RESEARCH ARTICLE

Eventual Convexity of Chance Constrained Feasible Sets

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In decision making problems where uncertainty plays a key role and decisions have to be
taken prior to observing uncertainty, chance constraints are a strong modelling tool for defining
safety of decisions. These constraints request that a random inequality system depending on a
decision vector has to be satisfied with a high probability. The characteristics of the feasible set
of such chance constraints depend on the constraint mapping of the random inequality system,
the underlying law of uncertainty and the probability level. One characteristic of particular
interest is convexity. Convexity can be shown under fairly general conditions on the underlying
law of uncertainty and on the constraint mapping, regardless of the probability-level. In some
situations convexity can only be shown when the probability-level is high enough. This is
defined as eventual convexity. In this paper we will investigate further how eventual convexity
can be assured for specially structured chance constraints involving Copulae. The Copulae
have to exhibit generalized concavity properties. In particular we will extend recent results
and exhibit a clear link between the generalized concavity properties of the various mappings
involved in the chance constraint for the result to hold. Various examples show the strength
of the provided generalization.

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1. Introduction

Chance-Constrained programming is the branch of Stochastic optimization dealing
with constraints of the type
\[ \mathbb{P}[h(x,\xi) \geq 0] \geq p, \]
where \( h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k \) is a constraint mapping, \( x \in \mathbb{R}^n \) the decision vector,
\( \xi \in \mathbb{R}^m \) a random variable, \( \mathbb{P} \) its associated probability measure and \( p \in (0,1] \) a
pre-specified probability level. Constraints of the form (1) express that the decision
vector \( x \) is feasible if and only if the random inequality system \( h(x,\xi) \geq 0 \) is
satisfied with high enough probability. Such constraints are encountered in many
engineering problems involving uncertain data. We can find applications in water
management, telecommunications, electricity network expansion, mineral blending,
A key question in Chance Constrained Programming is the convexity of the feasible set, i.e., of the set $M(p) = \{ x \in \mathbb{R}^n : P[h(x, \xi) \geq 0] \geq p \}$. It is well known ([6, 13, 14]) that if $\xi$ admits a density with specific generalized concavity properties, and $h$ is a (jointly) quasi-concave mapping, then indeed the feasible set is convex. In many practical applications, we have separable constraint mappings, i.e., $h(x, \xi) = g(x) - \xi$, where $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$. The requirement of joint quasi concavity of $h$ then can be asserted if $g$ is concave. Note that quasi concavity of $g$ is not sufficient.

The latter requirement on the $i$-th component $g_i$ of the mapping $g$ can be relaxed to $-\alpha_i$-concavity, $\alpha_i > 0$ if $\xi$ has independent components and each component $\xi_i$ has a so-called $\alpha_i + 1$-decreasing density ([8]). This comes at the cost of only being able to assert convexity of $M(p)$ for $p$ values larger than some threshold. Such convexity is called eventual convexity and is clearly sufficient in many practical applications when we are looking at large $p$ values. Independence of components of $\xi$ is a strong requirement, which was relaxed in a second work of the same authors [9]. Indeed, $\xi$ can be allowed to have a dependence structure induced by a specially structured Copula, called a log-exp-concave Copula. Some well-known Copulae (Maximum, Independent, Gumbel) are log-exp-concave, as shown in [9]. However, it turns out that the Clayton Copula is not log-exp-concave.

When examining very carefully the results of [9], it appears that the link between generalized concavity of $g$, individual distribution functions $F_i$ of each component $\xi_i, i = 1, \ldots, m$ and the Copula is not clearly exhibited. As a result all log-exp concave Copula actually provide the same probability threshold. Moreover this level depends in a way on one of the distribution functions $F_i$ only. One can thus set up two versions of a problem wherein only one component of $\xi$ has the same distribution and obtain convexity results for the same asserted probability threshold. In particular $\xi$ can have independent components, or components linked through an arbitrary log-exp concave Copula and the same probability threshold is obtained. This is intuitively puzzling.

In this paper, we will show that one can derive eventual convexity of the feasible set $M(p)$ for a larger class of Copulae. In particular, we will show that the Clayton Copula is in this extended class. We will moreover exhibit clearly the link between the generalized concavity properties of the mapping $g$, the individual distribution functions $F_i, i = 1, \ldots, m$ and that of the Copula. We will also show that by adding some additional explicit constraints to the optimization problem, defining a convex feasible set, the probability threshold can be made to depend on the Copula. We will provide several examples showing that one can obtain eventual convexity results for lower $p$ values than in the papers [8, 9]. Finally we provide a partial characterization of the Gaussian Copula in the upper tail.

This paper is organized as follows. We will begin by introducing some useful notations and generalized concavity in section 2. We will define a class of Copulae containing the class of log-exp-concave Copula and characterize this class in section 3. In section 4 we will provide our main Theorems proving eventual convexity for the feasible sets $M(p)$ under specific conditions on the Copula, the individual distribution functions and constraint mappings $g_i, i = 1, \ldots, m$. Section 5 provides a series of examples and results showing that sharper bounds on the probability threshold can be obtained and that this new class of Copulae contains strictly more Copulae than just log-exp-concave Copulae. In section 6 we derive a par-
tial characterization of the Gaussian Copula, showing that it has some underlying log-concavity in its upper-tail. A potential application for modelling probabilistic Constraints with Copula is provided in section 7. Conclusions and perspectives are provided last.

2. Notations and Generalized Concavity

Throughout the text, we will apply many algebraic operations on vectors. In order to have short notation, these are understood componentwise. As an example, for any $u \in \mathbb{R}^m$, $e^u$ will be defined as $(e^{u_1}, ..., e^{u_m})$. In a very similar way, we will define $u^{\frac{1}{\alpha}}$ for $\alpha \neq 0$. For a mapping $h : \mathbb{R}^m \to \mathbb{R}$, $u \in \mathbb{R}^m \mapsto h(e^u)$ is thus understood as $u \mapsto h(e^{u_1}, ..., e^{u_m})$. We will also extend this short notation to one-dimensional mappings applied to a vector. If $\varphi : \mathbb{R} \to \mathbb{R}$ is a mapping, we mean $\varphi(u) = (\varphi(u_1), ..., \varphi(u_m))$ when $u \in \mathbb{R}^m$.

We will make extensive use of generalized concavity and its properties. It is therefore useful to introduce the following function:

**Definition 2.1:** Let $\alpha \in [-\infty, \infty]$ and $m_\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ be defined as follows

$$m_\alpha(a,b,\lambda) = 0 \text{ if } ab = 0,$$

for $a > 0$, $b > 0$, $\lambda \in [0,1]$: 

$$m_\alpha(a,b,\lambda) = \begin{cases} 
  \lambda b^{1-\lambda} & \text{if } \alpha = 0 \\
  \max \{a, b\} & \text{if } \alpha = \infty \\
  \min \{a, b\} & \text{if } \alpha = -\infty \\
  (\lambda a^\alpha + (1-\lambda)b^\alpha)\frac{1}{\alpha} & \text{else}
\end{cases} \tag{3}$$

The following lemma will be used throughout this text and can be found in [6]

**Lemma 2.2:** Let $m_\alpha$ be the mapping as defined in Definition 2.1. The mapping $\alpha \mapsto m_\alpha$ is nondecreasing and continuous.

