Moreover we assume that, for each bounded subset C of $\mathrm{cl}\, U$, $\mathrm{cl}\, \Phi(C)$ is compact.

For any $\lambda \in \mathbf{R}$, let

$$U_\lambda := \{x \in E' \mid (x, \lambda) \in U\},$$

$$\Phi_\lambda := \Phi \circ i_\lambda$$

where $i_\lambda : \mathrm{cl}\, U_\lambda \to U$ is the inclusion. Finally, let $a \leq b$.

The following result easily follows from the homotopy property

5.29 Lemma *If, for each $\lambda \in [a, b]$, U_λ is bounded and $F(x) \notin \Phi(x, \lambda)$ for $x \in \mathrm{bd}\, \mathrm{U}_\lambda$, then*

$$\mathrm{ind_F}(\Phi_a, U_a) = \mathrm{ind_F}(\Phi_b, U_b).$$

Now let

$$\mathcal{S} := \{(x, \lambda) \in U \mid F(x) \in \Phi(x, \lambda),\ \lambda \in (a, b)\}.$$

5.30 Theorem *Suppose that U_a is bounded, $(\Phi_a, U_a) \in \widetilde{M}^F(E', E)$ and $\mathrm{ind_F}(\Phi_a, U_a) \neq 0$. Then there exists a connected subset Σ of $\mathcal{S}$ such that $\mathrm{cl}\, \Sigma \cap U_a \neq \emptyset$ and either:*

(i) $\mathrm{cl}\, \Sigma \cap \mathrm{bd}\, \mathrm{U} \neq \emptyset$*;*

(ii) *or* $\mathrm{cl}\, \Sigma \cap U_b \neq \emptyset$*;*

(iii) *or Σ is unbounded.*

The approach we use is rather standard: we shall recall a topological lemma due to J. C. Alexander [4, Prop. 5] and known also as Whyburn's lemma.

5.31 Whyburn's lemma Recall that nonempty subsets A, B of a space X are *separated (in X)* X if there are open neighborhoods $U_A \supset A$, $U_B \supset B$ in X such that $U_A \cap U_B = \emptyset$ and $U_A \cup U_B = X$. Two sets A, B are *connected (to each other)* in X if there is a connected set Y with $A \cap Y \neq \emptyset$ and $B \cap Y \neq \emptyset$. The following result holds.

Suppose A, B are closed and not separated in a compact space X. Then there is a connected set $D \subset X \setminus (A \cup B)$ such that $\operatorname{cl} D \cap A \neq \emptyset$, $\operatorname{cl} D \cap B \neq \emptyset$.

Proof (of Theorem 5.30) First observe that if $C \subset U \cap E' \times (a, b)$ is bounded and closed, then $C \cap \mathcal{S}$ is compact; hence $\mathcal{S}$ is locally compact.

Let $K = \{(x, a) \in U \mid F(x) \in \Phi(x, a)\}$ i.e. $K = \operatorname{cl} \mathcal{S} \cap U \cap E' \times \{a\}$. K is clearly compact. Let us adjoin a point ∞ at infinity to $E' \times [a, b]$. A neighborhood basis of ∞ consists of complements of bounded subsets of $E' \times [a, b]$. Set $\widetilde{U} := [E' \times [a, b] \cup \{\infty\}] \setminus [U \cap E' \times [a, b)]$ and define

$$Z = (K \cup \mathcal{S} \cup \widetilde{U})/\mathcal{U},$$

i.e. contract $\widetilde{U}$ to a point c. In this way Z becomes a one point compactification of the locally compact space $K \cup \mathcal{S}$.

Assume that K and $\{c\}$ can be separated in Z. Then there is an open bounded subset V of $E' \times [a, b]$ such that $K \subset V$, $\operatorname{cl} V \subset U \cap E' \times [a, b)$ and $\mathcal{S} \cap \operatorname{bd} \mathrm{W} = \emptyset$. Evidently $\operatorname{cl} \mathcal{S} \cap V$ is compact. If $\alpha := \sup \pi(\operatorname{cl} \mathcal{S} \cap V)$ and $\beta := \sup \pi(V)$, where $\pi : E' \times [a, b] \to [a, b]$ is the projection, then $a \leq \alpha < \beta$. Hence there is $\lambda \in (a, b)$ such that $V_\lambda \neq \emptyset$ and $F(x) \notin \Phi_\lambda(x)$ for all $x \in \operatorname{cl} V_\lambda$. However, in view of Lemma 5.29, it leads to a contradiction because $\operatorname{ind}_{\mathrm{F}}(\Phi_a, V_a) \neq \mathrm{o}$.

Therefore the sets K and $\{c\}$ cannot be separated. The Whyburn Lemma 5.31, implies the existence of the connected branch Σ of $\mathcal{S}$ such that the closure of Σ in Z contains c. This means that Σ satisfies at least one of conditions (i), (ii) or (iii). □

Employing Theorem 5.30 one may obtain a whole variety of the existence results for the inclusion of the form $F(x) \in \Phi(x)$ where Φ is an appropriate morphism from $\widetilde{M}$ (or M_{CE}) defined on a closure of an open subset of E'. The idea is clear: one should embed the studied inclusion into a family of inclusions indexed by $\lambda \in \mathbf{R}$ for which the continuation procedure established in Lemma 5.29 or in Theorem 5.30 is applicable.

5.32 Remark In this final remark let us indicate a possible generalization of the coincidence index. Again E', E and F has the same meaning as in Section 5.C. Let $U \subset E'$ be open. We say that a morphism $\Phi \in \widetilde{M}(U,E)$ (resp. $M_{CE}(U,E)$) is *admissible* if $\{x \in U \mid F(x) \in \Phi(x)\}$ is compact and Φ determines a *locally compact* set-valued map (i.e. each point $x \in U$ has a neighborhood N such that $\mathrm{cl}\,\Phi(N)$ is compact) and $\mathrm{i}(\Phi) \leq \dim E - 1$ if $\dim E < \infty$. Observe that if Φ is admissible, then there is a bounded open $V \subset U$ such that $\mathrm{cl}\,V \in U$ and such that $(\Phi, V) \in \widetilde{M}^F(E', E)$ (resp. $M^F_{CE}(E', E)$). Hence we may put

$$\mathrm{ind}_{\mathrm{F}}(\Phi, U) := \mathrm{ind}_{\mathrm{F}}(\Phi|\mathrm{cl}\,V, V).$$

This definition is correct since it does not depend on the choice of V in view of the localization property – see Theorem 5.24. Given two admissible morphisms $\Phi_0, \Phi_1 \in \widetilde{M}(U,E)$ (resp. M_{CE}) we say that they are *admissibly homotopic* if there is $\Phi \in \widetilde{M}(U \times I, E)$ (resp. $\Phi \in M_{CE}(U \times I, E)$) determining a locally compact map such that $\Phi \circ i_j = \Phi_j$, $j = 0, 1$, $\{x \in U \mid F(x) \in \Phi(x,t)$ for some $t \in I\}$ is comapct and $\mathrm{i}(\Phi) \leq \dim E$ if $\dim E < \infty$. It is clear that if the admissible morphisms Φ_0, Φ_1 are homotopic in this way, then $\mathrm{ind}_{\mathrm{F}}(\Phi_0, U) = \mathrm{ind}_{\mathrm{F}}(\Phi_1, U)$. Moreover, all other properties of ind collected in Sections 5.C and 5.D (save the boundary dependence) are still valid (with necessary adjustments).

In a similar way one can define $\deg(\Phi, U)$ where $U \subset \mathbf{R}^m$ is open, $\Phi \in \widetilde{M}_(U, \mathbf{R}^n)$ (or $M_{CE}(U, \mathbf{R}^n)$), $m \geq n$, and $\{x \in U \mid 0 \in \Phi(x)\}$ is compact.

This approach has obvious advanteges. However when it comes to applications, one usually encounters compact maps (or morphisms) defined *a priori* on closures of bounded opens sets. Moreover, in general it is easier to check that there are no coincidence points on the boundary. Therefore in the sequel we shall rather restrict ourselves to the index defined in Section 5.C.

Chapter 6.

BOUNDARY VALUE PROBLEMS

6.A. Differential inclusions

In this section we recall several definitions and concepts concerning differential inclusions that will be necessary in the sequel.

Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space.

6.1 Measurability A multivalued map $\varphi : \Omega \multimap \mathbf{R}^n$ with closed values is *measurable* if the set $\{x \in \Omega \mid \varphi(x) \cap U \neq \emptyset\} \in \mathfrak{A}$ for each open subset U of $\mathbf{R}^n$. The following important Kuratowski–Ryll-Nardzewski Theorem (see e.g [19, Th. 8.1.3]) holds.

If φ is measurable, then there exists a measurable (single-valued) selection of φ, i.e. a measurable function $f : \Omega \to \mathbf{R}^n$ such that $f(x) \in \varphi(x)$.

6.2 Carathéodory maps Now let $\varphi : \Omega \times \mathbf{R}^m \multimap \mathbf{R}^n$ be a multivalued transformation with closed values. We say that φ is a *Carathéodory* (resp. *strongly Carathéodory*) map if

(i) $\varphi(x, \cdot\,) : \mathbf{R}^m \multimap \mathbf{R}^n$ is upper semicontinuous (resp. continuous) for a.a. $x \in \Omega$;

(ii) $\varphi(\,\cdot\,, u) : \Omega \multimap \mathbf{R}^n$ is measurable for all $u \in \mathbf{R}^m$.

We say that φ is *product-measurable* if it is measurable with respect to $\mathfrak{A} \otimes \mathfrak{B}$ where $\mathfrak{B}$ denotes the family of the Borel subsets of $\mathbf{R}^m$ and $\mathfrak{A} \otimes \mathfrak{B}$ stands for the minimal σ-algebra generated by the sets $A \times B$, $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$.

We have the following important results (see the survey [12]): *if φ has compact values and is either a strongly Carathéodory map or product-measurable, then for any measurable (single-valued) function $u : \Omega \to \mathbf{R}^m$, the multivalued transformation $\Omega \ni x \mapsto \varphi(x, u(x))$ is measurable.*

In case φ is a Carathéodory map, then $\varphi(\,\cdot\,, u(\,\cdot\,))$ is not measurable in general (it is measurable provided Ω is a metric space, $\mathfrak{A}$ contains Borel subsets of Ω and u is continuous). However, *for any measurable $u : \Omega \to \mathbf{R}^n$, this map admits a measurable selection, i.e. there is a measurable multifunction $S : \Omega \multimap \mathbf{R}^n$ such that $S(x) \subset \varphi(x, u(x))$ for a.a. $x \in \Omega$.*

Given a measurable function $u : \Omega \to \mathbf{R}^m$, we consider the family $N_\varphi(u)$ of all "almost everywhere" measurable selections of $\varphi(\,\cdot\,, u(\,\cdot\,))$ (the so-called *Nemytski operator*), i.e.

$$N_\varphi(u) := \{w : \Omega \to \mathbf{R}^n \mid w(x) \in \varphi(x, u(x)) \text{ a.e. on } \Omega\,;\ w \text{ is measurable}\}.$$

According to the results presented above and the Kuratowski–Ryll-Nardzewski Theorem, $N_\varphi(u)$ is nonempty for all measurable u provided φ is either a product-measurable or a Carathéodory map.

6.3 L^p-spaces As usual, by $L^p(\Omega, \mathbf{R}^n)$, $1 \le p < \infty$, we denote the classical space of (equivalence classes of) Lebesque measurable functions $u : \Omega \to \mathbf{R}^n$ such that $|u|^p$ is integrable $\int_\Omega |u(x)|^p \,\mathrm{d}x < \infty$. $L^p(\Omega, \mathbf{R}^n)$ is a separable (reflexive for $p > 1$) Banach space when endowed with the standard norm $|u|_p = [\int_\Omega |u(x)|^p \,\mathrm{d}x]^{\frac{1}{p}}$. A sequence convergent in in L^p posesses a subsequence convergent almost everywhere. If $p = 2$, then L^p is a Hilbert space with the inner product $(u, v)_2 = \int_\Omega u(x) \cdot v(x) \,\mathrm{d}x$ where "$\cdot$" denotes the scalar product in $\mathbf{R}^n$. Recall that if $1 \le q \le p$, then

$$L^p(\Omega, \mathbf{R}^n) \hookrightarrow L^q(\Omega, \mathbf{R}^n)\ [1]$$

provided $\mu(\Omega) < \infty$.

If $p > 1$, then $X \subset L^p(\Omega, \mathbf{R}^n)$ is relatively weakly compact if and only

[1] Given Banach spaces E, F, we write $E \hookrightarrow F$ if $E \subset F$ and there is a constant $C > 0$ such that $\|x\|_F \le C\|x\|_E$ for any $x \in E$. If, additionally, any bounded sequence $(x_n)_{n=1}^\infty$ in E has a subsequence converging in F to some $y \in F$, then we write $E \hookrightarrow\!\hookrightarrow F$.

if it is bounded.

The Dunford-Pettis theorem (see e.g. [63, 4.21.2] and [156, (I.3.6)]) implies that $X \subset L^1(\Omega, \mathbf{R}^n)$ is relatively weakly compact in $L^1(\Omega, \mathbf{R}^n)$ if it is integrably bounded (i.e. there is $\mu \in L^1(\Omega, \mathbf{R})$ such that $|u(x)| \le \mu(x)$ for all $u \in X$ and a.a. $x \in \Omega$).

By $L^\infty(\Omega, \mathbf{R}^n)$ we denote the Banach space of *essentially bounded measurable* functions $u : \Omega \to \mathbf{R}^n$ with the norm $|u|_\infty = \sup\ \mathrm{ess}_{t\in\Omega}|u(t)|$.

6.4 The Convergence Theorem The following fundamental result, essentially due to A. Pliś [154], holds (see [18, Th. 3.2.6, Prop. 3.2.2] and comp. [123, 125]):

Let $\psi : \mathbf{R}^m \multimap \mathbf{R}^n$ be an upper semicontinuous multivalued map with closed convex values. Given sequences $(u_k)_{k=1}^\infty$ in $L^p(\Omega, \mathbf{R}^m)$ and $(w_k)_{k=1}^\infty$ in $L^q(\Omega, \mathbf{R}^n)$ $(1 \le p, q \le \infty)$ such that $u_k \to u$ in L^p and $w_k \to w$ weakly in L^q, if for almost all $x \in \Omega$ and all $\varepsilon > 0$ there is an integer $k_0 = k_0(x, \varepsilon)$ such that $d((u_k(x), w_k(x)), \mathrm{Gr}(\psi)) \le \varepsilon$ when $k \ge k_0$, then $w(x) \in \psi(u(x))$ a.e. on Ω.

We say that a map $\varphi : \Omega \times \mathbf{R}^m \times \Theta \multimap \mathbf{R}^n$, where Θ is a metric space, is a *Carathéodory* map if $\varphi(x, \cdot\,, \cdot\,)$ is upper semicontinuous for a.a $x \in \Omega$ and $\varphi(\,\cdot\,, u, \vartheta)$ is measurable for all $u \in \mathbf{R}^m$, $\vartheta \in \Theta$. It is clear that, for any $\vartheta \in \Theta$, $\varphi(\,\cdot\,, \cdot\,, \vartheta)$ is a Carathéodory map in the sense of 6.2.

Looking carefully onto the proof of Theorem 6.4 one easily obtains the folowing corollary.

6.5 Corollary *Let $\varphi : \Omega \times \mathbf{R}^m \times \Theta \multimap \mathbf{R}^n$ be a Carathéodory multimap with closed convex values. For any sequences (u_n) in $L^p(\Omega, \mathbf{R}^m)$, (ϑ_n) in Θ and (w_n) in $L^q(\Omega, \mathbf{R}^n)$ such that $u_n \to u$ in L^p, $\vartheta_n \to \vartheta$ in Θ and $w_n \to w$ weakly in L^q, we have that $w(x) \in \varphi(x, u(x), \vartheta)$ a.e. on Ω provided, for all n and a.a. $x \in \Omega$, $w_n(x) \in \varphi(x, u_n(x), \vartheta_n)$.*

6.6 Absolutely continuous functions Let $a, b \in \mathbf{R}$, $a < b$. A continuous function $x : [a, b] \to \mathbf{R}^n$ is said to be *absolutely continuous* whenever there is $y \in L^1([a, b], \mathbf{R}^n)$ such that, for all $t \in [a, b]$, $x(t) = x(0) + \int_a^t y(s)\,\mathrm{d}s$. It is easy to see that y is the weak derivative of x – see 6.31. Moreover x' exists almost everywhere on $[a, b]$ and $x' = y$ a.e. (see [158, Ch. 8]).

The collection of all absolutely continuous functions $[a, b] \to \mathbf{R}^n$ is denoted by $AC([a,b],\mathbf{R}^n)$. It is a Banach space when endowed with the norm $\|x\|_{AC} = \sup_{t\in[a,b]} |x(t)| + |x'|_1$.

Let $T > 0$ and Θ be a metric parameter space. Suppose that $\varphi : [0,T] \times \mathbf{R}^n \times \Theta \multimap \mathbf{R}^n$ is a Carathéodory map with compact convex values and φ has a "linear" growth, i.e. there is a nonnegative function $\mu \in L^1([0,T],\mathbf{R})$ such that

$$\sup_{y\in\varphi(t,x,\vartheta)} |y| \le \mu(t)(1 + |x|)$$

for all $x \in \mathbf{R}^n$, $\vartheta \in \Theta$ and almost all $t \in [0,T]$ and.

Recall that, for a fixed parameter $\vartheta \in \Theta$, by a *solution* to a Cauchy problem

$$\begin{cases} x' \in \varphi(t,x,\vartheta), \ x \in \mathbf{R}^n, \\ x(0) = \xi \in \mathbf{R}^n \end{cases} \tag{6.1$_\vartheta$}$$

we understand an absolutely continuous function $x : [0,T] \to \mathbf{R}^n$ such that $x(0) = \xi$ and $x'(t) \in \varphi(t,x(t),\vartheta)$ for almost all $t \in [0,T]$ (equivalently, for all $t \in [0,T]$, $x(t) = \xi + \int_0^t y(s)\,\mathrm{ds}$, $y \in L^1([0,T],\mathbf{R}^n)$ and $y(t) \in \varphi(t,x(t),\vartheta)$ for almost all $t \in [0,T]$).

It is well-known that, for each $\vartheta \in \Theta$ and $\xi \in \mathbf{R}^n$, solutions to $(6.1)_\vartheta$ exist – see [17] or e.g. [85] (see also an extensive bibliography therein).

We now consider a multivalued map $S_\varphi : U = [0,T] \times \mathbf{R}^n \times \Theta \multimap \mathbf{R}^n$ given by the formula

$$S_\varphi(t,\xi,\vartheta) := \{x(t) \mid x \text{ is a solution of } (6.1)_\vartheta\}$$

for $(t,\xi,\vartheta) \in U$. In particular, $S_\varphi(0,\xi,\vartheta) = \{\xi\}$ for all $\xi \in \mathbf{R}^n$ and $\vartheta \in \Theta$. The map S_φ is sometimes called the *Poincaré – Andronov operator of translation along trajectories* of a differential inclusion. Observe, that even when φ is a singlevalued Carathéodory map, i.e. in the case of a differential equation, the map S_φ is, in general, multivalued.

6.7 Proposition *The map S_φ is determined by a CE-morphism; in particular S_φ is a set-valued map.*

In order to provide a proof we need the following well-known results (see e.g. [17]).

6.8 Compactness Theorem *If a sequence $(x_k)_{k=1}^{\infty}$ in $AC([a,b],\mathbf{R}^n)$ is such that, for each $t \in [a,b]$, the sequence $(x_k(t))$ is bounded in $\mathbf{R}^n$ and there is $y \in L^1([a,b],\mathbf{R})$ such that $|x_k'(t)| \leq y(t)$ a.e. on $[a,b]$ for almost all k, then there is a function $x \in AC([a,b],\mathbf{R}^n)$ such that (passing to a subsequence if necessary) $x_k \to x$ uniformly on $[a,b]$ and $x_k' \to x'$ weakly in L^1.*

6.9 Gronwall Inequality *Let $x : [0,a] \to \mathbf{R}_+$ be a continuous function and let $p : [0,a] \to \mathbf{R}_+$ be an integrable function. If there is a nonnegative constant M such that*

$$x(t) \leq M + \int_0^t p(\xi)x(\xi)\,\mathrm{d}\xi \qquad \text{for } t \in [0,a],$$

then $x(t) \leq M\exp(\int_0^t p(\xi)\,\mathrm{d}\xi)$ for every $t \in [0,a]$.

Proof of Proposition 6.7 Let $W = \{(t,x,\vartheta) \in [0,T] \times AC([0,T],\mathbf{R}^n) \times \Theta \mid x'(s) \in \varphi(s,x(s),\vartheta)$ a.e. on $[0,T]\}$. On W we consider the topology inherited from $[0,T]\times C([0,T],\mathbf{R}^n)\times\Theta$ (as usual, $C([0,T],\mathbf{R}^n)$ denotes the Banach space of all (continuous) functions $[0,T] \to \mathbf{R}^n$ with the uniform convergence topology).

Define maps $p : W \to U$ and $q : W \to \mathbf{R}^n$ by the formulae $p(t,x,\vartheta) = (t,x(0),\vartheta)$, $q(t,x,\vartheta) = x(t)$. It is clear that $S_\varphi(t,\xi,\vartheta) = q(p^{-1}(t,\xi,\vartheta))$ and that both maps p,q are continuous.

First observe that p is a perfect map. To see that it is enough to show that p is proper since p acts between metric spaces. Let K be a compact subset of U and suppose that a sequence $((t_k,x_k,\vartheta_k))_{k=1}^{\infty}$ in $p^{-1}(K)$ is given, i.e. for each integer $k \geq 1$, $(t_k,x_k(0),\vartheta_k) \in K$. Passing to a subsequence, if necessary, we may assume that $t_k \to t_0$, $\vartheta_k \to \vartheta_0$ and $x_k(0) \to \xi_0$ as $k \to \infty$ and $(t_0,\xi_0,\vartheta_0) \in K$. The growth condition and the Gronwall inequality imply that the (functional) sequence (x_k) is uniformly bounded; hence so is the sequence (x_k'). By the Compactness Theorem, there is an absolutely continuous function $x_0 : [0,T] \to \mathbf{R}^n$ such that (again passing to sparser subsequences) $x_k \to x_0$ uniformly and $x_k' \to x_0'$ weakly in L^1. Therefore, by Corollary 6.5, we gather that $(t_0,x_0,\vartheta_0) \in p^{-1}(K)$. We have thus proved that $p^{-1}(K)$ is compact in W ([2]).

[2]Our argument implies, in particular, that the set $\{x \in AC([0,T],\mathbf{R}^n) \mid x'(t) \in \varphi(t,x(t),\vartheta)$ a.e. on $[0,T]\}$, $\vartheta \in \Theta$, is closed in $C([0,T],\mathbf{R}^n)$; apparently it was first proved in [154], [123, 128].

For any $(t, \xi, \vartheta) \in U$, the set $p^{-1}(t, \xi, \vartheta)$ is homeomorphic to the set of all solutions to $(6.1)_\vartheta$ on $[0, T]$. It is well-known – see e.g. [85, Th. 8.6] – that this is an R_δ-set in $C([0, T], \mathbf{R}^n)$ (comp. 1.12) and hence it is a cell-like set. □

6.10 Remark If Θ is a subspace of an ANR X, then all assumptions of Theorem 4.59 concerning W, p and q are satisfied: W is a subset of $Z = \mathbf{R} \times C([0, T], \mathbf{R}^n) \times X$ and q admits an extension $q' : Z \to \mathbf{R}^n$ given by the obvious formula $q'(t, x, \lambda) = x(t)$.

In view of 4.59, in order to produce a map $g : U \to \mathbf{R}^n$ such that $S_\varphi \simeq_{CE} g$ it is enough to have a sufficiently close graph approximation $g' : U \to Z$ of p^{-1} and put $g = q' \circ g'$.

6.B. Abstract boundary value problems

In this section we are going to present an abstract functional setting for general boundary problems for ordinary or partial differential inclusions. It allows to put these problems into a convenient topological framework which may be studied by means of topological methods such as the fixed point or the degree theory. In that we follow the ideas started in [126, 127], [139] and [156] (see also [71]). However, in the above mentioned papers , only problems of index 0 (the meaning of this will be explained later) have been studied . Our setting, modelled to some extent after [30], yields a sufficient generality also in case of problems of nonnegative index. Moreover, since our methods work also for maps with nonnecessarily convex values, we admit maps with more general structure of values.

6.11 Consider the following diagram

$$E' \hookleftarrow E \xrightarrow{A} Y$$

where E, E' and Y are Banach spaces, $A : E \to Y$ is a Fredholm operator of index k (see 3.23). Recall that notation $E \hookrightarrow E'$ means that $E \subset E'$ and, for any $x \in E$, $\|x\|_{E'} \leq C\|x\|_E$.

Since $\mathrm{Ker}(A)$ is finite-dimensional, there is a bounded linear projection $P : E' \to E'$ such that $\mathrm{R}(P) = \mathrm{Ker}(A)$. Clearly $\mathrm{Ker}(P) \oplus \mathrm{Ker}(A) = E'$. Let $Q : Y \to Y$ be a linear bounded projector such that $\mathrm{Ker}(Q) = \mathrm{R}(A)$. Then

$\mathrm{R}(Q) \oplus \mathrm{R}(A) = Y$. Evidently $\dim \mathrm{R}(P)$, $\dim \mathrm{R}(Q) < \infty$ and $\dim \mathrm{R}(P) - \dim \mathrm{R}(Q) = k$.

Since the restriction of A to $\mathrm{Ker}(P) \cap E$ is a linear homeomorphism onto $\mathrm{R}(A)$, it admits a right inverse, i.e. a linear continuous map $K_P : \mathrm{R}(A) \to E'$ such that $\mathrm{R}(K_P) = \mathrm{Ker}(P) \cap E$ and $K_P \circ A(x) = x$ for $x \in \mathrm{Ker}(P) \cap E$ and $A \circ K_P(y) = y$ for $y \in \mathrm{R}(A)$. For it is enough to put $K_P(y) = x$ if and only if $x \in \mathrm{Ker}(P) \cap E$ and $A(x) = y$.

Let $K_{PQ} : Y \to E'$, *the generalized inverse of* A, be defined by $K_{PQ} = K_P \circ (I - Q)$. Clearly $\mathrm{R}(K_{PQ}) = \mathrm{Ker}(P) \cap E$. The following relations are obvious

$$A \circ K_{PQ} = I - Q \tag{6.2}$$
$$K_{PQ} \circ A(x) = x - P(x) \qquad \text{for any } x \in E. \tag{6.3}$$

Observe that the projector Q is bounded and finite-dimensional; hence compact.

6.12 Proposition *If $E \hookrightarrow\hookrightarrow E'$, i.e. each bounded sequence $(x_n)_{n=1}^{\infty}$ in E posesses a subsequence which converges in E', then K_P and the generalized inverse K_{PQ} are compact linear maps; in particular these operators are strongly continuous (i.e. they map weakly convergent sequences into convergent ones).*

6.13 Definition (comp. [139]) We say that a multivalued transformation $\psi : X \multimap Y$, where X is a metric space, is *A-compact* if $Q \circ \psi : X \multimap Y$ and $K_{PQ} \circ \psi : X \multimap E'$ are compact set-valued maps (i.e. are upper semicontinuous, have compact values and relatively compact ranges $K_{PQ} \circ \psi(X)$, $Q \circ \psi(X)$).

For instance, if ψ is compact and upper semicontinuous (e.g. determined by a morphism $\Phi \in M(X, Y)$ (or $M_{CE}(X, Y)$)), then it is A-compact. However we have also another, perhaps more general, example.

6.14 Definition A multivalued transformation $\psi : X \multimap Y$ is said to be *weakly closed* if for all sequences (x_n) in X and (w_n) in Y the conditions $x_n \to x$ in X, $w_n \to w$ weakly in Y and $w_n \in \psi(x_n)$ for all n, imply that $w \in \psi(x)$. The map ψ is *weakly compact* if it is weakly closed and $\psi(X)$ is relatively weakly compact, i.e. each sequence (y_n) in $\psi(X)$ has a weakly convergent subsequence (see Preliminaries concerning weak compactness).

It is clear that a compact closed map is weakly compact.

6.15 Example Recall the Convergence Theorem from 6.5. It implies, for example, that if $\varphi : \Omega \times \mathbf{R}^m \multimap \mathbf{R}^n$ is a Carathéodory map with closed convex values and φ is L^q-bounded (i.e. there is $\mu \in L^q(\Omega, \mathbf{R})$ such that $\sup_{y\in\varphi(x,z)} |y| \le \mu(x)$ for a.a $x \in \Omega$ and all $z \in \mathbf{R}^m$), $1 \le q < \infty$, then the Nemytski operator N_φ (introduced in 6.2) is a well-defined weakly closed map from $L^p(\Omega, \mathbf{R}^m)$, $1 \le p \le \infty$, to $L^q(\Omega, \mathbf{R}^n)$; moreover $N_\varphi(L^p)$ is L^q-bounded, hence bounded. Therefore N_φ is also a weakly compact map on $L^p(\Omega, \mathbf{R}^m)$ – see 6.3.

Evidently values of N_φ are convex closed, hence weakly closed and, actually, weakly compact.

If $1 \le p, q < \infty$ and

$$\sup_{y\in\varphi(x,z)} |y| \le \mu(x) + C|z|^{p/q} \tag{6.4}$$

for a.a. $x \in \Omega$ and all $z \in \mathbf{R}^m$ where $\mu \in L^q$, C is a constant; or, if $p = \infty$ and

$$\sup_{y\in\varphi(x,z)} |y| \le \mu_h(x) \tag{6.5}$$

for a.a. $x \in \Omega$ and all $z \in \mathbf{R}^m$ with $|z| \le h$, $0 < h < \infty$ where $\mu_h \in L^q$ ([3])), then N_φ is weakly compact map with weakly compact convex values when restricted to any bounded subset of $L^p(\Omega, \mathbf{R}^m)$.

6.16 Proposition *If $\psi : X \multimap Y$ is weakly compact, $T : Y \to E'$ is a compact linear map, then the transformation $T \circ \psi : X \multimap E'$ is a compact set-valued map. In particular, if K_{PQ} is compact and ψ is weakly compact, then ψ is A-compact.*

Proof We are to check that $T \circ \psi$ is upper semicontinuous, its values are compact and $T \circ \psi(X)$ is relatively compact. To this end it is enough to show that the graph of $T \circ \psi$ is closed because $T(\psi(X))$ is clearly relatively compact (see Preliminaries concerning set-valued maps). Let $x_n \to x$ in X, $y_n \to y$ in E' and $y_n \in T(\psi(x_n))$. Then $y_n = T(w_n)$ where $w_n \in \psi(x_n)$.

[3] this holds, for instance, if $\sup_{y\in\varphi(x,z)} |y| \le \mu(x)(1+|z|)$ for a.a. $x \in \Omega$ and all $z \in \mathbf{R}^m$ where $\mu \in L^q$.

Without loss of generality we may assume that there is $w \in Y$ such that $w_n \to w$ weakly in Y. Thus $w \in \psi(x)$, $T(w_n) \to T(w) = y$ (recall that T, as a compact linear operator, maps weakly convergent sequences onto strongly convergent ones) and $y \in T \circ \psi(x)$. □

6.17 Most of boundary value problems for ordinary or partial differential equations or inclusions may be put into an abstract setting of the form

$$Ax \in \psi(x) \tag{6.6}$$

where $A : E \to Y$ is a Fredholm operator, $\psi : \operatorname{cl} U \multimap Y$ is a single- or multivalued transformation defined on the closure of an open subset U of E' (E, E', Y have the same meaning as in paragraph 6.11).

6.18 Definition We shall say that problem (6.6) is *well-posed* if A is a Fredholm operator of nonnegative index $\mathrm{ind(A)} \geq 0$ and ψ is A-compact. By the *index* of problem (6.6) we understand ind(A).

6.19 Remark More generally, one usually encounters the problem of the solvability of the inclusion

$$Lx \in \psi_1(x) \tag{6.7}$$

where $L : E \to Y$ is a linear bounded map and $\psi_1 : \operatorname{cl} U \multimap Y$, subject to some side – "boundary" – conditions of the form

$$lx \in \psi_2(x), \qquad x \in E, \tag{6.8}$$

where $l : E \to Z$ is a linear bounded map (Z is a Banach space) and $\psi_2 : \operatorname{cl} U \multimap Z$.

(i) Putting $A = (L, l) : E \to Y \oplus Z$ and $\psi = (\psi_1, \psi_2) : \operatorname{cl} U \multimap Y \times Z$, $\psi(x) = \psi_1(x) \times \psi_2(x)$ for $x \in \operatorname{cl} U$, we arrive problem (6.6). It is clear that in this case $\operatorname{Ker}(A) = \operatorname{Ker}(L) \cap \operatorname{Ker}(l)$. Therefore A is a Fredholm operator provided Coker(A) is finite-dimensional and $\dim \operatorname{Ker}(L) < \infty$ or $\dim \operatorname{Ker}(l) < \infty$. In particular, if $\dim Z < \infty$, then l is clearly compact; hence A is a Fredholm operator whenever L is so and then $\mathrm{ind(A)} = \mathrm{ind(L)} - \dim \mathrm{Z}$ (see 3.23).

