Mental Accounting for Multiple Outcomes: A Theoretical and Empirical Study

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Martín Egozcue
Department of Economics, University of Montevideo, Montevideo, Uruguay

Sébastien Massoni1
CES - Université Paris 1 & Paris School of Economics, Paris, France

Wing-Keung Wong
Department of Economics, Hong Kong Baptist University, Hong Kong

Ričardas Zitikis
Department of Statistical and Actuarial Sciences, University of Western Ontario, London, Ontario, Canada

Abstract. We develop a theoretical basis for integration and segregation of multiple outcomes with underlying S-shaped value functions. Our theoretical findings suggest that Thaler’ principles of mental accounting work as postulated when we deal with events having only positive outcomes (gains) or only negative ones (losses). In the case of ‘mixed’ events with positive and negative outcomes, determining preferences for integration and segregation is a complex task, whose thorough solution we provide in the case of three outcomes. This supplements the only so far in the literature developed case of two outcomes. Our theoretical research is illustrated with a number of examples, as well as with an experimental study conducted in Changchun, Hong Kong, and Paris.

Keywords: Integration, segregation, bundling, marketing, prospect theory, value function, decision making.

JEL codes: D03; D11; M31

1Corresponding author. Centre d’Economie de la Sorbonne, Office 402, 106-112 Bd de l’Hôpital 75647 Paris Cedex 13. Tel: +33 (0)1 44 07 82 06. Fax: +33 (0) 1 44 07 82 47. E-mail: sebastien.massoni@gmail.com
1 Introduction

Decision making in real-life situations is often done in a context of uncertainty and multiple outcomes. Economic theory models the first aspect within multiple frameworks. One of the most famous and the one we will consider in this paper is due to Kahneman and Tversky (1979) who propose prospect theory to reflect the subjective desirability of different decision outcomes and to provide possible explanations for behaviors of investors who maximize value functions instead of utility functions. However this value function measures only a single event. A question arises when it is used to evaluate multiple events aggregately or separately. Indeed, we can assume that a decision maker will not have the same behaviors facing a single event or a multiplicity of events. The main question that arises is to know whether individuals prefer to combine or to treat separately these outcomes. To investigate this issue we will deal with the mental accounting framework of Thaler (1985) which examines how consumers could behave when they face two outcomes. More precisely, mental accounting is defined as “the set of cognitive operations used by individuals and households to organize, evaluate, and keep track of financial activities” (Thaler, 1999). Thaler has defined a pattern of optimal behaviors depending of the type of outcomes (positive or negative) and the overall value of their combination in the case of two outcomes. Our main question is to know whether these principles hold for more than two outcomes. In a footnote of his seminal paper, Thaler (1985) wrote that “For simplicity I will deal only with two-outcome events, but the principles generalize to cases with several outcomes.” (p.202). Nevertheless, we can intuitively believe than it could be a disputable point. Indeed, treating a large number of events is a more complex and sophisticated cognitive process than only two events. Thus it is not sure that the mental accounting principles that work for two outcomes hold in the case of multiple outcomes. This paper tries to answer this question by a theoretical and an empirical investigation of consumers’ behaviors facing more than two outcomes. Many real-life examples can motivate this study: think to an individual that has to choose between to pay his annual tax in one single payment or to pay multiple payments in several times; or to a salary that could choice between to benefit from a bonus by his employer each month and then have to pay tax returns at the end of the year or could win a global bonus with tax included at the end of the year of the same amount. All in all, we can see that many decisions imply to choose if we prefer to segregate or integrate the benefits or the costs of multiple outcomes.

Thus suppose that we deal with $n \geq 2$ exposure units (or events), whose experiences (or outcomes) which are real numbers $x_1, \ldots, x_n$ can be negative and positive. For example, we
may think of the experiences as losses and gains, measured in monetary units, or recorded, say, on a Likert scale, as is the case in our experimental study reported later in this paper. The individual wants to know whether the exposure units should be segregated or integrated, partially or totally, so that the resulting combination of the \( n \) exposure units would offer a maximal value.

The (perceived) value of the outcomes is represented by a value function \( v : \mathbb{R} \to \mathbb{R} \), which we assume to be continuous and increasing, convex for non-positive outcomes \( (x \leq 0) \) and concave for non-negative outcomes \( (x \geq 0) \). In other words, our analysis is within the prospect theory (Kahneman and Tversky, 1979). If, for example, two exposure units with experiences \( x_m \) and \( x_n \) are integrated (i.e., bundled), then their value is \( v(x_m + x_n) \), but if they are kept segregated, then the value is \( v(x_m) + v(x_n) \). Thaler (1985) postulates four basic principles of integration and segregation:

P1. Segregate (two) positive outcomes

P2. Integrate (two) negative outcomes

P3. Integrate a smaller negative outcome with a larger positive outcome

P4. Segregate a larger negative outcomes from a smaller positive outcome

When there are only two exposure units (i.e., \( n = 2 \)), then there can be only two possibilities: either integrate both or segregate both. Nevertheless, even in this seemingly simple case, challenges arise because the value function \( v \) is neither concave everywhere nor convex everywhere – it is, as we have already noted above, convex on the non-positive half-line and concave on the non-negative half-line. For a detailed analysis of the case \( n = 2 \) within the prospect theory, we refer to Egozcue and Wong (2010). These authors find, for example, that when facing small positive experiences and large negative ones, loss averters sometimes prefer to segregate, sometimes to integrate, and at other times to stay neutral. Jarnebrant et al. (2009) provide detailed theoretical and experimental analyses of principles P2 and P3.

Exploring Thaler’s (1985) principles in the context of multiple exposure units is of both practical and theoretical interest. There are some empirical works studying decision maker’s behavior in the case of multiple units. For example, Loughran and Ritter (2002), Ljungqvist and Wilhelm (2005) examine how mental accounting of multiple outcomes affects the behavior of market participants in various contexts of finance. However, as far as we know, there has not been a theory that would completely sort out the behavior of investors for mental accounting in the case of multiple exposure units. The present paper aims at developing such results within the prospect theory.
We have organized the rest of the paper as follows. In Section 2 we present basic definitions, including the definition of a value function of our main interest, its properties, as well as a theorem that solves the integration/segregation problem when experiences are of same sign. In the same section we also consider the situation when only two choices are available to the value maximizing decision maker: either integrate all or segregate all. In the case of only two exposure units, such results provide optimal decisions for integration and segregation. In Section 3 we completely resolve the integration and segregation problem in the case of three exposure units. In Section 4, we offer a number of numerical examples illustrating our theoretical investigations of the earlier sections, showing in particular their optimality. The section also contains details of an experiment that we have conducted in Changchun, Hong Kong, and Paris. Section 5 concludes the paper.

2 Basics

2.1 Value functions

As we have noted in the introduction, we work within the prospect theory and thus deal with value functions \( v : \mathbb{R} \to \mathbb{R} \) that are continuous, increasing, convex on \(( -\infty, 0 ] \) and concave on \([0, \infty) \). Specifically, let

\[
v(x) = \begin{cases} 
  v_+(x) & \text{when } x \geq 0, \\
  -v_-(x) & \text{when } x < 0,
\end{cases} \tag{2.1}
\]

where \( v_-, v_+ : [0, \infty) \to [0, \infty) \) are two increasing and concave functions such that \( v_-(0) = 0 = v_+(0) \), \( v_-(x) > 0 \) and \( v_+(x) > 0 \) for all \( x > 0 \). For example, Kahneman and Tversky (1979) use a value function that is equal to \( x^{\gamma_G} \) when \( x \geq 0 \) and \( -\lambda(-x)^{\gamma_L} \) when \( x < 0 \), where \( \lambda > 0 \) is the degree of loss aversion, and \( \gamma_G \) and \( \gamma_L \in (0, 1) \) are degrees of diminishing sensitivity. Assuming preference for homogeneity and loss aversion, al-Nowaihi et al. (2008) demonstrate that \( \gamma_G \) and \( \gamma_L \) are equal; denote them by \( \gamma \in (0, 1) \). This naturally leads to the following class of value functions:

\[
v_\lambda(x) = \begin{cases} 
  u(x) & \text{when } x \geq 0, \\
  -\lambda u(-x) & \text{when } x < 0,
\end{cases} \tag{2.2}
\]

where \( u : [0, \infty) \to [0, \infty) \) is continuous, concave, and such that \( u(0) = 0 \) and \( u(x) > 0 \) for all \( x > 0 \), with \( \lambda > 0 \) reflecting loss aversion. We shall work with the latter value function throughout this paper.
Given a value function $v : \mathbb{R} \to \mathbb{R}$, loss aversion is sometimes defined (cf. Kahneman and Tversky, 1979) as the inequality $v(x) \leq -v(-x)$ for all $x \geq 0$, which in the case of value function (2.1) is equivalent to $v_+(x) \leq v_-(x)$ for all $x \geq 0$, and thus in the case of value function (2.2) becomes equivalent to $\lambda \geq 1$. For further information and references on loss aversion, we refer to Schmidt and Zank (2005).