We can now provide the definition of generalized concavity:

**Definition 2.3:** A non-negative function $f$ defined on some convex set $C \subseteq \mathbb{R}^n$ is called $\alpha$-concave ($\alpha \in [-\infty, \infty]$) if and only if for all $x, y \in C, \lambda \in [0,1]$: 

$$f(\lambda x + (1-\lambda)y) \geq m_\alpha(f(x), f(y), \lambda), \tag{4}$$

where $m_\alpha$ is as in Definition 2.1.

**Remark 2.1:** A function $f$ is 0-concave if its logarithm is concave. For $\alpha \neq 0$, $\alpha \in \mathbb{R}$, the function $f$ is $\alpha$-concave if either $f^\alpha$ is concave for $\alpha > 0$ or $f^\alpha$ is convex for $\alpha < 0$.

For some further calculus rules with $\alpha$-concavity we refer to Theorems 4.19-4.23 of [6].
Throughout this paper, $g : \mathbb{R}^n \to \mathbb{R}^m$ will be a constraint mapping, $\xi \in \mathbb{R}^m$ an $m$-dimensional random vector. The component $\xi_i$ is assumed to have one dimensional distribution function $z \in \mathbb{R} \mapsto F_i(z) := \mathbb{P}[\xi_i \leq z]$, $i = 1, ..., m$. Finally $C : [0, 1]^m \to [0, 1]$ is a Copula, such that

$$\mathbb{P}[\xi \leq g(x)] = C(F_1(g_1(x)), ..., F_m(g_m(x))).$$

(5)

We will assume that the mapping (5) defines a constraint of a Stochastic optimization problem in the following way:

$$\min_{x \in \mathbb{R}^n} f(x)$$

s.t. $\mathbb{P}[\xi \leq g(x)] \geq p,$

(6)

for some level $p$ and convex function $f$. This problem is assumed to be the "Stochastic" variant of the deterministic problem

$$\min_{x \in \mathbb{R}^n} f(x)$$

s.t. $b \leq g(x),$  

(7)

for an appropriately chosen vector $b \in \mathbb{R}^m$, e.g., $b = \mathbb{E}(\xi)$. Problem (7) is a convex optimization problem if and only if the mapping $g$ has some generalized concavity property. It is therefore natural to make such an assumption. Problem (6) arises, whenever problem (7) has turned out insufficiently robust and a decision vector $x$ accounting for uncertainty is looked for. Such left-hand side uncertainty arises in many practical applications such as [21, 22].

In this paper the set $M(p)$ will be defined as $M(p) := \{ x \in \mathbb{R}^n : \mathbb{P}[\xi \leq g(x)] \geq p \}$, where $\mathbb{P}[\xi \leq g(x)]$ is as in equation (5).

3. Copulae and generalized concavity

In [9], the authors call a Copula $C$ a log-exp concave Copula if and only if $u \in [0, 1]^m \mapsto \log(C(e^u))$ is a concave mapping. It therefore appears natural to provide the following extension of this concept. We will thus speak of a $\delta$-$\gamma$-concave Copula.

**Definition 3.1:** Let $\gamma \in \mathbb{R}$ be given, and let the set $X(\gamma)$ be defined as $X(\gamma) = [0, 1]^m$ for $\gamma > 0$, $X(0) = (-\infty, 0]^m$ and $X(\gamma) = [1, \infty)^m$ for $\gamma < 0$. Let $\delta \in [-\infty, \infty]$ be equally given. We call a Copula $C : [0, 1]^m \to [0, 1]$ $\delta$-$\gamma$-concave if the mapping $u \in X(\gamma) \mapsto C(u^{\gamma})$ is $\delta$-concave, whenever $\gamma \neq 0$ and $u \in X(0) \mapsto C(e^u)$ is $\delta$-concave whenever $\gamma = 0$.

The presence of the set $X(\gamma)$ is only to have the arguments mapped in $[0, 1]^m$, so that we can compose with the Copula afterwards.

**Remark 3.1:** This is indeed an extension of the notion of log-exp-concave Copulae as defined in the paper [9]. Indeed a log-exp-concave Copula is 0-0-concave in our setting. Examples of log-exp-concave Copulae are the independent, maximum and Gumbel Copula. The latter is an Archimedean Copula, a family of Copulae generated by a one-dimensional function called the generator. We refer to [11] for a full characterization of generators of Archimedean Copulae.
Remark 3.2: Copulae are such that \( C(u) = 0 \) if and only if there is some \( i = 1, \ldots, m \) with \( u_i = 0 \). Pick \( u \in [0,1]^m \) with \( C(u) = 0 \), some \( v \in [0,1]^m \) and \( \lambda \in [0,1] \) and form \( z = \lambda u + (1 - \lambda)v \). Pick moreover an arbitrary \( \gamma > 0 \), it is then clear from \( C \) being a Copula that \( C(z_{\gamma}) \geq 0 \) and from the definition of \( \delta \)-concavity that \( m_{\delta}(C(u_{\gamma}), C(v_{\gamma}), \lambda) = 0 \). It is therefore sufficient to verify \( \delta \)-concavity of a Copula with \( \gamma > 0 \) on \([0,1]^m\) only. This avoids the problem of degenerate arguments. Degenerate arguments are naturally avoided when \( \gamma \leq 0 \).

Remark 3.3: One could alternatively replace the \( \delta \)-concavity requirement with a \( g \)-concavity requirement as defined in [1] and [20]. We are however interested in eventual convexity, i.e., convexity of all level sets above some threshold. The notion of \( g \)-concavity then implies quasi-concavity on that (sub)-set.

Some Copulae might not be \( \delta \gamma \)-concave on the whole domain \( X(\gamma) \) as the following example shows:

Example 3.2 Let \( C : [0,1]^m \to [0,1] \) be the Gaussian Copula, i.e., \( C(u) = \Phi^R(\varphi^{-1}(u)) \), where \( \Phi^R \) is the probability distribution function of a centered Gaussian random variable in dimension \( m \) with correlation matrix \( R \) and \( \varphi^{-1} : (0,1) \to \mathbb{R} \) the inverse of a standard normal probability distribution function in dimension 1. The mapping \( \varphi^{-1} : (0,1) \to \mathbb{R} \) is concave on \((0,1/2)\) and \( \Phi^R \) is log-concave ([13]). It then follows that \( C \) is 0-1-concave on \([0,1/2]^m\). Indeed, pick \( x, y \in [0,1/2]^m \), \( \lambda \in [0,1] \). From concavity of \( \varphi^{-1} \) and monotonicity of the distribution function we get

\[
C(m_1(x,y,\lambda)) \geq \Phi^R(m_1(\varphi^{-1}(x_1), \varphi^{-1}(y_1), \lambda), \ldots, m_1(\varphi^{-1}(x_m), \varphi^{-1}(y_m), \lambda)) \\
\geq m_0(C(x), C(y), \lambda).
\]

Remark 3.4: The above example does not exclude a more general \( \delta \gamma \)-concavity property on other sets.