If $l \equiv 0$, then speak of the *purely nonlinear boundary condition* and (6.6) is well-posed if and only if $\dim Z < \infty$ and L is a Fredholm operator with $\mathrm{ind(L)} \geq \dim \mathrm{Z}$.

(ii) If above $\psi_2 \equiv \{0\}$, then we speak of the *homogenous boundary condition*. In such a case it makes sense to simplify the setting. Namely, one considers the restriction $A : E \cap \mathrm{Ker}(l) \to Y$ of L and easily proves that $\mathrm{ind(A)} = \mathrm{ind(L)} - \mathrm{codim}\,\mathrm{Ker(l)}$ (comp. [99, Cor. 1.3.3]).

Similarly, if $\psi_2(x) = \{z\}$ is a singleton for each $x \in \mathrm{cl}\,U$, then one may simplify the setting, too. Namely, suppose that L is a Fredholm operator of nonnegative index, P, Q and K_{PQ} have the same meaning as in paragraph 6.11 with A replaced by L and let $J : \mathrm{R}(Q) \to \mathrm{R}(P) = \mathrm{Ker}(L)$ be an arbitrary (but fixed) injection. For $v \in \mathrm{R}(P)$, let $S(v) := \{x \in \mathrm{cl}\,U \mid x \in v + (K_{PQ} + J \circ Q) \circ \psi_1(x)\}$. The set $S(v)$ may be empty for some v. But if $V := \{v \in \mathrm{R}(P) \mid S(v) \neq \emptyset\}$, then we easily prove that $S : V \multimap \mathrm{cl}\,U$ is a set-valued map provided ψ_1 is L-compact. Next we easily see that if the inclusion $z \in l \circ S(v)$ admits a solution $v_0 \in V$, then it can be easily checked that the system $Lx \in \psi_1(x)$, $l(x) = z$ also has a solution (see also 6.20 (ii) and 6.23). Such a setting will be used in section 6.F in the context of the controllability problems.

6.20 Remark

(i) In view of Propositions 6.16 and 6.12, if a transformation ψ is weakly compact and $E \hookrightarrow\!\!\hookrightarrow E'$ or K_{PQ} is compact, then problem (6.6) is well-posed if and only if A is a Fredholm operator of nonnegative index.

(ii) Suppose that problem (6.6) is well-posed. It is clear that the smaller its index is the better. The obvious question arises: is it possible to complement the problem in such a way its index becomes smaller? Assume that $\mathrm{ind(A)} = \mathrm{k} \geq 0$ and replace Y by $Y \times \mathbf{R}^k$. Now, regarding A as the operator $E \to Y \times \mathbf{R}^k$ (denote it, for a while by A'), we see that our problem is of index 0. However it appears that we got nothing new as concerns the solvability of (6.6).

Indeed, consider problem (6.6) from an other point of view: see paragraph 6.11 and let $P : E' \to E'$ be a (bounded linear) projection onto $\mathrm{Ker}(A)$, $Q : Y \to Y$ — a projection with $\mathrm{Ker}(Q) = \mathrm{R}(A)$ and let $K_{PQ} : Y \to E'$ be the generalized inverse of A with regard to the projectors P, Q. Recall that $\mathrm{R}(K_{PQ}) \subset E \cap \mathrm{Ker}(P)$. Now again let $J : \mathrm{R}(Q) \to \mathrm{R}(P)$ be an arbitary injective linear (and automatically continuous) map. Observe that $x \in \mathrm{cl}\,U \cap E$ is a solution to problem (6.6) if and only if

$$x - Px = (K_{PQ} + J \circ Q) \circ \psi(x), \tag{6.9}$$

i.e. in order to solve problem (6.6) we should look for coincidence points of $F = I - P : E' \to \mathrm{Ker}(P) \subset E'$ and the map $\Phi = (K_{PQ} + J \circ Q) \circ \psi : \mathrm{cl}\,U \multimap$

$\mathrm{Ker}(P) \oplus \mathrm{R}(J) \subset E'$. In fact, if $Ax = y \in \psi(x)$, then $Qy = Q \circ Ax = 0$ and, by (6.3), $Fx = x - Px = K_{PQ} \circ Ax = (K_{PQ} + J \circ Q)y \in \varPhi(x)$. Conversely if $Fx \in \varPhi(x)$, then $Fx = (K_{PQ} + J \circ Q)y$ where $y \in \psi(x)$. Clearly $Qy = 0$. Hence, by formula (6.2), $Ax = A \circ K_{PQ}y = y \in \psi(x)$. It is obvious that in this setting F regarded as the operator $E' \to \mathrm{Ker}(P) \oplus \mathrm{R}(J)$ is a Fredholm operator of index k and $\varPhi$ is a compact set-valued map. If F is regarded as the operator $E' \to E'$, then its index is 0, but still $F(x) - \varPhi(x) \subset \mathrm{Ker}(P) \oplus \mathrm{R}(J)$ for any $x \in \mathrm{cl}\, U$ and the Leray-Schauder theory fails to help. However, if the map $\varPhi$ is good enough (for instance $(\varPhi, U) \in \widetilde{M}^F(E', E')$ or $M^F_{CE}(E', E')$), then the theory developed in Chapter 5 is applicable. In fact this is the reason why we dealt with it.

If we proceed as above but regarding the operator $A' : E \to Y \times \mathbf{R}^k$ induced by A, then again we arrive at the same coincidence problem (6.9).

One would translate problem (6.6) into the problem of lower index successfully provided not only spaces are complemented but also one finds sufficiently "good" complements of maps involved (arguing above we have done it by putting zero complement to each map under question). In general such a complementing procedure might be difficult even if possible. Below we shall return to such and other methods to solve some abstract and concrete boundary value problems.

(iv) Sometimes it comes to consider problems of the form (6.6) with a Fredholm operator A of nonnegative index and ψ not A-compact but merely A-*completely continuous*, that is being A-compact when restricted to a bounded part of its domain $\mathrm{cl}\, U$. Since usually the existence procedures are performed on bounded sets it causes no problems to deal with such situations. Namely, one usually encounters the presence of *a priori* bounds for solutions and restricts oneself to a bounded set which *a priori* contains all possible solutions.

(v) Similar to (6.6) abstract boundary value problems have been studied by Pruszko in his dissertation [156]. In his setting $l \equiv 0$, ψ_2 is a completely continuous single-valued mapping and problems always have index 0. His attitude is modelled after the celebrated Mawhin's approach [139] towards boundary value problems of differential equations and has been succesfully applied to some boundary value problems of differential inclusions. Our setting involves problems of (possibly) positive index. Such problems arise quite often when studying general elliptic boundary value problems – see Section 6.E.

6.21 Example Let $\varphi : [0,T] \times \mathbf{R}^n \multimap \mathbf{R}^n$ be a Carathéodory multifunction with compact convex values and assume that there is a function $\mu \in L^1([0,T],\mathbf{R})$ such that

$$\sup_{y\in\varphi(t,x)} |y| \le \mu(t) \qquad \text{a.e. on } [0,T] \tag{6.10}$$

for $x \in \mathbf{R}^n$.

Let $E = AC([0,T],\mathbf{R}^n)$ be the space of absolutely continuous functions, $E' = Y = L^1([0,T],\mathbf{R}^n)$. We assume that $l : E \to \mathbf{R}^m$, $m \le n$, is a bounded linear map and $\psi_2 : E' \multimap \mathbf{R}^m$ is an arbitrary compact set-valued map.

Consider the following boundary-value problem for differential inclusions:

$$\begin{cases} x' \in \varphi(t,x), \ x \in \mathbf{R}^n, \ t \in [0,t] \\ l(x) \in \psi_2(x). \end{cases} \tag{6.11}$$

By a *solution* to (6.11) we mean here an absolutely continuous function $x : [0,T] \to \mathbf{R}^n$ such that $x'(t) \in \varphi(t,x(t))$ a.e. (almost everywhere) on $[0,T]$ and verifying the second condition in (6.11), i.e. the "boundary" condition.

Let $L : E \to Y$ be given by $Lx = x'$. Clearly L is a surjective Fredholm operator of index n. Consider the Nemytski map generated by φ, i.e. the map $\psi_1 : E' \multimap Y$ given by the formula

$$\psi_1 = N_\varphi(x) = \{w \in Y \mid w(t) \in \varphi(t,x(t)) \ \text{a.e. on} \ [0,T]\}.$$

The map ψ_1 is well-defined since, in view of 6.2, its values are nonempty. By (6.10) and and arguments from Example 6.15, ψ_1 is weakly compact and has convex weakly compact values.

We see immediately that the problem of the existence of solutions to (6.11) is equivalent to the following abstract b.v.p.

$$Ax \in \psi(x), \qquad x \in E' \tag{6.12}$$

where $A = (L,l) : E \to Y \times \mathbf{R}^m$, $\psi = (\psi_1,\psi_2)$. A is a Fredholm operator of index $k = n - m \ge 0$. Since $E = AC \hookrightarrow\hookrightarrow L^1 = E'$ (cf. 6.32 and 6.33 below) and ψ_1, ψ_2 are weakly compact, we see that problem 6.12 is well-posed in view of Propositions 6.12, 6.16.

If condition (6.10) is replaced by (6.4) with $p = q = 1$ and ψ_2 is *completely continuous*, i.e. compact on bounded subsets of E', then the same argument as above shows that the problem $Ax \in \psi(x)$, $x \in \operatorname{cl} U$, where U is bounded in E' is well-posed.

6.22 Remark

(i) For instance one may define $l : E \to \mathbf{R}^m$ by the formula

$$l(x) = Mx(0) + Nx(T)$$

M, N are real $(m \times n)$ matrices. The reader interested in a concrete construction of projectors P, Q and inverses K_P and K_{PQ} related to $A = (L, l)$ in this case is referred to the book of Mawhin [139, Section I.2].

(ii) Observe that the choice of E' above depends mainly on the nature of the problem. For instance, if $l \equiv 0$ and/or ψ_2 is defined only for continuous functions, then it is perhaps more natural to put $E' = C([0, T], \mathbf{R}^n)$. Then $Px = x(0)$ for $x \in E'$ and, since K_P is compact, the problem is well-posed again.

Similarly, the nature of the boundary value problem allows to relax the growth condition (6.10). For instance, if one studies the Cauchy initial value problem, then it is enough to assume that φ satisfies condition (6.5). In such cases it is necessary to restrict again the arising problem (6.12) to a suitable bounded subset of E'.

6.C. Abstract existence criteria

Now we are going to study more carefully the solvability of an abstract boundary value problem (6.6). Suppose E, E', Y and A are as in 6.11, $\psi : \mathrm{cl}\, U \multimap Y$ is a multivalued transformation where U is an open subset of E'.

6.23 As observed in Remark 6.20 (ii) problem (6.6) is equivalent to the following one

$$Fx \in \Phi(x) \tag{6.13}$$

where $F = I - P$ (recall that P is a projection of E' onto $\mathrm{Ker}(A)$), $\Phi := (K_{PQ} + J \circ Q) \circ \psi : \mathrm{cl}\, U \multimap E'' := \mathrm{Ker}(P) \oplus \mathrm{R}(J)$, $Q : Y \to Y$ is a projection with $\mathrm{Ker}(Q) = \mathrm{R}(A)$), K_{PQ} is the genearlized inverse with respect to P, Q and, finally, $J : \mathrm{R}(Q) \to \mathrm{R}(P)$ is an arbitrary (but fixed) injection. It is easy to see that F considered as a map from E' to E'' is a Fredholm operator of index $k = \mathrm{ind(A)}$.

6.24 Assumption *We assume additionally that*

(i) $\mathrm{ind(A)} = \mathrm{k} \geq 0$*;*

(ii) *there is a space* W, *a* $\widetilde{\mathcal{V}}_m$*-map* $p : W \to \mathrm{cl}\, U$, *where* $0 \leq m \leq \dim \mathrm{R}(Q) - 1$ (*and* $0 \leq m < \dim \mathrm{Ker}(P) \cap E$ *if* $\mathrm{R}(Q) = \{0\}$) (*or* $p \in CE$) *and a non-necessarily continuous mapping* $q : W \to Y$ *such that* $K_{PQ} \circ q$, $Q \circ q$ *are continuous and* $\psi(x) = q(p^{-1}(x))$ *for all* $x \in \mathrm{cl}\, U$.

(iii) *the maps* $K_{PQ} \circ q$, $Q \circ q$ *are compact, i.e.* $K_{PQ} \circ q(W)$, $Q \circ q(W)$ *are relatively compact.*

Condition (ii) implies that the set valued maps Φ, $K_{PQ} \circ \psi$ and both $J \circ Q \circ \psi$, $Q \circ \psi$ are determined by m-morphisms (or CE-morphisms). Precisely they are determined by morphisms (or CE-morphisms) $[(p, (K_{PQ} + J \circ Q) \circ q)]_\approx$, $[(p, K_{PQ} \circ q)]_\approx$, $[(p, J \circ Q \circ q)]_\approx$ and $[(p, Q \circ q)]_\approx$, respectively. All these morphisms belong to $\widetilde{M}_m(\mathrm{cl}\, U, E'')$ (resp. $M_{CE}(\mathrm{cl}\, U, E'')$).

If ψ is compact or it is weakly compact and K_{PQ} is compact (it holds, for instance, if $E \hookrightarrow\hookrightarrow E'$ – see Proposition 6.12), then, in view of Proposition 6.16, ψ is A-compact, so (iii) above is satisfied since clearly $K_{PQ} \circ q(W) = K_{PQ} \circ \psi(\mathrm{cl}\, U)$ and $Q \circ q(W) = Q \circ \psi(\mathrm{cl}\, U)$.

Condition (iii) in turn implies that Φ, $K_{PQ} \circ \psi$, $J \circ Q \circ \psi$ and $Q \circ \psi$ are compact set-valued maps.

Assumptions 6.24 imply, in particular, that problem (6.6) is well-posed.

In what follows, somewhat abusing the terminology, we shall refer to all the above maps as to the respective morphisms.

6.25 Example Consider the situation described in Example 6.21. Let $W = \mathrm{Gr}(\psi_1)$, i.e. $W = \{(x, w) \in E' \times Y \mid w(t) \in \varphi(t, x(t))$ a.e. on $[0, T]\}$ where $E' = Y = L^1([0, T], \mathbf{R}^n)$. On W we consider topology inherited from the product $E' \times Y_w$ where Y_w denotes the space Y endowed with the weak topology. Let $p : W \to E'$, $q : W \to Y$ are given by $p(x, w) = x$ and $q(x, w) = w$. We see that $\psi_1(x) = q(p^{-1}(x))$, fibers of p are convex and compact (in the topology of W) since ψ_1 has convex weakly compact values, p is continuous and closed. Hence p is a CE-map.

Clearly $q : W \to Y_w$ is continuous; thus the maps $K_{PQ} \circ q$, $Q \circ q$ are compact and continuous because $AC \hookrightarrow\hookrightarrow L^1$ (obviously K_{PQ}, P and Q are related to the operator A introduced in Example 6.21).

Similarly, the same holds if $E' = C([0, T], \mathbf{R}^n)$ and the boundary conditions are such that K_{PQ} is a compact linear map.

We easily obtain the first abstract result

6.26 Proposition *If an open set $V \subset U$ is bounded, $(\Phi, V) \in \widetilde{M}^F(E', E'')$ (resp. $M^F_{CE}(E', E'')$) (i.e. for $x \in \mathrm{bd\,V}$, $F(x) \notin \Phi(x)$) and $\mathrm{ind_F}(\Phi, V)$ is not trivial element of Π_k, then (6.6) has a solution* [4].

This is a simple consequence of the existence property of the generalized index $\mathrm{ind_F}$.

It is much more interesting to establish conditions implying the key hypothesis of the above proposition.

6.27 Theorem *Assume that $U = E'$ and suppose that there is a constant $M > 0$ such that*

(i) *for any $x \in E'$ and $y \in (I - Q) \circ \psi(x)$, $\|y\|_Y < M$;*

and, if $\mathrm{R}(Q) \neq \{0\}$, there is a constant $R > 0$ such that

(ii) *if $\|Px\| \geq R$ and $\|x - Px\| \leq \|K_P\|M$, then $0 \notin Q \circ \psi(x)$;*

(iii) *$\deg(Q \circ \psi, B_{E'}(0, R) \cap \mathrm{R}(P)) \in \Pi_k$ is nontrivial,*

then there is a solution to (6.6).

Proof First let us suppose that $Q \not\equiv 0$. Let $x \in E'$ be a solution to problem (6.13). Then there is $y \in \psi(x)$ such that $Fx = x - Px = K_{PQ}y$ and $Qy = 0$. Therefore, by (i), $\|x - Px\| < \|K_P\|M$ and, by (ii), $\|Px\| < R$. In other words we have obtained the *a priori* bounds for solutions to (6.13) (and hence to (6.6)): each solution is contained in an open bounded set $V = \{x \in E' \mid \|x - Px\| < \|K_P\|M, \|Px\| < R\}$. Hence $(\Phi, V) \in \widetilde{M}^F(E', E'')$ (resp. $M^F_{CE}(E', E''))$.

Let us consider a map Φ_λ, $\lambda \in [0, 1]$, given by

$$\Phi_\lambda(x) = (\lambda K_{PQ} + J \circ Q) \circ \psi.$$

Clearly Φ_λ is determined by a morphism; hence we may write $\Phi_{(\cdot)} \in \widetilde{M}_{m+1}(E', E'')$ (resp. $M_{CE}(E', E''))$. Due to the presence of *a priori* bounds, we easily get that, for any $x \in \mathrm{bd\,V}$ and $\lambda \in [0, 1]$, $Fx \notin \Phi_\lambda(x)$.

Our assumptions imply that the index theory developed in Chapter 5. may be employed. In particular, by Theorem 5.24 (iv),

$$\mathrm{ind_F}(\Phi_0, V) = \mathrm{ind_F}(\Phi_1, V) = \mathrm{ind_F}(\Phi, V).$$

[4] This result holds under much weaker assumptions than those stated in 6.17: we need only that Φ is determined by a sufficiently "good" morphism.

Observe that $\Phi_o = J \circ Q \circ \psi$; hence $\Phi_o(\mathrm{cl}\, V) \subset \mathrm{R}(J)$. Therefore, by Theorem 5.24 (vi), $\mathrm{ind_F}(\Phi_o, V) = \mathrm{ind_{F'}}(\Phi'_o, V \cap T')$ where $T' = F^{-1}(\mathrm{R}(J)) = \mathrm{R}(P)$ (for $\mathrm{R}(F) = \mathrm{Ker}(P)$ and $E'' = \mathrm{Ker}(P) \oplus \mathrm{R}(J)$), $F' = F|T' \equiv 0$ and $\Phi'_o = \Phi_o|T' \cap V$. Clearly $T' \cap V = B_{E'}(0, R) \cap \mathrm{R}(P)$ and, thus, Φ'_o is the restriction of $J \circ Q \circ \psi$ to this set. Now $\mathrm{ind_0}(\mathrm{J} \circ \mathrm{Q} \circ \psi, \mathrm{B_{E'}}(0, \mathrm{R}) \cap \mathrm{R(P)}) = \deg(-\mathrm{J} \circ \mathrm{Q} \circ \psi, \mathrm{B_{E'}}(0, \mathrm{R}) \cap \mathrm{R(P)})$ (comp. Remark 5.16 (iii)) is a nontrivial element of Π_k because of assumption (iii). This completes the proof when $Q \not\equiv 0$.

If $Q \equiv 0$, then $K_{PQ} = K_P$, $\Phi = K_P \circ \psi$ and, by (i), for all $y \in \Phi(E')$, $\|y\| \leq \|K_P\| M$. By the version of the Schauder Fixed Point Theorem (which follows e.g. from results established in [119, 42] – comp. [111]), there is $x \in E'$ such that $x \in \Phi(x)$ and hence $Fx \in \Phi(x)$. □

As an immediate application we get the following result.

6.28 Example Let $f : [0, 1] \times \mathbf{R}^n \to \mathbf{R}^m$ be a bounded Carathéodory function, $|f(t, x)| \leq M$ for all $t \in [0, 1]$, $x \in \mathbf{R}^n$. Suppose that $u : C([0, 1], \mathbf{R}^n) \to \mathbf{R}^m$, $n \geq m$, is a compact function such that there is $r > 0$ such that $u(x) \neq 0$ whenever x belongs to the closed ball of radius M centered at the constant function $[0, 1] \ni t \mapsto c \in \mathbf{R}^n$ where $|c| \geq r + M$.

We study the existence of solutions to the boundary value problem

$$\begin{cases} x'(t) = f(t, x(t)) & \text{for a.e. } t \in [0, T] \\ u(x) = 0 \end{cases}$$

Set $E' = C([0, 1], \mathbf{R}^n)$, $E = AC([0, 1], \mathbf{R}^n)$ and $Y = L^1([0, 1], \mathbf{R}^n)$; let $v(x) = f(\cdot, x(\cdot)) \in Y$, for $x \in E'$, $L : x \mapsto x' \in Y$, $A = (L, 0) : E \to Y \oplus \mathbf{R}^m$ and $\psi = (v, u)$, i.e $\psi : E' \to Y \oplus \mathbf{R}^m$. It is easy to check (comp. Examples 6.21, 6.25) that all assumptions of 6.24 are satisfied. If $E_c \subset E'$ denotes the subspace of constant functions $[0, 1] \to \mathbf{R}^n$ and $\deg(u, B) \neq 0$ where $B = \{x \in E_c \mid \|x\| < r\}$, then our problem admits a solution. This result follows as a direct consequence of Theorem 6.27.

For in our situation $Px = x(0)$ for $x \in E'$ and $K_P y(\cdot) = \int_0^{\cdot} y(s)\, \mathrm{ds}$ for $y \in Y$; hence $\|K_P\| = 1$. Moreover $Q(y, z) = z$ for $y \in Y$, $z \in \mathbf{R}^m$. Thus, if $y = (I - Q)\psi(x)$, $x \in E'$, then $y = v(x)$ and $\|y\| \leq M$. We gather that $0 \neq Q \circ \psi(x) = u(x)$ provided $\|x - Px\| \leq M = \|K_P\| M$ and $\|Px\| \geq R = r + M$. Since the degree over the open ball of radius R is equal to $\deg(u, B)$ we complete the proof.

The same result holds if we replace f by a L^1-bounded Carathéodory multifunction with compact convex values.

6.29 Remark Condition 6.24 (iii) may be relaxed. It is in fact enough to suppose that the maps $K_{PQ} \circ q$ and $Q \circ q$ are compact on $p^{-1}(B)$ where B is a bounded part of $\operatorname{cl} U$. In the context of Example 6.25 it means that growth condition (6.10) may be replaced by condition (6.4) (with $p = q = 1$). Similarly in the context of Example 6.28, it means that u has to be completely continuous and the boundedness of f may be replaced by condition analogous to (6.5).

6.D. Sobolev spaces

In order to proceed further we shall recall basic facts concerning Sobolev spaces and (partial) differential operators.

Let Ω be a bounded domain (i.e. open and connected set) in $\mathbf{R}^N$, $N \geq 1$, with the smooth boundary (i.e. the boundary $\operatorname{bd}\Omega$ is a C^∞-manifold). For instance, $N = 1$ and $\Omega = (a, b)$ where $-\infty < a < b < \infty$.

The following notation is widely used: for a C^∞-function $u : \Omega \to \mathbf{R}^n$, $j = 1, 2, \ldots, N$, let

$$\partial_j u := \frac{\partial}{\partial x_j} u, \qquad \partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_N^{\alpha_N}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{Z}_+^N$ is a multiindeks; $|\alpha| := \sum_{j=1}^N \alpha_j$.

6.30 Elliptic differential operators An arbitary partial (or ordinary if $N = 1$) differential operator D of order m with real C^∞-coefficients $a_\alpha(\,\cdot\,)$ defined on $\operatorname{cl}\Omega$ has the form

$$D = \sum_{|\alpha| \leq m} a_\alpha(\,\cdot\,)\partial^\alpha,$$

i.e. for a C^∞-function $u : \Omega \to \mathbf{R}^n$ and $x \in \Omega$ we have

$$Du(x) = \sum_{|\alpha| \leq m} a_\alpha(x)\partial^\alpha u(x).$$

Associating with D a homogeneous polynomial $D_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$, $\xi \in \mathbf{R}^N$, we say that D is *elliptic* provided $D_m(x, \xi) \neq 0$ for all $x \in \operatorname{cl}\Omega$,

$\xi \in \mathbf{R}^N \setminus \{0\}$. It may be shown that if D is elliptic and $N \geq 2$, then m is an even integer (it is a consequence of the Borsuk-Ulam theorem). For more details on differential operators – see e.g. [2, 70].

6.31 Weak derivatives Let a function $u : \Omega \to \mathbf{R}^n$ be locally integrable and $\alpha \in \mathbf{Z}_+^N$. A locally integrable function $v : \Omega \to \mathbf{R}^n$ is a *weak α-derivative* of u provided, for any test function $f \in C_0^\infty(\Omega, \mathbf{R}^n)$ (i.e. a smooth function with compact support contained in Ω), one has

$$\int_\Omega f \cdot v \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u \cdot \partial^\alpha f \, \mathrm{d}x.$$

In such a case we also write $v = \partial^\alpha u$. Consequently the equality

$$Du = v$$

for locally integrable functions u and v means that, for any test function f,

$$\int_\Omega f \cdot v \, \mathrm{d}x = \int_\Omega u \cdot D^* f \, \mathrm{d}x$$

where D^* is a formally adjoint differential operator

$$D^* u = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha (a_\alpha(\,\cdot\,) u).$$

6.32 Sobolev spaces Let m be a nonnegative integer and $1 \leq p < \infty$. The *Sobolev* space

$$H^{m,p}(\Omega, \mathbf{R}^n) := \{u \in L^p(\Omega, \mathbf{R}^n) \mid \partial^\alpha u \in L^p(\Omega, \mathbf{R}^n),\ |\alpha| \leq m\}$$

is a separable Banach space with the norm

$$\|u\|_{m,p} = \Big(\int_\Omega \sum_{|\alpha| \leq m} |\partial^\alpha u(x)|^p \, \mathrm{d}x \Big)^{\frac{1}{p}}.$$

In particular, if $N = 1$, then $H^{m,p}((a,b), \mathbf{R}^n)$ is (up to the a.e. equality) the collection of C^{m-1}-functions $u : [a,b] \to \mathbf{R}^n$ with absolutely continuous $v = u^{(m-1)}$ such that $v' \in L^p((a,b), \mathbf{R}^n)$. Hence, when $m = 1 = p$, we have that $H^{1,1}((a,b), \mathbf{R}^n) = AC([a,b], \mathbf{R}^n)$ and the norms $\|u\|_{AC} = \sup_{x \in [a,b]} |u(x)| + |u'|_1$ and $\|u\|_{1,1}$ are equivalent (this in fact follows from the so-called Poincaré inequality).

When $p = 2$, we write $H^m(\Omega, \mathbf{R}^n) := H^{m,2}(\Omega, \mathbf{R}^n)$. Clearly $H^m(\Omega, \mathbf{R}^n)$ is a separable Hilbert space with the scalar product

$$\langle u, v\rangle_m = \int_\Omega \sum_{|\alpha|\le m} \partial^\alpha u \cdot \partial^\alpha v \, \mathrm{d}x$$

and the norm

$$\|u\|_m = \langle u, u\rangle_m^{\frac{1}{2}}.$$

By $H_0^{m,p}(\Omega, \mathbf{R}^n)$ (resp. $H_0^m(\Omega, \mathbf{R}^n)$) we denote the closure of the set $C_0^\infty(\Omega, \mathbf{R}^n)$ in $H^{m,p}(\Omega, \mathbf{R}^n)$ (resp. $H^m(\Omega, \mathbf{R}^n)$) – comp. [1, 40]. Clearly $H^{0,p} = L^p$ and $\|u\|_{0,p} = |u|_p$ for any $u \in L^p(\Omega, \mathbf{R}^n)$.

By the very definition, if D is a partial differential operator of order $\le m$, then $D : H^{m,p}(\Omega, \mathbf{R}^n) \to L^p(\Omega, \mathbf{R}^n)$ is a continuous linear operator.

6.33 Sobolev embeddings There are several results concerning various relations between Sobolev spaces (see [2, 40, 83]). Let $m, p \ge 1$ be arbitrary integers. If $mp < N$, then

$$H^{m,p}(\Omega, \mathbf{R}^n) \hookrightarrow L^q(\Omega, \mathbf{R}^n)$$

provided $1 \le q \le \frac{Np}{N-mp}$; if $1 \le q < \frac{Np}{N-mp}$, then

$$H^{m,p}(\Omega, \mathbf{R}^n) \hookrightarrow\!\!\!\hookrightarrow L^q(\Omega, \mathbf{R}^n).$$

If $N = mp$, then the last result is valid for all $q \ge 1$.

If $mp > N$, then

$$H^{m,p}(\Omega, \mathbf{R}^n) \hookrightarrow\!\!\!\hookrightarrow C^k(\operatorname{cl}\Omega, \mathbf{R}^n) \ ({}^5)$$

for an integer $k \ge 0$ such that $kp < mp - N$.

In particular,

$$H^{m,p}(\Omega, \mathbf{R}^n) \hookrightarrow\!\!\!\hookrightarrow L^p(\Omega, \mathbf{R}^n)$$

and

$$H^{m,p}(\Omega, \mathbf{R}^n) \hookrightarrow\!\!\!\hookrightarrow H^{m-1,p}(\Omega, \mathbf{R}^n)$$

[5] This relation should be understand in the following way: if $u \in H^{m,p}(\Omega, \mathbf{R}^n)$, then there is $v \in C^k(\Omega, \mathbf{R}^n)$ such that $u = v$ a.e. on Ω and $\partial^\alpha v$ admits a continuous extension onto $\operatorname{cl}\Omega$ for all $|\alpha| \le k$.

for all $m, p, N \geq 1$.

When $N = 1$, it follows that

$$H^{1,1}((a,b), \mathbf{R}^n) \hookrightarrow C([a,b], \mathbf{R}^n)$$

(but *never* $\hookrightarrow\hookrightarrow$, see also 6.8 above);

$$H^{1,1}((a,b), \mathbf{R}^n) \hookrightarrow\hookrightarrow L^q((a,b), \mathbf{R}^n)$$

for all $q \geq 1$ and

$$H^{m,p}((a,b), \mathbf{R}^n) \hookrightarrow\hookrightarrow C^k([a,b], \mathbf{R}^n)$$

provided $0 \leq kp < mp - 1$. Taking above into account we write sometimes $H^{m,p}([a,b], \mathbf{R}^n)$.

6.34 Calderon Theorem The assumption concerning the boundary of Ω implies that functions from $H^{m,p}(\Omega, \mathbf{R}^n)$ admit extensions to an arbitrary bounded domain Ω' containing $\operatorname{cl}\Omega$ and these extensions belong to $H^{m,p}(\Omega', \mathbf{R}^n)$. Precisely the Calderon theorem – see [1, 83] – asserts that there is a continuous operator $E : H^{m,p}(\Omega, \mathbf{R}^n) \to H_0^{m,p}(\Omega', \mathbf{R}^n)$ such that $Eu = u$ (in the a.e. sense) on Ω for any $u \in H^{m,p}(\Omega, \mathbf{R}^n)$. Equivalently one may say that the set $C^\infty(\operatorname{cl}\Omega, \mathbf{R}^n)$ (i.e. the set of smooth functions on Ω having smooth extensions onto a neighborhood of $\operatorname{cl}\Omega$) is dense in $H^{m,p}(\Omega, \mathbf{R}^n)$.

6.35 Boundary value problems Taking 6.34 into account one may study boundary value problems. Let D be an elliptic differential operator of order $m \geq 1$ and let B_j, $j = 1, 2, \ldots, k$, be differential operators of order $m_j \leq m-1$ defined in a neighborhood of $\operatorname{bd}\Omega$ and consider the problem ([6]):

$$\begin{cases} Du = g(x, u, (\partial^\beta u)_{\beta \in J}), & x \in \Omega,\ u \in \mathbf{R}^n, \\ B_j u = 0 & \text{on } \operatorname{bd}\Omega,\ \mathrm{j} = 1, \ldots, \mathrm{k}, \end{cases}$$

where J is a (fixed) subset of $\{\beta \in \mathbf{Z}^N \mid 1 \leq |\beta| \leq m-1\}$, $\#J = M \geq 0$, $g : \operatorname{cl}\Omega \times \mathbf{R}^n \times \mathbf{R}^{nM} \to \mathbf{R}^n$ is a function satisfying certain conditions concerning regularity and growth (for instance, is bounded).

By a (*generalized*) *solution* to this problem one understands a function

[6] We restrict ourselves to homogeneous boundary value problems; otherwise we should introduce Sobolev spaces H^s with real exponents s.