### 2.2 Petrović’s inequality

From the mathematical point of view, integration and segregation are about superadditivity and subadditivity of the value function. However, decision makers ‘visualize’ value functions in terms of their shapes: e.g., concave or convex in one region or another. A link between additivity and concavity type notions is provided by the theory of functional inequalities, and in particular by Petrović’s inequality (see, e.g., Kuczma, 2008)

$$v\left(\sum_{k=1}^{n} x_k\right) \leq \sum_{k=1}^{n} v(x_k),$$  \hfill (2.3)

which holds for all continuous and concave functions $v : [0, \infty) \to \mathbb{R}$ such that $v(0) = 0$, and for $n \geq 2$ and $x_1, \ldots, x_n \in [0, \infty)$. In other words, inequality (2.3) means that the function $v$ is subadditive on $[0, \infty)$, which implies that the value maximizing decision maker prefers to segregate positive experiences. In the domain $(-\infty, 0]$ of losses, the roles of integration and segregation are reversed. In the case of the value function $v_\lambda(x)$, this follows from the easily checked equation

$$v_\lambda(x) = -\lambda v_{1/\lambda}(-x),$$  \hfill (2.4)

which holds for all $x \in \mathbb{R}$. We shall find this ‘reflection’ equation particularly useful later in the paper. Now, for convenient referencing, we collect the above observations concerning total integration and total segregation into a theorem.

**Theorem 2.1** The value maximizing decision maker with any value function $v$ given by equation (2.1) prefers to segregate any finite number of positive experiences, and integrate any finite number of negative experiences.

When there are at least one positive and at least one negative experiences, then deciding whether to integrate or segregate the experiences becomes a complex task well beyond the simplicity of Theorem 2.1. Indeed, as demonstrated by Egozcue and Wong (2010) in the case $n = 2$, the value maximizing decision maker may prefer to integrate some mixed experiences and segregate others, depending on how large or small they are. In the following sections we shall provide further results on the topic.
2.3 A threshold

In this subsection we tackle the problem of whether it is better to integrate all \( n \geq 2 \) exposure units or keep them segregated, assuming that – for whatever reason – these are the only two options available to the decision maker. From now on throughout this paper, we work with the value function \( v_\lambda(x) \).

The following quantity, denoted by \( T(x) \) and called threshold, plays a pivotal role in our considerations:

\[
T(x) = \frac{\sum_{k \in K_+} u(x_k) - u\left(\max\left\{0, \sum_{k=1}^{n} x_k\right\}\right)}{\sum_{k \in K_-} u(-x_k) - u\left(\max\left\{0, -\sum_{k=1}^{n} x_k\right\}\right)},
\]

where \( K_- = \{k : x_k > 0\} \) and \( K_+ = \{k : x_k < 0\} \) are subsets of \( \{1, \ldots, n\} \), and \( x = (x_1, \ldots, x_n) \) is the vector of exposures. (We shall usually have the coordinates of \( x \) arranged in a non-decreasing order.)

**Theorem 2.2** The threshold \( T(x) \) is always non-negative. It splits the set of \( \lambda > 0 \) values into two regions: integration and segregation. Namely, assuming that there is at least one exposure unit with a positive experience and at least one with a negative experience, then, given that only two options are available to the decision maker: either integrate or segregate all the exposure units, the value maximizing decision maker prefers the following decisions:

- **Integrate all exposure units if and only if** \( T(x) \leq \lambda \).
- **Segregate all exposure units if and only if** \( T(x) \geq \lambda \).

**Proof.** We start with the case \( \sum_{k=1}^{n} x_k \geq 0 \). The inequality \( v_\lambda(\sum_{k=1}^{n} x_k) \leq \sum_{k=1}^{n} v_\lambda(x_k) \) is equivalent to

\[
u\left(\sum_{k=1}^{n} x_k\right) \leq -\lambda \sum_{k \in K_-} u(-x_k) + \sum_{k \in K_+} u(x_k),
\]

which is equivalent to

\[
\lambda \leq T_+(x) = \frac{\sum_{k \in K_+} u(x_k) - u\left(\sum_{k=1}^{n} x_k\right)}{\sum_{k \in K_-} u(-x_k)}.
\]

(2.5)

Since \( \sum_{k=1}^{n} x_k \geq 0 \), we have that \( T_+(x) = T(x) \). We shall next show that \( T(x) \) is non-negative. This is equivalent to showing that the numerator on the right-hand side of bound...
(2.5) is non-negative. For this, we first note that since the function $u$ is non-decreasing and \( \sum_{k \in \mathcal{K}_-} x_k \leq 0 \), we have that
\[
\sum_{k \in \mathcal{K}_+} u(x_k) - u\left(\sum_{k=1}^n x_k\right) = \sum_{k \in \mathcal{K}_+} u(x_k) - u\left(\sum_{k \in \mathcal{K}_-} x_k + \sum_{k \in \mathcal{K}_+} x_k\right) 
\geq \sum_{k \in \mathcal{K}_+} u(x_k) - u\left(\sum_{k \in \mathcal{K}_+} x_k\right). \tag{2.6}
\]
Since the function $u : [0, \infty) \to \mathbb{R}$ is continuous, concave, and $u(0) = 0$, the right-hand side of bound (2.6) is non-negative. Hence, $T_+(\mathbf{x}) \geq 0$.

Consider now the case $\sum_{k=1}^n x_k \leq 0$. Then $v_\lambda(\sum_{k=1}^n x_k) \leq \sum_{k=1}^n v_\lambda(x_k)$ is equivalent to
\[
\lambda \sum_{k \in \mathcal{K}_-} u(-x_k) - \lambda u\left(-\sum_{k=1}^n x_k\right) \leq \sum_{k \in \mathcal{K}_+} u(x_k). \tag{2.7}
\]
To rewrite inequality (2.7) with only $\lambda$ on the left-hand side, we prove that the quantity $\sum_{k \in \mathcal{K}_-} u(-x_k) - u(-\sum_{k=1}^n x_k)$ is non-negative. Since the function $u$ is non-decreasing and $\sum_{k \in \mathcal{K}_+} x_k \geq 0$, we have that
\[
\sum_{k \in \mathcal{K}_-} u(-x_k) - u\left(-\sum_{k=1}^n x_k\right) = \sum_{k \in \mathcal{K}_-} u(-x_k) - u\left(-\sum_{k \in \mathcal{K}_-} x_k - \sum_{k \in \mathcal{K}_+} x_k\right) 
\geq \sum_{k \in \mathcal{K}_-} u(-x_k) - u\left(-\sum_{k \in \mathcal{K}_-} x_k\right). \tag{2.8}
\]
Since the function $u : [0, \infty) \to \mathbb{R}$ is continuous, concave, and $u(0) = 0$, the right-hand side of bound (2.8) is non-negative. Hence, inequality (2.7) is equivalent to
\[
\lambda \leq T_-(\mathbf{x}) \equiv \frac{\sum_{k \in \mathcal{K}_+} u(x_k)}{\sum_{k \in \mathcal{K}_-} u(-x_k) - u\left(-\sum_{k=1}^n x_k\right)}. \tag{2.9}
\]
Given the above, we have that $T_-(\mathbf{x}) \geq 0$. Furthermore, since $\sum_{k=1}^n x_k \leq 0$, we have that $T_-(\mathbf{x}) = T(\mathbf{x})$. This completes the proof of Theorem 2.2. \(\square\)

Since there can only be either complete integration or complete segregation when $n = 2$, the threshold $T(\mathbf{x})$ plays a decisive role in determining optimal strategies in the case of only two exposure units. For this reason, and also for having a convenient reference, we now reformulate Theorem 2.2 in the case $n = 2$. Namely, let $\mathbf{x} = (x_+, x_-)$ with $x_+ > 0$ and
Using the notation \( T(x_+, x_-) \) instead of \( T(x) \), we have the following expression for the threshold:

\[
T(x_+, x_-) = \frac{u(x_+) - u\left( \max\{0, x_- + x_+\} \right)}{u(-x_-) - u\left( \max\{0, -(x_- + x_+)\} \right)}.
\]  

(2.10)

Hence, having only the choices of either \( v_\lambda(x_- + x_+) \) or \( v_\lambda(x_-) + v_\lambda(x_+) \), the value maximizing decision maker prefers \( v_\lambda(x_- + x_+) \) if and only if \( T(x_+, x_-) \leq \lambda \), and prefers \( v_\lambda(x_-) + v_\lambda(x_+) \) if and only if \( T(x_+, x_-) \geq \lambda \).

Next we provide an additional insight into the case \( n = 2 \) with a result on the magnitude of \( T(x) \).

**Theorem 2.3** Let the conditions of Theorem 2.2 be satisfied, and let \( n = 2 \). (Thus, in particular, one among \( x_1 \) and \( x_2 \) is positive and another is negative.) If \( x_1 + x_2 \geq 0 \), then \( T(x) \leq 1 \), and if \( x_1 + x_2 \leq 0 \), then \( T(x) \geq 1 \).