This therefore motivates the following definition:

Definition 3.3: Let \( q \in (0,1)^m \) be some point and define the sets \( X(q, \gamma) \) as follows \( X(q, \gamma) = \prod_{i=1}^m [q_i, 1] \) for \( \gamma > 0 \), \( X(q, 0) = \prod_{i=1}^m [\log(q_i), 0] \) and \( X(q, \gamma) = \prod_{i=1}^m [1, q_i] \) for \( \gamma < 0 \). We call a Copula \( C : [0,1]^m \to [0,1] \) \( \delta \gamma q \)-concave if the mapping \( u \in X(q, \gamma) \mapsto C(u_{\gamma}) \) is \( \delta \)-concave, whenever \( \gamma \neq 0 \) and \( u \in X(q, 0) \mapsto C(e^u) \) is \( \delta \)-concave whenever \( \gamma = 0 \).

Remark 3.5: The reason the sets are of the specific form in Definition 3.3 and not their opposite will become apparent in Theorems 4.1,4.2 and 4.3.

We will introduce one last notion, wherein \( \delta \gamma \)-concavity holds in some asymptotic way.

Definition 3.4: We call a Copula \( C : [0,1]^m \to [0,1] \) asymptotically \( \delta \)-concave if for each \( \gamma > 0 \) there exists a point \( q(\gamma) \in (0,1)^m \) such that \( C \) is \( \delta \gamma \)-concave when restricted to the set \( \dot{X}(\gamma) = [0, q(\gamma)]^m \) and moreover \( \lim_{\gamma \downarrow 0} q(\gamma) = 1 \).
3.1. **Structure of the family of $\delta$-$\gamma$-concave Copulae**

At first it might appear that the family of $\delta$-$\gamma$-concave Copulae is rather loose. In particular it may appear that the $\delta$-$0$-Copulae fit in rather artificially. The following results show that this is not true and that the family is naturally ordered.

It follows from Lemma 2.2 that $\delta$-concavity implies some ”descending” order, i.e., a $\delta$-concave mapping is also $\beta$-concave whenever $\beta \leq \delta$. It turns out that the effect of $\gamma$ is ascending as the following lemma shows.

**Lemma 3.5:** Let $C : [0, 1]^m \to [0, 1]$ be a $\delta$-$\beta$-concave Copula and let $\alpha \in \mathbb{R}$ be given such that $\beta \leq \alpha$. Then $C$ is also $\delta$-$\alpha$-concave.

**Proof:** Pick any $x, y \in [0, 1]^m$ and $\lambda \in [0, 1]$ arbitrarily. We begin with the special case $\beta = 0$, and $\alpha > 0$. We derive from concavity of the log-function and monotonicity of the exp function that:

$$m_1(x, y, \lambda)^{\frac{1}{\alpha}} = \exp \left( \frac{1}{\alpha} \log m_1(x, y, \lambda) \right) \geq \exp \left( \frac{1}{\alpha} m_1(\log x, \log y, \lambda) \right)$$

$$= \exp \left( m_1 \left( \frac{1}{\alpha} \log x, \frac{1}{\alpha} \log y, \lambda \right) \right) = \exp \left( m_1(\log x^{\frac{1}{\alpha}}, \log y^{\frac{1}{\alpha}}, \lambda) \right),$$

where with $x \in (0, 1]^m$, it follows that $\log(x^{\frac{1}{\alpha}}) \in X(0)$. Now from monotonicity of the Copula we get

$$C(m_1(x, y, \lambda)^{\frac{1}{\alpha}}) \geq C(\exp (m_1(\log x^{\frac{1}{\alpha}}, \log y^{\frac{1}{\alpha}}, \lambda))) \geq m_\delta(C(x^{\frac{1}{\alpha}}), C(y^{\frac{1}{\alpha}}), \lambda),$$

which was to be shown by $\delta$-concavity of $u \mapsto C(e^u)$ and Remark 3.2. Now picking any $\beta \neq 0$ and $\alpha \geq \beta$, we define $z = x^{\frac{1}{\alpha}}$, $w = y^{\frac{1}{\alpha}}$, when $\alpha \neq 0$ and $z = \exp(x)$, $w = \exp(y)$ when $\alpha = 0$. Together with $x, y \in X(\alpha)$, this implies $z, w \in [0, 1]^m$. Now with $\beta \neq 0$, one obtains $z^\beta, w^\beta \in X(\beta)$. We then observe that $m_1(x, y, \lambda)^{\frac{1}{\alpha}} = m_\alpha(z, w, \lambda)$, when $\alpha \neq 0$ and $\exp(m_1(x, y, \lambda)) = m_0(z, w, \lambda)$ when $\alpha = 0$. It therefore follows from monotonicity of the Copula combined with Lemma 2.2 and $\delta$-$\beta$-concavity of the Copula that

$$C(m_1(x, y, \lambda)^{\frac{1}{\alpha}}) = C(m_\alpha(z, w, \lambda)) \geq C(m_\beta(z, w, \lambda)) \geq m_\delta(C(z), C(w), \lambda) = m_\delta(C(x^{\frac{1}{\alpha}}), C(y^{\frac{1}{\alpha}}), \lambda),$$

when $\alpha \neq 0$ and for $\alpha = 0$, we get

$$C(\exp (m_1(x, y, \lambda))) = C(m_0(z, w, \lambda)) \geq C(m_\beta(z, w, \lambda)) \geq m_\delta(C(z), C(w), \lambda) = m_\delta(C(e^z), C(e^y), \lambda),$$

as was to be shown. \qed

**Remark 3.6:** This is an extension of Proposition 3 of [9] where it is shown that $0$-$0$-concavity implies $0$-$1$-concavity.

One can also prove this same lemma with the local $\delta$-$\gamma$-concavity notion.
Lemma 3.6: Let \( q \in (0, 1)^m \) be given and let \( C : [0, 1]^m \to [0, 1] \) be a \( \delta-\beta-q \)-concave Copula and let \( \alpha \) be given such that \( 0 \leq \beta \leq \alpha \). Then \( C \) is also \( \delta-\alpha-q \)-concave.

Proof: The proof is identical to that of Lemma 3.5 except that we need to take care of the domains. To this end pick \( x \in X(q, \alpha) \). We begin by considering the case \( \beta = 0, \alpha > 0 \). It then follows from monotonicity of the log function and of \( x \mapsto x^\frac{1}{\alpha} \), that \( \log x^\frac{1}{\alpha} \geq \log(q) \). In a similar way it follows with \( \alpha > \beta > 0 \) that \( x^\beta \geq q \), so that \( \beta-q \)-concavity can be applied. \( \square \)

Since \( \delta \)-concavity of an arbitrary mapping implies weaker concavity properties for the same mapping, we trivially derive the following corollary:

Corollary 3.7: Let \( C : [0, 1]^m \to [0, 1] \) be a \( \delta-\gamma \)-concave Copula and let \( \alpha \geq \gamma \) and \( \beta \leq \delta \) be given. Then \( C \) is also \( \beta-\alpha \)-concave.