$u \in H^{m,p}(\Omega, \mathbf{R}^n)$, $1 \le p < \infty$, such that $Du(x) = g(x, u(x), (\partial^\beta u(x))_{\beta \in J})$ a.e on Ω and $B_j u = 0$ on $\mathrm{bd}\,\Omega$ for $j = 1, \ldots, k$ (meaning that there is a sequence $(u_\nu)_{\nu=1}^\infty$ in $C^\infty(\mathrm{cl}\,\Omega, \mathbf{R}^n)$ such that $u_\nu \to u$ in $H^{m,p}$ and $B_j u_\nu(x) = 0$ on $\mathrm{bd}\,\Omega$ for all $j = 1, \ldots, k$ and almost all ν).

The boundary operators $\{B_j\}_{j=1}^k$ should satisfy conditions assuring that the above problem is well-posed in a sense. Such conditions known as *ellipticity* or *coercivity* or *regularity* or the *Shapiro-Lopatynskij* conditions – see [2, 70, 133] – imply that:

(i) the space $\{u \in H^m(\Omega, \mathbf{R}^n) \mid Du = 0 \text{ on } \Omega, B_j u = 0 \text{ on } \mathrm{bd}\,\Omega, \mathrm{j} = 1, \ldots, \mathrm{k}\}$ is a finite-dimensional subspace of $C^\infty(\mathrm{cl}\,\Omega, \mathbf{R}^n)$.

(ii) Let $A := D|\{u \in H^m(\Omega, \mathbf{R}^n) \mid B_j u = 0 \text{ on } \mathrm{bd}\,\Omega, \mathrm{j} = 1, \ldots, \mathrm{k}\}$; then either $\mathrm{R}(A) = L^2(\Omega, \mathbf{R}^n)$ or there are functions $f_1, \ldots, f_l \in L^2(\Omega, \mathbf{R}^n)$ such that if $g \in L^2(\Omega, \mathbf{R}^n)$ and $\int_\Omega g(x) \cdot f_i(x)\,\mathrm{d}x = 0$ for $i = 1, \ldots, l$, then, there exits a solution $u \in H^m(\Omega, \mathbf{R}^n)$ to the problem $Du = g$ on Ω, $B_j u = 0$ on $\mathrm{bd}\,\Omega$ for $j = 1, \ldots, k$;

(iii) for any $u \in H^m(\Omega, \mathbf{R}^n)$,

$$\|u\|_m \le C(|Du|_2 + |u|_2).$$

6.E. Boundary value problems for partial differential inclusions

As stated in Section 6.C, we shall now apply results of this section to a general boundary value problem for elliptic partial differential inclusions.

First let us recall the following definitions.

6.36 Hausdorff limits Let (Y, d) be a metric space and $A, B \subset Y$ be nonempty. The *Hausdorff distance* between A and B (which may be equal to $+\infty$ when A or B is unbounded) is defined by

$$\mathfrak{H}(A, B) = \max\{\sup_{a \in A} d(a, B),\ \sup_{b \in B} d(b, A)\}.$$

If $\varphi : (a, \infty) \multimap Y$, $a < \infty$, is a multivalued map with compact values and there is $A \subset Y$ such that

$$\lim_{r \to \infty} \mathfrak{H}(\varphi(r), A) = 0,$$

(in this case we write also $A := \mathrm{Lim}_{r\to\infty}\varphi(r)$) then A is compact. If $\varphi(r)$ are convex (and Y is a normed space), then A is convex, too. There is also another description of $\mathrm{Lim}_{r\to\infty}$ by the so-called upper and lower (topological) limits of Kuratowski – see [19] for details.

It is easy to see that if $\varphi_n : X \multimap Y$, where X is a space, is a sequence of set-valued map such that $\varphi(x) = \mathrm{Lim}_{n\to\infty}\varphi_n(x)$ exists uniformly on X, then $\varphi : X \multimap Y$ is a set-valued map.

Note also that given a sequence $\varphi_k : \Omega \multimap \mathbf{R}^n$ of measurable multimaps with compact values, if for any $x \in \Omega$, $\varphi(x) = \mathrm{Lim}_{n\to\infty}\varphi_n(x)$ exists, then φ is measurable (see [19, Th. 8.2.5]).

Let now $m \geq 2$ and let M be the number of elements in an arbitrary (but fixed) subset J of the set $\{\beta \in \mathbf{Z}^N \mid 1 \leq |\beta| \leq m-1\}$, say $J = \{\beta_1, \ldots, \beta_M\}$.

6.37 Assumption *Let*

(i) Ω *be a bounded domain in* $\mathbf{R}^N$, $N \geq 1$, *having the smooth boundary* $\mathrm{bd}\,\Omega$*;*

(ii) $D = \sum_{|\alpha|\leq m} a_\alpha(\,\cdot\,)\partial^\alpha$ *be an elliptic partial differential operator of order* m *with smooth coefficients* $a_\alpha(\,\cdot\,)$ *defined on* $\mathrm{cl}\,\Omega$*;*

(iii) B_j, $j = 1, 2, \ldots, k$, *be differential operators with smooth coefficients defined in a neighborhood of* $\mathrm{bd}\,\Omega$ *and satisfying the coercivity* (*Shapiro-Lopatynskij*) *conditions* (*see* 6.35).

(iv) *Assume that the only solution to the system* $Du = 0$, $B_j u = 0$, $j = 1, \ldots, k$ *that vanishes on the set of positive measure in* Ω *is* $u \equiv 0$ ([7]).

(v) *Let* $\varphi : \mathrm{cl}\,\Omega \times \mathbf{R}^n \times \mathbf{R}^{nM} \multimap \mathbf{R}^n$ *be a Carathéodory multivalued map with compact convex values such that, for* $x \in \mathrm{cl}\,\Omega$, $z \in \mathbf{R}^{n(M+1)}$,

$$\sup_{y\in\varphi(x,z)} |y| \leq m(x)$$

where $m \in L^2(\Omega, \mathbf{R})$, *and*

(vi) *for all* $x \in \mathrm{cl}\,\Omega$, *there exists*

$$\mathrm{Lim}_{r\to\infty}\varphi(x, rz).$$

uniformly for $z \in S^{n(M+1)-1}$.

[7]This assumption has purely technical character. It is known that it is not always satisfied. However, as noted in [160] (comp. [147]), if all roots (real or complex) of D_m (see 6.30) are simple, then it holds.

We are going to study the following *boundary value problem for a partial differential inclusion*

$$\begin{cases} Du \in \varphi(x, u, (\partial^\beta u)_{\beta \in J}) & x \in \Omega,\ u \in \mathbf{R}^n \\ B_j u = 0 & \text{on } \operatorname{bd}\Omega,\ \mathrm{j} = 1, \dots, \mathrm{k}. \end{cases} \tag{6.14}$$

As in 6.35, by a (*generalized*) *solution* to (6.14) we mean a function $u \in H^m(\Omega, \mathbf{R}^n)$ such that $Du(x) \in \varphi(x, u(x), (\partial^\beta u(x))_{\beta \in J})$ ([8]) a.e. on Ω and $B_j u = 0$ on $\operatorname{bd}\Omega$ for $j = 1, \dots, k$ ([9]).

Let $E = \{u \in H^m(\Omega, \mathbf{R}^n) \mid B_j u = 0 \text{ on } \operatorname{bd}\Omega, \mathrm{j} = 1, \dots, \mathrm{k}\}$. Then E, as a closed subspace of $H^m(\Omega, \mathbf{R}^n)$, is a Banach space and

$$E \hookrightarrow\hookrightarrow E' := H^{m-1}(\Omega, \mathbf{R}^n). \tag{6.15}$$

We consider $A : E \to Y := L^2(\Omega, \mathbf{R}^n)$ given by

$$Au = Du \tag{6.16}$$

for $u \in E$. In view of Assumption 6.37 (iii) (see 6.35), A is a Fredholm operator.

6.38 Assumption $\operatorname{ind}(\mathrm{A}) \geq 0$.

For simplicity of notation we set

$$\partial u(x) := (u(x), (\partial^\beta u(x))_{\beta \in J}) = (u(x), \partial^{\beta_1} u(x), \dots, \partial^{\beta_M} u(x))$$

for $u \in E' = H^{m-1}(\Omega, \mathbf{R}^n)$ (recall that $J \subset \{\beta \in \mathbf{Z}^N \mid 1 \leq |\beta| \leq m-1\}$). Evidently $\partial : E' \to L^2(\Omega, \mathbf{R}^{n(M+1)})$ is a continuous operator.

For each $u \in E'$, let

$$\psi(u) := \{w \in Y \mid w(x) \in \varphi(x, \partial u(x)) \text{ a.e. on } \Omega\}, \tag{6.17}$$

i.e. $\psi : E' \multimap Y$ is the Nemytskij operator associated to φ – see 6.2. In virtue of our Assumptions 6.37 (v) and results from 6.2, we get that, $\psi(u)$ is

[8] This notation, being common for partial differential equations, should be understand in the following way: for a.a. $x \in \Omega$, $Du(x) \in \varphi(x, u(x), \partial^{\beta_1} u(x), \dots, \partial^{\beta_M} u(x))$.

[9] in the sense that there is a sequence $(u_\nu)_{\nu=1}^\infty \subset C^\infty(\operatorname{cl}\Omega, \mathbf{R}^n)$ converging to u in $H^{m,p}$ and $B_j u_\nu(x) = 0$ for all $x \in \operatorname{bd}\Omega$, $j = 1, \dots, k$ and almost all ν.

nonempty for any $u \in E'$. Again, by 6.37 (v), ψ is weakly compact. Indeed: it is weakly closed in view of Corollary 6.5 and since ∂ is continuous; $\psi(E')$ is bounded in the reflexive space Y.

Arguing exactly as in Example 6.25 and taking Assumption 6.38 into account we see that Assumption 6.24 (i), (ii) are satisfied. In view of (6.15) and Propositions 6.12, 6.16, the problem

$$Au \in \psi(u),\ u \in E, \tag{6.18}$$

equivalent to (6.14), is well-posed and condition 6.24 (iii) also holds.

Now we shall discuss further implications of our assumptions.

Let $P : E' \to E'$ be a projector such that $\mathrm{R}(P) = \mathrm{Ker}(A)$. Assumption 6.37 (iv) implies that if $u \in \mathrm{Ker}(A)$ and $u \neq 0$, then $u(x) \neq 0$ a.e. on Ω.

In view of Assumption 6.37 (iv), we get the following lemma.

6.39 Lemma

$$\lim_{c\to 0} \mu(c) = 0$$

where

$$\mu(c) := \sup_{u\in \mathrm{R}(P), \|u\|_{m-1}=1} \mu\{x \in \Omega \mid |\partial u(x)| \le c\}$$

and μ is the Lebesgue measure.

Proof Suppose to the contrary that there is a sequence $c_k \searrow 0$ and $u_k \in \mathrm{R}(P)$, $\|u_k\|_{m-1} = 1$ such that $\mu(A_k) \ge \varepsilon > 0$ where $A_k = \{x \in \Omega \mid |\partial u_k(x)| \le c_k\}$. Since $\dim \mathrm{R}(P) < \infty$, we may assume without loss of generality that $u_k \to u$ in E', $\|u\|_{m-1} = 1$ and, moreover, that $d_k = \|u - u_k\|_{m-1} \searrow 0$.

Since $L^2(\Omega, \mathbf{R}^{n(M+1)}) \hookrightarrow L^1(\Omega, \mathbf{R}^{n(M+1)})$, there is $C > 0$ such that $|v|_1 \le C|v|_2$ for any $v \in L^2$ (recall that $|\cdot|_p$, $p \ge 1$, is the norm in L^p).

Now we see that

$$A_k \subset \{x \in \Omega \mid |\partial u(x)| \le c_k + \sqrt{d_k}\} \cup \{x \in \Omega \mid |\partial(u - u_k)(x)| \ge \sqrt{d_k}\}.$$

But, by the Chebyshev inequality,

$$\mu\{x \in \Omega \mid |\partial(u - u_k)(x)| \ge \sqrt{d_k}\} \le \frac{|\partial(u - u_k)|_1}{\sqrt{d_k}} \le \frac{C|\partial(u - u_k)|_2}{\sqrt{d_k}}.$$

Since $E' \hookrightarrow L^2$, we gather that $|\partial(u - u_k)|_2/\sqrt{d_k} \to 0$.

We easily see that, by Assumption 6.37 (iv), $\mu\{x \in \Omega \mid |\partial u(x)| \leq c_k + \sqrt{d_k}\} \to 0$, and thus $\mu(A_k) \to 0$, a contradiction. □

In view of Assumption 6.37 (vi) and 6.36, we have defined the multivalued transformation $\widetilde{\varphi} : \mathrm{cl}\,\Omega \times S^{n(M+1)-1} \multimap \mathbf{R}^n$ with nonempty compact convex values given by

$$\widetilde{\varphi}(x, z) = \mathrm{Lim}_{\,\mathrm{r}\to\infty}\varphi(\mathrm{x}, \mathrm{rz}) \tag{6.19}$$

for all $z \in S^{n(M+1)-1}$ and for $x \in \mathrm{cl}\,\Omega$.

Since the limit in (6.19) exists uniformly on $S^{n(M+1)}$, we infer that $\widetilde{\varphi}(x, \cdot\,)$ is upper semicontinuous for almost all $x \in \mathrm{cl}\,\Omega$ – comp. 6.36. Moreover, for all $z \in S^{n(M+1)}$, $\widetilde{\varphi}(\,\cdot\,, z)$ is measurable – comp. 6.36. By the very definition of the limit Lim we also have that, for all $x \in \mathrm{cl}\,\Omega$ and $\varepsilon > 0$, there is $\eta > 0$ such that, for any $r \geq \eta$,

$$\widetilde{\varphi}(x, z) \subset B(\varphi(x, rz), \varepsilon), \qquad \varphi(x, rz) \subset B(\widetilde{\varphi}(x, z), \varepsilon) \tag{6.20}$$

for all $z \in S^{n(M+1)-1}$.

Let us now define a multivalued map $\overline{\varphi} : \mathrm{cl}\,\Omega \times \mathbf{R}^{n(M+1)} \multimap \mathbf{R}^n$ by the formula

$$\overline{\varphi}(x, z) = \widetilde{\varphi}\left(x, \frac{z}{|z|}\right) \qquad \text{for } z \neq 0 \tag{6.21}$$

and

$$\overline{\varphi}(x, 0) = D^n(0, m(x)) \qquad \text{for all } x \in \mathrm{cl}\,\Omega.$$

It is easy to check that $\overline{\varphi}$ is a Carathéodory multimap positively homogeneous (in the second variable) of degree 0 and $|y| \leq m(x)$ for a.a $x \in \mathrm{cl}\,\Omega$ where $y \in \overline{\varphi}(x, z)$, $z \in \mathbf{R}^{n(M+1)}$.

For any $u \in E'$, the set

$$\overline{\psi}(u) := \{w \in L^2(\Omega, \mathbf{R}^n) \mid w(x) \in \overline{\varphi}(x, \partial u(x)) \text{ a.e. on } \Omega\}$$

is nonempty and the multivalued map

$$\overline{\psi} : E' \multimap Y = L^2(\Omega, \mathbf{R}^n) \tag{6.22}$$

is weakly closed. It is weakly compact as well because $\overline{\psi}(E')$ is L^2-bounded.

Let $Q : Y \to Y$ be a (linear bounded) projector such that $\mathrm{Ker}(Q) = \mathrm{R}(A)$. Taking into account 6.35, if $Q \not\equiv 0$, then there are functions $f_1, \ldots, f_l \in Y$ such that $\mathrm{R}(Q) = \mathrm{span}\{f_1, \ldots, f_s\}$, thus

$$Qv = \sum_{i=1}^{s} (v, f_i)_2 f_i$$

for $v \in Y$ (recall that $(\,\cdot\,,\,\cdot\,)_2$ denotes the scalar product in L^2).

We shall now need the following assumption:

6.40 Assumption *If $Q \not\equiv 0$, then*

$$0 \notin Q \circ \overline{\psi}(u)$$

for any $u \in \mathrm{R}(P)$ with $\|u\|_{m-1} = 1$.

By arguments similar to those provided in Example 6.25, the map $\overline{\psi}$ is determined by a morphism from $M_{CE}(E', Y_w)$ (recall that Y_w denotes Y endowed with the weak topology), $Q \circ \overline{\psi}|_{\mathrm{R}(P)}$ is determined by a morphism from $M_{CE}(\mathrm{R}(P), \mathrm{R}(Q))$.

Since $\infty > \dim \mathrm{R}(P) \geq \dim \mathrm{R}(Q)$, in virtue of Assumption 6.40, we may define $\deg(Q \circ \overline{\psi}, D)$, where D is the unit (closed) ball in $\mathrm{R}(P)$. Observe that this degree is (at least theoretically) computable for we deal here with a totally finite-dimensional situation.

Let us make a final assumption.

6.41 Assumption *The degree $\deg(Q \circ \overline{\psi}, D)$ is not trivial in Π_l where $l = \mathrm{ind}(\mathrm{A})$.*

Now we shall proceed to apply Theorem 6.27. To this end we shall first check that its assumptions are satisfied (the notation in the present section is in accordance with that of Section 6.C.).

- We have already established that Assumption 6.24 is satisfied;
- For any $u \in E'$ and $w \in \psi(u)$, we have

$$|(I - Q)w|_2 \leq \|I - Q\||m|_2 < L,$$

where $L = \|i - q\||m|_2 + 1$; thus assumption (i) of Theorem 6.27 holds.

Unfortunately it is much harder to verify the remaining hypotheses of Theorem 6.27. For this reason we have to introduce some auxiliary objects.

Suppose $\rho > 0$ and, for each integer $k \geq 1$, consider the multivalued map

$$\psi_k : \{u \in \mathrm{R}(P) \mid \|u\|_{m-1} = 1\} \times D_{E'}(0,\rho) \to Y$$

given by

$$\begin{aligned} \psi_k(u,v) &= \psi(ku+v) \\ &= \{w \in Y \mid w(x) \in \varphi(x, \partial(ku(x)+v(x))) \text{ a.e. on } \Omega\}. \end{aligned} \tag{6.23}$$

This map is again well-defined and weakly compact.

We are now ready to prove the key lemma.

6.42 Lemma *For each $0 < \varepsilon < \frac{1}{2}$ and $\rho > 0$, there is an integer $k_0 > 0$ such that for any positive integer $k \geq k_0$, any $u \in \mathrm{R}(P)$, $\|u\|_{m-1} = 1$ and $v \in E'$, $\|v\|_{m-1} \leq \rho$,*

$$\psi_k(u,v) \subset B_Y(\overline{\psi}(B_{E'}(u,\varepsilon)),\varepsilon).$$

Thus ψ_k is, in a sense, an ε-graph approximation of $\overline{\psi}$ provided k is large enough.

Proof There is $\delta > 0$ such that, for any measurable set $\Gamma \subset \Omega$ with $\mu(\Gamma) < 2\delta$,

$$\int_\Gamma m^2(x)\,\mathrm{d}x < \frac{1}{8}\varepsilon^2.$$

In view of 6.33, we have $C > 0$ such that $|\partial v|_1 \leq C\|v\|_{m-1}$ for $v \in E'$. By (6.20) and (6.21), for any $x \in \mathrm{cl}\,\Omega$, there is $\eta > 0$ such that if $|z| \geq \eta$,

$$\overline{\varphi}(x,z) \subset B(\varphi(x,z),\varepsilon_0) \qquad \text{and} \qquad \varphi(x,z) \subset B(\overline{\varphi}(x,z),\varepsilon_0) \tag{6.24}$$

where $\varepsilon_0 = \varepsilon/\sqrt{2\mu(\Omega)}$.

Recall Lemma 6.39 and take an integer $k_0 > 0$ such that

$$\mu\left(\frac{\eta + C\rho\delta^{-1}}{k_0}\right) < \delta \qquad \text{and} \qquad \frac{\rho}{k_0} < \varepsilon.$$

Let $k \geq k_0$, $u \in \mathrm{R}(P)$, $\|u\|_{m-1} = 1$ and $v \in E'$ with $\|v\|_{m-1} \leq \rho$.

By Lemma 6.39, if $A_u = \{x \in \Omega \mid k|\partial u(x)| \leq \eta + C\rho\delta^{-1}\}$, then $\mu(A_u) <$

δ. By the Chebyshev inequality, if $A_v = \{x \in \Omega \mid |\partial v(x)| \geq C\rho\delta^{-1}\}$, then $\mu(A_v) \leq \frac{|\partial v|_1}{C\rho\delta^{-1}} \leq \delta$. Thus if $\Gamma = A_u \cup A_v$, then $\mu(\Gamma) < 2\delta$.

Consider the sequence $(f_n)_{n=1}^{\infty}$ where $f_n : \mathrm{cl}\,\Omega \to \mathbf{R}$ is a function given by $f_n(x) := \sup_{|z|\geq n} \mathfrak{H}(\overline{\varphi}(x,z), \varphi(x,z))$, $x \in \mathrm{cl}\,\Omega$. It is not difficult to show that each f_n is a mesurable function (see e.g. [19, Th. 8.1.4, 8.2.11]) and $\lim_{n\to\infty} f_n(x) = 0$ a.e. on $\mathrm{cl}\,\Omega$. Therefore, by the Egorov theorem, f_n converges uniformly to 0 outside a set of measure $< 2\delta$. Without loss of generality we may assume that actually $f_n \to 0$ uniformly on $\mathrm{cl}\,\Omega \setminus \Gamma$. Again without loss of generality we may therefore assume that (6.24) holds uniformly for $x \in \mathrm{cl}\,\Omega \setminus \Gamma$, i.e. in particular, if $|z| \geq \eta$, then

$$\varphi(x,z) \subset B(\overline{\varphi}(x,z), \varepsilon_0) \tag{6.25}$$

for any $x \in \mathrm{cl}\,\Omega \setminus \Gamma$.

Suppose that $x \in \Omega \setminus \Gamma$. Then

$$|\partial(ku+v)(x)| \geq k|\partial u(x)| - |\partial v(x)| \geq \eta.$$

Therefore, for each $x \notin \Gamma$, we have in view of (6.25)

$$\varphi(x, k\partial u(x) + \partial v(x)) \subset B\Big(\overline{\varphi}\Big(x, \partial u(x) + \frac{1}{k}\partial v(x)\Big), \varepsilon_0\Big).$$

Hence if we take $w \in \psi_k(u,v)$, then $w(x) \in B(\overline{\varphi}(x, \partial u(x) + \frac{1}{k}\partial v(x)), \varepsilon_0)$ for a.a. $x \in \Omega \setminus \Gamma$. The multivalued map

$$\Omega \setminus \Gamma \ni x \mapsto B(w(x), \varepsilon_0) \cap \overline{\varphi}\Big(x, \partial u(x) + \frac{1}{k}\partial v(x)\Big) \neq \emptyset$$

(defined a.e. on $\Omega \setminus \Gamma$) is measurable (see [19, Cor. 8.2.13, Th. 8.2.4]). Take any its L^2-selection $\overline{y} : \Omega \setminus \Gamma \to \mathbf{R}^n$. If $\widetilde{y} \in L^2(\Omega, \mathbf{R}^n)$ is an arbitrary selection of $\overline{\varphi}(\,\cdot\,, \partial u(\,\cdot\,) + \frac{1}{k}\partial v(\,\cdot\,))$, i.e. $\widetilde{y} \in \overline{\psi}(u + \frac{1}{k}v)$, then

$$y = \chi_\Gamma \widetilde{y} + \chi_{\Omega\setminus\Gamma}\overline{y},$$

where χ_Γ and $\chi_{\Omega\setminus\Gamma}$ are characteristic functions of Γ, $\Omega \setminus \Gamma$, respectively, is still a member of $\overline{\psi}(u + \frac{1}{k}v)$ and $y = \overline{y}$ on $\Omega \setminus \Gamma$. Now we have

$$\begin{aligned} |y-w|_2^2 &= \int_{\Omega\setminus\Gamma} |w(x) - \overline{y}(x)|^2 \,\mathrm{d}x + \int_\Gamma |w(x) - y(x)|^2 \,\mathrm{d}x \\ &< \mu(\Omega)\varepsilon_0^2 + 4\int_\Gamma m^2(x)\,\mathrm{d}x < \varepsilon^2. \end{aligned}$$

Since evidently $u + \frac{1}{k}v \in B_{E'}(u, \varepsilon)$, we complete the proof. $\square$

Now we are in a position to proof the following result which generalizes the main theorem of [147].

6.43 Theorem *Under all above assumptions, problem* (6.14) *has a solution.*

Proof If $Q \equiv 0$, then the assertion follows from Theorem 6.27 (it has been already stated that hypothesis (i) of this theorem is satisfied).

Suppose therefore that $Q \not\equiv 0$. First observe that, in view of 5.25 (see also 4.59), there is $\beta > 0$ such that

$$\deg(Q \circ \overline{\psi}, D) = \deg(g, D)$$

where $g : D \to \mathrm{R}(Q)$ is an arbitrary β-graph approximation of $Q \circ \overline{\psi}$.

Next, by assumption 6.40, there is a small $\varepsilon > 0$ such that $\|Q\|\varepsilon \leq \beta/2$ and $0 \notin B_Y(Q \circ \overline{\psi}(B_{E'}(u, \varepsilon) \cap \mathrm{R}(P)), \|Q\|\varepsilon)$ for $u \in \mathrm{R}(P)$, $\|u\|_{m-1} = 1$. This and Lemma 6.42 shows that, for any $\rho > 0$ and, in particular for $\rho = \|K_P\|L$, there is $R > 0$ such that, for every $u \in \mathrm{R}(P)$, $\|u\|_{m-1} \geq R$ and any $v \in E'$, $\|v\| \leq \rho$,

$$0 \notin Q \circ \psi(u + v),$$

i.e. assumption (ii) of 6.27 is satisfied.

Clearly $\deg(Q \circ \psi, D_R)$, where $D_R = D_{E'}(0, R) \cap \mathrm{R}(P)$, is well-defined. Again, by paragraph 5.25, there is a number $\nu > 0$ such that $\deg(Q \circ \psi, D_R) = \deg(f, D_R)$ where $f : D_R \to \mathrm{R}(Q)$ is a ν-graph approximation of $Q \circ \psi$. Of course we may assume that $\nu \leq \beta/2$. Now consider a map $g : D \to \mathrm{R}(Q)$ given by $g(u) = f(Ru)$. In virtue of Lemma 6.42, it now takes a little efort to see that g is nothing else but a β-graph approximation of $Q \circ \overline{\psi}$. Since clearly $\deg(f, D_R) = \deg(g, D)$, in view of assumption 6.41, we get that $\deg(Q \circ \psi, D_R)$ is a nontrivial element of Π_l. This shows that assumption (iii) of Theorem 6.27 also holds in our case. The existence of a solution to problem (6.14) follows. $\square$

6.44 Remark Let $M = 0$ and $n = 1$.

(i) First let us consider the case $\dim \mathrm{R}(P) = 1 = \dim \mathrm{R}(Q)$ (i.e. in particular ind(A) = 0). Then, for $z \in \mathbf{R}$, $z \neq 0$,

$$\overline{\varphi}(x,z) = \begin{cases} \varphi_+(x) & \text{when } z > 0 \\ \varphi_-(x) & \text{when } z < 0, \end{cases}$$

where $\varphi_\pm(x) := \mathrm{Lim}_{\,\mathrm{z}\to\pm\infty}\varphi(\mathrm{x},\mathrm{z})$, and

$$\overline{\varphi}(x,0) = [-m(x), m(x)].$$

Hence

$$\begin{aligned} \overline{\psi}(u) \;\; = \;\; & \{w \in Y \mid \text{ for a.a. } x \in \operatorname{cl}\Omega,\ w(x) \in \overline{\varphi}_+(x) \\ & \qquad \text{if } u(x) > 0 \text{ and } w(x) \in \varphi_-(x) \text{ if } u(x) < 0\} \end{aligned}$$

for any $u \in E'$.

Suppose that $\mathrm{R}(P) = \operatorname{span}\{g\}$ and $\mathrm{R}(Q) = \operatorname{span}\{f\}$. Let $\Omega_+ = \{x \in \operatorname{cl}\Omega \mid g(x) > 0\}$ and $\Omega_- = \{x \in \operatorname{cl}\Omega \mid g(x) < 0\}$. Consider the (closed) intervals

$$\begin{aligned} I_+ \;\; &= \;\; \left\{ \int_{\Omega_+} wf\,\mathrm{d}x + \int_{\Omega_-} w'f\,\mathrm{d}x \,\middle|\, w(x) \in \varphi_+(x) \right. \\ & \qquad\qquad \left. \text{a.e. on } \Omega_+;\ w'(x) \in \varphi_-(x) \text{ a.e. on } \Omega_- \right\}, \\ I_- \;\; &= \;\; \left\{ \int_{\Omega_+} wf\,\mathrm{d}x + \int_{\Omega_-} w'f\,\mathrm{d}x \,\middle|\, w(x) \in \varphi_-(x) \right. \\ & \qquad\qquad \left. \text{a.e. on } \Omega_+;\ w'(x) \in \varphi_+(x) \text{ a.e. on } \Omega_- \right\}. \end{aligned}$$

Assumptions 6.40 and 6.41 mean that $0 \notin I_+ \cup I_-$ and members of I_+ and I_- have different signs.

In this special case we also have that

$$\varphi_\pm(x) = [m_\pm(x), M_\pm(x)]$$

where $m_\pm$ and $M_\pm$ are measurable functions on $\operatorname{cl}\Omega$ and

$$\sup\{|m_\pm(x)|, |M_\pm(x)|\} \le m(x)$$

a.e. on $\operatorname{cl}\Omega$. Hence

$$\begin{aligned} I_+ \;\; &= \;\; \left[\int_{\Omega_+} m_+ f\,\mathrm{d}x + \int_{\Omega_-} m_- f\,\mathrm{d}x, \int_{\Omega_+} M_+ f\,\mathrm{d}x + \int_{\Omega_-} M_- f\,\mathrm{d}x \right] \\ I_- \;\; &= \;\; \left[\int_{\Omega_+} m_- f\,\mathrm{d}x + \int_{\Omega_-} m_+ f\,\mathrm{d}x, \int_{\Omega_+} M_- f\,\mathrm{d}x + \int_{\Omega_-} M_+ f\,\mathrm{d}x \right] \end{aligned}$$

and our assumptions may thus be easily verified.

(ii) If the operator A is self-adjoint (it holds, for instance, when $D = \Delta$ (Laplacian), $k = 1$ and $Bu = B_1 u = u|_{\mathrm{bd}\,\Omega}$ or $Bu = \frac{\partial u}{\partial \vec{n}}$ where $\frac{\partial}{\partial \vec{n}}$ denotes the normal derivative), then ind(A) = 0 and we may assume that $\mathrm{R}(P) = \mathrm{R}(Q)$.

If, for each $u \in \mathrm{R}(P)$, $\|u\|_{m-1} = 1$,

$$\int_{\{x|u(x)>0\}} m_+ u \,\mathrm{d}x + \int_{\{x|u(x)<0\}} m_- u \,\mathrm{d}x > 0,$$

then assumtions 6.40 and 6.41 are satisfied.

Indeed, the last condition is equivalent to the following statement: for any $u \in \mathrm{R}(P)$ with $\|u\|_{m-1} = 1$ and $v \in Q \circ \overline{\psi}(u)$, $(v, u)_2 > 0$. Therefore the ("linear") multimap

$$D \times [0,1] \ni (u,t) \mapsto (1-t) Q \circ \overline{\psi}(u) + tu \subset \mathrm{R}(Q)$$

(as before D is the unit closed ball in $\mathrm{R}(P)$) provides a homotopy that joins $Q \circ \overline{\psi}$ with the identity, proving that $\deg(Q \circ \overline{\psi}, D) = 1$.

6.F. Controllability of nonlinear systems

In this section we shall show that problems of the controllability of (non)linear processes governed by differential equations or inclusions may be set and solved in the abstract framework introduced in Section 6.B. There is a vast literature on the subject and many different approaches are available. It also appears that topological methods provide strong and useful tools to deal with these problems.

We are interested in the controllability of the system

$$x' \in A(t)x + b(t)u(t) + \varphi(t, x, u(t)), \qquad t \in J,\ x \in \mathbf{R}^n \tag{6.26}$$

where $J = [0,1]$, $A(\,\cdot\,)$ is an $n \times n$-matrix with $L^1(J, \mathbf{R})$ entries, $b \in L^1(J, \mathbf{R}^n)$ and $\varphi : J \times \mathbf{R}^n \times \mathbf{R} \multimap \mathbf{R}^n$ is a Carathéodory map with compact convex values, i.e. for all $z \in \mathbf{R}^{n+1}$, $\varphi(\,\cdot\,, z)$ is measurable and, for almost all $t \in J$, $\varphi(t, \,\cdot\,)$ is upper semicontinuous, and finally, the *control* $u(\,\cdot\,)$ belongs to the space $L^\infty(J, \mathbf{R})$ of *admissible* controls. Moreover we assume that φ is *integrably bounded*, i.e. there is a function $\mu \in L^1(J, \mathbf{R})$ such that

$$\sup_{y \in \varphi(t,z)} |z| \le \mu(t)$$

for almost all $t \in J$ and all $z \in \mathbf{R}^{n+1}$ ([10]).