**Proof.** We start with the case \( x_1 + x_2 \geq 0 \). Then \( T(x) \) is equal to \( T_+(x) \), which is defined on the right-hand side of inequality (2.5). Hence, the bound \( T(x) \leq 1 \) is equivalent to

\[
u(x_+) - u(x_- + x_+) \leq u(-x_-).
\]  

(2.11)

With the notation \( y_1 = -x_- \geq 0 \) and \( y_2 = x_- + x_+ \geq 0 \), we rewrite inequality (2.11) as

\[
u(y_1 + y_2) \leq u(y_1) + u(y_2).
\]  

(2.12)

By Theorem 2.1, inequality (2.12) holds because \( u : [0, \infty) \to \mathbb{R} \) is continuous, concave, and \( u(0) = 0 \). This proves that \( T(x) \leq 1 \).

Consider now the case \( x_1 + x_2 \leq 0 \). Then \( T(x) \) is equal to \( T_-(x) \), which is defined on the right-hand side of inequality (2.9). Hence, the bound \( T(x) \geq 1 \) is equivalent to

\[
u(x_+) + u(-x_- - x_+) \geq u(-x_-).
\]  

(2.13)

With the notation \( z_1 = x_+ \geq 0 \) and \( z_2 = -x_- - x_+ \geq 0 \), inequality (2.13) becomes

\[
u(z_1) + u(z_2) \geq u(z_1 + z_2).
\]  

(2.14)

By Theorem 2.1, inequality (2.14) holds, and so we have \( T(x) \geq 1 \). This completes the proof of Theorem 2.3. \( \square \)
3 Case $n = 3$: which ones to integrate, if any?

Complete integration and complete segregation may not result in the maximal value, and thus partial integration or segregation may be desirable. We shall next present a complete solution to this problem in the case of three exposure units, that is, when $n = 3$. Unless explicitly noted otherwise, throughout this section we denote the three experiences by $x$, $y$ and $z$, and assume that

$$x + y + z \geq 0. \quad (3.1)$$

Furthermore, without loss of generality we assume that

$$x \geq y \geq z, \quad (3.2)$$

since every other case can be reduced to (3.2) by simply changing the notation. Finally, we can and thus do assume that

$$x \neq 0, \; y \neq 0, \; z \neq 0, \quad (3.3)$$

because if at least one of the three experiences is zero, then the currently investigated case $n = 3$ reduces to $n = 2$, which has been discussed earlier in this paper and also thoroughly investigated by Egozcue and Wong (2010).

There are five possibilities for integration and segregation in the case of three exposure units:

(A) $v_\lambda(x) + v_\lambda(y) + v_\lambda(z)$

(B) $v_\lambda(x) + v_\lambda(y + z)$

(C) $v_\lambda(y) + v_\lambda(x + z)$

(D) $v_\lambda(z) + v_\lambda(x + y)$

(E) $v_\lambda(x + y + z)$

We want to determine which of these five possibilities, and under what conditions, produces the largest value. We also want to know which, and under what conditions, produces the smallest value. This makes the contents of Theorems 3.1–3.5 below.

In Theorems 3.1–3.5 and their proofs below, we shall use notation such as $(A) \succ (E)$, the latter meaning that $v_\lambda(x) + v_\lambda(y) + v_\lambda(z) \geq v_\lambda(x + y + z)$. That is, $(A) \succ (E)$ is a concise way of saying that the value maximizing decision maker prefers (A) to (E).
Note 3.1 The reason for including minimal values when only the maximal ones seem to be of true interest is based on the fact that finding maximal ones in the case $x + y + z \leq 0$ can be reduced to finding minimal ones under the condition $x + y + z \geq 0$. Indeed, note that $x + y + z \leq 0$ is equivalent to $x^- + y^- + z^- \geq 0$ with the notation $x^- = -x$, $y^- = -y$, and $z^- = -z$. Next, since $\lambda > 0$, equation (2.4) implies that finding the maximal value among (A)–(E) is equivalent to finding the minimal value among the following five ones:

\[ v_{1/\lambda}(x^-) + v_{1/\lambda}(y^-) + v_{1/\lambda}(z^-) \]
\[ v_{1/\lambda}(x^-) + v_{1/\lambda}(y^- + z^-) \]
\[ v_{1/\lambda}(y^-) + v_{1/\lambda}(x^- + z^-) \]
\[ v_{1/\lambda}(z^-) + v_{1/\lambda}(x^- + y^-) \]
\[ v_{1/\lambda}(x^- + y^- + z^-) \]

The minimal values among these five possibilities can easily be derived from Theorems 3.1–3.5 below, where we only need to replace $x$, $y$, and $z$ by $x^-$, $y^-$, and $z^-$, respectively, and also the parameter $\lambda$ by $1/\lambda$. □

Hence, throughout this section we are only concerned with the case $x + y + z \geq 0$, and thus have that at least one of the three exposure units has a non-negative experience. Furthermore, every triplet $x$, $y$, and $z$ falls into one of the following five cases:

\[ x \geq y \geq z \geq 0 \]  
(3.4)
\[ x \geq y \geq 0 \geq z \quad \text{and} \quad y \geq -z \]  
(3.5)
\[ x \geq y \geq 0 \geq z \quad \text{and} \quad x \geq -z \geq y \]  
(3.6)
\[ x \geq y \geq 0 \geq z \quad \text{and} \quad -z \geq x \]  
(3.7)
\[ x \geq 0 \geq y \geq z \]  
(3.8)

**Theorem 3.1** Let the value function be $v_\lambda$, and let $x \geq y \geq z \geq 0$. Then we have the following two statements:

Max: (A) gives the maximal value among (A)–(E).

Min: (E) gives the minimal value among (A)–(E).
Proof. Since the three exposure units have non-negative experiences \(x, y,\) and \(z,\) Theorem 2.1 implies that complete segregation maximizes the value. Hence, \((A)\) attains the maximal value among \((A)–(E)\). Same theorem also implies that complete integration, which is \((E)\), attains the minimal value. □

The following four theorems dealing with cases (3.5)–(3.8) will be considerably more complicated than Theorem 3.1. In their proofs we shall use the following special case of the Hardy-Littlewood-Pólya (HLP) majorization principle (e.g., Kuczma, 2009, p. 211). Namely, given two vectors \((x_1, x_2)\) and \((y_1, y_2)\), and also a continuous and concave function \(v\), we have the implication:

\[
\begin{align*}
    x_1 \geq x_2, y_1 \geq y_2
    \quad \Rightarrow \\
    x_1 + x_2 = y_1 + y_2
    \quad \Rightarrow \\
    x_1 \leq y_1
\end{align*}
\]  

\[
\left\{ \begin{array}{c}
x_1 \geq x_2, y_1 \geq y_2 \\
x_1 + x_2 = y_1 + y_2 \\
x_1 \leq y_1
\end{array} \right\} \Rightarrow v(x_1) + v(x_2) \geq v(y_1) + v(y_2).
\] (3.9)

Now we are ready to formulate and prove our next theorem.

**Theorem 3.2** Let the value function be \(v_\lambda\), and let \(x \geq y \geq 0 \geq z\) with \(y \geq -z\).

Max: With the threshold \(T_{AC} = T(x, z)\), the following statements specify the two possible maximal values among \((A)–(E)\):

- When \(T_{AC} \geq \lambda\), then \((A)\).
- When \(T_{AC} \leq \lambda\), then \((C)\).

Min: With the threshold \(T_{DE} = T(x + y, z)\), the following statements specify the two possible minimal values among \((A)–(E)\):

- When \(T_{DE} \geq \lambda\), then \((E)\).
- When \(T_{DE} \leq \lambda\), then \((D)\).

Proof. Since \(x\) and \(y\) are non-negative, from Theorem 2.1 we have that \((A) \succeq (D)\), and since \(x\) and \(y + z\) are non-negative, the same theorem implies that \((B) \succeq (E)\). The proof of \((C) \succeq (B)\) is more complex. Note that \((C) \succeq (B)\) is equivalent to

\[
v_\lambda(y) + v_\lambda(x + z) \geq v_\lambda(x) + v_\lambda(y + z),
\] (3.10)

which we establish as follows:

- When \(x + z \geq y\), then we apply the HLP principle on the vectors \((x + z, y)\) and \((x, y + z)\) and get \(v_\lambda(x + z) + v_\lambda(y) \geq v_\lambda(x) + v_\lambda(y + z)\), which is (3.10).
• When $x + z \leq y$, then we apply the HLP principle on the vectors $(y, x + z)$ and $(x, y + z)$, and get $v_\lambda(y) + v_\lambda(x + z) \geq v_\lambda(x) + v_\lambda(y + z)$, which is (3.10).

This completes the proof of inequality (3.10). Hence, in order to establish the ‘max’ part of Theorem 3.2, we need to determine whether (A) or (C) is maximal, and for the ‘min’ part, we need to determine whether (D) or (E) is minimal.

The ‘max’ part. Since $x \geq 0$ and $z \leq 0$, whether (A) or (C) is maximal is determined by the threshold $T_{AC}$: when $T_{AC} \leq \lambda$, then $(C) \succ (A)$, and when $T_{AC} \geq \lambda$, then $(A) \succ (C)$. This concludes the proof of the ‘max’ part.