3.2. Tools for deriving \( \delta-\gamma \)-Concavity of Copulae

Copulae are in particular probability distribution functions, some of them admitting densities. The following result is therefore a trivial consequence of Theorem 4.15 [6]:

Lemma 3.8: Let \( C : [0, 1]^m \to [0, 1] \) be a Copula, admitting a density \( c : [0, 1]^m \to [0, 1] \). If the density \( c \) is \( \alpha \)-concave for some \( \alpha \geq -\frac{1}{m} \), then \( C \) is a \( \delta \)-1-concave Copula for \( \delta = \frac{\alpha}{1+\alpha m} \).

The following Corollary follows immediately from this result

Corollary 3.9: Let \( C : [0, 1]^m \to [0, 1] \) be a Copula, admitting a density \( c : [0, 1]^m \to [0, 1] \). If the density \( c \) is such that \( u \in (-\infty, 0]^m \mapsto c(e^{\exp(u)}) \) is log-concave then the Copula is 0-0-concave.

Proof: We begin by remarking that \( h : (-\infty, 0]^m \to [0, 1] \), defined as \( h(u) = C(e^{\exp(u)}) \) is also a distribution function. Since \( C \) admits a density, the density of \( h \) is given by

\[
\frac{\partial^m}{\partial u_1 \cdots \partial u_m} h = c(e^u) \prod_{i=1}^{m} e^{u_i}.
\]

Now the logarithm of the latter function is \( \log(c(e^u)) + \sum_{i=1}^{m} u_i \), i.e., the sum of concave functions, i.e., concave. It follows from Lemma 3.8 that \( h \) is log-concave, i.e., \( C \) is 0-0-concave. \( \square \)

3.3. Estimates with Copulae

Before moving to our main result, we first derive some useful auxiliary results.

Lemma 3.10: Let \( C : [0, 1]^m \to [0, 1] \) be a \( \delta-\gamma \)-concave Copula and \( \infty \geq \gamma_i \geq \gamma \), \( i = 1, \ldots, m \) any sequence of numbers. For any \( x, y \in [0, 1]^m \), \( \lambda \in [0, 1] \), the following
inequality holds
\[ C(m_{\gamma_1}(x_1, y_1, \lambda), ..., m_{\gamma_m}(x_m, y_m, \lambda)) \geq m_\delta(C(x_1, ..., x_m), C(y_1, ..., y_m), \lambda). \]

Proof: Let \( \gamma_i \geq \gamma \) for all \( i = 1, ..., m \) be given and pick \( x, y \in [0, 1]^m \) and \( \lambda \in [0, 1] \) in an arbitrary way. We begin by noting that \( m_{\gamma_i}(x_i, y_i, \lambda) \geq m_{\gamma_i}(x, y, \lambda) \) for any \( i \in \{1, ..., m\} \), since this mapping is non-decreasing by Lemma 2.2. Copulae are distribution functions and hence are increasing in increasing arguments, so we get
\[ C(m_{\gamma_1}(x_1, y_1, \lambda), ..., m_{\gamma_i}(x_m, y_m, \lambda)) \geq C(m_{\gamma_1}(x_1, y_1, \lambda), ..., m_{\gamma_m}(x_m, y_m, \lambda)). \] (8)

Now, if \( \gamma \neq 0 \), the right-hand side of (8) is equal to
\[ C(m_\gamma(x_1, y_1, \lambda), ..., m_\gamma(x_m, y_m, \lambda)) = C(m_\gamma(x_1^\gamma, y_1^\gamma, \lambda)^{1/\gamma}, ..., m_\gamma(x_m^\gamma, y_m^\gamma, \lambda)^{1/\gamma}) \] (9)
and in the case \( \gamma = 0 \), we get
\[ C(m_0(x_1, y_1, \lambda), ..., m_0(x_m, y_m, \lambda)) = \]
\[ C(\exp (m_1 (\log x_1, \log y_1, \lambda)), ..., \exp (m_1 (\log x_m, \log y_m, \lambda))). \] (10)

For \( z, w \in (0, 1) \), we can also derive that
\[ m_1(z^\gamma, w^\gamma, \lambda) \in [\min \{z^\gamma, w^\gamma\}, \max \{z^\gamma, w^\gamma\}], \]
when \( \gamma \neq 0 \) and \( m_1(\log z, \log w, \lambda) \in [\min \{\log(z), \log(w)\}, \max \{\log(z), \log(w)\}] \).
This shows that
\[ (m_1(x_1^\gamma, y_1^\gamma, \lambda), ..., m_1(x_m^\gamma, y_m^\gamma, \lambda)) \in X(\gamma), \]
when \( \gamma \neq 0 \) and
\[ (m_1(\log x_1, \log y_1, \lambda), ..., m_1(\log x_m, \log y_m, \lambda)) \in X(0). \]

Hence, since the mappings on the right-hand side of (9) and (10) are \( \delta \)-concave we obtain the estimates
\[ C(m_{\gamma_1}(x_1, y_1, \lambda), ..., m_{\gamma_m}(x_m, y_m, \lambda)) \geq m_\delta(C(x_1, ..., x_m), C(y_1, ..., y_m), \lambda), \]
as was to be shown. \( \square \)

Lemma 3.11: Let \( q \in (0, 1)^m \) be a given point and let \( C : [0, 1]^m \rightarrow [0, 1] \) be a \( \delta \)-\( \gamma \)-\( q \)-concave Copula. Assume furthermore that \( \infty \geq \gamma_i \geq \gamma, i = 1, ..., m \) is any sequence of numbers. Then for any \( x, y \in [q, 1]^m \) and \( \lambda \in [0, 1] \) the following inequality holds:
\[ C(m_{\gamma_1}(x_1, y_1, \lambda), ..., m_{\gamma_m}(x_m, y_m, \lambda)) \geq m_\delta(C(x_1, ..., x_m), C(y_1, ..., y_m), \lambda). \]
Theorem 4.1: Let $\gamma_i \geq \gamma$ for all $i = 1, \ldots, m$ be given and pick $x, y \in [q, 1]^m$ and $\lambda \in [0, 1]$ in an arbitrary way. Using the arguments of the proof of Lemma 3.10 we can derive equations (9) and (10). It remains to show that

$$\left( m_1(x_1^1, y_1^1, \lambda), \ldots, m_1(x_m^\gamma, y_m^\gamma, \lambda) \right) \in X(q, \gamma),$$

when $\gamma \neq 0$ and

$$\left( m_1(\log x_1, \log y_1, \lambda), \ldots, m_1(\log x_m, \log y_m, \lambda) \right) \in X(q, 0).$$

Then we can apply again the $\delta$-concavity inequality to derive the final estimate of the Lemma. To this end, pick $z, w \in [q, 1]$ arbitrarily and consider the case $\gamma \geq 0$. From monotonicity of the mapping $u \in [0, 1] \mapsto u^\gamma$ (or $u \in (0, 1) \mapsto \log(u)$), we obtain for $u \geq q$ that $u^\gamma \geq q^\gamma$ (or $\log(u) \geq \log(q)$). Altogether this implies $m_1(z^\gamma, w^\gamma, \lambda) \geq \min \{z^\gamma, w^\gamma\} \geq q^\gamma$, i.e., $m_1(x^\gamma, y^\gamma, \lambda) \in X(q, \gamma)$. When $\gamma < 0$, the map $u \in (0, 1) \mapsto u^\gamma$ is decreasing, so $u \geq q$ yields $u^\gamma \leq q^\gamma$. This in turn implies $m_1(z^\gamma, w^\gamma, \lambda) \leq \max \{z^\gamma, w^\gamma\} \leq q^\gamma$, i.e., $m_1(x^\gamma, y^\gamma, \lambda) \in X(q, \gamma)$.