In the last years there has been an increasing interest in investigating two different types of controllability properties. The first one deals with the possibility of reducing the control space to a finite dimensional one or at least one parametrized by finite dimensional space, for instance by fixing the switching times and taking piecewise constant controls [73]; this is done in order to allow a topological approach in proving the local or global controllability of systems (see [74, 117, 148]). The second one, which goes back to [131], concerns finding an upper bound on the number of switches of bang-bang controls cf. [130, 167]; this last property is deeply related to the regularity of time-optimal controls.

Here we shall consider the notion of controllability, called *relay controllability*, introduced by Arronson [15] in 1977 and obtained by taking piecewise constant controls whose absolute value (*amplitude*) is constant. This notion seems to give a partial answer to both questions at the same time. We are going to give sufficient conditions assuring that the controllability property is preserved when relay controls admit n switching times and a constant amplitude.

Let us study the following example.

6.45 Example Consider the linear control process

$$x' = A(t)x + b(t)u(t), \qquad \text{a.e. on } t \in J$$

where

$$A(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad b(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and $u \in [-1, 1]$. The reachable set for this process is $W = \{x = (x_1, x_2) \in \mathbf{R}^2 \mid -\frac{1}{2} - \left(\frac{1-x_1}{2}\right)^2 \leq x_2 \leq \frac{1}{2} + \left(\frac{1-x_1}{2}\right)^2, -1 \leq x_1 \leq 1\}$. Since the system is normal, any point on the boundary of W can be reached by a unique bang-bang control (that is also a relay), which in this case has only one switch, and then the whole set W can be reached by relay controls with only one switch and with amplitude less or equal to 1. On the other hand if we fix the switching times, then we may reach only a polygon with vertices on the boundary that will never be the whole set W.

[10] obviously, the growth condition may be suitably relaxed.

Thus if one wants to maintain the same reachable set, then the relay controls represent a better choice than finite dimensional linear spaces although they do not form a linear space but rather a manifold.

6.46 ***D*-controllability** Let $D : C(J, \mathbf{R}^n) \to \mathbf{R}^m$ be a bounded linear surjective operator. Given a set $U \subset L^\infty(J, \mathbf{R})$ such that $0 \in U$, we say, following [136], that the system (6.26) is *D-controllable by means of U* provided, for every $y \in \mathbf{R}^m$, the problem

$$\begin{cases} x'(t) \in A(t)x(t) + b(t)u(t) + \varphi(t, x(t), u(t)) & \text{a.e. on } t \in J \\ Dx = y \end{cases} \tag{6.27}$$

is solvable for some $u \in U$.

If $D = D_0 : C(J, \mathbf{R}^n) \to \mathbf{R}^{2n}$ is given by

$$Dx = (x(0), x(1)),$$

then the notion of D-controllability reduces to the usual one of complete controllability [50].

Let us now remark that when using the concept of D-controllability, we can use the initial conditions of (6.27) as control parameters. To separate the effects of the controls and that of the initial conditions on the control system let us consider a bounded linear surjective differential operator $L : AC(J, \mathbf{R}^n) \to L^1(J, \mathbf{R}^n)$ given by

$$Lx(t) = x'(t) - A(t)x(t), \qquad x \in AC(J, \mathbf{R}^n). \tag{6.28}$$

Observe that $\dim \operatorname{Ker} L = n$.

Given a linear continuous surjection $C : C(J, \mathbf{R}^n) \to \mathbf{R}^k$, $k \geq 0$, we say that the system (6.26) is *C-controllable by means of U* provided for every $\xi \in \mathbf{R}^n$ and $z \in \mathbf{R}^k$, there is $u \in U$ such that the system

$$\begin{cases} x'(t) \in A(t)x(t) + b(t)u(t) + \varphi(t, x(t), u(t)), & \text{a.e. on } J \\ x(0) = \xi \end{cases} \tag{6.29}$$

has a solution $x \in AC(J, \mathbf{R}^n)$ satisfying the condition

$$Cx = z. \tag{6.30}$$

Note that, for a fixed control $u \in U$, the right hand side of the system is a Carathéodory map and thus, for every $\xi \in \mathbf{R}^n$, there are solutions to

(6.29); therefore it makes sense to ask whether there are solutions to (6.29) such that (6.30) is satisfied.

It is always possible to reduce D-controllability to C-controllability of (6.26). Depending on how D acts on $\operatorname{Ker} L$, there are two possible cases:

Case 1: $\operatorname{Ker} L \cap \operatorname{Ker} D = \{0\}$, i.e. $D|\operatorname{Ker} L : \operatorname{Ker} L \to \mathbf{R}^m$ is injective. This implies that $m \geq n$ and it can be easily shown that D may be decomposed as the direct sum of two operators

$$D = C_0 \oplus C : C(J, \mathbf{R}^n) \to \mathbf{R}^n \oplus \mathbf{R}^{m-n},$$

where $C_0|\operatorname{Ker} L \to \mathbf{R}^n$ is an isomorphism and $\operatorname{Ker} C_0 \oplus \operatorname{Ker} L = C(J, \mathbf{R}^n)$. This happens for instance when $D = D_0$ and in this case $C_0 : x \mapsto x(0)$. Moreover, without loss of generality, we may assume that $C_0(x) = x(0)$ since C_0 and the operator $C(J, \mathbf{R}^n) \ni x \mapsto x(0)$ are equal up to an isomorphism. Observe that now D-controllability of (6.26) is equivalent to the C-controllability of (6.26). Finally note that if $k = m - n = 0$, then (6.26) is D-controllable by means of $\{0\}$ (i.e. $u \equiv 0$).

Case 2: If $\dim(\operatorname{Ker} D \cap \operatorname{Ker} L) = s > 0$, then by the Hahn-Banach theorem, there exists a bounded linear operator

$$S : C(J, \mathbf{R}^n) \to \mathbf{R}^s$$

such that $S|(\operatorname{Ker} D \cap \operatorname{Ker} L)$ is onto. It is easy to see now that the $S \oplus D$-controllability of (6.26) implies its D-controllability. But $\operatorname{Ker}(S \oplus D) \cap \operatorname{Ker} L = \{0\}$ so we are back in Case 1.

Therefore in the sequel we confine ourselves to the problem of C-controllability (with $k \geq 1$) of (6.26).

6.47 Relay controllability Let us consider the following set of *relay control functions*,

$$U_{rel} := \bigcup_{l=1}^{\infty} U^l$$

where $u \in U^l$ if and only if $u \in L^\infty(J, \mathbf{R})$, $u(\,\cdot\,)$ has at most l discontinuities on J and there is a number $\lambda \geq 0$ such that $|u(t)| = \lambda$, for almost all $t \in J$. We refer to l as the *number of switches* and to λ as the *amplitude* of the control. Obviously, for each $l \geq 1$, U^l (and thus U_{rel}), is a metric space (with the metric inherited from $L^\infty(J, \mathbf{R})$).

We say that system (6.26) is *C-relay controllable* if it is C-controllable by means of U_{rel}. It is known [14, 15] that if $C(x) = x(1)$ (i.e. $k = n$), then relay controllability and complete controllability are equivalent; the same reasoning can be adapted to prove that the same holds for C-controllability. Here we shall be interested in minimizing the number of switches, i.e. we are concerned with C-controllability by means of U^l with the minimal possible l.

6.48 Paramerization The space U^l, $l \geq 1$, may be parametrized. Namely, let

$$\Delta_l := \{(t_1, t_2, \ldots, t_l) \in \mathbf{R}^l \mid 0 \leq t_1 \leq \ldots \leq t_l\}.$$

Observe that Δ_l is nothing else but the standard l-dimensional simplex (written in a bit different way). For any $\lambda \in \mathbf{R}$ and $T = (t_1, \ldots, t_l) \in \Delta_l$, consider a function

$$v(\lambda, T)(\,\cdot\,) = \lambda \sum_{j=0}^{l} (-1)^j \chi_{[t_j, t_{j+1}]}(\,\cdot\,)$$

where we have set $t_0 = 0$ and $t_{l+1} = 1$ and $\chi_{[t_j, t_{j+1}]}$ stands for the characteristic function of the interval $[t_j, t_{j+1}]$. It is clear that $v(\lambda, T)$ has at most l-switches and $|v(\lambda, T)|_\infty = |\lambda|$; thus $v(\lambda, T)$ is well-defines a continuous map

$$v : \mathbf{R} \times \Delta_l \to U^l$$

which we call the *parametrization of U^l*.

Observe finally that $T \in \operatorname{bd} \Delta_l$ if and only if, for some $0 \leq j \leq l$, $t_j = t_{j+1}$ and if $T \in \operatorname{bd} \Delta_l$, then $v(\lambda, T) \in U^{l-1}$.

It is clear that the problem of the C-relay controllability of (6.26) is equivalent to the solvability (with respect to $x \in AC(J, \mathbf{R}^n)$, $u \in U^l$ for some $l \geq 1$) for every $\xi \in \mathbf{R}^n$, $z \in \mathbf{R}^k$, of the following system

$$\begin{cases} L(x) \in N_\varphi(x, u) \\ x(0) = \xi, \; C(x) = z, \end{cases} \tag{6.31}$$

where $N_\varphi(x, u) := \{v \in L^1 \mid v(t) \in b(t)u(t) + \varphi(t, x(t), u(t)) \text{ a.e. on } J\}$ is the Nemytskij operator associated to the right hand side of the system.

If we take the parametrization v introduced above into account, we see that the solvability of (6.31) is equivalent to the problem of the solvability (with respect to $x \in AC(J, \mathbf{R}^n)$, $\lambda \in \mathbf{R}$ and $T \in \Delta_l$), for every $\xi \in \mathbf{R}^n$, $z \in \mathbf{R}^k$, of the problem

$$\begin{cases} \widetilde{L}(x, \lambda, T) \in N_\varphi(x, v(\lambda, T)) \\ x(0) = \xi,\ C(x) = z \end{cases} \tag{6.32}$$

where $\widetilde{L}(x, y) = L(x)$ for $x \in AC(J, \mathbf{R}^n)$ and $y \in \mathbf{R}^{l+1}$. Clearly $\widetilde{L} : AC(J, \mathbf{R}^n) \oplus \mathbf{R}^{l+1} \to L^1(J, \mathbf{R}^n)$ is a Fredholm operator of index $n + l + 1$ onto $L^1(J, \mathbf{R}^n)$. Thus problem (6.32) is of the form (6.7), (6.8) and the method discussed in Remark 6.19 (ii) is applicable, i.e. in order to show the C-controllablity of (6.26) by means of U^l we have to show that, for every $\xi \in \mathbf{R}^n$, $z \in \mathbf{R}^k$, there are $\lambda \in \mathbf{R}$ and $T \in \Delta_l$ such that

$$0 \in C \circ S_\lambda(T) - z \tag{6.33}$$

where

$$\begin{aligned} S_\lambda(T) \quad = \quad & \{x \in AC(J, \mathbf{R}^n) \mid x(0) = \xi \text{ and} \\ & x'(t) = A(t)x(t) + b(t)u(t) + \varphi(t, x(t), v(\lambda, T)(t)) \\ & \text{a.e. on } J\}. \end{aligned} \tag{6.34}$$

The operator L has a unique right continuous inverse

$$K : L^1(J, \mathbf{R}^n) \to AC(J, \mathbf{R}^n)$$

such that $K(v)(0) = 0$.

6.49 Theorem *Assume that*

$$C(K(bu)) = 0, \ \textit{for } u \in U^{k-1} \ \Leftrightarrow \ u \equiv 0. \tag{6.35}$$

Then system (6.26) *is C-controllable by means of U^k.*

Proof Let us fix $\xi \in \mathbf{R}^n$, $z \in \mathbf{R}^k$. For $\lambda \in \mathbf{R}$, we define a map $p_\lambda : W \to \Delta_k \times [0, 1]$, where $W := \{(x, T, s) \in AC(J, \mathbf{R}^n) \times \Delta_k \times [0, 1] \mid x(0) = s\xi, x'(t) = A(t)x(t) + b(t)v(\lambda, T)(t) + s\varphi(t, x(t), v(\lambda, T)(t))$ a.e. on $J\}$ is endowed with the topology inherited from $C(J, \mathbf{R}^n) \times \Delta_k \times [0, 1]$, given by the formula

$$p_\lambda(x, T, s) = (T, s).$$

Exactly as in Proposition 6.7 we prove that p_λ is a perfect map since v is continuous. Moreover, for $T \in \Delta_k$ and $s \in [0,1]$, $p_\lambda^{-1}(T,s)$ is the set of all solutions of the system

$$\begin{cases} x' = A(t)x + b(t)u(t) + s\varphi(t,x,u(t)) \\ x(0) = s\xi, \end{cases} \tag{6.36}$$

where $u(\,\cdot\,) = v(\lambda, T)(\,\cdot\,)$. Hence $p_\lambda^{-1}(T,s)$ is an R_δ-set.

Next, for a fixed $\lambda \in \mathbf{R}$, we consider a CE-morphism Φ_λ represented by the cotriad

$$\Delta_k \times [0,1] \xleftarrow{p_\lambda} W \xrightarrow{q_\lambda} \mathbf{R}^k,$$

where $q_\lambda(x,T,s) = C(x) - sz$. It is clear that Φ_λ is a homotopy joining $(s=1)$ the map $C \circ S_\lambda - z$ (S_λ was defined above in (6.F.)) to $(s=0)$ the map $\Delta_k \ni T \mapsto F_\lambda(T) = C(\overline{x}) \in \mathbf{R}^k$ where $\overline{x}(t) = K(bv(\lambda,T))$, $t \in J$ is the unique solution to the system

$$\begin{cases} x' = A(t)x(t) + b(t)v(\lambda,T)(t) \\ x(0) = 0. \end{cases} \tag{6.37}$$

Claim 1: There is $r > 0$ such that, for $|\lambda| \geq r$,

$$0 \notin \Phi_\lambda(T,s)$$

for any $T \in \mathrm{bd}\,\Delta_\mathrm{k}$, $s \in [0,1]$.

To this end fix $T \in \mathrm{bd}\,\Delta_\mathrm{k}$, $s \in [0,1]$, $\lambda \in \mathbf{R}$ and let x and $\overline{x}$ be such that $(x,T,s) \in p_\lambda^{-1}(T,s)$ and $(\overline{x},T,0) \in p_\lambda^{-1}(T,0)$, i.e.

$$x(t) = s\xi + \int_0^t [A(\tau)x(\tau) + b(\tau)v(\lambda,T)(\tau) + sy(\tau)]\,\mathrm{d}\tau,$$

where $y(\tau) \in \varphi(\tau, x(\tau), v(\lambda,T)(\tau))$ for almost all $\tau \in J$, is a solution to (6.36) and

$$\overline{x}(t) = \int_0^t [A(\tau)\overline{x}(\tau) + b(\tau)v(\lambda,T)(\tau)]\,\mathrm{d}\tau$$

is again the solution to (6.37).

We are to show that $|q_\lambda(x,T,s)| > 0$ for sufficiently large $|\lambda|$. Clearly

$$|x(t) - \overline{x}(t)| \leq \int_0^t \|A(\tau)\||x(\tau) - \overline{x}(\tau)|\,\mathrm{d}\tau + \mathrm{s}\int_0^\mathrm{t} |\mathrm{y}(\tau)|\,\mathrm{d}\tau + \mathrm{s}|\xi|,$$

so by the Gronwall inequality – see 6.9,

$$\|x-\overline{x}\|_C = \sup_{t\in J}|x(t)-\overline{x}(t)| \le M \tag{6.38}$$

where $M = (|\xi| + \int_0^1 \mu(\tau)\,\mathrm{d}\tau)\exp(\int_0^1 \|\mathrm{A}(\tau)\|\,\mathrm{d}\tau)$. Hence

$$\begin{aligned} |q_\lambda(x,T,s) - C(\overline{x})| &= |C(x) - sz - C(\overline{x})| \\ &\le \|C\|\|x-\overline{x}\|_C + |z| \le \|C\|M + |z| = R_1 \end{aligned}$$

and

$$|q_\lambda(x,T,s)| \ge |C(\overline{x})| - R_1. \tag{6.39}$$

Observe that, by (6.38), R_1 is independent of λ.

Note that $C(\overline{x}) = F_\lambda(T) = C(K(bv(\lambda,T)) = \lambda C(K(bv(1,T))$; hence $|C(\overline{x})| = |\lambda||C(K(bv(1,T))|$. In view of assumption (6.35) and since $v(1,T) \in U^{k-1}$ (recall that $T \in \mathrm{bd}\,\Delta_\mathrm{k}$), $|C(K(bv(1,T))| > 0$. Hence, for sufficiently large λ, say $|\lambda| \ge r$, $|C(\overline{x})| > R_1$ and, by (6.39),

$$|q_\lambda(x,T,s)| > 0.$$

This completes the proof of Claim 1.

Choose $\lambda \in \mathbf{R}$ with $|\lambda| \ge r$. Claim 1 entails that, for each $s \in [0,1]$, $\deg(\Phi_\lambda(\,\cdot\,,s),\Delta_k)$ is a well-defined element of $\mathbf{Z}$ and

$$\deg(C\circ S_\lambda - z, \Delta_k) = \deg(F_\lambda, \Delta_k). \tag{6.40}$$

Claim 2: $\deg(F_1,\Delta_k) \ne 0$ and hence $\deg(F_\lambda,\Delta_k) \ne 0$.

First observe that a homotopy $F_{(1-s)+s\lambda} = [(1-s)+s\lambda]C(K(bv(1,T))$, $s \in [0,1]$, joins F_1 to F_λ and $F_{(1-s)+s\lambda}(T) \ne 0$ on $\mathrm{bd}\,\Delta_\mathrm{k}$ for all $s \in [0,1]$ in view of assumption (6.35). Hence $\deg(F_1,\Delta_k) = \deg(F_\lambda,\Delta_k)$.

Now observe that there is a suitable $(k\times n)$-matrix $H(\,\cdot\,)$ with $L^\infty(J,\mathbf{R})$ entries such that, for any $y \in L^1(J,\mathbf{R}^n)$,

$$C(K(y)) = \int_0^1 H(\tau)y(\tau)\,\mathrm{d}\tau.$$

This follows from the duality $L^1(J,\mathbf{R})^* \cong L^\infty(J,\mathbf{R})$. Therefore, for all $T \in \Delta_k$,

$$F_1(T) = \int_0^1 H(\tau)b(\tau)v(1,T)(\tau)\,\mathrm{d}\tau = \sum_{\mathrm{j}=0}^{\mathrm{k}}(-1)^\mathrm{j}\int_{\mathrm{t_j}}^{\mathrm{t_{j+1}}} \mathrm{H}(\tau)\mathrm{b}(\tau)\,\mathrm{d}\tau. \tag{6.41}$$

Let us indicate some easily checked properties of the map F_1:

For any $0 \le t_1 \le t_2 \le \ldots \le t_{k-1} \le 1$,

(i) $F_1(0, t_1, \ldots, t_{k-1}) = -F(t_1, \ldots, t_{k-1}, 1)$;

(ii) if $k \ge 2$, then for any s_j, $j = 1, 2, \ldots, k-1$ with $t_{j-1} \le s_j \le t_j$ $(t_0 = 0)$, $F_1(s_1, s_1, t_1, \ldots, t_{k-2}) = F_1(t_1, s_2, s_2, t_2, \ldots, t_{k-1}) = \ldots = F_1(t_1, \ldots, t_{k-2}, s_{k-1}, s_{k-1})$; if $k = 2$, then for any $s \in [0,1]$, $F_1(s,s) = const.$

By Lemma 6.50 below, $\deg(F_1, \Delta_k) \neq 0$. This ends the proof of Claim 2.

In view of Claim 2 and (6.40), $\deg(C \circ S_\lambda - z, \Delta_k) \neq 0$ and hence there are $T \in \Delta_k$ and $x \in p_\lambda^{-1}(T, 1)$ such that $C(x) = z$. This completes the proof of the C-controllability of (6.26) by means of U^k. □

6.50 Lemma *Suppose that $F : \Delta_k \to \mathbf{R}^k$ is a continuous map such that properties* (i), (ii) *above are satisfied* (*with F_1 replaced by F*). *Then either $0 \in F(\operatorname{bd} \Delta_\mathrm{k})$ or otherwise $\deg(F, \Delta_k)$ is an odd number.*

A quite complicated proof of this Lemma, based on the Borsuk theorem has been given in [118].

Finally, following [118], we shall consider the case $\varphi \equiv 0$, i.e. we study the relay controllability of the linear system

$$x' = A(t)x + b(t)u(t). \tag{6.42}$$

6.51 Theorem *Suppose that Assumption* (6.35) *is satisfied. Then the system* (6.42) *is C-controllable by means of U^{k-1}.*

Proof We are to show that, for any $\xi \in \mathbf{R}^n$, $z \in \mathbf{R}^k$, there is $T \in \Delta_{k-1}$ and $\lambda \in \mathbf{R}$ such that

$$z = C(x_0) + C(K(bv(\lambda, T))$$

where $x_0(t) = \xi$ for all $t \in J$. To this end it is enough to show that the map $\mathbf{R} \times \Delta_{k-1} \ni (\lambda, T) \mapsto C(K(bv(\lambda, T)) = \lambda C(K(bv(1, T)) \in \mathbf{R}^k$ is a surjection. To facilitate the notation recall (6.41) and put $F_1(T) = C(K(bv(1, T))$ for $T \in \Delta_{k-1}$.

The set $\{\lambda F_1(T) \mid \lambda \in \mathbf{R},\, T \in \Delta_{k-1}\}$ is exactly the cone generated by $F_1(\Delta_{k-1})$. Therefore in order to prove the C-controllability of (6.42) we must show that F_1 generates every direction in $\mathbf{R}^k$. Assumption (6.35) implies that the possibility of reaching any direction in $\mathbf{R}^k$ by means of F_1 can be stated in an equivalent but more convenient way: for any orthonormal set V of $(k-1)$ vectors $\{v_i \in \mathbf{R}^k \mid i = 1, 2, \ldots, k-1\}$ there exists $T \in \Delta_{k-1}$ such that

$$\langle F_1(T), v_i \rangle = 0 \qquad \text{for } i = 1, \ldots, k-1.$$

For a fixed system V, we define $F_V : \Delta_{k-1} \to \mathbf{R}^{k-1}$ by the formula

$$F_V(T) = (\langle F_1(T), v_1 \rangle, \ldots, \langle F_1(T), v_{k-1} \rangle), \qquad T \in \Delta_{k-1}.$$

Our controllability problem reduces to proving that F_V has a zero on Δ_{k-1}, for each V. But it is easily verified that the mapping F_V satisfies all the assumptions of Lemma 6.50 (with obvious adjustments) and the statement follows immediately. □

6.52 Example Let us consider the linear control process

$$(x_1', x_2') = A(t)(x_1, x_2) + b(t)u(t), \qquad \text{a.e. on } t \in [0, 2\pi]$$

where

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{and} \qquad b(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This system is completely controllable, but it is not completely relay controllable because in this case $k - 1 = 1$ and, for any $t \in [0, 1]$, $F_1(t) = (2\cos t - 2, 2\sin t)$; hence we can not reach any point on the x_2-axis except the origin. In this case assumption (6.35) is not satisfied for $F_1(1) = F_1(0) = 0$.

Chapter 7.

BIFURCATION

In this chapter we are going to introduce an invariant responsible for the bifurcation of solutions of a parametrized inclusion of the form

$$x \in \varphi(\lambda, x)$$

where $\varphi : U \multimap E$, U is an open subset of $\mathbf{R}^k \times E$ and E is a Banach space. Assuming that $0 \in \varphi(\lambda, 0)$ for all $\lambda \in \{\lambda \in \mathbf{R}^k \mid (\lambda, 0) \in U\}$, we study the existence of *bifurcation* points and *connected branches* of nontrivial solutions (i.e. of the form (λ, x) with $x \neq 0$) emanating from these points. Our approach was to some extent motivated by the papers of Bartsch [22], Izé [105] and Gęba, Massabo and Vignoli [81] where the authors study bifurcation invariants in the context of singlevalued maps. However, the machinery we developed in the above chapters allows us to simplify Bartsch's approach and to extend it to set-valued maps.

First we shall study finite-dimensional bifurcation phenomena.

7.A. Finite-dimensional bifurcation

To simplify the technicalities we now replace the inclusion $x \in \varphi(\lambda, x)$, by $0 \in \Phi(\lambda, x)$ where $\Phi = \pi - \varphi$ and π stands for the projection $\mathbf{R}^k \times \mathbf{R}^n \to \mathbf{R}^n$. Therefore we shall study the bifurcation of solutions to the inclusion

$$0 \in \Phi(\lambda, x). \tag{7.1}$$

7.1 Let us make the following standing assumptions:

(i) U is an open subset of $\mathbf{R}^m = \mathbf{R}^k \times \mathbf{R}^n$, i.e. $m = n + k$, $n, k \geq 1$; first k coordinates are considered to represent a parameter $\lambda \in \mathbf{R}^k$;

(ii) $\Phi \in \widetilde{M}_n(U, \mathbf{R}^n)$ (or $M_{CE}(U, \mathbf{R}^n)$) is a morphism such that

$$0 \in \Phi(\lambda, \mathrm{o})$$

for all $\lambda \in \Lambda := \{\lambda \in \mathbf{R}^k \mid (\lambda, 0) \in U\}$.

Additionally we define the set of *nontrivial solutions* to (7.1)

$$\mathcal{S} := \{(\lambda, x) \in U \setminus \Lambda \times \{0\} \mid 0 \in \Phi(\lambda, x)\}$$

and suppose that

(iii) the set of *bifurcation points*

$$\mathcal{B}(\Phi) := \{(\lambda, \mathrm{o}) \in \Lambda \times \{\mathrm{o}\} \mid (\lambda, \mathrm{o}) \in \mathrm{cl}\, \mathcal{S}\}$$

is compact (observe that $\mathcal{B}(\Phi)$ is always closed in U).

In order to define the *bifurcation index* of Φ we shall need some auxiliary objects. Let us consider an arbitrary continuous function $\alpha : \Lambda \to [0, \infty)$ such that, for $(\lambda, 0) \notin \mathcal{B}(\Phi)$,

$$0 < \alpha(\lambda) < d((\lambda, 0),\ \mathrm{bd\, U} \cup \mathrm{cl}\, \mathcal{S}) \qquad \text{and} \qquad \alpha(\lambda) = 0$$

for $(\lambda, 0) \in \mathcal{B}(\Phi)$. For instance we may put

$$\alpha(\lambda) = \min\left\{1, \frac{1}{2} d((\lambda, 0), \mathrm{bd\, U} \cup \mathrm{cl}\, \mathcal{S})\right\}.$$

Next we let

$$\begin{aligned} X &:= \{(\lambda, x) \in \mathbf{R}^m \mid \lambda \in \Lambda,\ |x| = \alpha(\lambda)\}, \\ X^+ &:= \{(\lambda, x) \in \mathbf{R}^m \mid \lambda \in \Lambda,\ |x| < \alpha(\lambda)\}. \end{aligned}$$

Observe that $X^+ \cup X \subset U$ and put

$$X^- := U \setminus \mathrm{cl}\, X^+.$$

It is easy to see that $\mathcal{S} \subset X^-$ and $\mathcal{B}(\Phi) \subset X$.

Let $f : U \to \mathbf{R}$ be a continuous function such that

$$f(\lambda, x) \begin{cases} < 0 & \text{for } (\lambda, x) \in X^- \\ = 0 & \text{for } (\lambda, x) \in X \\ > 0 & \text{for } (\lambda, x) \in X^+. \end{cases} \tag{7.2}$$

Now we consider a morphism Ψ from $\widetilde{M}_n(U, \mathbf{R}^{n+1})$ (resp. $M_{CE}(U, \mathbf{R}^{n+1})$) such that, for all $(\lambda, x) \in U$, $\Psi(\lambda, x) = \Phi(\lambda, x) \times \{f(\lambda, x)\}$. As before (see 4.15 and Remark 4.16) we write $\Psi := (\Phi, f)$.

Observe that $0 \in \Psi(\lambda, x)$ if and only if $0 \in \Phi(\lambda, x)$ and $f(\lambda, x) = 0$, i.e. iff $x = 0$ and $(\lambda, 0) \in \mathcal{B}(\Phi)$. Since, by 7.1 (iii), the set of zeros of Ψ is compact, there is an open bounded set V such that $\operatorname{cl} V \subset U$ and, for $(\lambda, x) \in U \setminus V$, $0 \notin \Psi(\lambda, x)$. Therefore $(\Psi, V) \in \widetilde{M}(m, n+1)$ (resp. $M_{CE}(m, n+1)$) – see 5.25 – and we are in position to give the following

7.2 Definition The *bifurcation* (of zeros) *index* $\mathrm{BI}(\Phi)$ of Φ is defined by the formula

$$\mathrm{BI}(\Phi) := \deg(\Psi, V) \in \Pi_{k-1}.$$

The above definition is correct, i.e. it does not depend on the choice of α, f and V. The independence of the choice of V follows from the localization property and the independence of the choice of α and f follows from the homotopy invariance property of the degree – see Theorem 5.24 (ii), (iv).

7.3 Remark

(i) It is possible to define $\mathrm{BI}(\Phi) := \deg(\Psi, U)$ where here deg stands for the degree defined in Remark 5.32.

(ii) Clearly, in place of a morphism Φ we may take an arbitrary (singlevalued) map $U \to \mathbf{R}^n$ satisfying conditions analogous to that stated in 7.1. Thus our bifurcation index works in the singlevalued context and, thus, corresponds to the index introduced by Bartsch [22].

(iii) The function f defined above plays a role of a *complementing function* of Izé [105] (used also in many other papers - see e.g. [81]). In our setup its definition is a bit more complicated; however it enables us to simplify some technicalitiues in the sequel.

(iv) Definition 7.2 reminds that given in [91]. We worked there with acyclic set-valued maps.

Now we are going to collect some relevant properties of the defined invariant $\mathrm{BI}(\Phi)$. Here comes the main result of this section.

7.4 Theorem (Existence and structure of solutions) *Let* $\mathrm{BI}(\Phi) \neq 0 \in \Pi_{k-1}$. *Then* $\mathcal{B}(\Phi) \neq \emptyset$. *Moreover, assume that* K *is a compact subset of* U

such that $\mathcal{B}(\Phi) \subset K$*. There exists a connected subset* $\mathcal{C}$ *of* $\mathcal{S} \setminus K$ *such that* $\mathrm{cl}\,\mathcal{C} \cap K \neq \emptyset$ *and* $\mathcal{C}$ *is unbounded or* $\mathrm{cl}\,\mathcal{C} \cap \mathrm{bd\,U} \neq \emptyset$*, i.e.* $\mathcal{C}$ *is not contained in any compact subset of* U*.*

Proof That $\mathcal{B}(\Phi) \neq \emptyset$ follows immediately from the existence property of deg.

Now we are going to work in S^m treated as a one-point compactification of $\mathbf{R}^m$, i.e $S^m = \mathbf{R}^m \cup \{\infty\}$. We shall show that the compact sets $A := S^m \setminus U$ and K cannot be separated in a space $Z := A \cup \mathcal{S} \cup K$ (observe that Z is compact). If so, then in virtue of Lemma 5.31, there is a connected subset $\mathcal{C} \subset Z \setminus (A \cup K) = \mathcal{S} \setminus K$ such that $\mathrm{cl}\,\mathcal{C} \cap K \neq \emptyset$ and $\mathrm{cl}\,\mathcal{C} \cap A \neq \emptyset$. That is what we actually require.

Suppose to the contrary that A and K can be separated in Z. Hence there is an open set $W \subset U$ such that

$$K \subset W, \qquad \mathrm{cl}\,W \cap A = \emptyset \qquad \text{and} \qquad \mathcal{S} \cap \mathrm{bd\,W} = \emptyset.$$

Thus $\mathrm{cl}\,W \subset U$ and $\mathrm{cl}\,W$ is compact (in $\mathbf{R}^m$).

Let $W' := X^+ \cup W$ and $X' = \mathrm{bd\,W'}$. It is easy to see that $X' = (X^- \cap \mathrm{bd\,W}) \cup [\mathrm{X} \cap (\mathrm{U} \setminus \mathrm{cl\,W})] \cup [\mathrm{X} \cap \mathrm{bd\,W}]$. We define a continuous function $f' : U \to \mathbf{R}$ such that

$$f'(\lambda, x) \begin{cases} > 0 & \text{for } (\lambda, x) \in W' \\ = 0 & \text{for } (\lambda, x) \in X' \\ < 0 & \text{for } (\lambda, x) \in U \setminus \mathrm{cl}\,W'. \end{cases}$$

and let $\Psi' = (\Phi, f')$. Clearly $\Psi' \in \widetilde{M}_n(U, \mathbf{R}^{n+1})$ (resp. $M_{CE}(U, \mathbf{R}^{n+1})$).