The ‘min’ part. Since $x + y \geq 0$ and $z \leq 0$, the threshold $T_{DE} = T(x + y, z)$ plays a decisive role: if $T_{DE} \leq \lambda$, then $(E) \succ (D)$, and if $T_{DE} \geq \lambda$, then $(D) \succ (E)$. This concludes the proof of the ‘min’ part and of Theorem 3.2 as well. □

**Theorem 3.3** Let the value function be $v_\lambda$, and let $x \geq y \geq 0 \geq z$ with $x \geq -z \geq y$.

Max: With the threshold $T_{AC} = T(x, z)$, the following statements specify the two possible maximal values among (A)–(E):

- When $T_{AC} \geq \lambda$, then (A).
- When $T_{AC} \leq \lambda$, then (C).

Min: With the thresholds $T_{BE} = T(x, y + z)$, $T_{DE} = T(x + y, z)$, and

$$
T_{BD} = \frac{u(x + y) - u(x)}{u(-z) - u(-y - z)},
$$

the following statements specify the three possible minimal values among (A)–(E):

- When $T_{BE} \leq \lambda$ and $T_{BD} \geq \lambda$, then (B).
- When $T_{DE} \leq \lambda$ and $T_{BD} \leq \lambda$, then (D).
- When $T_{BE} \geq \lambda$ and $T_{DE} \geq \lambda$, then (E).

**Proof.** Since $x$ and $y$ are non-negative, we have $(A) \succ (D)$, and since $y$ and $x + z$ are non-negative, we have $(C) \succ (E)$. Hence, only (A), (B), and (C) remain to consider for the ‘max’ part of the theorem, and (B), (D), and (E) for the ‘min’ part.
The ‘max’ part. First we show that $T_{AC} \leq T_{AB}$. Since $x + z \geq 0$, from Theorem 2.3 we have $T_{AC} \leq 1$, and since $y + z \leq 0$, the same theorem implies $T_{AB} \geq 1$. Hence, $T_{AC} \leq T_{AB}$.

To establish that (A) is maximal when $T_{AC} \geq \lambda$, we check that (A) $\succ$ (B) and (A) $\succ$ (C). The former statement holds when $T_{AB} = T(y, z) \geq \lambda$, and the latter when $T_{AC} = T(x, z) \geq \lambda$. But we already know that $T_{AC} \leq T_{AB}$. Therefore, when $T_{AC} \geq \lambda$, then $T_{AB} \geq \lambda$. This proves that when $T_{AC} \geq \lambda$, then (A) gives the maximal value among (A), (B), (C), and thus, in turn, among all (A)–(E).

To establish that (C) is the maximal when $T_{AC} \leq \lambda$, we need to check that (C) $\succ$ (A) and (C) $\succ$ (B). First we note that when $T_{AC} \leq \lambda$, then (C) $\succ$ (A). Furthermore,

$$v_\lambda(x) + v_\lambda(y + z) \leq v_\lambda(y) + v_\lambda(x + z) \iff u(x) - \lambda u(-y - z) \leq u(y) + u(x + z) \iff T_{BC} \leq \lambda,$$

where $T_{BC}$ is defined by the equation

$$T_{BC} = \frac{u(x) - u(x + z) - u(y)}{u(-y - z)}.$$

Hence, when $T_{BC} \leq \lambda$, then (C) $\succ$ (B). Simple algebra shows that the bound $T_{BC} \leq T_{AB}$ is equivalent to $T_{AC} \leq T_{AB}$, and we already know that the latter holds. Hence, $T_{BC} \leq T_{AB}$ and so $T_{BC} \leq \lambda$ when $T_{AC} \leq \lambda$. In summary, when $T_{AC} \leq \lambda$, then (C) gives the maximal value among all cases (A)–(E). This concludes the proof of the ‘max’ part.

The ‘min’ part. We first establish conditions under which (B) is minimal. We have (E) $\succ$ (B) when $T_{BE} \leq \lambda$. To have (D) $\succ$ (B), we need to employ the threshold $T_{BD}$, which is defined in the formulation of the theorem. The role of the threshold is seen from the following equivalence relations:

$$v_\lambda(x) + v_\lambda(y + z) \leq v_\lambda(z) + v_\lambda(x + y) \iff u(x) - \lambda u(-y - z) \leq -\lambda u(-z) + u(x + y) \iff \lambda \leq T_{BD}.$$

Hence, if $T_{BD} \geq \lambda$, then (D) $\succ$ (B). In summary, when $T_{BE} \leq \lambda$ and $T_{BD} \geq \lambda$, then (B) gives the minimal value among all (A)–(E).

We next establish conditions under which (D) is minimal. First, we have (E) $\succ$ (D) when $T_{DE} \leq \lambda$. Next, we have (B) $\succ$ (D) when $T_{BD} \leq \lambda$. In summary, when $T_{DE} \leq \lambda$ and $T_{BD} \leq \lambda$, then (D) gives the minimal value among all (A)–(E).

Finally, we have (B) $\succ$ (E) when $T_{BE} \geq \lambda$, and (D) $\succ$ (E) when $T_{DE} \geq \lambda$. Hence, when $T_{BE} \geq \lambda$ and $T_{DE} \geq \lambda$, then (E) is minimal among all (A)–(E). This finishes the proof of the ‘min’ part, and thus of Theorem 3.3 as well. □
Theorem 3.4 Let the value function be \( v_\lambda \), and let \( x \geq y \geq 0 \geq z \) with \(-z \geq x\).

Max: With the threshold
\[ T_{AE} = \frac{u(x) + u(y) - u(x + y + z)}{u(-z)}, \]
the following statements specify the two possible maximal values among (A)–(E):

- When \( T_{AE} \geq \lambda \), then (A).
- When \( T_{AE} \leq \lambda \), then (E).

Min: With the thresholds \( T_{AC} = T(x, z) \), \( T_{BE} = T(x, y + z) \), \( T_{CE} = T(y, x + z) \), \( T_{DE} = T(x + y, z) \), and

\[
T_{BC} = \frac{u(x) - u(y)}{u(-y - z) - u(-x - z)},
\]
\[
T_{BD} = \frac{u(x + y) - u(x)}{u(-z) - u(-y - z)},
\]
\[
T_{CD} = \frac{u(x + y) - u(y)}{u(-z) - u(-x - z)},
\]
the following statements specify the four possible minimal values among (A)–(E):

- When \( T_{BE} \leq \lambda \), \( T_{BC} \leq \lambda \), and \( T_{BD} \geq \lambda \), then (B).
- When \( T_{CE} \leq \lambda \), \( T_{BC} \geq \lambda \), and \( T_{CD} \geq \lambda \), then (C).
- When \( T_{DE} \leq \lambda \), \( T_{BD} \leq \lambda \), and \( T_{CD} \leq \lambda \), then (D).
- When \( T_{BE} \geq \lambda \), \( T_{CE} \geq \lambda \), and \( T_{DE} \geq \lambda \), then (E).

Proof. Since both \( x \) and \( y \) are non-negative, we have \( (A) \succ (D) \). This eliminates (D) from the ‘max’ part of Theorem 3.4 and (A) from the ‘min’ part.

The ‘max’ part. We first eliminate (B). When \( T_{BE} \leq \lambda \), then \( (E) \succ (B) \). If, however, \( T_{BE} \geq \lambda \), then by Theorem 2.2 we have \( (B) \succ (E) \). We shall next show that in this case we also have \( (A) \succ (B) \), thus making (B) unattractive to the value maximizing decision maker. Since \( y + z \leq 0 \) and \( x + y + z \geq 0 \), we have from Theorem 2.3 that \( T_{BE} \leq 1 \). Theorem 2.3 also implies that \( T_{AC} \geq 1 \) because \( x + z \leq 0 \). Hence, \( T_{BE} \leq T_{AB} \). Since \( T_{BE} \geq \lambda \), we conclude that \( T_{AB} \geq \lambda \). By Theorem 2.2, the latter bound implies \( (A) \succ (B) \). Therefore, the value maximizing decision maker will not choose (B). Analogous arguments but with \( T_{CE} \) and \( T_{AC} \) instead of \( T_{BE} \) and \( T_{AB} \), respectively, show that the value maximizing decision
maker will not choose (C) either. Hence, in summary, we are left with only two cases: (A) and (E). Which of the two maximizes the value is determined by the equivalence relations:

\[ v_\lambda(x) + v_\lambda(y) + v_\lambda(z) \leq v_\lambda(x + y + z) \iff u(x) + u(y) - \lambda u(-z) \leq u(x + y + z) \iff T_{AE} \leq \lambda. \]

This concludes the proof of the ‘max’ part.