4. Eventual Convexity of the Feasible Set

As in the result by Kataoka ([10]), convexity of the feasible set $M(p)$ cannot always be obtained for any probability level $p$. From a practical perspective this is not necessarily a problem since we are naturally looking for high $p$ levels in problems of type (6). In some cases, we can show that the feasible set is convex if $p$ is large enough. Convexity of feasible sets with high enough $p$ is known as eventual convexity. In this section we will show that we can derive such eventual convexity if the Copulae, individual probability distribution functions and constraint mappings have some specific generalized concavity properties.

In the following theorem, we will provide conditions on Copulae, individual probability distribution functions and constraint mappings $g$ such that eventual convexity of the feasible set can be asserted.

Theorem 4.1: Let $\xi \in \mathbb{R}^m$ be a random vector with associated Copula $C$, and let $g_i: \mathbb{R}^n \to \mathbb{R}$ be functions such that

$$\mathbb{P}[\xi \leq g(x)] = C(F_1(g_1(x)), \ldots, F_m(g_m(x))),$$

(11)

where $F_i$ are the marginal distribution functions of component $i$ of $\xi$, $i = 1, \ldots, m$. Assume that we can find $\alpha_i \in \mathbb{R}$, such that the functions $g_i$ are $\alpha_i$-concave and a second set of parameters $\gamma_i \in (-\infty, \infty], b_i > 0$ such that either one of the following conditions holds:

1. $\alpha_i < 0$ and $z \mapsto F_i(z^{b_i})$ is $\gamma_i$-concave on $(0, b_i^{\alpha_i}]$
2. $\alpha_i = 0$ and $z \mapsto F_i(\exp z)$ is $\gamma_i$-concave on $[\log b_i, \infty)$
3. $\alpha_i > 0$ and $z \mapsto F_i(z^{b_i})$ is $\gamma_i$-concave on $[b_i^{\alpha_i}, \infty)$,

where $i \in \{1, \ldots, m\}$ is arbitrary. If the Copula is either $\delta$-$\gamma$-concave or $\delta$-$\gamma$-$F(b)$-concave for $\gamma \leq \gamma_i \leq \infty$, $i = 1, \ldots, m$, then the set $M(p) := \{x \in \mathbb{R}^n : \mathbb{P}[\xi \leq g(x)] \geq p\}$ is convex for all $p > p^* := \max_{i=1,\ldots,m} F_i(b_i)$. Convexity can moreover be derived for all $p \geq p^*$ if each individual distribution function
$F_i$, $i = 1, \ldots, m$ is strictly increasing. In the specific case that $\alpha_i \geq 0$, $\gamma_i$-concavity of the distribution functions holding everywhere, for all $i \in \{1, \ldots, m\}$ and $C$ being a $\delta$-$\gamma$-concave Copula, the set $M(p)$ is convex for all $p$.

**Proof:** Pick any $p > p^*$, $x, y \in M(p)$, $\lambda \in [0, 1]$ and $i \in \{1, \ldots, m\}$ arbitrarily. Define $x^\lambda := m_1(x, y, \lambda)$. Since all Copulae are dominated by the maximum-Copula, we get:

$$F_i(g_i(x)) \geq \min_{j=1,\ldots,m} F_j(g_j(x)) \geq C(F_1(g_1(x)), \ldots, F_m(g_m(x))) \geq p > p^* \geq F_i(b_i).$$

(12)

Now the latter entails

$$g_i(x) \geq b_i.$$  

(13)

Estimate (13) also holds whenever $p \geq p^*$ and $F_i$ is strictly increasing for each $i = 1, \ldots, m$. A similar estimate is obtained for $y$ clearly. We make a case distinction

1. $\alpha_i < 0$: In this case $\lambda g_i(x)^{\alpha_i} + (1 - \lambda)g_i(y)^{\alpha_i} \leq \max\{g_i(x)^{\alpha_i}, g_i(y)^{\alpha_i}\} \leq b_i^{\alpha_i}$
2. $\alpha_i = 0$: In this case $\lambda \log g_i(x) + (1 - \lambda) \log g_i(y) \geq \min\{\log g_i(x), \log g_i(y)\} \geq \log b_i$.
3. $\alpha_i > 0$: In this case $\lambda g_i(x)^{\alpha_i} + (1 - \lambda)g_i(y)^{\alpha_i} \geq \min\{g_i(x)^{\alpha_i}, g_i(y)^{\alpha_i}\} \geq b_i^{\alpha_i}$.

From monotonicity of the probability distribution function $F_i$, and $\alpha_i$-concavity of $g_i$ we obtain

$$F_i(g_i(x^\lambda)) \geq F_i(m_{\alpha_i}(g_i(x), g_i(y), \lambda)) = F_i((\lambda g_i(x)^{\alpha_i} + (1 - \lambda)g_i(y)^{\alpha_i})^{1/\alpha_i}),$$

(14)

whenever $\alpha_i \neq 0$ and

$$F_i(g_i(x^\lambda)) \geq F_i(m_0(g_i(x), g_i(y), \lambda)) = F_i(\exp(\lambda \log g_i(x) + (1 - \lambda) \log g_i(y))),$$

(15)

when $\alpha_i = 0$. The mappings in the right-hand side are $\gamma_i$ concave by assumption on a specific domain. Since we have shown that our arguments map in this domain, we can apply $\gamma_i$-concavity and obtain:

$$F_i(g_i(x^\lambda)) \geq m_{\gamma_i}(F_i(g_i(x)), F_i(g_i(y), \lambda)).$$

(16)

Since $i$ was fixed but arbitrary, the above equation holds for all $i = 1, \ldots, m$.