First, observe that $\{(\lambda, x) \in U \mid 0 \in \Psi'(\lambda, x)\} = \emptyset$. Indeed, if $0 \in \Psi'(\lambda, x)$, then $(\lambda, x) \in X'$ and $0 \in \Phi(\lambda, x)$; if $(\lambda, x) \in X^- \cap \mathrm{bd\,W}$, then $x \neq 0$, hence $(\lambda, x) \in \mathcal{S}$; if $(\lambda, x) \in [X \cap (U \setminus \mathrm{cl}\,W)] \cup [X \cap \mathrm{bd\,W}]$, then $x = 0$ and $\alpha(\lambda) = 0$, i.e. $(\lambda, 0) \in \mathcal{B}(\Phi)$. In both cases we get a contradiction.

Consider a morphism $\Theta = (\Phi \circ \pi, h) \in \widetilde{M}_{n+1}(U \times [0,1], \mathbf{R}^{n+1})$ (resp. $M_{CE}(U \times [0,1], \mathbf{R}^{n+1})$) where $\pi : U \times [0,1] \to U$ is the projection and

$$h(\lambda, x, t) = (1-t) f(\lambda, x) + t f'(\lambda, x)$$

for $(\lambda, x) \in U$ and $t \in [0,1]$. We easily see that if $0 \in \Theta(\lambda, x, t)$ for some $t \in [0,1]$, then $(\lambda, x) \in \mathrm{cl}\,X^- \cap \mathrm{cl}\,W$; hence, without loss of generality we may assume that $0 \notin \Theta(\lambda, x, t)$ on $U \setminus V$ (V comes from the definition of $\mathrm{BI}(\Phi)$). By the homotopy invariance of the degree, we infer that

$$\deg(\Psi', V) = \deg(\Psi, V) = \mathrm{BI}(\Phi) \neq 0,$$

a contradiction since there are no $(\lambda, x) \in V$ such that $0 \in \Psi'(\lambda, x)$. □

7.5 Corollary
(i) (Bifurcation) *If* $\mathrm{BI}(\Phi) \neq \mathrm{o}$*, then there is a connected branch* $\mathcal{C}$ *of nontrivial solutions* (*i.e.* $\mathcal{C} \subset \mathcal{S}$) *such that* $\mathrm{cl}\,\mathcal{C} \cap \Lambda \times \{0\} = \mathrm{cl}\,\mathcal{C} \cap \mathcal{B}(\Phi) \neq \emptyset$ *and* $\mathcal{C}$ *is unbounded or* $\mathrm{cl}\,\mathcal{C} \cap \mathrm{bd\,U} \neq \emptyset$*.*
(ii) (Compactness) *If the set* $\mathrm{cl}\,\mathcal{S}$ *is compact* (*in* U)*, then* $\mathrm{BI}(\Phi) = \mathrm{o}$.

Proof (i) follows immediately from Theorem 7.4 if we set $K = \mathcal{B}(\Phi)$ and (ii) is just a restatement of (i). □

The localization and homotopy invariance of the index imply that

7.6 Proposition (Localization) *If* $U' \subset U$ *is open and* $\mathcal{B}(\Phi) \subset U'$*, then* $\mathrm{BI}(\Phi) = \mathrm{BI}(\Phi|U')$*. Thus* $\mathrm{BI}(\Phi)$ *depends only on the behaviour of* Φ *on a neighborhood of* $\mathcal{B}(\Phi)$*. In particular, if* Φ *is defined on a larger open set* $U'' \supset U$ *such that* $(U'' \setminus U) \cap \mathbf{R}^k \times \{0\} = \emptyset$*, then* $\mathrm{BI}(\Phi) = \mathrm{BI}(\Phi|U)$*.*

Having all these we are in a position to state and prove one of the next main result of this chapter.

In addition to assumptions from 7.1 let us suppose that:

(iv) there is an open set $U_1 \supset U$ and a morphism $\Phi_1 \in \widetilde{M}_n(U_1, \mathbf{R}^n)$ (resp. $M_{CE}(U_1, \mathbf{R}^n)$) such that $\Phi_1|U = \Phi$ and $0 \in \Phi_1(\lambda, \mathrm{o})$ for all $(\lambda, 0) \in U_1 \cap \mathbf{R}^k \times \{0\}$. Let

$$\mathcal{S}_1 := \{(\lambda, x) \in U_1 \mid x \neq 0,\ 0 \in \Phi_1(\lambda, x)\}.$$

7.7 Corollary (Global Bifurcation) *Let* K *be a compact subset of* U_1 *such that* $\mathcal{B}(\Phi) \subset K$ *and* $K \cap (\mathbf{R}^k \setminus \Lambda) \times \{0\} = \emptyset$ (*e.g.* $K = \mathcal{B}(\Phi)$)*. If* $\mathrm{BI}(\Phi) \neq \mathrm{o}$*, then there is a nonempty connected set* $\mathcal{C} \subset \mathcal{S}_1 \setminus K$ *such that* $\mathrm{cl}\,\mathcal{C} \cap K \neq \emptyset$ *and at least one of the following occurs:*
(i) $\mathcal{C}$ *is unbounded;*
(ii) $\mathrm{cl}\,\mathcal{C} \cap \mathrm{bd\,U_1} \neq \emptyset$*;*
(iii) *there is a point* $\lambda_0 \in \mathbf{R}^k \setminus \Lambda$ *such that* $(\lambda_0, 0) \in U_1$ *and* $(\lambda_0, 0) \in \mathrm{cl}\,\mathcal{C}$*. Thus* Φ_1 *has bifurcation points outside* U *connected to* K *in* $\mathrm{cl}\,\mathcal{S}_1$*.*

Proof Let $U' = U_1 \setminus (\mathbf{R}^k \setminus \Lambda) \times \{0\}$. It is clear that U' is open and $K \subset U'$. Moreover $\{(\lambda, x) \in U' \mid x \neq 0,\ 0 \in \Phi_1(\lambda, x)\} = \mathcal{S}_1$ and $\mathcal{B}(\Phi_1|U') = \mathcal{B}(\Phi)$. In view of Proposition 7.6, $\mathrm{BI}(\Phi_1|U') = \mathrm{BI}(\Phi) \neq \mathrm{o}$. Hence, by Theorem 7.4, we get the existence of the required set $\mathcal{C}$. □

Now, we shall proceed with collecting the properties of the bifurcation index.

7.8 Proposition (Additivity) *If U_1, U_2 are open disjoint, $U = U_1 \cup U_2$, then*

$$\mathrm{BI}(\Phi) = \mathrm{BI}(\Phi|U_1) + \mathrm{BI}(\Phi|U_2).$$

In particular, if U_1, U_2 are open subsets of U, $\mathcal{B}(\Phi) \subset U_1 \cup U_2$ and $\mathcal{B}(\Phi) \cap$ $\mathrm{bd}\,(\mathrm{U}_1 \cap \mathrm{U}_2) = \emptyset$*, then* $\mathrm{BI}(\Phi|U_1 \cap U_2)$, $\mathrm{BI}(\Phi|U_1)$ *and* $\mathrm{BI}(\Phi|U_2)$ *are defined and*

$$\mathrm{BI}(\Phi) = \mathrm{BI}(\Phi|U_1) + \mathrm{BI}(\Phi|U_2) - \mathrm{BI}(\Phi|U_1 \cap U_2).$$

Proof The first assertion follows easily from the additivity property of deg and the second one is just a restatement of the first one. □

As always, the homotopy invariance of the bifurcation index is of importance. We shall give two versions of this fact.

7.9 Proposition (Homotopy invariance I) *Let Φ be a morphism from $\widetilde{M}_n(U \times [0,1], \mathbf{R}^n)$ (resp. $M_{CE}(U \times [0,1], \mathbf{R}^n)$) such that $0 \in \Phi(\lambda, \mathrm{o}, t)$ for all $\lambda \in \Lambda$ and $t \in [0,1]$. If the set $\bigcup_{t \in [0,1]} \mathcal{B}(\Phi_t)$, where $\Phi_t := \Phi \circ i_t$ ($i_t : U \ni x \mapsto (x,t) \in U \times [0,1]$) is compact in $\Lambda \times \{0\}$, then* $\mathrm{BI}(\Phi_0) = \mathrm{BI}(\Phi_1)$.

Proof follows immediately from the homotopy invariance of deg. □

The above homotopy invariance result is not sufficient since, in general, one cannot avoid that during homotopy the bifurcation points escape through the boundary of $\Lambda \times \{0\}$. For instance it is the case when Φ_0 and Φ_1 determine single-valued maps and one considers the linear homotopy joining them. Hence we have to state another homotopy invariance property.

7.10 Theorem (Homotopy invariance II) *Let again Φ be a morphism from $\widetilde{M}_n(U \times [0,1], \mathbf{R}^n)$ (resp. $M_{CE}(U \times [0,1], \mathbf{R}^n)$) such that* $\mathrm{BI}(\Phi_i)$ *is* *defined for* $i = 0, 1$. *Suppose that there are open sets* $\Lambda_1, \Lambda_2 \subset \Lambda$ *such that* $\mathcal{B}(\Phi_0), \mathcal{B}(\Phi_1) \subset \Lambda_1 \times \{0\} \subset \mathrm{cl}\,\Lambda_1 \times \{0\} \subset \Lambda_2 \times \{0\}$ *and* $\mathrm{cl}\,\Lambda_1$ *is compact. Let* $\Omega = \Lambda_2 \setminus \Lambda_1$. *If there is* $\varepsilon > 0$ *such that* $\Omega \times D^n_\varepsilon \subset U$ *and, for any* $t \in [0,1]$, $(\lambda, x) \in \Omega \times S^{n-1}_\varepsilon$, $0 \notin \Phi(\lambda, x, t)$ *and, for any* $\lambda \in \Omega$, $0 < |x| \leq \varepsilon$, $i = 0, 1$, $0 \notin \Phi(\lambda, x, i)$, *then* $\mathrm{BI}(\Phi_0) = \mathrm{BI}(\Phi_1)$.

Observe that Proposition 7.9 follows easily as a consequence of Theorem 7.10. In fact under the assumptions of Proposition 7.9, $\mathcal{B} = \bigcup_{t \in [0,1]} \mathcal{B}(\Phi_t)$ is compact in Λ; hence taking any open sets Λ_1, Λ_2 such that $\mathcal{B} \subset \Lambda_1 \times \{0\} \subset \mathrm{cl}\,\Lambda_1 \times \{0\} \subset \Lambda_2 \times \{0\} \subset \mathrm{cl}\,\Lambda_2 \times \{0\} \subset \Lambda \times \{0\}$ and $\mathrm{cl}\,\Lambda_2$ is compact, we can set $\varepsilon = \frac{1}{2}\mathrm{dist}((\mathrm{cl}\,\Lambda_2 \setminus \Lambda_1) \times \{0\}, \mathrm{cl}\,\mathcal{S} \cup \mathrm{bd}\,\mathrm{U})$, where here $\mathcal{S} := \{(\lambda, x) \in U \setminus \Lambda \times \{0\} \mid 0 \in \Phi(\lambda, x, t) \text{ for some } t \in [0, 1]\}$.

Proof (of Theorem 7.10) Without loss of generality we may assume that Λ_2 is also bounded and that actually $\mathrm{cl}\,\Omega \times D^n_\varepsilon \cup \Lambda_1 \times \{0\} \subset U$.

Let V be an open bounded and such that

$$\mathrm{cl}\,\Omega \times D^n_\varepsilon \cup \Lambda_1 \times \{0\} \subset V \subset \mathrm{cl}\,V \subset U.$$

For $i = 0, 1$, set

$$\mathcal{S}_i = \{(\lambda, x) \in U \setminus \Lambda \times \{0\} \mid 0 \in \Phi_i(\lambda, x)\},$$

take a function $\alpha_i : \Lambda \to [0, \infty)$ such that $\alpha_i(\lambda) = 0$ for $\lambda \in \mathcal{B}(\Phi_i)$ and $0 < \alpha_i(\lambda) < d((\lambda, 0), \mathrm{bd}\,\mathrm{U} \cup \mathrm{cl}\,\mathcal{S}_\mathrm{i})$ for $\lambda \in \Lambda \setminus \mathcal{B}(\Phi_i)$. Due to our setting and assumptions we may assume that $\alpha_i(\lambda) < d((\lambda, 0), \mathrm{bd}\,\mathrm{V})$ for all $\lambda \in \Lambda_2$ and $\alpha_i(\lambda) = \varepsilon$ for $\lambda \in \mathrm{cl}\,\Omega$. Let

$$X_i := \{(\lambda, x) \in \mathbf{R}^m \mid \lambda \in \Lambda,\ |x| = \alpha_i(\lambda)\}$$

and define a continuous $f_i : U \to \mathbf{R}$ such that

$$f_i(\lambda, x) \begin{cases} > 0 & \text{if } |x| < \alpha_i(\lambda),\ \lambda \in \Lambda \\ = 0 & \text{if } (\lambda, x) \in X_i \\ < 0 & \text{if } \lambda \notin \Lambda \text{ or } |x| > \alpha_i(\lambda). \end{cases}$$

According to the definition,

$$\mathrm{BI}(\Phi_i) = \deg((\Phi_i, f_i), V), \qquad i = 0, 1.$$

Let $\alpha := \min\{\alpha_0, \alpha_1\}$, $X := \{(\lambda, x) \in \mathbf{R}^m \mid |x| = \alpha(\lambda),\ \lambda \in \operatorname{cl}\Lambda_2\}$ and $V' = V \cap \Lambda_2 \times \mathbf{R}^n$. We see that $X \subset \operatorname{cl} V'$. Define a continuous function $f : \operatorname{cl} V' \to \mathbf{R}$ such that

$$f(\lambda, x) \begin{cases} < 0 & \text{if } |x| < \alpha(\lambda) \\ = 0 & \text{for } (\lambda, x) \in X \\ > 0 & \text{if } |x| > \alpha(\lambda). \end{cases}$$

Clearly $((\Phi_i, f), V') \in \widetilde{M}(m, n+1)$ (resp. $M_{CE}(m, n+1)$). Using the homotopy and localization property of deg, it is not difficult to see that $\mathrm{BI}(\Phi_i) = \deg((\Phi_i, f), V')$, $i = 0, 1$.

Consider a morphism $\Psi \in \widetilde{M}_n(\operatorname{cl} V' \times I, \mathbf{R}^{n+1})$ (resp. $M_{CE}(\operatorname{cl} V' \times I, \mathbf{R}^{n+1})$) given by $\Psi = (\Phi, F)$ where $F : V \times I \to \mathbf{R}$, $F(\lambda, x, t) = f(\lambda, x)$ for $(\lambda, x) \in \operatorname{cl} V'$ and $t \in I$. It is clear that, for $(\lambda, x) \in \operatorname{cl} V'$, $t \in I$, if $\lambda \in \operatorname{cl}\Omega$, then $0 \notin \Psi(\lambda, x, t)$. Therefore the set $\{(\lambda, x) \in V \mid 0 \in \Psi(\lambda, x, t)$ for some $t \in I\}$ is contained in $X \cap [\operatorname{cl}\Lambda_1 \times \mathbf{R}^n] \subset V'$ and is compact. Since Ψ furnishes a homotopy joining (Φ_0, f) to (Φ_1, f) in $\widetilde{M}(m, n+1)$ (or $M_{CE}(m, n+1)$), we gather that

$$\mathrm{BI}(\Phi_0) = \deg((\Phi_0, f), V) = \deg((\Phi_1, f), V) = \mathrm{BI}(\Phi_1)$$

in view of the homotopy invariance of deg [1]. □

7.B. The J. C. Alexander invariant

Now we are going to relate our bifurcation index with another bifurcation invariant γ_Φ which generalizes the so-called Alexander bifurcation invariant γ_f defined (in the context of a single-valued map f) in [5] (comp. [6]) and hence to provide a different description of it.

We shall start with some abstract investigations.

7.11 Loop spaces For $m \geq 1$, let $\Omega(m) := \mathcal{C}(S^{m-1}, S^{m-1})$ be the space of all (continuous) maps $S^{m-1} \to S^{m-1}$ with the the compact-open topology. It is clear that the suspension generates the embedding

[1] The proof might be simplified if we employ Remark 7.3 (i) and the degree from Remark 5.32.

$\Omega(m) \subset \Omega(m+1)$. Set $\Omega := \bigcup_{m\geq 1} \Omega(m)$ and endow it with the topology compatible with the family $\{\Omega(m)\}$.

Now let $i \geq 0$. We shall consider the homotopy group $\pi_i(\Omega)$. It is known that this group is well-defined, i.e. the base point may be skipped and, actually, $\pi_i(\Omega) \cong \Pi_i$. Since the author does not know any direct reference to this fact, we provide a short proof below.

First observe that the family $\{\pi_i(\Omega(m))\}_{m\geq 1}$ together with maps $\pi_i(\Omega(m)) \to \pi_i(\Omega(m+1))$ induced by the above inclusions forms a direct system with the direct limit equal to $\pi_i(\Omega)$ (see [103, Prop. 1.4.3]). Hence, let us first study the group $\pi_i(\Omega(m), \omega)$, $i \geq 1$, where $\omega \in \Omega(m)$ and the set of (free) homotopy classes $[S^i; \Omega(m)]$.

Evidently
$$[S^i \times S^{m-1}; S^{m-1}] \cong [S^i; \Omega(m)]$$
where we identify the class $[\alpha]$ of a map $\alpha : S^i \times S^{m-1} \to S^{m-1}$ with the class of the *adjoint map* $\widetilde{\alpha} : S^i \to \Omega(m)$, i.e. given by *adjointness*, i.e. $\widetilde{\alpha}(\lambda)(x) = \alpha(\lambda, x)$ for any $\lambda \in S^i$ and $x \in S^{m-1}$. The set $[S^i \times S^{m-1}; S^{m-1}]$ has a group structure; namely that of $\pi^{m-1}(S^i \times S^{m-1})$ provided $m > i+2$. However we shall use some other isomorphisms.

Let us fix a base point $x_0 \in S^{m-1}$, denote (as usual) by $\Omega^{m-1}S^{m-1}$ the pointed space of maps $S^{m-1} \to S^{m-1}$ preserving the base point x_0 with the base point at the constant map $\omega_0 \equiv x_0$. Of course we identify

$$\pi_i(\Omega^{m-1}S^{m-1}, \omega_0) \cong \pi_{m+i-1}(S^{m-1}, x_0). \tag{7.3}$$

There is a (weak) fibration

$$\mathfrak{p} : \Omega(m) \to S^{m-1}$$

where $\mathfrak{p}(h) = h(x_0)$ for $h \in \Omega(m)$ with the basic fibre $\mathfrak{p}^{-1}(x_0) = \Omega^{m-1}S^{m-1}$ (see [102, Cor. 13.2]).

For $i \geq 1$, we have the exact homotopy sequence of this fibration

$$\pi_{i+1}(S^{m-1}, x_0) \xrightarrow{\partial} \pi_i(\Omega^{m-1}S^{m-1}, \omega_0) \xrightarrow{j_*} \pi_i(\Omega(m), \omega_0) \xrightarrow{\mathfrak{p}_*} \pi_i(S^{m-1}, x_0)$$

where j_* is induced by the inclusion $j : \Omega^{m-1}S^{m-1} \hookrightarrow \Omega(m)$. If $0 \leq i < m-2$ (or $i = 0$ and $m = 2$) then,

$$j_* : \pi_i(\Omega^{m-1}S^{m-1}, \omega_0) \to \pi_i(\Omega(m), \omega_0) \tag{7.4}$$

is an isomorphism.

Now, since $\Omega^{m-1}S^{m-1}$ is a H-space (see [102, Prop. III.11.3]), $\pi_0(\Omega^{m-1}S^{m-1}, \omega_0)$ has the group structure; for $i = 0$, j_* is an isomorphism and, thus, for $m \geq 2$, path components of $\Omega^{m-1}S^{m-1}$ and $\Omega(m)$ are in 1–1 correspondence with each other (and with integers $\mathbf{Z}$ via degree).

Clearly (see [166, Th. 7.3.5]) each path component of $\Omega^{m-1}S^{m-1}$ is simple (i.e. the homotopy groups do not depend on the choice of the base point). Moreover, all the components of $\Omega^{m-1}S^{m-1}$ have the same homotopy type and, hence, the homotopy groups of different components can be naturally identified.

This implies that also (each path component of) $\Omega(m)$ is i-simple and different components have the same i-th homotopy groups for $1 \leq i < m-2$, since the fibration above is a fibration on each path component of $\Omega(m)$, and so the exactness of the above sequence holds for each component.

Finally it is easy to see that above arguments do not depend on the choice of the base point x_0 (for if we change x_0 to some other x_1, then it is easy to show the eqiuvalence between the homotopy sequences of the corresponding fibrations).

Therefore, for $0 \leq i < m-2$, when speaking of the i-th homotopy group of $\Omega(m)$, base points may be skipped in the sense that, for each $\omega \in \Omega(m)$,

$$\pi_i(\Omega(m), \omega) = \pi_i(\Omega(m)_d, \omega) = [S^i; \Omega(m)_d]$$

where $\Omega(m)_d$ is the path component containing ω (precisely if the degree of ω is equal to $d \in \mathbf{Z}$, then $\Omega(m)_d$ consists of maps $S^{m-1} \to S^{m-1}$ of the degree d) and the group does not depend on the choice of ω.

This also implies that the group $\pi_i(\Omega)$ is well-defined and the base points may be omitted.

By (7.3), (7.4), for $0 \leq i < m-2$, there is an isomorphism

$$E_{i,m} : \pi_i(\Omega(m)) \to \pi_{m+i-1}(S^{m-1}). \tag{7.5}$$

It is also clear that this isomorphism is stable in the following sense: the diagram

$$\begin{array}{ccc} \pi_i(\Omega(m)) & \xrightarrow{E_{i,m}} & \pi_{m+i-1}(S^{m-1}) \\ \downarrow & & \downarrow \Sigma \\ \pi_i(\Omega(m+1)) & \xrightarrow{E_{i,m+1}} & \pi_{m+i}(S^m), \end{array}$$

where the left vertical arrow is induced by the the embedding $\Omega(m) \hookrightarrow \Omega(m+1)$ and Σ is the suspension isomorphism (for $i < m-2$), is commutative. Hence this left arrow is also an isomorphism and $\{E_{i,m}\}$ gives rise to a (stable) isomorphism

$$E_i : \pi_i(\Omega) \to \Pi_i. \tag{7.6}$$

Finally, for $0 \le i < m-2$, we have the commutative diagram

$$\begin{array}{ccc} \pi_i(\Omega(m)) & \xrightarrow{E_{i,m}} & \pi_{m+i-1}(S^{m-1}) \\ \downarrow & & \downarrow \Sigma \\ \pi_i(\Omega) & \xrightarrow{E_i} & \Pi_i \end{array} \tag{7.7}$$

7.12 γ-invariant Let us return to the setting from paragraph 7.1 but now suppose that:

(i) there are $\lambda_0 \in \mathbf{R}^k$ and positive numbers r, ε such that $D^k(\lambda_0, r) \times D^n_\varepsilon \subset U$, $\mathcal{B}(\Phi) \subset B^k(\lambda_0, r) \times \{0\}$ and $0 \notin \Phi(\lambda, x)$ for all $\lambda \in S^{k-1}(\lambda_0, r)$ and x with $0 < |x| \le \varepsilon$.

(ii) $\mathrm{i}(\Phi) \le n-1$.

To simplify our further considerations, we rescale and, without loss of generality, assume that $\lambda_0 = 0$, $\varepsilon = r = 1$.

Condition (i) implies that the set $\mathcal{B}(\Phi)$ is compact and we are in position to define $\mathrm{BI}(\Phi) \in \Pi_{k-1}$ as in Definition 7.2 taking $V = B^k \times B^n$.

Suppose that $\Phi|\mathrm{bd\,V}$ is represented by a cotriad

$$\mathrm{bd\,V} \xleftarrow{\mathrm{p}} \mathrm{W} \xrightarrow{\mathrm{q}} \mathbf{R}^{\mathrm{n}}$$

from $\widetilde{D}_{n-1}$ (or D_{CE}).

Clearly, in view of assumption 7.12 (i), we may regard $\Phi' = \Phi|S^{k-1} \times S^{n-1}$ as a morphism from $\widetilde{M}_{n-1}(S^{k-1} \times S^{n-1}, \mathbf{R}^n \setminus \{0\})$ (or $M_{CE}(S^{k-1} \times S^{n-1}, \mathbf{R}^n \setminus \{0\})$). Then Φ' is represented by the cotriad

$$S^{k-1} \times S^{n-1} \xleftarrow{p'} W' \xrightarrow{q'} \mathbf{R}^n \setminus \{0\}$$

where $W' = p^{-1}(S^{k-1} \times S^{n-1})$, $p' = p|W'$ and $q' = q|W'$. Obviously $(p', q') \in \widetilde{D}_{n-1}$ (or $(p', q') \in D_{CE}$).

In view of Theorem 2.17 (or 2.19), there is a unique (up to homotopy) map $g' : S^{k-1} \times S^{n-1} \to S^{n-1}$ such that

$$g' \circ p' \simeq q' : W' \to \mathbf{R}^n \setminus \{0\}, \tag{7.8}$$

that is

$$\Phi' \simeq_n g'. \tag{7.9}$$

Now we are in a position to state the following definition.

7.13 Definition Let $\widetilde{g}' : S^{k-1} \to \Omega(n)$ be the adjoint of g'. By γ_Φ we denote the class $E_{k-1}(\alpha) \in \Pi_{k-1}$ where $\alpha \in \pi_{k-1}(\Omega)$ is the homotopy class $\widetilde{g}'$ considered as a map $\widetilde{g}' : S^{k-1} \to \Omega$; or equivalently, $\gamma_\Phi = E_{k-1,m}(\alpha)$ where $\alpha \in \pi_{k-1}(\Omega(m))$ is the homotopy class of $\widetilde{g}'$ considered as a map $\widetilde{g}' : S^{k-1} \to \Omega(m)$ where $m \geq n$ is in the stable area, i.e. $m - 1 > k$.

The invariant γ_Φ is well-defined because it obviously does not depend on the choice of λ_0, r and ε in paragraph 7.12 satisfying the assumptions stated there. It only depends on Φ.

The following result holds true.

7.14 Theorem *The following equality* (*up to an isomorphism*) *holds*

$$\gamma_\Phi = \mathrm{BI}(\Phi).$$

Proof Without loss of generality we may assume that actually $k < n-1$. According to Definition 7.13, $\gamma_\Phi = E_{k-1,n}[\widetilde{g}']$. Let us put

$$S^{k,n} := D^k \times S^{n-1} \cup S^{k-1} \times D^n.$$

Clearly $S^{k,n} = \mathrm{bd}\,\mathrm{V}$ (recall that $V = B^k \times B^n$).

In virtue of the construction from the previous section we have a function $f : U \to \mathbf{R}$ defined by (7.2). In view of assumption (i) from paragraph 7.12, we may actually assume that

$$f(\lambda, x) \begin{cases} > 0 & \text{if } (\lambda, x) \in S^{k-1} \times B^n; \\ < 0 & \text{if } (\lambda, x) \in B^k \times S^{n-1}; \end{cases}$$

This implies that $f(\lambda, x) = 0$ when $(\lambda, x) \in S^{k-1} \times S^{n-1}$. Then $\mathrm{BI}(\Phi) = \deg(\Psi, V)$ where $\Psi = (\Phi, f)$. Moreover, if $f' : D^k \times D^n \to \mathbf{R}$ is another function satisfying the above conditions, then it may be taken in place of f because of the boundary dependence of deg – see Proposition 5.4 and 5.25.

In view of the construction from 5.25, we have a unique (up to homotopy) map $g : S^{k,n} \to \mathbf{R}^{n+1} \setminus \{0\}$ such that

$$\Psi | S^{k,n} \simeq_n g \tag{7.10}$$

(or $\Psi | S^{k,n} \simeq_{CE} g$) and

$$\mathrm{BI}(\Phi) = \deg(g, V) \in \pi^n(S^{n+k-1}) \cong \Pi_{k-1} \tag{7.11}$$

(or rather, in place of g one should take its extension onto $\mathrm{cl}\, V$).

By Remark 4.16, $\Psi | S^{k,n}$ is represented by the triad

$$S^{k,n} \xrightarrow{\quad p \quad} W \xrightarrow{(q, f\circ p)} \mathbf{R}^{n+1} \setminus \{0\}.$$

Hence (7.10) means that

$$g \circ p \simeq (q, f \circ p). \tag{7.12}$$

It is easy to see that if $G : S^{k,n} \to D^n$ denotes an arbitary extension of g' (see (7.8) or (7.9)) onto $S^{k,n}$, then $(G, f) \circ p \simeq (q, f \circ p) : W \to \mathbf{R}^{n+1} \setminus \{0\}$. Therefore $g \simeq (G, f) : S^{k,n} \to \mathbf{R}^{n+1} \setminus \{0\}$ because of (7.12) and the (homotopy) uniqueness.

Hence without loss of gerality we may suppose that

$$g = (G, f) \tag{7.13}$$

where G is an extension of g' onto $S^{k,n}$.

Recall now the so-called Hopf construction (see [103, Def. 3.3]). It yields a homomorphism $\mathcal{H} : \pi_{k-1}(\Omega(n)) \to \pi_{n+k-1}(S^n)$ which admits the following (equivalent) description. It is enough to define $\mathcal{H}([v])$ for $[v] \in [S^{k-1} \times S^{n-1}; S^{n-1}]$ where $v : S^{k-1} \times S^{n-1} \to S^{n-1}$. Define

$$H_v : S^{k,n} \to \mathbf{R}^n \cup \{\infty\} \cong S^n \ (^2)$$

by

$$H_v(\lambda, x) = \begin{cases} 0 & \text{if } \lambda = 0 \\ |\lambda| v(\frac{\lambda}{|\lambda|}, x) & \text{if } \lambda \neq 0,\ x \in S^{n-1} \\ |x|^{-1} v(\lambda, \frac{x}{|x|}) & \text{if } \lambda \in S^{k-1},\ x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

In other words one takes the extension of v onto $S^{k,n}$ with values in S^n which maps $D^k \times S^{n-1}$ into the south and $S^{k-1} \times D^n$ into the north hemisphere of S^n.

Let $h : S^{n+k-1} \to S^{k,n}$ be the radial homeomorphism. Then

$$\mathcal{H}([v]) := [H_v \circ h] \in \pi_{n+k-1}(S^n).$$

It is known that $\mathcal{H}$ is an isomorphism (since $k < n - 1$) – see [103, 15.6].

If we now apply $\mathcal{H}$ to $[\widetilde{g}']$, then we easily see that $\mathcal{H}([\widetilde{g}'])$ is nothing else but the class $[(G, f) \circ h]$ because of the definition of f.

On the other hand, in view of (7.11) and Remark 5.3 (i), we have $\mathrm{BI}(\Phi) = \deg(g, V) = [g \circ h] \in \pi_{n+k-1}(S^n)$. By (7.13), $g = (G, f)$, so $\mathrm{BI}(\Phi) = [(G, f) \circ h]$. Therefore we obtained that

$$\mathcal{H}([\widetilde{g}']) = \mathrm{BI}(\Phi).$$

Hence finally $\mathcal{H} \circ E^{-1}_{k-1,n}(\gamma_\Phi) = \mathrm{BI}(\Phi)$. □

It is clear that neither $\mathrm{BI}(\Phi)$ nor γ_Φ can be easily computed. Nevertheless, there are quite general situations in which one can present concrete hypotheses which guarantee the (common) nontriviality of these invariants.

[2] We identify $\mathbf{R}^n \cup \{\infty\}$ with S^n via the stereographic projection; in particular, ∞ corresponds to the north pole and 0 to the south pole of S^n.

7.15 Example Recall paragraph 7.12. If $k = 1$, then $\gamma_\Phi \in \pi_{n-1}(S^{n-1}) \cong \Pi_0 \cong \mathbf{Z}$ is induced by the map $g' : \{-1,1\} \times S^{n-1} \to S^{n-1}$ or the adjoint $\widetilde{g}' : \{-1,1\} \to \Omega(n)$. Hence γ_Φ is nontrivial if and only if $\deg(\widetilde{g}'(1)) - \deg(\widetilde{g}'(-1)) = \deg(g'(1,\,\cdot\,), B^n) - \deg(g'(-1,\,\cdot\,), B^n) = \deg(\Phi(1,\,\cdot\,), B^n) - \deg(\Phi(-1,\,\cdot\,), B^n)$. Thus, in this case, the ordinary degree data yields the global bifurcation.

If $k > 1$, then the situation becomes more complicated but we still can provide some computations.