The ‘min’ part. To prove the ‘min’ part, we only need to deal with (B)–(E), because we already know that (A) \( \succ (D) \). Case (E) gives the minimal value when \( T_{BE} \geq \lambda, T_{CE} \geq \lambda, \) and \( T_{DE} \geq \lambda \). If, however, there is at least one among \( T_{BE}, T_{CE}, \) and \( T_{DE} \) not exceeding \( \lambda \), then the minimum is achieved by one of (B), (C), and (D). To determine which of them and when is minimal, we employ simple algebra and obtain the equivalence relationships:

\[
\begin{bmatrix}
(C) \succ (B) \iff T_{BC} \leq \lambda \\
(D) \succ (B) \iff T_{BD} \geq \lambda \\
(E) \succ (B) \iff T_{BE} \leq \lambda \\
\end{bmatrix}, \quad \begin{bmatrix}
(B) \succ (C) \iff T_{BC} \geq \lambda \\
(D) \succ (C) \iff T_{CD} \geq \lambda \\
(E) \succ (C) \iff T_{CE} \leq \lambda \\
\end{bmatrix}, \quad \begin{bmatrix}
(B) \succ (D) \iff T_{BD} \leq \lambda \\
(C) \succ (D) \iff T_{CD} \leq \lambda \\
(E) \succ (D) \iff T_{DE} \leq \lambda \\
\end{bmatrix}.
\]

This finishes the proof of Theorem 3.4. \( \square \)

**Theorem 3.5** Let the value function be \( v_\lambda \), and let \( x \geq 0 \geq y \geq z \).

Max: With the thresholds \( T_{BE} = T(x, y + z), T_{CE} = T(x + z, y), T_{DE} = T(x + y, z) \), and

\[
T_{BC} = \frac{u(x) - u(x + z)}{u(-y - z) - u(-y)},
\]
\[
T_{BD} = \frac{u(x) - u(x + y)}{u(-y - z) - u(-z)},
\]
\[
T_{CD} = \frac{u(x + y) - u(x + z)}{u(-z) - u(-y)},
\]

the following statements specify the four possible maximal values among (A)–(E):

- When \( T_{BE} \geq \lambda, T_{BC} \geq \lambda, \) and \( T_{BD} \geq \lambda \), then (B).
- When \( T_{CE} \geq \lambda, T_{BC} \leq \lambda, \) and \( T_{CD} \leq \lambda \), then (C).
- When \( T_{DE} \geq \lambda, T_{BD} \leq \lambda, \) and \( T_{CD} \geq \lambda \), then (D).
- When \( T_{BE} \leq \lambda, T_{CE} \leq \lambda, \) and \( T_{DE} \leq \lambda \), then (E).
Min: With the thresholds $T_{AC} = T(x, z)$, $T_{AD} = T(x, y)$,

$$T_{AE} = \frac{u(x) - u(x + y + z)}{u(-y) + u(-z)},$$

and the other ones defined in the 'max' part of this theorem, the following statements specify the four possible minimal values among (A)–(E):

- When $T_{AC} \leq \lambda$, $T_{AD} \leq \lambda$, and $T_{AE} \leq \lambda$, then (A).
- When $T_{AC} \geq \lambda$, $T_{CD} \geq \lambda$, and $T_{CE} \leq \lambda$, then (C).
- When $T_{AD} \geq \lambda$, $T_{CD} \leq \lambda$, and $T_{DE} \leq \lambda$, then (D).
- When $T_{AE} \geq \lambda$, $T_{CE} \geq \lambda$, and $T_{DE} \geq \lambda$, then (E).

Proof. Since $-y \geq 0$ and $-z \geq 0$, we have from inequality (2.3) that $u(-y) + u(-z) \geq u(-(y + z))$ and thus $-\lambda u(-y) - \lambda u(-z) \leq -\lambda u(-(y + z))$. The latter is equivalent to $v_{\lambda}(y) + v_{\lambda}(z) \leq v_{\lambda}(y + z)$, which means that $(B) \succ (A)$.

The 'max' part. We have four cases (B)–(E) to deal with. To determine which of them and when is maximal among (B)–(E), we employ simple algebra and obtain the equivalence relationships:

$$
\begin{bmatrix}
(B) \succ (C) & \iff & T_{BC} \geq \lambda \\
(B) \succ (D) & \iff & T_{BD} \geq \lambda \\
(B) \succ (E) & \iff & T_{BE} \geq \lambda \\
\end{bmatrix},
\begin{bmatrix}
(C) \succ (B) & \iff & T_{BC} \leq \lambda \\
(C) \succ (D) & \iff & T_{CD} \leq \lambda \\
(C) \succ (E) & \iff & T_{CE} \leq \lambda \\
\end{bmatrix},
\begin{bmatrix}
(D) \succ (B) & \iff & T_{BD} \leq \lambda \\
(D) \succ (C) & \iff & T_{CD} \geq \lambda \\
(D) \succ (E) & \iff & T_{DE} \geq \lambda \\
\end{bmatrix}.
$$

This finishes the proof of the 'max' part.

The 'min' part. To prove the 'min' part of the theorem, we verify the following four sets of orderings:

$$
\begin{array}{c}
\begin{bmatrix}
(C) \succ (A) & \iff & T_{AC} \leq \lambda \\
(D) \succ (A) & \iff & T_{AD} \leq \lambda \\
(E) \succ (A) & \iff & T_{AE} \leq \lambda \\
\end{bmatrix},
\begin{bmatrix}
(A) \succ (C) & \iff & T_{AC} \geq \lambda \\
(D) \succ (C) & \iff & T_{CD} \geq \lambda \\
(E) \succ (C) & \iff & T_{CE} \leq \lambda \\
\end{bmatrix},
\end{array}
\begin{array}{c}
\begin{bmatrix}
(A) \succ (D) & \iff & T_{AD} \geq \lambda \\
(C) \succ (D) & \iff & T_{CD} \leq \lambda \\
(E) \succ (D) & \iff & T_{DE} \leq \lambda \\
\end{bmatrix},
\begin{bmatrix}
(A) \succ (E) & \iff & T_{AE} \geq \lambda \\
(C) \succ (E) & \iff & T_{CE} \geq \lambda \\
(D) \succ (E) & \iff & T_{DE} \geq \lambda \\
\end{bmatrix}.
\end{array}
$$

This concludes the proof of the 'min' part and of Theorem 3.5 as well. □

The following proposition is a simplification of Theorem 3.5 that we have achieved in the case of the value function $v_{\lambda, \gamma}$.
Proposition 3.1 Let the value function be \( v_{\lambda, \gamma} \) with \( \gamma \in (0, 1) \), and let \( x \geq 0 \geq y \geq z \) with \( x + y + z > 0 \). Furthermore, let the thresholds be same as in Theorem 3.5 but now with the function \( u(x) = x^\gamma \).

Max: The following statements specify the two possible maximal values among (A)–(E):

- When \( T_{BE} \geq \lambda, T_{BC} \geq \lambda, \) and \( T_{BD} \geq \lambda \), then (B).
- When \( T_{BE} \leq \lambda, T_{CE} \leq \lambda, \) and \( T_{DE} \leq \lambda \), then (E).

Min: The following statements specify the four possible minimal values among (A)–(E):

- When \( T_{AC} \leq \lambda, T_{AD} \leq \lambda, \) and \( T_{AE} \leq \lambda \), then (A).
- When \( T_{AC} \geq \lambda, T_{CD} \geq \lambda, \) and \( T_{CE} \leq \lambda \), then (C).
- When \( T_{AE} \geq \lambda, T_{CE} \geq \lambda, \) and \( T_{DE} \geq \lambda \), then (E).

Proof. To prove the ‘max’ part of the proposition, we need to show that the following two cases are impossible to realize:

- When \( T_{CE} \geq \lambda, T_{BC} \leq \lambda, \) and \( T_{CD} \leq \lambda \), then (C).
- When \( T_{DE} \geq \lambda, T_{BD} \leq \lambda, \) and \( T_{CD} \geq \lambda \), then (D).

We shall accomplish this task by establishing the inequalities:

\[
T_{CE} < T_{BC}, \quad \tag{3.11}
\]
\[
T_{DE} < T_{BD}. \quad \tag{3.12}
\]

Indeed, when (3.11) holds, then the two conditions \( T_{CE} \geq \lambda \) and \( T_{BC} \leq \lambda \) cannot be fulfilled simultaneously, and when (3.12) holds, then the two conditions \( T_{DE} \geq \lambda \) and \( T_{BD} \leq \lambda \) cannot be fulfilled simultaneously.

To prove inequality (3.11), we employ simple algebra and obtain that the inequality is equivalent to

\[
T(x + z, y) < T(x, y + z), \quad \tag{3.13}
\]

which can be rewritten as \( T_{CE} < T_{BE} \). Since \( u(x) = x^\gamma \), bound (3.13) is equivalent to

\[
\left( \frac{x + z}{-y} \right)^\gamma - \left( \frac{x + z}{-y} - 1 \right)^\gamma < \left( \frac{x}{-y - z} \right)^\gamma - \left( \frac{x}{-y - z} - 1 \right)^\gamma.
\]
To prove that the latter holds, we observe that $v^\alpha - (v - 1)^\alpha$ is a strictly decreasing function in $v$ whenever $\gamma \in (0, 1)$, and that the bounds
\[
\frac{x + z}{-y} > \frac{x}{-y - z} > 1
\]
hold because $x + y + z > 0$. This proves inequality (3.11).