A Copula is strictly increasing in its arguments, so we get from (16):

$$C(F_1(g_1(x^\lambda)), \ldots, F_m(g_m(x^\lambda))) \geq C(m_{\gamma_i}(F_i(g_1(x)), F_i(g_1(y), \lambda)), \ldots, m_{\gamma_m}(F_m(g_m(x)), F_m(g_m(y), \lambda))).$$

(17)

If the Copula is $\delta$-$\gamma$-concave everywhere then we can apply Lemma 3.10 to obtain:

$$C(m_{\gamma_i}(F_1(g_1(x)), F_1(g_1(y), \lambda), \ldots, m_{\gamma_m}(F_m(g_m(x)), F_m(g_m(y), \lambda))) \geq m_{\delta}(C(F_1(g_1(x)), \ldots, F_m(g_m(x))), C(F_1(g_1(y)), \ldots, F_m(g_m(y))), \lambda) \geq p,$$

which together with (17) gives $C(F_1(g_1(x^\lambda)), \ldots, F_m(g_m(x^\lambda))) \geq p$, i.e., $x^\lambda \in M(p)$. If the Copula is $\delta$-$\gamma$-$F(b)$-concave, we have $F_i(g_i(x)) \geq F_i(b_i)$ and we can apply
Theorem 4.3: Let \( \xi \in \mathbb{R}^m \) be a random vector with associated Copula \( C \), and let \( g_i : \mathbb{R}^n \to \mathbb{R} \) be functions such that

\[
\mathbb{P}[\xi \leq g(x)] = C(F_1(g_1(x)), ..., F_m(g_m(x))), \tag{18}
\]

where \( F_i \) are the marginal distribution functions of component \( i \) of \( \xi \) with unbounded support. Assume that we can find \( \alpha_i \in \mathbb{R} \), such that the functions \( g_i \) are \( \alpha_i \)-concave and a second set of parameters \( \gamma_i \in [\bar{\gamma}, \infty) \), \( b_i > 0 \), \( \bar{\gamma} > 0 \) such that either one of the following conditions holds:

1. \( \alpha_i < 0 \) and \( z \mapsto F_i(\frac{1}{z^{\alpha_i}}) \) is \( \gamma_i \)-concave on \( (0, b_i^{\alpha_i}] \)
2. \( \alpha_i = 0 \) and \( z \mapsto F_i(\exp z) \) is \( \gamma_i \)-concave on \( [\log b_i, \infty) \)
3. \( \alpha_i > 0 \) and \( z \mapsto F_i(\frac{1}{z^{\alpha_i}}) \) is \( \gamma_i \)-concave on \( [b_i^{\alpha_i}, \infty) \),

for all \( i \in \{1, ..., m\} \). If the Copula is asymptotically \( \delta \)-concave, then set \( M(p) := \{ x \in \mathbb{R}^n : \mathbb{P}[\xi \leq g(x)] \geq p \} \) is convex for all \( p > p^* := \max_{i=1, ..., m} F_i(b_i) \).

Proof: Pick \( x, y \in M(p) \), \( \lambda \in [0, 1] \) and \( i \in \{1, ..., m\} \) arbitrarily. Since the distribution functions \( F_i \) are assumed to have unbounded support, it follows that \( F_i(g_i(x)) < 1 \), \( F_i(g_i(y)) < 1 \) and \( F_i(g_i(\lambda x + (1 - \lambda)y)) < 1 \). From the definition of asymptotic \( \delta \)-concavity, it follows that one can find a \( \bar{\gamma} > 0 \) such that
\( F_i(g_i(z)) \leq q_i(\gamma) \) for all \( \gamma < \bar{\gamma}, \ z = x, y \) or \( z = \lambda x + (1 - \lambda)y \) and for all \( i = 1, \ldots, m \).

One can pick \( \bar{\gamma} \) moreover such that \( \bar{\gamma} \leq \bar{\gamma} \). This shows that all arguments in the proof of Theorem 4.1 are in the set \( X(q, \gamma) \) for \( \gamma < \bar{\gamma} \) and one can therefore apply that theorem to conclude the proof. \( \square \)

**Remark 4.1:** Our main Theorem 4.1 provides a link between generalized concavity requirements on the constraint mapping \( g_i \), that of the one dimensional distribution functions \( F_i \), \( i = 1, \ldots, m \) and that of the Copula. This link was less apparent in the earlier results [9] since it was required to have \( \alpha_i < 0, \ \gamma_i = 1 \) together with \( \delta = \gamma = 0 \). The results presented here show that already in the setting of [9] an improvement could be obtained by remarking that concavity of \( z \mapsto F_i(z) \) for one dimensional probability distribution functions \( F \) on a set \( K \) implies log-concavity on the same set. The result is clearly an extension of those obtained in [8], since the independent Copula is 0-0-concave.

**Remark 4.2:** The advantage of Theorem 4.2 over Theorem 4.1 is that it reinserts the dependence of \( p^* \) on the Copula and not just its generalized concavity property. We will illustrate this effect in later examples. If we call \( p_1^* \) the critical level obtained in Theorem 4.1 and \( p_2^* \) that of Theorem 4.2, then it follows always \( p_2^* \leq p_1^* \). Indeed picking \( p \geq p_1^* \), \( x \in \hat{M}(p) \) one derives from equation (13) that \( x \in \hat{D} \), i.e., \( x \in M(p) \cap \hat{D} \). The latter set is shown to be convex whenever \( p \geq p_2^* \), implying \( p_1^* \geq p_2^* \).

An extension of interest is obtained only if either

1. More constraint mappings \( g \) can be allowed for. This is not truly the case, since any \( \alpha \)-concave mapping is in particular \( \beta \)-concave for some \( \beta < 0 \). However we might be able to exploit better the true concavity properties of the mappings. This is shown in section 5.1.
2. The obtained probability level \( p^* \) is lower. This is shown in the examples of section 5.2.
3. The class of \( \delta \)-\( \gamma \)-concave Copulae is larger than that of the log-exp-concave Copulae. This is shown in section 5.3.

These points are illustrated in section 5 through examples and results.

One key question if probability distribution functions with the required properties of Theorem 4.1 exist is answered positively already in the papers [8, 9]. It is shown that a specific property of the density known as \( r \)-decreasingness is sufficient.

**Definition 4.4:** The function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is called \( r \)-decreasing (\( r \in \mathbb{R} \)) if it is continuous on \((0, \infty)\) and there exist some \( t^* > 0 \) such that \( t \mapsto t^* f(t) \) decreases strictly for \( t > t^* \).

The authors [8] have then shown

**Lemma 4.5:** Let \( F : \mathbb{R} \rightarrow [0, 1] \) be a probability distribution function with a \( \gamma + 1 \)-decreasing density for some \( \gamma > 0 \). Then the function \( z \mapsto F(z^{-\frac{1}{\gamma}}) \) is concave on \((0, (t^*)^{-\gamma})\), with \( t^* \) as in Definition 4.4. Moreover \( F(t) < 1 \) for all \( t \in \mathbb{R} \).