7.16 First we shall consider single-valued maps of the special form. Suppose that $f : U = \Lambda \times \mathbf{R}^n \to \mathbf{R}^n$, where Λ is a neighborhood of $\lambda_0 \in \mathbf{R}^k$, is a continuous map such that, for each $\lambda \in \Lambda$, $x \in \mathbf{R}^n$, $f(\lambda, x) = L(\lambda)x$ where $L(\lambda)$ is a linear map, i.e. $L(\lambda) \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ and $L(\lambda)$ depends continuously on λ. Assume that, for some $r > 0$, $D^k(\lambda_0, r) \subset \Lambda$ and that for $\lambda \in \Lambda \setminus B^k(\lambda_0, r)$, $L(\lambda)$ belongs to the general linear group $GL(n)$ of all linear automorphisms of $\mathbf{R}^n$. Hence, if $\lambda \in \Lambda$, $|\lambda - \lambda_0| \geq r$ and $x \neq 0$, then $f(\lambda, x) \neq 0$. This also shows that $\mathcal{B}(f) \subset B^k(\lambda_0, r) \times \{0\}$. Clearly all assumptions of 7.12 are satisfied, i.e. there is $\varepsilon > 0$ such that $f(\lambda, x) \neq 0$ for all $\lambda \in S^{k-1}(\lambda_0, r)$ and $x \in \mathbf{R}^n$ with $0 < |x| \leq \varepsilon$. Thus, the Alexander invariant $\gamma_f \in \Pi_{k-1}$ is defined.

If $k = 1$, then, by Example 7.15, $\gamma_f = \deg(f(\lambda_0 + r,\,\cdot\,), B^n_\varepsilon) - \deg(f(\lambda_0 - r,\,\cdot\,), B^n_\varepsilon)$ and $\deg(f(\lambda_0 \pm r,\,\cdot\,), B^N_\varepsilon) = \mathrm{sgn}\det \mathrm{L}(\lambda_0 \pm \mathrm{r})$ or, what is equivalent, $\deg(f(\lambda_0 \pm r,\,\cdot\,), B^N_\varepsilon) = (-1)^{\beta(\lambda_0 \pm r)}$ where $\beta(\lambda)$ is the number of negative eigenvalues of $L(\lambda)$ counted with multiplicities. Therefore the bifurcation occurs whenever $\beta(\lambda)$ changes parity when λ passes through 0.

In particular, when

$$L(\lambda) = I - \lambda A, \tag{7.14}$$

where $A \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$, our assumption guarantees that if $\mu \in \Lambda$ is a characteristic number (i.e. the reciprocal of an eigenvalue) of A, then $\mu \in (\lambda_0 - r, \lambda_0 + r)$. Hence the bifurcation index $\gamma_f \neq 0$ if and only if the number of characteristic numbers of A in $(\lambda_0 - r, \lambda_0 + r)$ counted with multiplicities is odd.

If $k \geq 1$, then it is much harder to compute $\gamma_f \in \Pi_{k-1}$. For simplicity assume now that $L(\lambda)$ is invertible for $\lambda \in \Lambda$, $\lambda \neq \lambda_0$ and that $L(\,\cdot\,)$ is differentiable at λ_0. Further assume that in a neighborhood of λ_0,

$$|L(\lambda)x| \geq c|\lambda - \lambda_0||x|. \tag{7.15}$$

After [7, Prop.2.1], we have the following result. Let $k = 1, 2, 4, 8$ modulo 8 and let

$$c_k = \begin{cases} k & \text{if } k = 1, 2, 4 \text{ or } 8 \\ 16c_{k-8} & \text{if } k > 8 \end{cases}$$

Then $\dim \mathrm{Ker} L(\lambda_0)$ is an integer multiple of c_k. If it is an odd multiple of c_k, then γ_f is nontrivial ([3]). Let us add that cited Proposition 2.1 from [7] contains an error: the authors do not assume that L is differentiable.

There are other means to compute γ_f in the above case.

7.17 ***J*-homomorphism** Suppose that f is as in the above paragraph. To simplify the notation we rescale again and suppose that $\lambda_0 = 0$ and $r = 1$. Without loss of generality we may assume that, for all $\lambda = S^{k-1}$, $L(\lambda)$ belongs to the orthogonal group $O(n)$ ($O(n)$ is a deformation retract of $GL(n)$ – [169, Prop. 11.44]). Moreover we may suppose that $\det L(*) = 1$ ($*$ is the base point in S^{k-1}); hence $L(\lambda) \in SO(n)$ for all $\lambda \in S^{k-1}$. Thus the adjoint $\widetilde{f} : S^{k-1} \to SO(n)$ and $\widetilde{f} = L$.

After [103, 15.6.3] recall the *classical J-homomorphism* of G. W. Whitehead: $J_{m-1} : \pi_{k-1}(SO(m-1)) \to \pi_{m+k-2}(S^{m-1})$, $m \geq 2$, given by the formula $J_{m-1}([\alpha]) = \mathcal{H}([\beta]) \in \pi_{m+k-2}(S^{m-1})$ for $[\alpha] \in \pi_{k-1}(SO(m-1))$, where $\beta(\lambda, x) = \alpha(\lambda)(x)$ ($SO(m-1)$ as a path connected topological group is a H-space and, hence, is simple and base points may be skipped). There is a commutative diagram

$$\begin{array}{ccc} \pi_{k-1}(SO(m-1)) & \xrightarrow{J_{m-1}} & \pi_{m+k-2}(S^{m-1}) \\ \Big\downarrow i_* & & \Big\downarrow \Sigma \\ \pi_{k-1}(SO(m)) & \xrightarrow{J_m} & \pi_{m+k-1}(S^m) \end{array} \tag{7.16}$$

where i_* is a homomorphism induced by the natural embedding $i : SO(m-1) \to SO(m)$ ($SO(m-1)$ operates on $\mathbf{R}^m$ by operating on the first $(m-1)$ coordinates) and Σ is the suspension homomorphism.

[3]This result is not difficult for $k = 1$ — one easy checks that in this case, $\beta(\lambda)$ changes parity exactly when the dimension $\dim \mathrm{Ker} L(\lambda_0)$ is odd; observe moreover that if L is of the form (7.14), then assumption (7.15) implies that if λ_0 is a characteristic number, then its geometric and algebraic multiplicities coincide.

Let $SO = \bigcup_{m\geq 1} SO(m)$. Since the SO is a path connected topological group, it is simple and $\pi_{k-1}(SO) = [S^{k-1}; SO]$ is well defined.

If in (7.16) we pass to direct limits we obtain the *stable J-homomorphism*

$$J : \pi_{k-1}(SO) = \varinjlim_{l\geq 0} \pi_{k-1}(SO(l)) \to \Pi_{k-1}.$$

The last equality follows again from [103, Prop. 1.4.3]. Observe that, for $m > k+1$, $\pi_{k-1}(SO(m-1)) \cong \pi_{k-1}(SO)$ (see [103, Th.7.4.1]). Therefore, vertical arrows in diagram (7.16) are in fact isomorphims provided $m > k+1$.

The map

$$S^{k-1} \ni \lambda \mapsto L(\lambda) \in SO(n) \subset SO$$

induces an element $\overline{\gamma}_f \in \pi_{k-1}(SO)$; or equivalently $\overline{\gamma}_f \in \pi_{k-1}(SO(m))$ is induced by $L : S^{k-1} \to SO(m)$ where $m > k+1$.

7.18 Proposition *The following equality* (*up to isomorphism*) *holds*

$$\gamma_f = J(\overline{\gamma}_f).$$

Proof If $j : SO \to \Omega$ is the embedding given by the embeddings $j_m : SO(m) \subset \Omega(m)$, then $\gamma_f = E_{k-1}(j_*(\overline{\gamma}_f))$ where E_{k-1} was defined in (7.6).
Suppose that $m > k+1$. Then $\gamma_f = E_{k-1,m}(j_{m*}(\overline{\gamma}_f))$.

Husemoller [103, 15.6.4] shows that

$$J_{m-1} : \pi_{k-1}(SO(m-1)) \to \pi_{m+k-2}(S^{m-1})$$

splits as the composition

$$\pi_{k-1}(SO(m-1)) \xrightarrow{\varepsilon_*} \pi_{k-1}(\Omega(m)) \xrightarrow{E_{k-1,m}} \pi_{m+k-2}(S^{m-1})$$

where the homomorphism ε_* is induced by the natural embedding $\varepsilon = j_m \circ i : SO(m-1) \to \Omega(m)$.
From the above composition and diagram (7.16), we get that

$$J_m(\overline{\gamma}_f) = \Sigma(\gamma_f).$$

Thus, passing to the stable groups, we get that indeed

$$\gamma_f \quad = \quad J(\overline{\gamma}_f).$$

□

7.19 Computation of the J-homomorphism The advantage of the above setting is twofold. Although the structure of Π_{k-1} is largely unknown, the structure of $\pi_{k-1}(SO)$ is completely known by the results of Bott [36]. It can be read off the following table:

k modulo 8	1	2	3	4	5	6	7	8
$\pi_{k-1}(O)$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$

Moreover, the kernel of J has been computed by Adams [1]. In particular, it is known that if $k = 1, 2 \,\text{modulo}\, 8$, then J is monomorphic; if $4|k$, then the kernel of J consists of elements that are divisible by b_k where b_k is the denominator of $B_{\frac{k}{4}}/k$ expressed in lowest terms and B_l is the l-th Bernoulli number.

Hence, in particular $\gamma_f \neq 0$ if $k = 1, 2$ modulo 8 and $\overline{\gamma}_f \neq 0$ or if $4|k$ and $\overline{\gamma}_f$ is not divisible by b_k.

Under circumstances this also allows to compute γ_Φ.

7.20 Let again $U = \Lambda \times \mathbf{R}^n$, where Λ is an open neighborhood of $\lambda_0 \in \mathbf{R}^k$, and let $\Phi \in \widetilde{M}_{n-1}(U, \mathbf{R}^n)$ (or $M_{CE}(U, \mathbf{R}^n)$) be such that $0 \in \Phi(\lambda, o)$ for all $\lambda \in \Lambda = \{\lambda \in \mathbf{R}^k \mid (\lambda, 0) \in U\}$. Further we suppose that there are:

(i) a number $r > 0$, such that $D^k(\lambda_0, r) \subset U$;

(ii) a cotriad

$$U \xleftarrow{p} W \xrightarrow{q} \mathbf{R}^n$$

in $\widetilde{D}_{n-1}$ (or D_{CE}) representing Φ;

(iii) a continuous map $L : \Lambda \to \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ such that, for $\lambda \in \Lambda \setminus B^k(\lambda_0, r)$, $L(\lambda) \in GL(n)$;

(iv) a number $\beta > 0$ and a neighborhood Λ_0 of $S^{k-1}(\lambda_0, r)$, say $\Lambda_0 = \{\lambda \in \Lambda \mid r - \eta \leq |\lambda - \lambda_0| \leq r + \eta\}$, such that, for $w \in p^{-1}(\Lambda_0 \times B^n_\beta)$,

$$|f \circ p(w) - q(w)| = o(|p_1(w)|) \qquad \text{as} \quad p_1(w) \to 0$$

where $p_1(w)$ is the x-coordinate of $p(w)$ and $f(\lambda, x) = L(\lambda)x$.

It is clear that in the case Φ is singlevalued, our assumption (iii) implies that, for each $\lambda \in \Lambda_0$, $\varphi(\lambda, \cdot\,)$ is Fréchet differentiable at $x = 0$ and $Dg(\lambda, 0) = L(\lambda)$.

In virtue of 7.16, the invariants $\gamma_f \in \Pi_{k-1}$ and $\overline{\gamma}_f \in \pi_{k-1}(SO)$ are defined.

Suppose that $\Lambda_2 := B^k(\lambda_0, r+\eta)$ and $\Lambda_1 := B^k(\lambda_0, r-\eta)$.

7.21 Proposition *Under the above assumptions* $\mathcal{B}(\Phi|\Lambda_2 \times B^n_r) \subset B^k(\lambda_o, r) \times\{0\}$, $\mathrm{BI}(\Phi|\Lambda_2 \times B^n_\beta)$ *is defined and equal to* γ_f.

Proof Without loss of generality we may assume that, for $\lambda \in \Omega := \Lambda_2 \backslash \Lambda_1$, $L(\lambda)$ is invertible. Observe that $\Omega \subset \Lambda_0$.

The continuity of L implies that there is $c > 0$ such that $|f(\lambda, x)| \geq c|x|$ for all $\lambda \in \Omega$, $x \in \mathbf{R}^n$, i.e. $|f \circ p(w)| \geq c|p_1(w)|$ for each $w \in p^{-1}(\Omega \times B^n_\beta)$.

By assumption 7.20 (iv), there is $0 < \varepsilon < \beta$ such that $|f(p(w)) - q(w)| \leq \frac{c}{2}|p_1(w)|$ provided $|p_1(w)| \leq \varepsilon$ and $p(w) \in \Omega \times B^n_\beta$.

By Ψ we denote a morphism represented by a cotriad

$$U \times [0,1] \xleftarrow{P} W \times [0,1] \xrightarrow{Q} \mathbf{R}^n$$

where $P(w,t) = (p(w), t)$ and $Q(w,t) = (1-t) f \circ p(w) + t q(w)$ for $t \in [0,1]$, $w \in W$. Clearly $(P, Q) \in \widetilde{D}_n$ (or D_{CE}).

Now it is easy to show that if $t \in [0,1]$, $(\lambda, x) \in \Omega \times S^{n-1}_\varepsilon$, then $0 \notin \Psi(\lambda, x, t)$; and if $\lambda \in \Omega$, $0 < |x| \leq \varepsilon$, then $0 \notin \Psi(\lambda, x, i)$, $i = 0, 1$. This means, in particular, that $\mathcal{B}(\Phi|\Lambda_2 \times B^n) \subset \Lambda_1$. Moreover, this implies that $\mathrm{BI}(\Phi)$ is defined. In view of the homotopy invariance II (see Theorem 7.10) and Theorem 7.14, we complete the proof. □

7.22 Remark Proposition 7.21 implies that $\mathrm{BI}(\Phi) = \gamma_f$ provided condition 7.20 (iv) holds on $W = p^{-1}(U)$. Moreover if $L(\lambda)$ is nonsingular when $\lambda \in \Lambda$, $\lambda \neq \lambda_0$, then in fact $\mathcal{B}(\Phi) \subset \{(\lambda_o, o)\}$ since we may take r above as small as we wish. Hence, in this situation it is clear that Propositions 7.21, 7.18 together with 7.19 give means to compute $\mathrm{BI}(\Phi)$. Similarly on may use results from 7.16 combined with Proposition 7.21 to establish nontriviality of γ_Φ.

7.C. Infinite-dimensional bifurcation

Let us now return to the problem sketched in the beginning of this chapter: we shall study the bifurcation of fixed points of a set-valued map (morphism) defined in a possibly infinite-dimensional Banach space E. An invariant we are going to introduce again simplifies and generalizes that of Bartsch [22].

7.23 Assume that U is an open subset of $\mathbf{R}^k \times E$ and a morphism $\Phi \in \widetilde{M}(U, E)$ (resp. $M_{CE}(U, E)$) determining a set-valued map being compact on bounded subsets of U (i.e. for each bounded subset $G \subset U$, $\mathrm{cl}\, \Phi(G)$ is compact) such that $0 \in \Phi(\lambda, \mathrm{o})$ for all $\lambda \in \Lambda := \{\lambda \in \mathbf{R}^k \mid (\lambda, 0) \in U\}$. If $\dim E = m < \infty$, then we additionally assume that $\mathrm{i}(\Phi) \leq m$.

As usual $\mathcal{S} := \{(\lambda, x) \in U \setminus \Lambda \times \{0\} \mid x \in \Phi(\lambda, x)\}$ is the set of nontrivial fixed points and $\mathcal{B}(\Phi) := \mathrm{cl}\, \mathcal{S} \cap \Lambda \times \{\mathrm{o}\}$. As in the finite-dimensional case we assume $\mathcal{B}(\Phi)$ to be compact. Observe that, for any closed bounded subset G of U, $\mathrm{cl}\, \mathcal{S} \cap G$ is compact.

Now we again introduce a continuous function $\alpha : \Lambda \to [0, +\infty)$ such that $0 < \alpha(\lambda) < d((\lambda, 0), \mathrm{bd}\, \mathrm{U} \cup \mathrm{cl}\, \mathcal{S})$ when $(\lambda, 0) \notin \mathcal{B}(\Phi)$ and $\alpha(\lambda) = 0$ when $(\lambda, 0) \in \mathcal{B}(\Phi)$, define $X := \{(\lambda, x) \in \mathbf{R}^k \times E \mid \lambda \in \Lambda, \|x\| = \alpha(\lambda)\}$, $X^+ := \{(\lambda, x) \in \mathbf{R}^k \times E \mid \lambda \in \Lambda, \|x\| < \alpha(\lambda)\}$ and $X^- = U \setminus \mathrm{cl}\, X^+$ and consider a function $f : U \to \mathbf{R}$ being positive (resp. negative) on X^+ (resp. X^-).

Let $F : \mathbf{R}^k \times E \to E \times \mathbf{R}$ be given by $F(\lambda, x) = (x, 0)$. Clearly F is a Fredholm operator of index $k - 1$. It is clear that $F(\lambda, x) \in (\Phi, f)(\lambda, x)$ if and only if $x = 0$ and $(\lambda, 0) \in \mathcal{B}(\Phi)$. Hence there is an open bounded set V, $\mathrm{cl}\, V \subset U$, such that $((\Phi, f), V) \in \widetilde{M}^F(\mathbf{R}^k \times E, E \times \mathbf{R})$ (resp. $M^F_{CE}(\mathbf{R}^k \times E, E \times \mathbf{R})$) – see 5.21. Hence we are in a position to give the following definition.

7.24 Definition By the *bifurcation* (of fixed points) *index* $\mathrm{BIF}(\Phi)$ of Φ we understand

$$\mathrm{BIF}(\Phi) := \mathrm{ind_F}((\Phi, f), V) \in \Pi_{k-1}.$$

Again this definition is correct, i.e. it does not depend on the choice of α, f and V.

All the properties (existence, additivity and the generalized homotopy invariance) continue to hold in the infinite-dimensional situation (with obvious changes concerning assumptions). The proofs we gave go through almost without changes. As an example we sketch the proof of the existence property assuming that $\mathrm{BIF}(\Phi) \neq$ o.

Using methods already employed in Section 5.D and as in the corresponding proof of Section 7.A, we adjoin a point at infinity to $\mathbf{R}^k \times E$. A neighborhood basis of ∞ consists of complements of bounded subsets of $\mathbf{R}^k \times E$. Set $\widetilde{U} := [\mathbf{R}^k \times E \cup \{\infty\}] \setminus U$ and define

$$Z = (\mathcal{B}(\Phi) \cup \mathcal{S} \cup \widetilde{U})/\widetilde{U},$$

i.e. we contract $\widetilde{U}$ to a point. Let $c \in Z$ correspond to $\widetilde{U}$. In this manner Z becomes the one point compactification of the locally compact space $\mathcal{B}(\Phi) \cup \mathcal{S}$. The claim now is that $\mathcal{B}(\Phi)$ and c cannot be separated in Z. Then, again by [4, Prop. 5], there is a connected subset $\mathcal{C}$ of $\mathcal{S}$ such that $\mathrm{cl}\, \mathcal{C} \cap \mathcal{B}(\Phi) \neq \emptyset$ and $\mathcal{C}$ is not contained in a compact subset of U. The proof of nonseparation is carried through analogously as in the proof of Theorem 7.4.

Chapter 8.

BIFURCATION OF PERIODIC SOLUTIONS

A classical problems in the study of periodic solutions to differential equations is the following: given an equation

$$x' = g(t, x, \vartheta),$$

where $\vartheta \in \mathbf{R}^k$, $k \geq 1$, is a parameter, suppose that for a fixed value of ϑ, say $\vartheta = 0$, there is a periodic solution (we do not exclude the trivial, i.e. the constant solution). Then does the equation have a periodic solution for small $|\vartheta|$ and is this solution close to the given one?

Usually, the phenomenon described above (if occurs) is called the *branching of periodic solutions* if the equation is nonautonomous. If the equation is autonomous, then it is called the *bifurcation of periodic solutions.* Both problems have been intensively studied by many authors and there is an enormous literature on the subject – see e.g. [8, 78] and the recent paper [82]. However it seems that the case of a differential inclusion (replacing the above equation) will be dealt with here for the first time (save for [67] where a different type of bifurcation is studied under different assumptions). The difficulty is that methods used to study bifurcation or branching of periodic orbits of differential equations involve smooth analysis. Although non-smooth analysis (also for multivalued maps) is quite developed – see

[19], it does not seem to suit well the problem we want to deal with. Moreover, in general, differential inclusions admit nonuniqueness of solutions. This causes serious problems, since solutions do not behave as in case of differential equations with, say, smooth right hand sides. We propose an attitude and results that are not as precise as those valid for differential equations having C^1 right hand sides due to already discussed reasons; another reason is that we have not developed the so-called S^1-degree for set-valued maps ([1]), thus we could not have obtained results in the spirit of Gęba-Marzantowicz [82].

8.A. Periodic points of differential inclusions

First we are going to study periodic orbits of a parametrized differential inclusion.

8.1 Consider a differential inclusion of the form

$$x' \in \varphi(\vartheta, x), \qquad x \in \mathbf{R}^N,\ \vartheta \in \Theta \tag{8.1}$$

where Θ is a closed interval in $\mathbf{R}$, $\Theta = [a, b]$, $a < b$, say, and $\varphi : \Theta \times \mathbf{R}^N \multimap \mathbf{R}^N$ is an upper semicontinuous map with compact convex values such that

- $0 \in \varphi(\vartheta, 0)$;
- there is $M > 0$ such that

$$\sup_{y \in \varphi(\vartheta, x)} |y| \le M$$

for all $\vartheta \in \Theta$ and $x \in \mathbf{R}^N$.

Of course, one could study the situation when φ is defined on an open subset of $\Theta \times \mathbf{R}^N$ and satisfies less restrictive growth conditions; however it would lead to more tedious technical arguments in the sequel.

Recall that, for a fixed $\vartheta \in \Theta$, by a solution to a Cauchy problem

$$\begin{cases} x' \in \varphi(\vartheta, x),\ x \in \mathbf{R}^N, \\ x(0) = \xi \in \mathbf{R}^N \end{cases} \qquad (8.2)_\vartheta$$

[1] a version of an equivariant fixed point theory has been studied in [61], see also [62].

we understand an absolutely continuous, i.e. continuous function $x : J \to \mathbf{R}^N$, where J is an interval containing 0, for which there is a locally integrable $y : J \to \mathbf{R}^N$ such that, for all $t \in J$, $x(t) = \xi + \int_0^t y(s)\,\mathrm{d}s$ (i.e. y is a weak derivative of x) and $x'(t) := y(t) \in \varphi(\vartheta, x(t))$ for almost all $t \in J$. In virtue of the Gronwall inequality all solutions to problem $(8.2)_\vartheta$ actually exist on the whole line $\mathbf{R}$.

8.2 Definition We say that a point (t, ϑ, ξ), $t > 0$, $\vartheta \in \Theta$ and $\xi \in \mathbf{R}^N$, is a *periodic point* of the problem (8.1) if there is a solution $x : [0, +\infty) \to \mathbf{R}^N$ to $(8.2)_\vartheta$ such that $x(t) = \xi$, i.e. if

$$\begin{cases} x'(\tau) \in \varphi(\vartheta, x(\tau)), & \text{a.e. on } [0, +\infty), \\ x(0) = \xi = x(t). \end{cases}$$

We also say that a solution $x : [0, +\infty) \to \mathbf{R}^N$ *corresponds* to the periodic point (t, ϑ, ξ) ([2]).

For instance, any point $(t, \vartheta, 0)$, $t > 0$, $\vartheta \in \Theta$, is periodic and the constant solution $x \equiv \xi$ corresponds to it. Clearly, due to the generic nonuniqueness of solutions, many different solutions may correspond to the same periodic point (t, ϑ, ξ).

Let a solution x correspond to a periodic point (t, ϑ, ξ) of (8.1). There are two possibilities: x is constant, i.e. $x \equiv \xi$ (in this case $0 \in \varphi(\vartheta, \xi)$), or x is a nonconstant function. In the latter case x may not be a periodic function. However it gives rise to a periodic solution $x_t : \mathbf{R} \to \mathbf{R}^N$ of (8.1) with period t, i.e. such that $x(0) = \xi$ and $x_t(\tau + kt) = x_t(\tau)$ for any $\tau \in \mathbf{R}$ and $k \in \mathbf{Z}$. Indeed it is enough to put

$$x_t(\tau) = x(\tau - kt)$$

for $\tau \in [kt, (k+1)t]$, $k \in \mathbf{Z}$. In view of this observation it makes sense to say that t is a *period* of (t, ϑ, ξ).

As we see a special care is necessary when one speaks of solutions to the periodic problem for differential inclusions. Unlike a differential equation (with uniqueness of solutions) to periodic points of which correspond

[2] note that we consider periodic points in the extended, i.e. the so-called Fuller phase space $(0, +\infty) \times \Theta \times \mathbf{R}^N$ of the problem.

either (unique) constant solutions or (also unique) nonconstant and periodic ones, a differential inclusion may possess periodic points to which correspond simultaneuosly constant and nonconstant nonperiodic solutions (nonuniqueness of solutions occurs!).

Generally speaking there are two different types of solutions corresponding to periodic points.

8.3 Definition We say that a solution $x : [0, +\infty) \to \mathbf{R}^N$ corresponding to a periodic point (t, ϑ, ξ) of (8.1) is:

- *nontrivial* if the number of points $\tau \in [0, t]$ such that $x(\tau) = \xi$ is finite; and
- *trivial* if it is not nontrivial.

Accordingly, we make the following classification: a periodic point (t, ϑ, ξ) is

- *trivial* if all solutions corresponding to it are trivial; and
- *nontrivial* if there is a nontrivial solution corresponding to it.

8.4 Lemma *If there is a trivial solution* $x : [0, +\infty) \to \mathbf{R}^N$ *corresponding to a periodic point* (t, ϑ, ξ) *of* (8.1)*, then* $0 \in \varphi(\vartheta, \xi)$ *and, hence, also the constant solution* $x \equiv \xi$ *corresponds to* (t, ϑ, ξ)*.*

Proof Since x is trivial, there is an infinite sequence (t_n) of points of $[0, t]$ such that $x(t_n) = \xi$. Without loss of generality we may assume that $t_n \to t_0 \in [0, t]$. Fix an integer n and set $t'_n = \min\{t_n, t_0\}$, $t''_n = \max\{t_n, t_0\}$ and define a function $x_n : [0, +\infty) \to \mathbf{R}^N$ by the formula

$$x_n(\tau) = x(\tau + (k+1)t'_n - kt''_n) \text{ for } \tau \in [k(t''_n - t'_n), (k+1)(t''_n - t'_n)],\ k = 0, 1, \dots$$

Evidently x_n is a solution to $(8.2)_\vartheta$. Moreover, for $\tau \in [k(t''_n - t'_n), (k+1)(t''_n - t'_n)]$,

$$|x_n(\tau) - \xi| = |x(\tau - k(t''_n - t'_n) + t'_n) - x(t'_n)| \le \int_{t'_n}^{t''_n} |x'(s)|\,\mathrm{d}s \le M(t''_n - t'_n).$$

This shows that $x_n \to 0$ uniformly. Since, for all n and $\tau \geq 0$, $|x_n'(\tau)| \leq M$, in virtue of the Compactness Theorem 6.8, (a subsequence) $x_n \to 0$ weakly in $L^1(J, \mathbf{R}^N)$ for each bounded interval J. Next, by the Convergence Theorem 6.4, $0 \in \varphi(\vartheta, \xi)$. □

It is not difficult to provide an example of a trivial periodic point to which corresponds a nonconstant (trivial) solution. Consider a map $\varphi : \Theta \times \mathbf{R}^2 \multimap \mathbf{R}^2$ given, for $\vartheta \in \Theta$, $x, y \in \mathbf{R}$, by

$$\varphi(\vartheta, (x, y)) = \begin{cases} \{0\} \times [0, 1] & \text{if } x = 1,\ y = 0 \\ (-y, x) & \text{otherwise,} \end{cases}$$

modify it for large $|(x, y)|$ as to satisfy the growth condition and take its periodic point $((\frac{5}{2}\pi, \vartheta, (1, 0))$. There is no nontrivial solution, but of course there are nonconstant (trivial) solutions corresponding to it.

Let $x : [0, +\infty) \to \mathbf{R}^N$ be a nontrivial solution corresponding to a periodic point (t, ϑ, ξ) and let $\{\tau_1, \ldots, \tau_n\} = \{\tau \in (0, t) \mid x(\tau) = \xi\}$, $\tau_1 < \ldots < \tau_n$, and let $\tau_0 = 0$, $\tau_{n+1} = t$. Set $T_x := \min\{\tau_i - \tau_{i-1} \mid 1 \leq i \leq n+1\}$. Clearly $T_x \leq t$. We say that T_x is the *minimal period* of the solution x. This name is justified since it is easy to construct a periodic solution $\overline{x}$ of period T_x to (8.1) with $\overline{x}(0) = \xi$. Moreover it is easy to see that if (t, ϑ, ξ) is a periodic point such that $0 \notin \varphi(\vartheta, \xi)$, then the infimum of numbers T_x where x varies over the set of all nontrivial solutions corresponding to (t, ϑ, ξ) (in view of Lemma 8.4, there are no trivial solutions corresponding to it) is positive.

Finally, observe that a periodic point (t, ϑ, ξ) of (8.1) may also be nontrivial when $0 \in \varphi(\vartheta, \xi)$. In particular, points of the form $(t, \vartheta, 0)$ may be nontrivial! However, in view of Lemma 8.4, if $0 \notin \varphi(\vartheta, \xi)$, then (t, ϑ, ξ) is purely nontrivial, i.e. all solutions corresponding to it are nontrivial.

It is clear that if (8.1) is a differential equation (with uniqueness of solutions), then a periodic point (t, ϑ, ξ) is trivial if and only if $0 \in \varphi(\vartheta, \xi)$. If it is nontrivial, then the unique solution x corresponding to it is nontrivial and periodic; moreover, there is a positive integer l such that $\{\tau \in [0, t] \mid x(\tau) = \xi\} = \{\frac{jt}{l} \mid 0 \leq j \leq l\}$. Hence $T_x = t/l$ is the least period of x.

Let

$$\mathcal{P}(\varphi) := \{(t, \vartheta, \xi) \in (0, +\infty) \times \Theta \times \mathbf{R}^N \mid \xi \neq 0,\ (t, \vartheta, \xi) \text{ is a nontrivial periodic point of (8.1)}\}.$$

We consider $\mathcal{P}(\varphi)$ to be a topological subspace of $[0,+\infty) \times \Theta \times \mathbf{R}^N$.

8.5 Consider a multi-valued map $S_\varphi : [0,+\infty) \times \Theta \times \mathbf{R}^N \multimap \mathbf{R}^N$ given by the formula

$$S_\varphi(t,\vartheta,\xi) := \{x(t) \mid x : [0,+\infty) \to \mathbf{R}^N \text{ is a solution of } (8.2)_\vartheta\}$$

for $t \geq 0$, $\vartheta \in \Theta$ and $\xi \in \mathbf{R}^N$. In particular, $S_\varphi(0,\vartheta,\xi) = \{\xi\}$ and $0 \in S_\varphi(t,\vartheta,0)$ for all $t > 0$, $\vartheta \in \Theta$ and $\xi \in \mathbf{R}^N$.

One shows easily that S_φ is a set-valued map (comp. Example 6.7).

Clearly if $\xi \in S_\varphi(t,\vartheta,\xi)$ for some $t > 0$, $\vartheta \in \Theta$ and $\xi \in \mathbf{R}^N$, then (t,ϑ,ξ) is a periodic point of (8.1). If $\xi \in S_\varphi(t,\vartheta,\xi)$ and $0 \notin \varphi(\vartheta,\xi)$, then (t,ϑ,ξ) is a purely nontrivial periodic point.

8.6 Proposition *If* $(t,\vartheta,\xi) \in \operatorname{cl}\mathcal{P}(\varphi) \setminus \mathcal{P}(\varphi)$, *then* $0 \in \varphi(\vartheta,\xi)$.

Proof Let (t_n,ϑ_n,ξ_n) be a sequence from $\mathcal{P}(\varphi)$ converging to (t,ϑ,ξ). The upper semicontinuity of S_φ implies that $\xi \in S_\varphi(t,\vartheta,\xi)$. If $t > 0$, then all solutions corresponding to (t,ϑ,ξ) are trivial since $(t,\vartheta,\xi) \notin \mathcal{P}$. Hence by Lemma 8.4, $0 \in \varphi(\vartheta,\xi)$. Now suppose that $t = 0$ and take any $t' > 0$. Of course $t_n > 0$ for all n by the definition of $\mathcal{P}$. Hence there is $k_n = \min\{k \in \mathbf{Z} \mid kt_n \geq t'\}$. Then $k_n t_n \to t'$ as $n \to \infty$. Observe that, for all n, $\xi_n \in S_\varphi(k_n t_n,\vartheta_n,\xi_n)$. Indeed, there is a solution $x_n : [0,\infty) \to \mathbf{R}^N$ corresponding to (t_n,ϑ_n,ξ_n); although this solution is not a periodic function in general, one constructs a periodic solution $\overline{x}_n : [0,\infty) \to \mathbf{R}^N$ with period t_n such that $\overline{x}_n|[0,t_n] = x_n|[0,t_n]$. Hence $\overline{x}_n(0) = \xi_n = \overline{x}_n(k_n t_n)$.