The proof of inequality (3.12) is analogous. First, simple algebra shows that the inequality is equivalent to
\[
T(x + y, z) < T(x, y + z),
\]
which is $T_{DE} < T_{BE}$. Bound (3.14) is equivalent to
\[
\left(\frac{x + y}{-z}\right)^\gamma - \left(\frac{x + y - 1}{-z}\right)^\gamma < \left(\frac{x}{-y - z}\right)^\gamma - \left(\frac{x}{-y - z - 1}\right)^\gamma.
\]
Since $v^\alpha - (v - 1)^\alpha$ is strictly decreasing in $v$, to complete the proof of inequality (3.12), we only note that
\[
\frac{x + y}{-z} > \frac{x}{-y - z} > 1.
\]
The ‘max’ part of Proposition 3.1 is finished.

To prove the ‘min’ part of the proposition, we shall show that the following case is impossible:

- When $T_{AD} \geq \lambda$, $T_{CD} \leq \lambda$, and $T_{DE} \leq \lambda$, then (D).

For this, we establish the inequality
\[
T_{AD} < T_{DE},
\]
which is equivalent to
\[
\left(\frac{x}{-y}\right)^\gamma - \left(\frac{x}{-y - 1}\right)^\gamma < \left(\frac{x + y}{-z}\right)^\gamma - \left(\frac{x + y}{-z - 1}\right)^\gamma.
\]
It now remains to observe that
\[
\frac{x}{-y} > \frac{x + y}{-z} > 1.
\]
This finishes the proof of the ‘min’ part, and thus of Proposition 3.1 as well. □

We conclude this section with a corollary to Theorem 3.5 based on a different value function than any one noted so far. It offers a somewhat ‘pathological’ result helping us to understand difficulties associated with our general results above.
Corollary 3.1 Let the value function be $v_{\lambda}$ with $u(x) = 1 - \exp\{-\varrho x\}$ and parameter $\varrho > 0$, and let $x \geq 0 \geq y \geq z$. Then all the thresholds noted in the ‘max’ part of Theorem 3.5 are equal to $T_{\max} = \exp\{-\varrho(x+y+z)\}$. Hence, the following statements specify the four possible maximal values among (A)–(E): when $T_{\max} \geq \lambda$, then (B), when $T_{\max} = \lambda$, then (C) and (D), and when $T_{\max} \leq \lambda$, then (E). Moreover, when $T_{\max} = \lambda$, then all cases (B)–(E) produce the same value $1 - \exp\{-\varrho(x+y+z)\}$.

4 Illustrations

Numerical examples that we shall present in the next subsection are hypothetical and designed to illustrate our earlier theoretical results, especially their optimality. Though we have not made a special effort to put the examples into a real life context, one can easily do so, say within the context of marketing (e.g., Drumwright, 1992; Heath et al., 1995; and references therein). Indeed, following Drumwright (1992), let $R$ denote the reservation price of a product, which is the largest price that the consumer is willing to pay in order to acquire the product. Furthermore, let $M$ be the market price of the product. The consumer buys the product if the consumer surplus is non-negative: $R - M \geq 0$.

Assume that a company is manufacturing two products, $A$ and $B$. Let their reservation prices be $R_A = 21$ and $R_B = 10$, and the market prices $M_A = 15$ and $M_B = 15$. The economic theory would predict that the consumer buys only the product $A$, because the consumer surplus is positive only for this product. However, bundling can make the consumer also buy the product $B$, thus increasing the company’s revenue. Namely, suppose that a bundle of the two products $A$ and $B$ sells at a price of 30. Then, according to mental accounting, the consumer will buy the bundle. To demonstrate this rigorously, assume that the consumer is loss averse in the sense that $\lambda \geq 1$.

Using our adopted terminology, the two experiences (consumer surpluses) corresponding to $A$ and $B$ are $x = R_A - M_A = 6$ and $y = R_B - M_B = -5$, respectively. The total experience is positive: $x + y = 1$. The value of the individually purchased products is $v_{\lambda}(6) + v_{\lambda}(-5)$. If they are bundled, then the consumer surplus is $(R_A + R_B) - (M_A + M_B) = 31 - 30 = 1$, and the value is $v_{\lambda}(1)$. Theorems 2.2 and 2.3 imply $v_{\lambda}(1) \geq v_{\lambda}(6) + v_{\lambda}(-5)$ because $\lambda \geq 1$ and $T(x) \equiv T(6, -5) \leq 1$, thus implying that $T(x) \leq \lambda$, which means ‘integration’ for the value maximizing decision maker. Hence, the company is better off when the two products are bundled: the revenue is 30 by selling both products as a bundle, whereas the revenue is just 15 when the two products are sold separately, because in the latter case the consumer...
buys only the product A.

In the case of more than two products, the bundling strategy becomes much more complex, with five possibilities (A)–(E) in the case $n = 3$ as we have seen above. We shall next illustrate our earlier developed theory with numerical examples.

### 4.1 Examples

In the examples of this subsection we use the $S$-shaped value function (al-Nowaihi et al., 2008)

$$v_{\lambda, \gamma}(x) = \begin{cases} 
  x^\gamma & \text{when } x \geq 0, \\
  -\lambda(-x)^\gamma & \text{when } x < 0.
\end{cases} \quad (4.1)$$

Obviously, $v_{\lambda, \gamma} = v_\lambda$ with $u(x) = x^\gamma$.

The following two numerical examples illustrate the validity of principles P1 and P2.

**Example 4.1** (cf. principle P1) Let the value function be $v_{\lambda, \gamma}$ with the parameters $\lambda = 2.25$ and $\gamma = 0.88$. Suppose that we have three exposure units with positive experiences 5, 10, and 20. Principle P1 suggests segregating them, and this is mathematically confirmed by the inequality: $v_{\lambda, \gamma}(\sum x_k) = 22.8444 < \sum v_{\lambda, \gamma}(x_k) = 25.6683$. (We use $\sum$ instead of $\sum_{k=1}^3$ to simplify notation.) Our general results say that the value maximizing decision maker prefers segregating any number of positive exposures. □

**Example 4.2** (cf. principle P2) Let the value function be $v_{\lambda, \gamma}$ with the parameters $\lambda = 2.25$ and $\gamma = 0.88$. Suppose that we have three exposure units with negative experiences $-5$, $-10$, and $-20$. Principle P2 suggests integrating them, and this is confirmed by the inequality: $v_{\lambda, \gamma}(\sum x_k) = -51.3999 > \sum v_{\lambda, \gamma}(x_k) = -57.7537$. Our general results say that the value maximizing decision maker prefers integrating any number of negative exposures. □

The following two examples show that principles P3 and P4 can be violated.

**Example 4.3** (cf. principle P3) Let the value function be $v_{\lambda, \gamma}$ with the parameters $\lambda = 2.25$ and $\gamma = 0.88$. Suppose that we have three exposure units with mixed experiences $-0.5$, 10, and 20, whose total (positive) experience is $\sum x_k = 29.5$. Principle P3 would suggest integrating the exposure units into one, but the following inequality implies the opposite: $v_{\lambda, \gamma}(\sum x_k) = 19.6537 < \sum v_{\lambda, \gamma}(x_k) = 20.3239$. In fact, we see from our theoretical analysis of the case $n = 3$ that neither complete segregation nor complete integration of three (or more) experiences with mixed exposures may lead to a maximal value, which may be achieved only by a partial integration and segregation. □
Example 4.4 (cf. principle P4) Let the value function be $v_{\lambda, \gamma}$ with the parameters $\lambda = 2.25$ and $\gamma = 0.88$. Suppose that we have three exposure units with mixed experiences 0.5, $-10$, and $-20$, whose total (negative) experience is $\sum x_k = -29.5$. Principle P4 suggests segregating the exposure units, but the following inequality says the opposite: $v_{\lambda, \gamma}(\sum x_k) = -44.2207 > \sum v_{\lambda, \gamma}(x_k) = -47.9361$. Our theory developed above says that neither complete segregation nor complete integration may lead to a maximal value when $n \geq 3$.

The following two examples illustrate Theorem 2.2 in the case of three exposure units and assuming that the decision maker is given only two options: either integrate all exposure units or keep them segregated.

Example 4.5 (cf. Theorem 2.2) Let $x_1 = 25$, $x_2 = 10$, and $x_3 = -0.5$, with the positive total sum $x_1 + x_2 + x_3 = 34.5$. Let the value function be $v_{\lambda, \gamma}$ with $\gamma = 0.88$. The threshold is $T(x) = 3.7149$. Thus, facing the dilemma of integrating or segregating all exposure units, the decision maker prefers segregating them when $\lambda \leq 3.7149$ and integrating them when $\lambda \geq 3.7149$. An additional illustration is provided in Table 4.1.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$v_{\lambda, \gamma}(\sum x_k)$</th>
<th>$\sum v_{\lambda, \gamma}(x_k)$</th>
<th>Preference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>22.557</td>
<td>24.3039</td>
<td>Segregate</td>
</tr>
<tr>
<td>1.50</td>
<td>22.557</td>
<td>23.7605</td>
<td>Segregate</td>
</tr>
<tr>
<td>4.00</td>
<td>22.557</td>
<td>22.4021</td>
<td>Integrate</td>
</tr>
</tbody>
</table>

Table 4.1: Total integration/segregation decisions with $T(x) = 3.7149$.