It turns out that many distribution functions have this property and the dependence of \( t^* \) on \( \gamma \) can be analytically computed for these distributions. We refer to table 1 of [9] for those results.
5. A generalization of Results

5.1. Improved use of Generalized Concavity of the Constraint Mappings

We begin by showing that Theorem 4.1 allows us to exploit the true generalized concavity properties of the constraint mappings \( g \) better. To this end consider the following example:

**Example 5.1** Consider log-concave mappings \( g_i \) and a log-exp-concave Copula together with a random vector having exponentially distributed components. Let \( \lambda \) denote the parameter of these exponential distributions. We can for instance take 
\[
g_1(x, y) = \log(x + y) + 0.1 \quad \text{and} \quad g_2(x, y) = \log(2x + 3y) + 0.2, \quad \text{where} \quad g_i : \mathbb{R}^2 \to \mathbb{R}, \ i = 1, 2.
\]
In order to be able to apply Theorem 4.1 and derive convexity of the set \( M(p) \) for chance constraints structured as (5), we have to show that the mappings \( z \mapsto F(z^2) \) are log-concave, i.e., \( z \mapsto h(z) := \log(1 - \exp(-\lambda e^z)) \) has to be concave. This can be shown to hold on \( \mathbb{R} \). Pick \( x, y \in \mathbb{R} \) and \( \mu \in [0, 1] \) arbitrarily. From convexity of the exponential function we get 
\[
-\exp(-\lambda m_1(x, y, \mu)) \geq -\lambda m_1(\exp(x), \exp(y), \mu).
\]
A second application then yields 
\[
-\exp(-\lambda m_1(\exp(x), \exp(y), \mu)) \geq m_1(\exp(-\lambda \exp(x)), \exp(-\lambda \exp(y)), \mu).
\]
Altogether we get from strict increasingness of the log 
\[
\log(1 + (-\exp(-\lambda \exp(m_1(x, y, \mu)))))) \geq
\log(m_1(1 - \exp(-\lambda \exp(x)), 1 - \exp(-\lambda \exp(y)), \mu)) \geq m_1(h(x), h(y), \mu),
\]
as was required. We can therefore apply Theorem 4.1 to show that convexity of \( M(p) \) holds for all \( p \) levels. Since any log-concave mapping is in particular \( r \)-concave for all \( r < 0 \), we could have also applied the earlier results obtained by \cite{9}. Then we would obtain convexity of the feasible set for \( p \geq 1 - e^{-r-1} \). And this would hold for all \( r < 0 \), yielding convexity for \( p > 1 - e^{-1} = 0.63 \).

5.2. Improved Estimates of \( p^* \)

Following Remark 4.1, Theorem 4.1 appears in a somewhat weaker form in \cite{8, 9}. In particular, in those papers the authors show that if \( h \) is \( r+1 \)-decreasing, then the constraint mapping \( z \mapsto F(z^{\frac{1}{2}}) \) is concave on some set \((0, t^*_r)\). Now this corresponds to picking \( \gamma_i = 1 \) in Theorem 4.1, but log-exp concavity of Copulae implies \( \gamma = 0 \), leaving room for a gap to be filled. Indeed the following Lemma is a trivial consequence of generalized concavity:

**Lemma 5.2**: Let \( F : \mathbb{R} \to [0, 1] \) be a probability distribution function with a \( \gamma+1 \)-decreasing density for some \( \gamma > 0 \). Then the function \( z \mapsto F(z^{\frac{1}{2}}) \) is \( \alpha \)-concave on \((0, (t^\#)^{-\gamma})\), with \( t^\# \leq t^* \), where \( t^* \) is as in Definition 4.4. Moreover this holds for all \( \alpha \leq 1 \).

**Proof**: This follows trivially since \( \alpha \)-concavity implies \( \beta \)-concavity for all \( \beta \leq \alpha \) on the same set, so one can pick \( t^\# \leq t^* \), potentially degenerate \( t^\# = t^* \). \( \square \)

An example shows that weaker concavity is actually obtained on a larger set, i.e., \( t^\# < t^* \).
Example 5.3 We can get back to [8, example 4.1]. To this end we pick $-1$-concave mappings $g_i$. In the above cited example the specific mappings $g_1(x, y) = \frac{1}{x^2+y^2+0.1}$ and $g_2(x, y) = \frac{1}{(x+y)^2+0.1}$ from $\mathbb{R}^2$ to $\mathbb{R}$ are chosen. We moreover pick any 0-0-concave Copula, i.e., some log-exp concave Copula, such as for instance the independent, Gumbel or maximum Copula. In order to be able to apply Theorem 4.1 and derive convexity of the set $M(p)$ for chance constraints structured as (5), we have to show that $z \mapsto F(1/z)$ is log-concave on some set $(0, (t^\#)^{-1})$. Assume that $F$ is the distribution function of an exponential random variable with parameter $\lambda$. Then upon defining the mapping $h(z) = \log F(1/z) = \log (1 - \exp (-\frac{\lambda}{z}))$, we have to show that this mapping is concave. To this end we will compute the first and second derivative and we obtain for any $z > 0$:

$$h'(z) = (1 - (1 - \exp (-\frac{\lambda}{z}))^{-1})\lambda z^{-2}$$

$$h''(z) = h'(z)\left[-2z^{-1} + \lambda z^{-2}(1 - \exp (-\frac{\lambda}{z})^{-1}\right].$$

Now $z > 0$ implies $0 < \exp (-\frac{\lambda}{z}) < 1$, yielding $(1 - \exp (-\frac{\lambda}{z}))^{-1} > 1$ and therefore $h'(z) < 0$. So the sign of $h''(z)$ depends on that of $[-2z^{-1} + \lambda z^{-2}(1 - \exp (-\frac{\lambda}{z})^{-1}]$. From the above estimate we get

$$-2z^{-1} + \lambda z^{-2}(1 - \exp (-\frac{\lambda}{z})^{-1}\geq -2z^{-1} + \lambda z^{-2},$$

so indeed $h''(z) < 0$ for small $z$ (but we knew this already, since the exponential density is 2-decreasing). It turns out that $h''(z) = 0$ if and only if $\exp (-\frac{\lambda}{z}) = 1 - \frac{\lambda}{z}$. We can compute this $z^\#$ by using for instance a dichotomy procedure (picking $\lambda = 1$, yielding $z^\# = 0.62750048$, i.e., $t^\# = 1.59362426$). The obtained $p^*$ of Theorem 4.1 is then equal to $p^* = 1 - \exp (-\lambda t^\#) = 0.7968121$, which is significantly better than the earlier obtained $p^* = 0.864$. Empirically varying $\lambda$ yields the result that the improved $p^*$ does not depend on $\lambda$, similarly to the results obtained in [8].

We can also look at Theorem 4.2 and look at the set $M(p) \cap \{x \in \mathbb{R}^n : g_i(x) \geq t^\# \}$ for $p \geq p^* = C(F_1(t^\#), ..., F_n(t^\#))$. Now this results depends on the Copula. Picking the Maximum Copula we get $p^* = 0.7968121$ as before. Picking the Independent Copula we get $p^* = 0.7968121^2 = 0.6349$ and the Gumbel Copula with $\theta = 1.1$ yields $p^* = 0.652770$. The latter result does not depend on $\theta$, at least as found empirically.