Therefore, again by the upper semicontinuity of S_φ, $\xi \in S_\varphi(t',\vartheta,\xi)$. Since it is true for any $t' > 0$, reasoning as in the proof of Lemma 8.4, we prove that $0 \in \varphi(\vartheta,\xi)$. □

8.B. Bifurcation of periodic points

Let us again consider problem (8.1).

8.7 Definition We say that a point $(t,\vartheta,0)$, $t > 0$, $\vartheta \in \Theta$, is a *p-bifurcation point* of our problem if it belongs to the closure of the set $\mathcal{P}(\varphi)$.

Observe that the above notion of a p-bifurcation point differs from the abstract definition of the set $\mathcal{B}(S_\varphi)$. Recall that, according to Section 7.A., $(t, \vartheta, 0) \in \mathcal{B}(S_\varphi)$, $t > 0$, if and only if there is a sequence $(t_n, \vartheta_n, \xi_n)$, $\vartheta_n \in \Theta$, $\xi_n \neq 0$, such that $\xi_n \in S_\varphi(t_n, \vartheta_n, \xi_n)$ and $(t_n, \vartheta_n, \xi_n) \to (t, \vartheta, 0)$. Saying that $(t, \vartheta, 0)$ is a p-bifurcation point we demand that any of its neighborhoods contains not only points (t', ϑ', ξ) such that $\xi \in S_\varphi(t', \vartheta', \xi)$ and $\xi \neq 0$, but also such that there exists a nontrivial solution corresponding to (t', ϑ', ξ).

We shall study the existence of p-bifurcation points. Let us collect some assumptions.

8.8 Assumption *Let ϑ_0 belong to the interior of Θ (i.e. $a < \vartheta_0 < b$) and let*

$$L : \Theta \to \mathcal{L}(\mathbf{R}^N, \mathbf{R}^N)$$

be a continuous transformation. Assume the following four hypotheses.

(i) *$L(\vartheta_0)$ is non-singular and has a conjugate pair of purely imaginary eigenvalues $\pm i\beta$, $\beta > 0$.*

Let $\mathcal{N}$ be the set of *all* eigenvalues of $L(\vartheta_0)$ which are positive integer multiples of $i\beta$ (including $i\beta$ itself). Since the spectrum $\sigma(L(\vartheta_0))$ of $L(\vartheta_0)$ is isolated, there is a positive number, say $\varepsilon > 0$, such that $|z - z'| \geq 2\varepsilon$ for $z, z' \in \sigma(L(\vartheta_0))$, $z \neq z'$. In particular, any eigenvalue of $L(\vartheta_0)$ ε-close to $\mathcal{N}$ belongs to $\mathcal{N}$. By the upper semicontinuity of the (multivalued transformation) $\vartheta \mapsto \sigma(L(\vartheta))$, there is $\delta > 0$ such that $\sigma(L(\vartheta)) \subset B(\sigma(L(\vartheta_0)), \varepsilon)$ for $\vartheta \in \Theta$ with $|\vartheta - \vartheta_0| < \delta$ (the open ball over $\sigma(L(\vartheta_0))$ is understood in the sense of $\mathbf{C}$). Let $\mathcal{N}_\vartheta := \sigma(L(\vartheta)) \cap B(\mathcal{N}, \varepsilon)$.

(ii) *For $\vartheta \in \Theta$, $0 < |\vartheta - \vartheta_0| < \delta$, none of the eigenvalues of $L(\vartheta)$ in N_ϑ has zero real part. Without loss of generality (diminishing δ if necessary) we may assume that, for such ϑ, $L(\vartheta)$ is nonsingular. To simplify the notation, we assume that $[\vartheta_0 - \delta, \vartheta_0 + \delta] \subset \Theta$ and let $\Theta_0 = (\vartheta_0 - \delta, \vartheta_0 + \delta)$.*

(iii) *The* crossing number *$c(L, i\beta)$ of L through $i\beta$ is an odd number (below we give a rigorous definition of the crossing number).*

(iv) $d(L(\vartheta)x, \varphi(\vartheta, x)) = o(|x|)$ *as* $x \to 0$ *uniformly with respect to* ϑ *from a neighborhood of* ϑ_0*. Again, without loss of generality, we may assume that the above condition holds uniformly with respect to* $\vartheta \in [\vartheta_0 - \delta, \vartheta_0 + \delta]$ ([3]).

8.9 Remark Some words of explanation are due.

(i) Assumption 8.8 (iv), in particular, implies that a map $(\vartheta, x) \mapsto L(\vartheta)x$ is a local arbitrarily close graph approximation of φ. If φ is single-valued, then it means that $\varphi(\vartheta, \cdot)$ is Fréchet differentiable at $x = 0$ and its derivative is equal to $L(\vartheta)$. Hence we may interpret our assumption as follows: $L(\vartheta)$ serves as a surrogate of a derivative of $\varphi(\vartheta, \cdot)$ at $x = 0$. We write *a derivative* since there are many other concepts of derivation in set-valued analysis – see [19]; however the concept introduced here seems to be just well-designed for our purposes. It is rather clear that 8.8 (iv) is not a strong assumption.

(ii) It is well-known (see [82, Prop. 1.1]) that if φ is C^1-singlevalued and either $D_x\varphi(\vartheta, 0)$ has no eigenvalue with zero real part or there are no integer $k \geq 1$ and $\beta > 0$ such that $i\beta \in \sigma(L(\vartheta))$ and $t = 2k\pi\beta^{-1}$, then $(t, \vartheta, 0)$ has a neighborhood containing no periodic points; hence $(t, \vartheta, 0)$ is not a p-bifurcation point. In other words p-bifurcation at $(2k\pi\beta^{-1}, \vartheta, 0)$ may occur only when $i\beta \in \sigma(D_x\varphi(\vartheta, 0)) \cap i\mathbf{R}$. It is also known that this is not a necessary condition for bifurcation in the phase space $\Theta \times \mathbf{R}^N$, i.e. even if $\sigma(L(\vartheta)) \cap i\mathbf{R} = \emptyset$, then a sequence $(t_n, \vartheta_n, \xi_n)$ of nontrivial periodic points such that $\vartheta_n \to \vartheta$ and $\xi_n \to 0$ (no control over (t_n) is granted) may exist. This explains the nature of Assumption 8.8 (i) and answers the question why we decide to work in the extended phase space.

Condition 8.8 (i) is shortly expressed by saying that $(\vartheta_0, 0)$ is a *center* for φ and is usually stated in a bit stronger form: authors require that $(\vartheta_0, 0)$ is an *isolated* center, i.e. there are no other centers in a neigborhood of $(\vartheta_0, 0)$.

In the case of a differential inclusion the situation is more complicated; there may exist p-bifurcation points which are not traced by the behaviour of $L(\vartheta)$ and its spectrum. In order to make these assumptions work in the course of the proof of our main result – Theorem 8.10, we have to modify the map φ (see paragraph 8.13) in order to avoid the nonuniqueness of solutions at $(\vartheta, 0)$.

(iii) We now give a precise definition of the *crossing number* of L (comp.

[3] $d(L(\vartheta)x, \varphi(\vartheta, x))$ denotes the distance from $L(\vartheta)x$ to the set $\varphi(\vartheta, x)$. Condition (iv) means that $\lim_{x\to 0} \frac{d(L(\vartheta)x, \varphi(\vartheta, x))}{|x|} = 0$ uniformly with respect to $\vartheta \in [\vartheta_0 - \delta, \vartheta_0 + \delta]$.

e.g. [82]) through $i\beta$. Assume that $\vartheta_0 - \delta < \vartheta_- < \vartheta_0 < \vartheta_+ < \vartheta_0 + \delta$. Set

$$c(L, i\beta) = \mu^+(L(\vartheta_+)) - \mu^+(L(\vartheta_-))$$

where $\mu^+(L(\vartheta))$, $\vartheta \in \Theta_0$, stands for the number of eigenvalues of $L(\vartheta)$ in $B(i\beta, \varepsilon)$, counted with multiplicity, with positive real part. It is easily seen that the crossing number is well-defined, i.e. it does not depend on the choice of $\vartheta_\pm$ with small $|\vartheta_0 - \vartheta_\pm|$.

Thus $c(L, i\beta)$ is the net number of changes of sign of real parts of elements of $\mathcal{N}_\vartheta \cap B(i\beta, \varepsilon)$ as ϑ passes through ϑ_0. For instance, if $i\beta$ is a simple eigenvalue, then in virtue of Assumption 8.8 (ii), $\mathcal{N}_\vartheta \cap B(i\beta, \varepsilon)$, $\vartheta \in \Theta_0$, is a singleton $\{\alpha_\vartheta + i\beta_\vartheta\}$, $\vartheta \mapsto \alpha_\vartheta$, $\vartheta \mapsto \beta_\vartheta$ are continuous real-valued functions with $\alpha_{\vartheta_0} = 0$, $\beta_{\vartheta_0} = \beta$ and $\alpha_\vartheta \neq 0$ for $\vartheta \neq \vartheta_0$. Hence $c(L, i\beta) = \pm 1$ or 0 depending whether α_ϑ changes sign at ϑ_0 or not.

(iv) The assumptions we made concerning the behaviour of eigenvalues seem to be slightly weaker than those made in [8] (apart from the fact the authors deal with a differential equation with C^1 single-valued φ there and then $L(\vartheta) = D_x\varphi(\vartheta, 0)$). However, below we shall see that when we actually apply our assumptions to their situation it does not change anything. Let us also observe that in [82] (where the situation analogous to that of [8] is studied), the requirement is that $c(L, i\beta) \neq 0$. The main advantage of the approach of [82] is that instead of the Brouwer (or Leray-Schauder) degree argument (used throughout [8]), the authors use the appropriate version of the S^1-equivariant degree. In our setting such assumption would suffice provided the S^1-degree is introduced for set-valued maps. It is possible that it can be done within the framework developed in this paper. However, as long as we have no S^1-degree at our disposal, we have to content ourselves with results that rely on assumptions just stated.

(v) Let k be the greatest among positive integers such that $ik\beta$ is an eigenvalue of $L(\vartheta_0)$ (i.e. $ik\beta \in N$) with the *odd* crossing number $c(L, ik\beta)$. Let $\omega = k\beta$. Then the crossing number of L through any positive (proper) multiple of $i\omega$ which belongs to the spectrum of $L(\vartheta_0)$ is even. For if $c(L, ir\omega)$, $r \geq 2$, is odd, then by the definition of k, $kr \leq k$, a contradiction. Hence, the *index* (in the terminology of [8]) $\sum_r c(L, ir\omega)$ of $i\omega$ (summation goes over integers $r \geq 1$ such that $ir\omega$ is an eigenvalue of $L(\vartheta_0)$) is odd. Therefore, our hypothesis is equivalent to the assumption that $L(\vartheta_0) = D_x\varphi(\vartheta_0, 0)$ has an eigenvalue with odd index which was actually posed by the authors in [8].

The following main result of this Section gives a version of the Hopf bifurcation theorem for a differential inclusion.

8.10 Theorem *Let*

$$t_0 = 2\pi\omega^{-1} \qquad \text{and} \qquad T > t_0 := 2\pi\omega^{-1} \tag{8.3}$$

and suppose that Assumptions 8.8 *are satisfied* ([4]).

(i) *There is a component $\mathcal{N}$ of $\mathcal{P}$ (i.e. a maximal connected subset) such that $(t_0, \vartheta_0, 0) \in \mathrm{cl}\mathcal{N}$ and, additonally, at least one of the following conditions is satisfied:*

- *$\mathcal{N}$ is not contained in any compact subset of $(0,T) \times \Theta_0 \times \mathbf{R}^N$;*
- *$\mathrm{cl}\mathcal{N}$ connects $(t_0, \vartheta_0, 0)$ to a point $(\bar{t}, \overline{\vartheta}, \bar{\xi}) \in (0,T) \times \Theta_0 \times \mathbf{R}^N$ such that $0 \in \varphi(\overline{\vartheta}, \bar{\xi})$ (i.e. $(\bar{t}, \overline{\vartheta}, \bar{\xi}) \in \mathrm{cl}\mathcal{N}$).*

(ii) *In particular, the point $(t_0, \vartheta_0, 0)$ is a p-bifurcation point. There is a sequence $((t_m, \vartheta_m, \xi_m))_{m=1}^{\infty}$ of points from $\mathcal{P}$ converging to $(t_0, \vartheta_0, 0)$ with the following property: for any m, there is a nontrivial solution x_m to* (8.1) *corresponding to $(t_m, \vartheta_m, \xi_m)$, such that if T_m is the minimal period of x_m, then $T_m \to 2s\pi\mu^{-1}$ where $s \in \mathbf{N}$ and $i\mu \in \sigma(L(\vartheta_0))$.*

The proof will be preceded by a number of lemmas.

8.11 Let us put $d(\vartheta, x) = d(L(\vartheta)x, \varphi(\vartheta, x))$, $\vartheta \in \Theta$, $x \in \mathbf{R}^N$. By Assumption 8.8 (iv), $d(\vartheta, x) = o(|x|)$ as $x \to 0$ uniformly with respect to $\vartheta \in [\vartheta_0 - \delta, \vartheta_0 + \delta]$. Unfortunately d is only a lower semicontinuous function. However it is not difficult to construct a continuous function $\mu : D^N(0,r) \to [0,\infty)$, defined on the closed ball with radius $r > 0$, such that $\mu(0) = 0$ and $d(\vartheta, x) \le \mu(x)|x|$ for all $\vartheta \in [\vartheta_0 - \delta, \vartheta_0 + \delta]$ and $x \in D^N(0,r)$.

8.12 Lemma (comp. [8, Lemma 3.2]) *Let $K : [0,T] \times \Theta \to \mathcal{L}(\mathbf{R}^N, \mathbf{R}^N)$ be defined by*

$$K(t,\vartheta)\xi = e^{tL(\vartheta)}\xi - \xi \qquad \text{where } t \in [0,T],\ \vartheta \in \Theta,\ \xi \in \mathbf{R}^N.$$

There is $0 < \rho < r$ such that $(t_0 - \rho, t_0 + \rho) \times (\vartheta_0 - \rho, \vartheta_0 + \rho) \subset (0,T) \times \Theta_0$ and the linear map $K(t,\vartheta)$ is nonsingular for $0 < |t - t_0|$, $|\vartheta - \vartheta_0| < \rho$.

[4] ω was defined in Remark 8.9 (v).

Proof If $\gamma_{1\vartheta}, \ldots, \gamma_{n\vartheta}$ are the eigenvalues of $L(\vartheta)$, then $e^{t\gamma_{j\vartheta}} - 1$, $j = 1, \ldots, n$ are the eigenvelues of $K(t, \vartheta)$. One of these eigenvalues is zero if and only if either one of $\gamma_{j\vartheta}$ is zero (which is impossible for $\vartheta \in \Theta_0$) or one of $\gamma_{j\vartheta}$ is purely imaginary, i.e. $\gamma_{j\vartheta} = i\beta_j$, $\beta_j > 0$, and $\beta_j t = 2l\pi$ for some integer $l \geq 1$. Hence $t = 2l\pi\beta_j^{-1}$. If t is sufficiently close to t_0, then $\gamma_{j\vartheta}$ is close to $il\omega$, so by Assumption 8.8 (ii), $\gamma_{j\vartheta}$ cannot be purely imaginary. □

8.13 Let us now define an auxiliary map $\psi : [\vartheta_0 - \delta, \vartheta_0 + \delta] \times \mathbf{R}^N \multimap \mathbf{R}^N$ by the formula

$$\psi(\vartheta, x) = \begin{cases} D^N(L(\vartheta)x, \mu(x)|x|) \cap \varphi(\vartheta, x) & \text{for } |x| < r \\ \varphi(\vartheta, x) & \text{for } |x| \geq r \end{cases} \tag{8.4}$$

for $x \in \mathbf{R}^N$, $\vartheta \in [\vartheta_0 - \delta, \vartheta_0 + \delta]$. It is clear that the set $\psi(\vartheta, x)$ is nonempty convex and compact; moreover the map ψ is upper semicontinuous since its graph is closed. Moreover, $\psi(\vartheta, x) \subset \varphi(\vartheta, x)$ for all $\vartheta \in [\vartheta_0 - \delta, \vartheta_0 + \delta]$ and $x \in \mathbf{R}^N$.

Along with (8.1) we shall study a differential inclusion

$$x' \in \psi(\vartheta, x),\ x \in \mathbf{R}^N,\ \vartheta \in [\vartheta_0 - \delta, \vartheta_0 + \delta] \tag{8.5}$$

and the Cauchy problem:

$$\begin{cases} x' \in \psi(\vartheta, x) \\ x(0) = \xi \in \mathbf{R}^N. \end{cases} \tag*{$(8.6)_\vartheta$}$$

As before for all data $(\vartheta, \xi) \in [\vartheta_0 - \delta, \vartheta_0 + \delta] \times \mathbf{R}^N$, problem $(8.5)_\vartheta$ has solutions existing on $[0, +\infty)$.

It is clear that any periodic point (resp. nontrivial periodic point) (t, ϑ, ξ), $t > 0$, $\vartheta \in [\vartheta_0 - \delta, \vartheta_0 + \delta]$, $\xi \in \mathbf{R}^N$, of (8.5) is a periodic point (resp. nontrivial periodic point) of (8.1). However, there certainly may exist trivial periodic points of (8.5) being nontrivial for (8.1).

Let

$$U_1 := (0, T) \times \Theta_0 \times \mathbf{R}^N,$$

and consider a map $\Phi_1 : U_1 \multimap \mathbf{R}^N$ be given by

$$\Phi_1(t, \vartheta, \xi) = \{x(t) - \xi \mid x : [0, +\infty) \to \mathbf{R}^N \text{ is a solution to } (8.5)_\vartheta\}.$$

Next let

$$\begin{aligned}\Lambda &:= (t_0-\rho, t_0+\rho)\times(\vartheta_0-\rho,\vartheta_0+\rho),\\ U &:= \Lambda\times\mathbf{R}^N\subset U_1\end{aligned}$$

and let $\Phi : U \multimap \mathbf{R}^N$ be given by

$$\Phi = \Phi_1|U.$$

Obviously $0\in\Phi_1(t,\vartheta,\mathrm{o})$ for all $t\in(0,T)$, $\vartheta\in\Theta_0$. Moreover, if $0\in\Phi_1(t,\vartheta,\xi)$, then (t,ϑ,ξ) is a periodic point of (8.5).

In view of Example 6.7, Φ_1 is determined by a CE-morphisms which, in turn, is represented by the cotriad

$$U_1 \xleftarrow{p} W_1 \xrightarrow{q} \mathbf{R}^N$$

where $W_1=\{(t,\vartheta,x)\in U_1\times AC([0,T],\mathbf{R}^N)\mid x'(s)\in\psi(s,x(s),\vartheta)$ a.e. on $[0,T]\}$ (with topology inherited from $U_1\times C([0,T],\mathbf{R}^N)$),

$$p(t,\vartheta,x)=(t,\vartheta,x(0))\qquad\text{and}\qquad q(t,\vartheta,x)=x(t)-x(0)$$

for $(t,\vartheta,x)\in W_1$. Obviously also $\Phi\in M_{CE}(U,\mathbf{R}^N)$ and is represented by $(p|W,q|W)$ where $W=p^{-1}(U)\subset W_1$.

Finally, let $p_1(t,\vartheta,x)=x(0)$ for $(t,\vartheta,x)\in W_1$.

8.14 Lemma *Let $f : U\to\mathbf{R}^N$ be given by*

$$f(t,\vartheta,\xi)=K(t,\vartheta)\xi\qquad \textit{for } (t,\vartheta)\in\Lambda,\ \xi\in\mathbf{R}^N.$$

The following condition is satisfied

$$|f\circ p(w)-q(w)|=o(|p_1(w)|)\qquad \textit{as } p_1(w)\to 0$$

on W.

Proof Take an arbitary $\varepsilon>0$. We are to show that there is $\delta>0$ such that if $|\xi|\le\delta$, then for any $(t,\vartheta)\in\Lambda$ and any solution $x:[0,T]\to\mathbf{R}^N$ to $(8.5)_\vartheta$, we have

$$|f(t,\vartheta,\xi)-x(t)+\xi| = |e^{tL(\vartheta)}\xi - x(t)|\le\varepsilon|\xi|. \tag{8.7}$$

Let

$$C = \max\{\sup_{\vartheta\in\Theta_0} \|L(\vartheta)\|, \sup_{y\in D_r^N} \mu(y)\}$$

and choose $\overline{\varepsilon} > 0$ such that

$$\sup_{|y|\le\overline{\varepsilon}} \mu(y) \le \frac{\varepsilon}{Te^{3TC}}.$$

Finally we let

$$\delta = \min\left\{\frac{r}{2}, \frac{r}{2}e^{-2TC}, \overline{\varepsilon}e^{-2TC}\right\}.$$

Let $|\xi| \le \delta$, $(t,\vartheta) \in \Lambda$ and let $x : [0,T] \to \mathbf{R}^N$ be a solution to $(8.6)_\vartheta$. One easily shows that

$$\|x\| = \sup_{\tau\in[0,T]} |x(\tau)| \le r.$$

For the reader's convenience we provide a proof. Suppose to the contrary that $T_0 = \sup\{\tau \in [0,T] \mid \sup_{s\in[0,\tau]} |x(s)| \le r\} < T$. Clearly $T_0 > 0$ since $|x(0)| = |\xi| \le r/2$. Thus $r = |x(T_0)| \le |\xi| + \int_0^{T_0} |x'(s)|\,\mathrm{d}s$. Since $|x(s)| \le r$ and $x'(s) \in \psi(\vartheta, x(s))$, we gather that $|x'(s) - L(\vartheta)x(s)| \le \mu(x(s))|x(s)| \le C|x(s)|$. This implies that $|x'(s)| \le (\|L(\vartheta)\| + C)|x(s)| \le 2C|x(s)|$; thus by the Gronwall inequality

$$|x(T_0)| \le |\xi|e^{2TC} \le r/2,$$

a contradiction. The same argument shows actually that

$$\|x\| \le |\xi|e^{2TC} \le \overline{\varepsilon}. \tag{8.8}$$

Now let $y(s) = e^{sL(\vartheta)}\xi$ for $s \in [0,T]$. Hence, $y'(s) = L(\vartheta)y(s)$, $s \in [0,T]$, and $y(0) = \xi$. If $z(s) = y(s) - x(s)$, then by (8.8),

$$\begin{aligned}
|z(t)| \le \int_0^t |z'(s)|\,\mathrm{d}s &= \int_0^t |L(\vartheta)z(s) + L(\vartheta)x(s) - x'(s)|\,\mathrm{d}s \\
&\le \int_0^t C|z(s)|\,\mathrm{d}s + \int_0^t \mu(x(s))|x(s)|\,\mathrm{d}s \\
&\le \int_0^t C|z(s)|\,\mathrm{d}s + T\sup_{|y|\le\overline{\varepsilon}} \mu(y)|\xi|e^{2TC};
\end{aligned} \tag{8.9}$$

hence, by the Gronwall inequality,

$$|e^{tL(\vartheta)}\xi - x(t)| \quad = \quad |z(t)| \leq \varepsilon|\xi|. \qquad \square$$

8.15 Remark We have actually proven that estimate (8.7) holds for all $t \in [0, T]$.

8.16 Recall the setting and notation introduced in Section 7A. We already know that $\Phi \in M_{CE}(U, \mathbf{R}^N)$, $0 \in \Phi(\lambda, \mathrm{o})$ where $\lambda = (t, \vartheta) \in \Lambda$. Next we recall paragraph 7.20. Lemma 8.14 actually shows that all assumptions from 7.20 are satisfied; hence (see Remark 7.22) $\mathrm{BI}(\Phi)$ may be computed by means of f. Namely $\mathrm{BI}(\Phi) = \gamma_f$. Next, we also see that f satisfies assumptions stated in paragraph 7.16. Namely $f(\lambda, \xi) = f(t, \vartheta, \xi) = e^{tL(\vartheta)}\xi - \xi = K(t, \vartheta)\xi$ where $K(\lambda) = K(t, \vartheta) = e^{tL(\vartheta)} - I$. In our case the parameter $\lambda = (t, \vartheta)$ is now 2-dimesional. By Proposition 7.18, $\gamma_f = J(\overline{\gamma}_f) \in \Pi_1$. Moreover, $\pi_1(SO) = \mathbf{Z}_2$ and J is monomorphic (see 7.19). Hence in order to prove that $\mathrm{BI}(\Phi) \neq \mathrm{o}$ it is enough to prove that $\overline{\gamma}_f$ is a generator of $\mathbf{Z}_2$. But actually in [8] (see also [9]) Alexander and Yorke have shown that under our Assumption 8.8 (iii) this is the case.

8.17 Corollary $\mathcal{B}(\Phi) = \{(t_\mathrm{o}, \vartheta_\mathrm{o}, \mathrm{o})\}$ *and* $\mathrm{BI}(\Phi) \neq \mathrm{o}$.

8.18 Lemma *There is a sufficiently small $\varepsilon > 0$ such that if $(t, \vartheta) \in \Lambda$, $0 < |\xi| < \varepsilon$, $|t - t_0|, |\vartheta - \vartheta_0| < \varepsilon$ and $0 \in \Phi(t, \vartheta, \xi)$, then (t, ϑ, ξ) is a nontrivial periodic point of* (8.5)*; thus it is also a nontrivial periodic point of* (8.1) (*i.e.* $(t, \vartheta, \xi) \in \mathcal{P}$).

Proof Suppose to the contrary that there is a sequence $((t_m, \vartheta_m, \xi_m))_{m=1}^{\infty}$ converging to $(t_0, \vartheta_0, 0)$ and consisting of trivial periodic points of (8.5) only. In view of Lemma 8.4, (stated in terms of the problem $(8.6)_{\vartheta_m}$) the constant solution $\overline{x}_m \equiv \xi_m$ corresponds to $(t_m, \vartheta_m, \xi_m)$ for every m. Choose $t \in (t_0 - \rho, t_0 + \rho)$ such that $t < t_m$ for all m. In view of Lemma 8.14,

$$\frac{|e^{tL(\vartheta_m)}\xi_m - \xi_m|}{|\xi_m|} \to 0.$$

Passing to a subsequence, if necessary, we may assume that $\xi_m/|\xi_m| \in S^{N-1}$ converges to some $\xi \in S^{N-1}$. The above implies that $K(t,\vartheta_0)\xi = 0$, i.e. $K(t,\vartheta_0)$ is singular, a contradiction (with Lemma 8.12). □

Proof of Theorem 8.10 (i) Let $V \subset U$ be the open ball around $(t_0,\vartheta_0,))$ of radius ε (ε was determined in Lemma 8.18). In view of Proposition 7.6, $\mathrm{BI}(\Phi|V) = \mathrm{BI}(\Phi) \neq$ o. Hence, by Theorem 7.4, there is a connected set $\mathcal{D} \subset \{(t,\vartheta,\xi) \in V \mid \xi \neq 0, 0 \in \Phi(t,\vartheta,\xi)\}$ such that $(t_0,\vartheta_0,)) \in \mathrm{cl}\,\mathcal{D}$. By Lemma 8.18, $\mathcal{D} \subset \mathcal{P}$.

Let $\mathcal{N}$ be a maximal connected subset of $\mathcal{P}$ such that $(t_0,\vartheta_0,0) \in \mathrm{cl}\,\mathcal{N}$. In view of the above, such a set exists. Suppose that $\mathcal{N}$ is contained in a compact subset of $U_1 = (0,T) \times \Theta_0 \times \mathbf{R}^N$. Hence also the (connected) set $K := \mathrm{cl}\,\mathcal{N}$ is compact and $K \subset U_1$. If $K \cap (\mathbf{R}^2 \setminus \Lambda) \times \{0\} \neq \emptyset$, then there is $\bar{t} \in (0,T)$, $|\bar{t} - t_0| \geq \rho$ and $\overline{\vartheta} \in \Theta_0$, $|\overline{\vartheta} - \vartheta_0| \geq \rho$ such that $(\bar{t},\overline{\vartheta},0) \in K$.

Suppose that $K \cap (\mathbf{R}^2 \setminus \Lambda) \times \{0\} = \emptyset$ and let $\mathcal{S}_1 := \{(t,\vartheta,\xi) \in U_1 \mid \xi \neq 0,\ 0 \in \Phi_1(t,\vartheta,\xi)\}$. In view of Theorem 7.7, there is a nonempty connected set $\mathcal{C} \subset \mathcal{S}_1 \setminus K$ such that $\mathrm{cl}\,\mathcal{C} \cap K \neq \emptyset$ and $\mathrm{cl}\,\mathcal{C}$ has points outside a neighborhood of K since it is unbounded or $\mathrm{cl}\,\mathcal{C} \cap \mathrm{bd}\,(\mathrm{U}_1 \setminus (\mathbf{R}^2 \setminus \Lambda) \times \{0\})$.

Claim. There is a sequence $((t_n,\vartheta_n,\xi_n))$ of trivial periodic points of (8.1) belonging to $\mathcal{C}$ and converging to some $(\bar{t},\overline{\vartheta},\bar{\xi}) \in K$.

Indeed, otherwise there is a sufficiently small $\varepsilon > 0$ such that $\{z \in \mathrm{cl}\,\mathcal{C} \mid d(z,K) < \varepsilon\} \subset \mathcal{P}$. Let $F = \{z \in S^{N+2} \mid d(z,K) \geq \varepsilon\}$ (here again $S^{N+2} = \mathbf{R}^{N+2} \cup \{\infty\}$), $Z = F \cup \mathrm{cl}\,\mathcal{C} \cup K$. Clearly F and K cannnot be separated in Z for they are connected by $\mathrm{cl}\,\mathcal{C}$. By Lemma 5.31, there is a connected set $\mathcal{D} \subset \mathrm{cl}\,\mathcal{C} \setminus (F \cup K)$, i.e. $\mathcal{D} \subset \mathcal{P}$, such that $\mathrm{cl}\,\mathcal{D} \cap K \neq \emptyset$. This contradicts the maximality of $\mathcal{N}$ and, thus, completes the proof of Claim.

Clearly $t_n, \bar{t} > 0$. Since $0 \in \varphi(\vartheta_n,\xi_n)$, we gather that also $0 \in \varphi(\overline{\vartheta},\bar{\xi})$. This ends the proof of part (i).

(ii) We have already proved the existence of a sequence $((t_m,\vartheta_m,\xi_m))$ of nontrivial periodic points of (8.5) converging to $(t_0,\vartheta_0,0)$. Let $T_m > 0$ be the minimal period of an arbitrary nontrivial solution of $(8.6)_{\vartheta_m}$ corresponding to (t_m,ϑ_m,ξ_m). We claim that there is $T_0 \in (0,T)$ such that, passing to a subsequence if necessary, $T_m \to T_0$ as $m \to \infty$. Suppose to the contrary that $T_m \to 0$. Take a positive integer r such that, for any $r' \geq r$, $ir'\beta \notin \sigma(L(\vartheta_0))$. Then, for $l_m = \frac{t_0}{T_m}$ $([r^{-1}l_m]T_m,\vartheta_m,\xi_m) \to (r^{-1}t_0,\vartheta_0,0)$ where $[r^{-1}l_m]$ stands for the integer part of $r^{-1}l_m$. As in the proof of

Lemma 8.18 (see also Remark 8.15), we gather that $K(r^{-1}t_0, \vartheta_0)$ is singular, hence it has 0 as an eigenvalue. That is, for some $\gamma \in \sigma(L(\vartheta_0))$ such that $e^{r^{-1}t_0\gamma} = 1$. Since, by 8.8 (i), $\gamma \neq 0$, γ must be purely imaginary, say $\gamma = i\mu$, and $r^{-1}t_0\mu = 2\pi s$ for some positive integer s. Therefore $\gamma = ir'\beta \in \sigma(L(\vartheta_0))$ where $r' = skr \geq r$ (k was defined in Remark 8.9 (v)), a contradiction.

Of course, again $K(T_0, \vartheta_0)$ is singular; hence $T_0 = 2s\pi\mu^{-1}$ where s is a positive integer and $i\mu \in \sigma(L(\vartheta_0))$. □

8.C. Branching of periodic solutions

Now we proceed to a general study of the branching problem. We consider the parametrized differential inclusion

$$x' \in \varphi(\lambda, t, x), \qquad \lambda \in \Lambda,\ t \in [0, T],\ x \in \mathbf{R}^N,$$

where Λ is an open subset of $\mathbf{R}^k$ and $\varphi : \Lambda \times [0, T] \times \mathbf{R}^N \multimap \mathbf{R}^N$ is a Carathéodory map, i.e. for almost all $t \in [0, T]$, $\varphi(\,\cdot\,, t, \,\cdot\,)$ is an upper semicontinuous map with compact convex values and, for all $(\lambda, x) \in \Lambda \times \mathbf{R}^N$, $\varphi(\lambda, \,\cdot\,, x)$ is measurable. Moreover, we assume that there is $M \geq 0$ such that

$$\sup_{y \in \varphi(\lambda, t, x)} |y| \leq M(1 + |x|)$$

for all $\lambda \in \Lambda$, $t \in [0, T]$ and $x \in \mathbf{R}^N$.