Example 4.6 (cf. Theorem 2.2) Let $x_1 = -6$, $x_2 = -5$, and $x_3 = 10$, with the negative total sum $x_1 + x_2 + x_3 = -1$. Let the value function be $v_{\lambda, \gamma}$ with $\gamma = 0.1$. The threshold is $T(x) = 0.9529$, and thus, facing the dilemma of integrating or segregating all exposure units, the decision maker prefers segregating when $\lambda \leq 0.9529$ and integrating when $\lambda \geq 0.9529$. An additional illustration is provided in Table 4.2.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$v_{\lambda, \gamma}(\sum x_k)$</th>
<th>$\sum v_{\lambda, \gamma}(x_k)$</th>
<th>Preference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>-0.50</td>
<td>3.1056</td>
<td>Segregate</td>
</tr>
<tr>
<td>0.98</td>
<td>-0.98</td>
<td>-1.1961</td>
<td>Integrate</td>
</tr>
<tr>
<td>1.50</td>
<td>-1.50</td>
<td>-5.8558</td>
<td>Integrate</td>
</tr>
</tbody>
</table>

Table 4.2: Total integration/segregation decisions with $T(x) = 0.9529$. 
Example 4.7 (cf. Theorem 3.2) Assume that the value function is $v_{\lambda, \gamma}$ with $\gamma = 0.88$. With the experiences $x = 5$, $y = 3$ and $z = -2$, we have that $T_{AC} = 0.8109$ and $T_{ED} = 0.7575$. Hence, we have the following statements:

Max: When $\lambda \leq 0.8109$, then (A) is maximal, and when $0.8109 \leq \lambda$, then (C) is maximal.

Min: When $\lambda \leq 0.7575$, then (E) is minimal, and when $0.7575 \leq \lambda$, then (D) is minimal.

Example 4.8 (cf. Theorem 3.3) Assume that the value function is $v_{\lambda, \gamma}$ with $\gamma = 0.88$. With the experiences $x = 10$, $y = 1$ and $z = -2$, we have that $T_{AC} = 0.7349$, $T_{BD} = 0.7897$, and $T_{BE} = 0.6717$. Hence, we have the following statements:

Max: When $\lambda \leq 0.7349$, then (A) is maximal, and when $0.7349 \leq \lambda$, then (C) is maximal.

Min: When $\lambda \leq 0.6717$, then (E) is minimal, when $0.6717 \leq \lambda \leq 0.7897$, then (B) is minimal, and when $0.7897 \leq \lambda$, then (D) is minimal.

Example 4.9 (cf. Theorem 3.4) Assume that the value function is $v_{\lambda, \gamma}$ with $\gamma = 0.88$. With the experiences $x = 4$, $y = 3$ and $z = -5$, we have that $T_{AE} = 0.8404$, $T_{BD} = 0.9446$, $T_{CE} = 0.7890$, and $T_{CB} = 0.9014$. Hence, the following statements:

Max: When $\lambda \leq 0.8404$, then (A) is maximal, and when $0.8404 \leq \lambda$, then (E) is maximal.

Min: When $\lambda \leq 0.7890$, then (E) is minimal, when $0.7890 \leq \lambda \leq 0.9014$, then (C) is minimal, when $0.9014 \leq \lambda \leq 0.9446$, then (B) is minimal, and finally when $0.9446 \leq \lambda$, then (D) is minimal.

Example 4.10 (cf. Proposition 3.1) Assume that the value function is $v_{\lambda, \gamma}$ with $\gamma = 0.88$. With the experiences $x = 36$, $y = -2$ and $z = -14$, we have that $T_{BE} = 0.8243$, $T_{AC} = 0.8074$, and $T_{CE} = 0.6636$. Hence, we have the following statements:

Max: When $\lambda \leq 0.8243$, then $B$ is maximal, and when $0.8243 \leq \lambda$, then $E$ is maximal.

Min: When $\lambda \leq 0.6636$, then $E$ is minimal, when $0.6636 \leq \lambda \leq 0.8074$, then $C$ is minimal, and when $0.8074 \leq \lambda$, then $A$ is minimal.

4.2 An experiment

We have proved mathematically that mental accounting of multiple outcomes for investors with extended value functions is not always verified. But in order to give credit to this fact we need to test it empirically by running some experiments.
4.2.1 Empirical evidence

Some studies test empirically the hedonic editing hypothesis for two outcomes. In his paper Thaler (1985) evaluates the happiness of subjects face to different scenarios meaning his four possible types of outcomes. While this experiment supports the hedonic editing models, two experiments based on temporal spacing of outcomes (Thaler and Johnson, 1990; Linville and Fischer, 1991) reject integration of losses and develop some variations of the initial model (quasi-hedonic editing model and renewable resources model). There are several works which focus on different aspects of mental accounting, like frame sensitivity (Heath et al., 1995) and hedonic limen (Morewedge et al., 2006), or on different applications of this hypothesis, like investors decisions (Lim, 2006) and tax returns (Moreno et al., 2006). Furthermore two recent studies (Wu and Markle, 2008; Jarnebrant et al., 2009) test the Thaler’s principle in the case of mix gambles. All these studies focus on the case of two outcomes (even if they may sometimes deal with multiple outcomes but without explicit references to this point) and to the best of our knowledge there is any experiment trying to test explicitly the mental accounting principles for more than two outcomes.

4.2.2 Experimental design

In the literature we can find two ways to treat mental accounting for sure outcomes. The first one in the line of Thaler (1985) proposes to evaluate the happiness of two fictitious situations in which the outcomes are divided on two events (segregation) or gathered on one single event (integration). The second one, due to Thaler and Johnson (1990) and Linville and Fisher (1991), is based on timing separation of events: subjects are asked if they prefer an outcome on one single moment (integration) or on different moments (segregation). In this experiment we use the second approach and so assume that we can test the hedonic editing hypothesis by eliciting preferences of subjects over the timing of events.

In our experiments we will only use hypothetical choices. There are several reasons supporting this choice. The main problem is link to the possibility of losses in our design. It will be hard to implement an experiment during which the subjects have to suffer some monetary losses. Furthermore almost all the paper in the field deal with hypothetical choices (even in the case of gambles) and the main paper which introduces monetary incentives (Thaler and Johnson, 1990) finds no significant differences between the treatments with hypothetical choices and with monetary choices. We can also find some papers arguing that there is no major quantitative difference between hypothetical choices and real monetary choices (Camerer and Hogarth, 1999) and then the question of monetary incentives should
not be the main aspect of the question (Read, 2005).

The experimental design is based on the following task:\footnote{see Appendix for a detail of the instructions given to subjects}: subjects have to fill a Likert scale of preferences (from -2 to +2) facing two possible situations with timing separation of sure events. We use different patterns of choices: gain, loss, mixed gain, mixed loss, and tie. We vary also the number of outcomes with 3 and 5 outcomes’ questions. Overall subjects have to answer 14 questions.

### 4.2.3 Results

These experiments were done on November 2010 with 55 undergraduate students in economics of the University of Paris 1 and 115 undergraduate students of the Hong Kong Baptist University. We first pool all the data to test if the hedonic hypothesis is robust to multiple outcomes. Then we check if the number of outcomes has an impact on our results (test results for 3 outcomes and 5 outcomes). Finally, we explore inter-cultural difference by comparing the results from Paris, Changchun and Hong Kong.

The Table 4.3 presents the main results for pooling data:

<table>
<thead>
<tr>
<th>Event type</th>
<th>Consistent according P1-P4</th>
<th>Mean Preferences</th>
<th>Actual (Prediction)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>All outcomes</strong></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Gains</td>
<td>41.4</td>
<td>−0.17 10%</td>
<td>Integration (Segregation)</td>
</tr>
<tr>
<td>Losses</td>
<td>44.7</td>
<td>−0.09</td>
<td>Indifference (Integration)</td>
</tr>
<tr>
<td>Mixed gains</td>
<td>60.4</td>
<td>−0.54 1%</td>
<td>Integration (Integration)</td>
</tr>
<tr>
<td>Mixed losses</td>
<td>37.7</td>
<td>−0.13 5%</td>
<td>Integration (Segregation)</td>
</tr>
<tr>
<td>Null</td>
<td>51.5</td>
<td>−0.34 1%</td>
<td>Integration (Integration)</td>
</tr>
</tbody>
</table>

Table 4.3: Results for pooled data: consistency with mental accounting (P1–P4) predictions (%), means of the preferences value (statistically significant departures from zero i.e. indifference), stated and predicted preferences.