For a fixed $\lambda \in \Lambda$, we shall study the periodic boundary value problem

$$\begin{cases} x' \in \varphi(\lambda, t, x) \\ x(0) = x(T) \end{cases} \tag{8.10$_\lambda$}$$

First observe that, in virtue of the Gronwall inequality, for each $\lambda \in \Lambda$ and $\xi \in \mathbf{R}^N$, the problem

$$\begin{cases} x' \in \varphi(\lambda, t, x) \\ x(0) = \xi \end{cases}$$

has a solution $x : [0, T] \to \mathbf{R}^N$. However the existence of solutions to $(8.10)_\lambda$ and their behaviour with respect to λ is not clear at all.

8.19 Assumption *Suppose $\lambda_0 \in \Lambda$ and $r > 0$ are such that $D^k(\lambda_0, r) \subset \Lambda$ and assume the following three hypotheses.*

(i) *For each $\lambda \in \Lambda$, $(8.10)_\lambda$ admits a solution $x : [0,T] \to \mathbf{R}^N$ with $x(0) = x(T) = 0$.*

(ii) *There is a function $V : \Lambda \times \mathbf{R}^N \to \mathbf{R}$ having continuous derivatives $\frac{\partial}{\partial x}V$ and $\frac{\partial^2}{\partial x^2}V$ on $\Lambda \times \mathbf{R}^N$ such that:*

$$\frac{\partial}{\partial x}V(\lambda, 0) = 0$$

for all $\lambda \in \Lambda$; and

$$\frac{\partial^2}{\partial x^2}V(\lambda, 0) \in GL(N)$$

for all $\lambda \in \Lambda$ with $|\lambda - \lambda_0| \geq r$.

(iii) *For all $\lambda \in \Lambda$ with $|\lambda - \lambda_0| \geq r$, $t \in [0,T]$ and $x \in \mathbf{R}^N$,*

$$\sup_{y \in \varphi(\lambda,t,x)} \left\langle y, \frac{\partial}{\partial x}V(\lambda, x) \right\rangle \geq 0,$$

i.e. there is $y \in \varphi(\lambda, t, x)$ such that $\langle y, \frac{\partial}{\partial x}V(\lambda, x)\rangle \geq 0$ ([5]).

In order to simplify the notation, let $U = \Lambda \times \mathbf{R}^N$ and let $F : U \to \mathbf{R}^N$, $L : \Lambda \to \mathcal{L}(\mathbf{R}^N, \mathbf{R}^N)$ be given by

$$F(\lambda, x) := \frac{\partial}{\partial x}V(\lambda, x) \qquad \text{and} \qquad L(\lambda) := \frac{\partial^2}{\partial x^2}V(\lambda, 0)$$

for $\lambda \in \Lambda$, $x \in \mathbf{R}^N$.

Observe that under Assumption 8.19 (ii), $\mathrm{BI(F)} \in \Pi_{\mathrm{k}-1}$ is defined. Indeed, the continuity of L and Assumption 8.19 (ii) imply that there is $\eta > 0$ such that $L(\lambda)$ is nonsingular for all $\lambda \in \Lambda$ with $r - \eta \leq |\lambda - \lambda_0|$. Let $\mathcal{S}_F = \{(\lambda, \xi) \in U \mid \xi \neq 0, 0 = F(\lambda, \xi)\}$. In virtue of the implict function theorem, we gather that $\mathrm{cl}\,\mathcal{S}_F \cap \Lambda \times \{0\} \subset D^k(\lambda_0, r - \eta) \times \{0\}$, i.e. $\mathcal{B}(F) \subset D^k(\lambda_0, r - \eta)$. Therefore $\mathrm{BI(F)} \in \Pi_{\mathrm{k}-1}$ is well-defined.

Moreover, without loss of generality, we may suppose that

$$0 \neq F(\lambda, \xi) \tag{8.11}$$

for all $0 < |\xi| \leq \eta$ and $r - \eta \leq |\lambda - \lambda_0| \leq r + \eta$.

We have the following result.

[5] here $\langle \cdot\,, \cdot \rangle$ denotes the scalar product in $\mathbf{R}^N$; some authors say that V is a guiding function for φ.

8.20 Theorem *If Assumptions* 8.19 *are satisfied and* BI(F) *is nontrivial in* Π_{k-1}, *then there either:*

(i) *exists a sequence* $((\lambda_n, \xi_n))_{n=1}^{\infty}$, $\lambda_n \to \overline{\lambda} \in S^{k-1}(\lambda_0, r)$, $\xi_n \in \mathbf{R}^N$, $\xi_n \neq \xi_m$ *for* $n \neq m$, *and a sequence* (x_n) *of solutions to* $(8.10)_{\lambda_n}$ *such that* $x_n(0) = x_n(T) = \xi_n \to 0$ *and* $x_n \to \overline{x}$ *in* $C([0,T], \mathbf{R}^N)$ *where* $\overline{x}$ *is a solution to* $(8.10)_{\overline{\lambda}}$ *such that* $\overline{x}(0) = \overline{x}(T) = 0$; *or*

(ii) *exits a connected set* $\mathcal{C}$ *of points* (λ, ξ) *with* $\xi \neq 0$ *such that*

- $(\overline{\lambda}, 0) \in \operatorname{cl}\mathcal{C}$, *where* $|\overline{\lambda} - \lambda_0| < r$,
- $\mathcal{C}$ *is unbounded or* $\operatorname{cl}\mathcal{C} \cap \operatorname{bd} \mathrm{U} \neq \emptyset$ *or* $(\widetilde{\lambda}, 0) \in \operatorname{cl}\mathcal{C}$ *where* $\widetilde{\lambda} \in \Lambda$, $|\widetilde{\lambda} - \lambda_0| > r$,
- *each point* $(\lambda, \xi) \in \operatorname{cl}\mathcal{C}$ *corresponds to a solution* $x : [0,T] \to \mathbf{R}^N$ *of* $(8.10)_\lambda$ *with* $x(0) = x(T) = \xi$. *In particular, there is a sequence* $(x_n)_{n=1}^{\infty}$ *of solutions to* $(8.10)_{\lambda_n}$, $x_n(0) = x_n(T) = \xi_n$, *where* $\lambda_n \to \overline{\lambda}$ *in* Λ *with* $|\overline{\lambda} - \lambda_0| < r$, *converging to a solution* $\overline{x}$ *to* $(8.10)_{\overline{\lambda}}$ *with* $\overline{x}(0) = \overline{x}(T) = 0$.

Proof Let us define a map $f : U \to \mathbf{R}^N$ by

$$f(\lambda, x) = \begin{cases} F(\lambda, x) & \text{if } |F(\lambda, x)| \leq 1 \\ \dfrac{F(\lambda, x)}{|F(\lambda, x)|} & \text{if } |F(\lambda, x)| > 1 \end{cases}$$

for $(\lambda, x) \in U$ and a map $P : U \times [0,T] \to \mathbf{R}^N$ by the formula

$$P(\lambda, \xi, t) = x(t) - \xi$$

where $x : [0,T] \to \mathbf{R}^N$ is the unique solution to the problem $x' = f(\lambda, x)$, $x(0) = \xi$. It is clearly a well-defined single-valued (continuous) map since f is bounded and it satisfies the Lipschitz condition with respect to the second variable.

Consider a map $A : U \multimap \mathbf{R}^N$ given by

$$A(\lambda, x) = \{y \in \mathbf{R}^N \mid \langle y, \alpha(\lambda) F(\lambda, x) \rangle \geq 0\}$$

where $\alpha(\lambda) = 0$ for $|\lambda - \lambda_0| \leq r$ and $\alpha(\lambda) = 1$ for $|\lambda - \lambda_0| > r$. Evidently, the graph of A is closed; hence $\psi : \Lambda \times [0,T] \times \mathbf{R}^N \multimap \mathbf{R}^N$ given by

$$\psi(\lambda, t, x) := A(\lambda, x) \cap \varphi(\lambda, t, x)$$

is a Carathéodory map with nonempty values in view of Assumption 8.19 (iii). Let $\Phi : U \times [o, T] \multimap \mathbf{R}^N$ be given by

$$\Phi(\lambda, \xi, t) := \{x(t) - \xi \mid x'(s) \in \psi(\lambda, s, x(s)) \text{ for a.a. } s \in [o, T],\ x(o) = \xi\}.$$

Assume that

($*$) *there is* $0 < \varepsilon \le \eta/2$ *such that, for all* $r + \varepsilon/2 \le |\lambda - \lambda_0| < r + \varepsilon$ *and* $0 < |\xi| \le \varepsilon$, $0 \notin \Phi(\lambda, \xi, T)$.

If condition ($*$) is not satisfied, then there are sequences $\lambda_n \to \overline{\lambda}$ in Λ and $\xi_n \to 0$ in $\mathbf{R}^N$ such that $\lambda_n \neq \lambda_m$, $\xi_n \neq \xi_m$ for $n \neq m$, $|\overline{\lambda} - \lambda_0| = r$ and and $0 \in \Phi(\lambda_n, \xi_n, T)$. Hence there is a sequnece of solutions $x_n : [0, T] \to \mathbf{R}^N$ to $(8.10)_{\lambda_n}$ such that $x_n(0) = x_n(T) = \xi_n$. In view of the Gronwall inequality, the Compactness Theorem 6.8 implies that, passing to a subsequence if necessary, $x_n \to \overline{x}$ in $C([0, T], \mathbf{R}^N)$. Then, by the Convergence Theorem 6.4, $\overline{x}$ is the solution of $(8.10)_{\overline{\lambda}}$ such that $\overline{x}(0) = \overline{x}(T) = 0$ and this completes the proof of part (i) of the theorem.

Let $\Lambda_1 = B^k(\lambda_0, r + \varepsilon/2)$, $\Lambda_2 = B^k(\lambda_0, r + \varepsilon)$ and $\Omega = \Lambda_2 \setminus \Lambda_1 = \{\lambda \in \Lambda \mid r + \varepsilon/2 \le |\lambda - \lambda_0| < \varepsilon\}$.

Claim 1. For any $\lambda \in \Omega$, $\xi \in \mathbf{R}^N$, $|\xi| = \varepsilon$ and $t \in [0, T]$, $0 \neq P(\lambda, \xi, t)$.

Indeed, take $t \in [0, T]$, $(\lambda, \xi) \in \Omega \times S_\varepsilon^{N-1}$ and the solution $x'(z) = f(\lambda, x(z))$, $x(0) = \xi$. Then

$$\begin{aligned} V(\lambda, x(t)) - V(\lambda, x(0)) &= \int_0^t \langle F(\lambda, x(s)), x'(s) \rangle \, \mathrm{d}s \\ &= \int_0^t \langle F(\lambda, x(s)), f(\lambda, x(s)) \rangle \, \mathrm{d}s > 0, \end{aligned}$$

since $x \not\equiv 0$ and so the function under the integral is positive on the set of positive measure (see also (8.11). Thus $x(t) \neq x(0) = \xi$ and $0 \neq P(\lambda, \xi, t)$. This ends the proof of Claim 1.

For $t \in [0, T]$, consider a map $h_t : U \times [0, 1] \to \mathbf{R}^N$,

$$h_t(\lambda, \xi, s) := (1 - s)F(\lambda, \xi) + sP(\lambda, \xi, t).$$

Claim 2. There is $\tau \in [0, T]$ such that, for $\lambda \in \Omega$, $|\xi| = \varepsilon$ and $s \in [0, 1]$,

$$0 \neq h_\tau(\lambda, \xi, s).$$

Indeed, the continuity of f and (8.11) imply that there is $\tau > 0$ such that if $\xi, \xi' \in \mathbf{R}^N$, $|\xi| = \varepsilon$ and $|\xi - \xi'| \le \tau$, then $\langle F(\lambda, \xi), f(\lambda, \xi') \rangle > 0$.

Now suppose that $(\lambda, \xi) \in \Omega \times S_\varepsilon^{N-1}$ and $s \in [0, 1]$. If $h_\tau(\lambda, \xi, s) = 0$, then by (8.11) and Claim 1, $s \in (0,1)$ and $x(\tau) - \xi = \frac{s-1}{s} F(\lambda, \xi)$ where $x' = f(\lambda, x)$ on $[0,T]$ and $x(0) = \xi$. Since $|f| \le 1$, it is clear that, for all $z \in [0,\tau]$, $|x(z) - \xi| \le \tau$. Thus

$$0 > \langle x(\tau) - \xi, F(\lambda,\xi)\rangle = \int_0^\tau \langle f(\lambda, x(z)), F(\lambda,\xi)\rangle \, \mathrm{d}z > 0,$$

a contradiction. This completes the proof of Claim 2.

Now let

$$k(s) = \begin{cases} 1 & \text{for } s \in [0,1/2) \\ 2 - 2s & \text{for } s \in [1/2, 1] \end{cases}$$

and

$$t(s) = \begin{cases} 2(T-\tau)s + \tau & \text{if } s \in [0,1/2) \\ T & \text{if } s \in [1/2,1]. \end{cases}$$

and consider a map $\Psi' : U \times [0,1] \multimap \mathbf{R}^N$ given by

$$\begin{aligned} \Psi'(\lambda,\xi,s) &= \{x(t(s)) - \xi \mid x'(z) \in k(s) f(\lambda, x(z)) \\ &\quad + (1-k(s))\psi(\lambda, z, x(z)) \text{ a.a. on } [0,T],\ x(0) = \xi\}. \end{aligned}$$

It is clear that $\Psi'(\lambda,\xi,0) = h_\tau(\lambda,\xi,1) = P(\lambda,\xi,\tau)$ and $\Psi'(\lambda,\xi,1) = \Phi(\lambda,\xi,T)$. In view of condition $(*)$ and Claim 2, if $\lambda \in \Omega$, $0 < |\xi| \le \varepsilon$, then $0 \notin \Psi(\lambda,\xi,i)$, $i = 0,1$. Now we shall see that also if $(\lambda,\xi) \in \Omega \times S_\varepsilon^{N-1}$ and $s \in (0,1)$, then for any $\zeta \in \Psi'(\lambda,\xi,s)$, $\zeta \ne 0$. By the definition, $\zeta = x(t(s)) - \xi$ where the function $x : [0,T] \to \mathbf{R}^N$ is such that $x(0) = \xi$ and $x'(z) \in k(s) f(\lambda, x(z)) + (1-k(s))\psi(\lambda, z, x(z))$ for a.a. $z \in [0,T]$, i.e. $x'(z) = k(s) f(\lambda, x(z)) + (1-k(s)) y(z)$ a.e. on $[0,T]$ where $y(z) \in \psi(\lambda, z, x(z))$.

Then, for $0 < s \le 1/2$,

$$\zeta = P(\lambda,\xi,t(s)) \ne 0$$

by Claim 1, and if $1/2 < s < 1$, then

$$V(\lambda, \zeta + \xi) - V(\lambda,\xi) \ge k(s) \int_0^T \langle F(\lambda, x(z)), f(\lambda, x(z))\rangle \, \mathrm{d}z > 0$$

since $|\lambda - \lambda_0| \ge r + \varepsilon/2$ so $\langle y(z), F(\lambda, x(z))\rangle \ge 0$ and the intergrand in the right-hand side is positive on a set of positive measure. Hence indeed $\zeta \ne 0$.

Finally, we consider a map $\Psi : U \times [0,1] \multimap \mathbf{R}^N$ given by

$$\Psi(\lambda,\xi,s) = \begin{cases} h_\tau(\lambda,\xi,2s) & \text{if } s \in [0,1/2] \\ \Psi'(\lambda,\xi,2s-1) & \text{if } s \in (1/2,1] \end{cases}$$

Taking into account (8.11), Claim 2, the above and condition $(*)$, we gather that, for $\lambda \in \Omega$, $0 < |\xi| \le \varepsilon$, $0 \notin \Psi(\lambda,\xi,i)$, $i = 0,1$ and if $(\lambda,\xi) \in \Omega \times S_\varepsilon^{N-1}$, $s \in [0,1]$, then $0 \notin \Psi(\lambda,\xi,s)$.

Observe that all maps considered above: P, h_τ, Ψ', Ψ and $\Phi_T := \Phi(\,\cdot\,,\,\cdot\,,T)$ are determined by CE-morphisms.

In view of Assumption 8.19 (i), for all $\lambda \in \Lambda$, $0 \in \Phi_T(\lambda,0)$ and, by condition $(*)$, $\mathcal{B}(\Phi_T|\Lambda_2 \times \mathbf{R}^N) \subset \Lambda_1 \times \{0\}$; hence $\mathrm{BI}(\Phi_T|\Lambda_2 \times \mathbf{R}^N)$ is defined. Theorem 7.10 and Proposition 7.6 imply that

$$\mathrm{BI}(\Phi_T|\Lambda_2 \times \mathbf{R}^N) = \mathrm{BI}(\mathrm{F}|\Lambda_2 \times \mathrm{R}^\mathrm{N}) = \mathrm{BI}(\mathrm{F}).$$

Since BI(F) is nontrivial, the assertion follows from Corollary 7.5 and Theorem 7.7. □

Observe that in virtue of Propositions 7.21, 7.18, $\mathrm{BI}(\mathrm{F}) = \gamma_\mathrm{g} = \overline{\gamma}_\mathrm{g}$ where $g(\lambda,x) = L(\lambda)x$, i.e. it only depends on the behaviour of L. Therefore, in particular, results from 7.16 give means to compute BI(F).

Now we shall provide an example showing the availability of Assumptions 8.19.

8.21 Example Let $\Lambda = (-1,1)$, $f_1, f_2 : \Lambda \times [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ be lower semicontinuous and $g_1, g_2 : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ upper semicontinuous functions such that, for all $\lambda \in \Lambda$, $0 \le t \le 1$ and $x \in \mathbf{R}^2$,

(i) $f_i(\lambda,t,x) \le \lambda g_i(t,x)$;

(ii) $f_i(\lambda,t,0) \le 0 \le \lambda g_i(t,0)$;

(iii) $\langle (g_1(t,x), g_2(t,x)), x\rangle \ge 0$.

Let $\varphi : \Lambda \times [0,1] \times \mathbf{R}^2 \multimap \mathbf{R}^2$ be given by

$$\varphi(\lambda,t,x) = [f_1(\lambda,t,x), \lambda g_1(t,x)] \times [f_2(\lambda,t,x), \lambda g_2(t,x)]$$

for $\lambda \in \Lambda$, $0 \le t \le 1$ and $x \in \mathbf{R}^2$. Clearly φ is a Carethéodory map with compact convex values. In view of (ii) above, $0 \in \varphi(\lambda,t,0)$ for all $(\lambda,t) \in \Lambda \times [0,1]$. Now let $V : \Lambda \times \mathbf{R}^2 \to \mathbf{R}$ be given by

$$V(\lambda,x) = \frac{1}{2}|\lambda| x_1^2 + \frac{1}{4}\lambda x_2^2.$$

We easily see that V satisfies Assumption 8.19 (ii) and, in view of (iii) above Assumption 8.19 is also true. Clearly,

$$L(\lambda) = \begin{bmatrix} |\lambda| & 0 \\ 0 & \frac{1}{2}\lambda \end{bmatrix}.$$

Hence $\det L(-1) < 0$ and $\det L(1) > 0$ and the bifurcation invariant γ_g (where $g(\lambda, x) = L(\lambda)x$) is nonzero. By Theorem 8.20, there is a sequence (λ_n, x_n), $\lambda_n \to 0$, $x_n : [0,1] \to \mathbf{R}^2$ such that $x_n'(t) \in \varphi(\lambda_n, t, x_n(t))$ a.e. on $[0,1]$ and $x_n(0) = x_n(1) \to 0$.

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Glossary of Notations

$\emptyset$	empty set
X	generic (Hausdorff paracompact) space
2^X	family of all subsets of X
(X, A)	pair of spaces (with A closed in X)
$\mathrm{cl}\, B$	closure of $B \subset X$
$\mathrm{int}\, \mathrm{B}$	interior of $B \subset X$
$\mathrm{bd}\, \mathrm{B}$	boundary of $B \subset X$
$X \oplus Y$ (or $X \vee Y$)	disjoint union of spaces X and Y
$f : X \to Y$	generic (continuous) map of spaces
1_X	identity map of space X
$f\|A : A \to Y$	restriction of f to $A \subset X$
$f : (X, A) \to (Y, B)$	map of pairs
$f : f_0 \simeq f_1$	generic homotopy $f : (X, A) \times [0, 1] \to (Y, B)$ joining maps $f_0, f_1 : (X, A) \to (Y, B)$
$[X, A; Y, B]$	set of homotopy classes of maps $(X, A) \to (Y, B)$
$g_\# : [X, A; Y, B] \to [X, A; Y', B']$	set transformation induced by $g : (Y, B) \to (Y', B')$ (i.e. $g_\#[f] = [g \circ f]$)
$g^\# : [X, A; Y, B] \to [X', A'; Y, B]$	set transformation induced by $g : (X', A') \to (X, A)$ (i.e. $g^\#[f] = [f \circ g]$)
$\pi_n(X, A; x_0)$	n-th homotopy group of (X, A) at $x_0 \in A$
$\pi^n(X, A)$	n-th cohomotopy group of (X, A)
AE (resp. ANE)	absolute (resp. neighborhood) extensor
AR (resp. ANR)	metric absolute (resp. neighborhood) retract
$Z(f)$	cylinder of a map $f : X \to Y$, 10
$X \cup_g Y$	union of spaces X, Y along a map $g : A \to Y$, 10
$\check{H}^*(X, A; G)$	Čech cohomology of pair (X, A) with coefficients in group G
$B^X(A, \varepsilon)$ (resp. $D^X(A, \varepsilon)$)	open (resp. closed) ball of radius $\varepsilon > 0$ around subset A of metric space X, 10
$\mathbf{R}$	set of real numbers
(a, b) (resp. $[a, b]$)	open (resp. closed) interval in $\mathbf{R}$
$\mathbf{Z}$	group of integers
$\mathbf{N} = \{1, 2, \ldots\}$	
$\mathbf{R}^n$	Euclidean (real) n-dimensional space
$\| \cdot \|$	norm in $\mathbf{R}^n$
$x \cdot y$ (or $\langle x, y \rangle$)	scalar product of $x, y \in \mathbf{R}^n$
$B^n(x, r)$ (resp. $D^n(x, r)$)	open (resp. closed) ball at $x \in \mathbf{R}^n$ of radius $r > 0$, 11
$S^{n-1}(x, r) = \mathrm{bd}\, \mathrm{D}^n(\mathrm{x}, \mathrm{r})$	
$D^n_r = D^n(0, r)$, $D^n = D^n_1$	
$B^n_r = B^n(0, r)$, $B^n = B^n_1$	
$S^{n-1}_r = S^{n-1}(0, r)$, $S^{n-1} = S^{n-1}_1$	
$\varphi : X \to 2^Y \setminus \{\emptyset\}$	multivalued transformation, 11

$\varphi : X \multimap Y$	set-valued map (compact values upper semicontinuous), 11
$\varphi^{-1}(U)$	("small") preimage of $U \subset Y$ through φ, 11
$\varphi(A) = \bigcup_{a\in A} \varphi(a)$	image of $A \subset X$ through φ
Gr(f) (resp. Gr(φ))	graph of (resp. set-valued) map f (resp. φ), 11
UV^n, $0 \le n \le \infty$, $n = \omega$, R_δ	set property, 17
$\hookrightarrow$	embedding
$(E, \\|\cdot\\|)$	generic Banach space
$E \hookrightarrow\hookrightarrow E'$	compact embedding of Banach spaces, 140, 145
$\dim X$	(covering) dimension of space X
$\mathrm{rd_X(A)}$	relative dimension of A in X, 44
$\mathrm{def\,dim\,X}$	deformation dimension of space X, 20
$\mathrm{Ind\,X}$	large inductive (Brouwer-Čech) dimension of X
$\mathcal{U}$	generic (open) relation in $X \times Y$
$\mathcal{U}(A) = \{y \mid \exists\, x \in A\ (x,y) \in \mathcal{U}\}$	
$\mathcal{U}^{-1}$	inverse relation (in $Y \times X$), 24
$\mathcal{U} \circ \mathcal{V}$	composition of relations $\mathcal{V}, \mathcal{U}$, 25
$\mathfrak{A}$	generic (open) covering of space
$\mathrm{st\,(A, \mathfrak{A})}$	star of $A \subset X$ with respect to cover $\mathfrak{A}$ of X, 24
K	generic (abstract) simplicial complex
$\lvert K\rvert$	geometric realization of complex (with Whitehead topology)
$v \in \sigma \prec K$	v is a vertex of a simplex σ of complex K
$\partial\sigma$	boundary of simplex, 25
$\lvert\sigma\rvert$ (resp. $\langle\sigma\rangle$)	closed (resp. open) simplex in $\lvert K\rvert$ spanned by σ
$X(\mathfrak{A})$	nerve of a covering $\mathfrak{A}$ of X
$X_{\mathfrak{A}} = \lvert X(\mathfrak{A})\rvert$	
$X/\mathfrak{D}$	quotient space of X through a decomposition $\mathfrak{D}$
$\mathcal{A}$	generic sheaf of abelian groups
$\mathcal{A}^*$	inverse image of a sheaf, 43
$\mathcal{A}_y$	fibre of $\mathcal{A}$ over y
$\mathrm{s^k(f)}$	singular set of f in dimension k, 43, 44
$\mathrm{i^N(f)}$, $\mathrm{i(f)}$	Vietoris indices of f, 44
$\mathbf{E}$	typical CW-spectrum
$\mathbf{K}(G)$	Eilenberg-MacLane spectrum
$\mathbf{E}^*(X)$	spectral (extraordinary reduced) cohomology of a pointed space X, 47
$\mathbf{S}$	spherical spectrum, 49
$\pi_s^n(X)$	n-th stable cohomotopy group of X, 49
$\pi_n^s(X)$	n-th stable homotopy group of X
Π_n	n-th stable homotopy group of spheres, 115
$\Sigma^k X$ (resp. $S^k X$)	k-th (resp. reduced) suspension of (resp. pointed) space X
$\mathcal{LS}(E)$	Leray-Schauder category, 66
$g_0 \simeq_u g_1$	u-homotopy of u-fields, 67

$[g]_u$	u-homotopy class of a u-field g, 67
$\pi^E(X, A; u)$	collection of u-homotopy classes, 67
$\mathcal{O}_L$ (resp. $\mathcal{O}$)	orientation of a (finite-dimensional) linear space L (resp. of E), 67
Δ_{LN}	Mayer-Vietoris coboundary transformation, 68
δ	coboundary transformation, 78
$\Sigma^E(X, A; u)$	direct limit of $\Sigma^L(X_L, A_L)$, 69
$\mathcal{L}(E, E')$	space of bounded linear operators $E \to E'$
$\mathrm{Ker}(F)$	null-space of $F \in \mathcal{L}(E, E')$, 72
$\mathrm{R}(F)$	range of F, 72
$\mathrm{Coker(F) = E'/R(f)}$	
$\mathcal{F}(E, E')$	family of Fredholm operators, 72
$\mathrm{ind(F)}$	index of a Fredholm operator, 72
$\mathcal{A}_m(X, Y)$	family of m-acyclic set-valued maps $X \multimap Y$, 82
$\mathcal{CE}(\mathcal{X}, \mathcal{Y})$	family of $\mathcal{CE}$-valued maps, 82
$\mathcal{V}_m$, $\widetilde{\mathcal{V}}_m$	classes of Vietoris maps, 83
D_m (resp. $\widetilde{D}_m$)	class of cotriads $X \overset{p}{\leftarrow} W \overset{q}{\rightarrow} Y$ with $p \in \mathcal{V}_m$ (resp. $\widetilde{\mathcal{V}}_m$), 84
$\approx$	relation of equivalence of cotriads, 85
M, M_m, $\widetilde{M}$, $\widetilde{M}_m$	classes of morphisms, 85, 86
$\mathrm{i}(\varPhi)$	Vietoris index of a morphism $\varPhi$, 86
$W_1 \boxtimes W_2$	fibre-product of spaces, 84
$\simeq_n$	relation of homotopy of morphisms in M_n, 90
$M_m[X, X'; Y, Y']_n$	set of homotopy classes of morphisms, 92
$\check{H}^*(\varPhi)$	homomorphism on cohomology induced by a morphism, 88
$\deg(\varPhi)$	degree of a morphism of spheres, 98
D_{CE}	class of CE-cotriads, 102
M_{CE}	class of CE-morphisms, 103
$\simeq_{CE}$	relation of homotopy of CE-morphisms, 104
$M_{CE}[X, X'; Y, Y']$	set of homotopy classesof CE-morphisms, 105
$\mathcal{C}(m, n)$	class of pairs (f, U), $U \subset \mathbf{R}^m$, $f : (\mathrm{cl}\, U, \mathrm{bd\, U}) \to (\mathbf{R}^n, \mathbf{R}^n \setminus \{0\})$, 115
$\deg(f, U)$	degree of (f, U), 117
$\widetilde{M}(m, n)$, $M_{CE}(m, n)$	classes of morphisms $\mathbf{R}^m \multimap \mathbf{R}^n$, 133
$\deg(\varPhi, U)$	degree of $(\varPhi, U)$, 133
$\mathcal{C}^F(E', E)$	class of compact F-fields, 125
$\mathrm{ind_F(g, U)}$	coincidence index of (g, U), 126
$\widetilde{M}^F(E', E)$, $M^F_{CE}(E', E)$	classes of compact morphisms, 131
$\mathrm{ind_F}(\varPhi, U)$	coincidence index of $(\varPhi, U)$, 131
Ω, μ	generic domain in $\mathbf{R}^n$ with Lebesgue measure
N_φ	Nemytski map associated to φ, 140
$L^p(\Omega, \mathbf{R}^n)$	space of integrable with power $p \geq 1$ functions $\Omega \to \mathbf{R}^n$, 140

$AC([a,b],\mathbf{R}^n)$	space of absolutely continuous maps $[a,b]\to\mathbf{R}^n$, 141
S_φ	set of solutions to a differential inclusion with the right hand side φ, 142
P, Q	generic projectors onto $\mathrm{Ker}(F)$ and Coker(F), 144
K_{PQ}	generic generalized inverse to a Fredholm operator F, 144
∂^α	weak derivative of order $\lvert\alpha\rvert$, 156
D, L	typical differential operators
$H^{m,p}(\Omega,\mathbf{R}^n)$, $\lVert\cdot\rVert_{m,p}$	Sobolev space and its norm, 156
$\langle\cdot,\cdot\rangle_m$	scalar product in $H^m(\Omega,\mathbf{R}^n)=H^{m,2}(\Omega,\mathbf{R}^n)$, 156
$\mathfrak{H}(A,B)$	Hausdorff distance, 159
Lim	limit with respect to Hausdorff distance, 159
U_{rel}	set of relay controls, 172
Δ_k	standard k-dimensional simplex, 173
$\mathcal{B}(\varPhi)$	set of bifurcation points, 180
$\mathrm{BI}(\varPhi)$	bifurcation (of zeros) index, 181
$\Omega(m)$	space of maps $S^{m-1}\to S^{m-1}$, 186
$\gamma_\varPhi$	Alexander invariant, 190
$\mathcal{H}$	Hopf construction map, 191, 192
$GL(n)$	general linear group
$O(n)$, $SO(n)$	orthogonal group and special orthogonal group
J_m, J	classical and stable J-homomoprphism, 194
$\mathrm{BIF}(\varPhi)$	bifurcation (of fixed points) index, 198
$\mathcal{P}(\varphi)$	set of nontrivial periodic points, 204
$\sigma(L)$	spectrum of a linear operator L
$c(L,i\beta)$	crossing number of L through $i\beta$, 206, 207

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About the author

Wojciech Kryszewski was born on August 15, 1958 in Toruń. He studied at the University of Łódź in 1976–1981. In 1980 he won the second prize at the J. Marcinkiewicz Contest for Students Research Papers. In 1981–84 he worked as an assistant and in 1984–88 as a senior assistant at the Institute of Mathematics of the University of Łódź. He got his Ph. D. Degree at the Institute of Mathematics of Łódź University in 1988. In 1988 he won the prize for Young Mathematicians of Polish Mathematical Society. In 1990–92 he was the Research Fellow of the Alexander von Humboldt Foundation in Munich and Heidelberg, Germany. In 1992 he started his research and academic activity as an adjoint professor at the Department of Mathematics and Informatics of the Nicholas Copernicus University in Toruń. He was a research fellow of Centro Nazionale dello Richerche at the University of Firenze (3 months 1990), the Swedish Royal Academy of Sciences at the University of Stockholm (2 months 1995). He is a member of Polish and American Mathematical Societies.

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