We can see that the mental accounting predictions are not supported by three of our five patterns: gains, losses and mixed losses. The mental accounting seems only work for mixed gains and tie. Subjects’ behaviors support our hypothesis that mental accounting can not be extend to the case of multiple outcomes. Indeed our results show that: people prefer to integrate gains, are indifferent between integrate and segregate losses, prefer to integrate a
small loss into big gains, integrate a small gain into losses, and integrate gains and losses in the case of tie.

It is now interesting to see if the number of outcomes implied in the events plays a role. Table 4.4 presents the results for three and five outcomes:

<table>
<thead>
<tr>
<th>Event type</th>
<th>Consistent according P1-P4</th>
<th>Mean</th>
<th>Preferences: Actual (Prediction)</th>
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</thead>
<tbody>
<tr>
<td><strong>Three outcomes</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gains</td>
<td>39.6</td>
<td>−0.21 10%</td>
<td>Integration (Segregation)</td>
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<td>Losses</td>
<td>43.5</td>
<td>−0.01</td>
<td>Indifference (Integration)</td>
</tr>
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<td>Mixed gains</td>
<td>62.6</td>
<td>−0.57 1%</td>
<td>Integration (Integration)</td>
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<tr>
<td>Mixed losses</td>
<td>38.2</td>
<td>−0.09</td>
<td>Indifference (Segregation)</td>
</tr>
<tr>
<td>Null</td>
<td>53.0</td>
<td>−0.34 1%</td>
<td>Integration (Integration)</td>
</tr>
<tr>
<td><strong>Five outcomes</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gains</td>
<td>42.9</td>
<td>−0.12</td>
<td>Indifference (Segregation)</td>
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<td>Losses</td>
<td>45.9</td>
<td>−0.16 10%</td>
<td>Integration (Integration)</td>
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<td>Mixed gains</td>
<td>56.2</td>
<td>−0.47 1%</td>
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<td>Mixed losses</td>
<td>36.9</td>
<td>−0.19 10%</td>
<td>Indifference (Segregation)</td>
</tr>
<tr>
<td>Null</td>
<td>49.7</td>
<td>−0.35 1%</td>
<td>Integration (Integration)</td>
</tr>
</tbody>
</table>

Table 4.4: Results for three and five outcomes

We find some differences of behaviors but only the level of preferences in the case of mixed gains is significant: subjects have a stronger preferences for integration in the case of three outcomes (p-value of mean difference significant at 10%).

We can now compare the results of three different countries: Paris, Changchun and Hong Kong. Table 4.5 presents these results:

This comparison offers robust differences. The two most important are the cases of gains and losses. For gains, French students prefer to integrate gains and students from Changchun are indifferent. The values of preferences are significantly different (p-value of mean difference significant at 5%). For losses, we find reverse preferences: French prefer to segregate losses whereas Changchun subjects prefer to integrate losses (p-value of mean difference significant at 1%). This implies that the percentage of answers violating mental accounting is significantly higher for French subjects (p-value of percentage difference significant at 5%). Note that the results of Hong Kong students seem to show a different pattern with a lot of indifferences. But as the number of subjects is too small (15 students) we are not able to
<table>
<thead>
<tr>
<th>Event type</th>
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<th>Mean</th>
<th>Preferences: Actual (Prediction)</th>
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<tr>
<td><strong>CHANGCHUN</strong></td>
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<tr>
<td>Gains</td>
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<td>Indifference (Segregation)</td>
</tr>
<tr>
<td>Losses</td>
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<td>Integration (Integration)</td>
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<tr>
<td>Mixed losses</td>
<td>34.3</td>
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<td>Integration (Segregation)</td>
</tr>
<tr>
<td>Null</td>
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<td>Integration (Integration)</td>
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<td><strong>HONG KONG</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Gains</td>
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<td>Indifference (Segregation)</td>
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<tr>
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<tr>
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<td>+0.25</td>
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<td><strong>PARIS</strong></td>
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<td></td>
<td></td>
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<tr>
<td>Gains</td>
<td>37.0</td>
<td>-0.37</td>
<td>Integration (Segregation)</td>
</tr>
<tr>
<td>Losses</td>
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<td>+0.42</td>
<td>Segregation (Integration)</td>
</tr>
<tr>
<td>Mixed gains</td>
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<td>-0.62</td>
<td>Integration (Integration)</td>
</tr>
<tr>
<td>Mixed losses</td>
<td>44.2</td>
<td>-0.07</td>
<td>Indifference (Segregation)</td>
</tr>
<tr>
<td>Null</td>
<td>50.9</td>
<td>-0.25</td>
<td>Segregation (Integration)</td>
</tr>
</tbody>
</table>

Table 4.5: Results for Paris, Changchun and Hong Kong data

find significant statistical tests.

All these results show that the patterns of preferences due to Thaler’s prediction is experimentally violated. However, our theoretical predictions do not fit perfectly to the data and are also partially violated by our experimental results.

5 Conclusions

Our study has shown that, within the class of value functions specified by the prospect theory, Thaler’s principles can be established rigorously in the case of only non-negative experiences, or only non-positive experiences, irrespectively of the number of exposure units. When exposure units carry both negative and positive experiences, then the principles may break down. Our theory provides a complete solution to the integration/segregation problem in the case of three exposure units and demonstrates in particular that the transition from
two to three exposure units increases the complexity of decisions enormously, thus showing
the challenges that we encounter when dealing with multiple exposure units.

In addition to our analyses of various cases in the previous sections, note also that there
might be situations when the decision maker needs or wishes for one reason or another to
use partial integration of non-negative experiences. In such cases, a generalization of Lim’s
(1971) inequality (cf. Kuczma, 2008) plays a decisive role. Namely, for any continuous and
concave function \( v : [0, \infty) \rightarrow \mathbb{R} \), and for any triplet \((x_1, x_2, x_3)\) of non-negative real numbers
such that \( x_3 \geq x_1 + x_2 \), we have that

\[
v(x_1) + v(x_2 + x_3) \leq v(x_1 + x_2) + v(x_3).
\] (5.1)

Inequality (5.1) implies that if we have three exposure units with positive experiences and if
for some reason we can only integrate two of them, then in order to decide whether, say, \( x_2 \)
should be integrated with \( x_1 \) or \( x_3 \), we need to verify \( x_3 \geq x_1 + x_2 \): if the inequality holds,
then the value maximizing decision maker should integrate \( x_2 \) with \( x_1 \), leaving \( x_3 \) segregated.

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Kong, and the Natural Sciences and Engineering Research Council (NSERC) of Canada.

**References**


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of investors with extended value functions. *Advances in Decision Sciences*, 2010, Article
ID 302895, 8 pages, doi:10.1155/2010/302895


Appendix

Instruction of the experiment

The written instructions given to the subjects were the following:

Below you find two scenario (named A and B) in which some events occur. The events are the same but they are either aggregated (scenario B) on one single day or segregated on different consecutive days (scenario A). You are asked to rate your level of preference between the two scenarios on a scale taken values from -2 (strongly prefer the integrated offer, i.e. scenario B) to +2 (strongly prefer the segregated offer, i.e. scenario A).

Note: Having the events occur together does not imply that they occur sooner or later than if they were apart. That is not the question. You are only asked to judge whether it is better to have the events separately or together.

In order to rate their level of preferences for integration (values of -2 or -1), segregation (values of +1 or +2), or indifference (value of 0) subjects have to fill the following Likert scale:

Please rate you level of preferences from -2 (scenario B) to +2 (scenario A)

| -2 | -1 | 0 | +1 | +2 |

Table 5.1: Likert scale of preferences

The list of questions given to subjects was the following:
### Three outcomes

1. **seg)** Three gains of $50 each
   - **int)** Gain of $150
2. **seg)** Three losses of $50 each
   - **int)** Loss of $150
3. **seg)** Two gains of $30 each and a loss of $10
   - **int)** Gain of $50
4. **seg)** Two gains of $25 each and a loss of $100
   - **int)** Loss of $50
5. **seg)** Two gains of $10 each and a loss of $20
   - **int)** No gains and no losses
6. **seg)** Gain of $50 and two losses of $10 each
   - **int)** Gain of $30
7. **seg)** A gain of $50 and two losses of $50 each
   - **int)** Loss of $50
8. **seg)** A gain of $50 and two losses of $25 each
   - **int)** No gains and no losses

### Five outcomes

9. **seg)** Five gains of $20 each
   - **int)** A gain of $100
10. **seg)** Three losses of $10 each and two losses of $35 each
    - **int)** A loss of $100
11. **seg)** Three gains of $40 each and two losses of $10 each
    - **int)** A gain of $100
12. **seg)** Two gains of $25 each and three losses of $50 each
    - **int)** A loss of $100
13. **seg)** Three gains of $40 each and two losses of $60 each
    - **int)** No gains and no losses
14. **seg)** Two gains of $60 each and three losses of $40 each
    - **int)** No gains and no losses

Table 5.2: Fourteen hypothetical scenarios.
Statistical Results

The following table contains data in the form of percentages for each scenario and for each level on the Likert scale.
### Changchun (100 students)

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<th>+2</th>
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### Hong Kong (15 students)

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