Example 5.4 Returning once again to [8, example 4.1] and example 5.3 above, we can likewise stipulate that the components of $\xi$ follow a standard normal distribution. These components are linked through a log-exp concave Copula. In the paper [8] the authors have shown that the map $z \mapsto \Phi(1/z)$ is concave on $(0, 1/\sqrt{2})$. This then yields a bound of $p^* = \Phi(1/\sqrt{2}) = 0.921$, where $\Phi$ is the probability distribution function of a standard normal random variable in dimension 1. In order to be able to apply Theorem 4.1 and derive convexity of the set $M(p)$ for chance constraints structured as (5), we have to show that $z \mapsto \Phi(1/z)$ is log-concave only. Calling $h(z) = \log(\Phi(1/z))$, we can compute and obtain for $z > 0$:

$$h'(z) = -z^{-2}\Phi'(z^{-1})\Phi(z^{-1})^{-1}$$

$$h''(z) = h'(z^{-1})(-2z^{-1} + z^{-3} + z^{-2}\Phi'(z^{-1})\Phi(z^{-1})^{-1}),$$
where we have used that $\Phi''(z^{-1}) = -z^{-1}\Phi'(z^{-1})$. Indeed, $h'(z) < 0$ for $z > 0$ and $h''(z) < 0$ for $z > 0$ small enough. We can again numerically compute $z^\#$ such that $h''(z^\#) = 0$, i.e., $-2z^{-1} + z^{-3} + z^{-2}\Phi'(z^{-1})\Phi(z^{-1})^{-1} = 0$ at $z = z^\#$, and obtain $z^\# = 0.754205$. Convexity of the feasible set is then obtained for $p \geq p^* = \Phi(\frac{1}{z^\#}) = 0.90756$, from Theorem 4.1 which is only marginally better.

Applying Theorem 4.2 however and distinguishing Copulae allows us to obtain an improved $p^*$ level for the Gumbel and Independent Copula. Indeed, then we get $p^* = 0.8334$ and $p^* = 0.824$ respectively.

5.3. More Copulae

The last extension that Theorem 4.1 allows for is that we can use more Copulae, if the class of $\delta$-$\gamma$-concave Copulae contains more Copulae than just log-exp-concave Copulae. This turns out to be the case. Indeed the following Lemma shows that the Clayton Copula is $\delta$-$\gamma$-concave for specific $\delta$ values. It was however shown in [9] that it was not log-exp-concave.

**Lemma 5.5:** Let $\theta > 0$ be the parameter of the strict generator $\psi : [0,1] \rightarrow \mathbb{R}_+$, $\psi(t) = \theta^{-1}(t^{-\theta} - 1)$ of the Clayton Copula. This Copula is $\delta$-$\gamma$-concave for all $\gamma > 0$ provided that $\delta \leq -\theta < 0$.

**Proof:** The inverse of the generator is given by $\psi^{-1}(s) = (\theta s + 1)^{-\frac{1}{\theta}}$, so the Copula is defined as $C(u) = \psi^{-1}(\sum_{i=1}^m \psi(u_i))$, where $u \in [0,1]^m$. We begin by computing the derivatives of the generator

$$
\frac{d\psi}{dt}(t) = -t^{-\theta-1}
$$

$$
\frac{d\psi^{-1}}{ds}(s) = (\theta s + 1)^{-\frac{1}{\theta}-1} = -\psi^{-1}(s)(\theta s + 1)^{-1}.
$$

We will consider the mapping $h(u) = C(u^\delta)^\gamma$ and have to show that it is convex. From Remark 3.2 it follows that it is sufficient to consider $u \in (0,1]^m$ only. This implies that $C(u^\gamma) > 0$ and in particular that $h(u)$ and its derivatives are well defined. We therefore fix $u \in (0,1]^m$ arbitrarily and compute the first and second order derivatives of $h$. A computation gives

$$
\frac{\partial h}{\partial u_k}(u) = \delta C(u^\gamma)^{\gamma-1} \frac{\partial C}{\partial z_k}(u^\gamma)^{\frac{1}{\gamma}} u_k^{-\frac{1}{\gamma}}, k = 1, ..., m.
$$

(20)

When computing the derivative of the Copula, we obtain:

$$
\frac{\partial C}{\partial u_k}(u) = \frac{d\psi^{-1}}{ds}(\sum_{i=1}^m \psi(u_i)) \frac{d\psi}{dt}(u_k) = C(u)(\theta \sum_{i=1}^m \psi(u_i) + 1)^{-1} u_k^{-\theta-1}, k = 1, ..., m
$$

(21)

Substituting altogether we get:

$$
\frac{\partial h}{\partial u_k}(u) = \delta \frac{\partial C}{\partial u_k}(u) C(u^\gamma)^{\gamma-1} \sum_{i=1}^m \psi(u_i)^{\frac{1}{\gamma}} (\theta \sum_{i=1}^m \psi(u_i)^{\frac{1}{\gamma}} + 1)^{-1} u_k^{-\frac{2\theta+1}{\gamma}}, k = 1, ..., m.
$$

(22)
Deriving a second time, we get
\[
\frac{\partial^2 h}{\partial u_k^2}(u) = \delta \left( \theta \sum_{i=1}^{m} \psi(u_i) + 1 \right)^{-1} u_k^{-\frac{\theta+\gamma}{\gamma}} \frac{\partial}{\partial u_k} C(u^\gamma) \delta \\
+ \frac{\delta}{\gamma} C(u^\gamma) \delta u_k^{-\frac{\theta+\gamma}{\gamma}} \frac{\partial}{\partial u_k} \left( \theta \sum_{i=1}^{m} \psi(u_i) + 1 \right)^{-1} \frac{\partial^2 h}{\partial u_k^2}(u) \\
+ \frac{\delta}{\gamma} C(u^\gamma) \delta \left( \theta \sum_{i=1}^{m} \psi(u_i) + 1 \right)^{-1} \frac{\partial}{\partial u_k} u_k^{-\frac{\theta+\gamma}{\gamma}}, k = 1, \ldots, m.
\]

In the first line we can just substitute (22), in the second line we obtain
\[
\frac{\partial}{\partial u_k} \left( \theta \sum_{i=1}^{m} \psi(u_i) + 1 \right)^{-1} = -\left( \theta \sum_{i=1}^{m} \psi(u_i) + 1 \right)^{-2} \frac{\partial^2 h}{\partial u_k^2}(u) \\
= \frac{1}{\gamma} \left( \theta \sum_{i=1}^{m} \psi(u_i) + 1 \right)^{-1} u_k^{-\frac{\theta+\gamma}{\gamma}}, k = 1, \ldots, m.
\]

and in the third line,
\[
\frac{\partial}{\partial u_k} u_k^{-\frac{\theta+\gamma}{\gamma}} = -\frac{\theta + \gamma}{\gamma} u_k^{-\frac{\theta+\gamma}{\gamma}-1}, k = 1, \ldots, m. \tag{23}
\]

For the cross-derivative, \( j, k = 1, \ldots, m \), the expression becomes:
\[
\frac{\partial^2 h}{\partial u_j \partial u_k}(u) = \frac{\delta}{\gamma} \left( \theta \sum_{i=1}^{m} \psi(u_i) + 1 \right)^{-1} u_k^{-\frac{\theta+\gamma}{\gamma}} \frac{\partial}{\partial u_j} C(u^\gamma) \delta \\
+ \frac{\delta}{\gamma} C(u^\gamma) \delta u_k^{-\frac{\theta+\gamma}{\gamma}} \frac{\partial}{\partial u_j} \left( \theta \sum_{i=1}^{m} \psi(u_i) + 1 \right)^{-1}, j \neq k
\]

In the first line one substitutes (22) with \( k \) replaced by \( j \) and the second is dealt with similarly as before. Combining these expression we obtain for all \( u \in (0, 1]^m \), \( k, j = 1, \ldots, m \):
\[
\frac{\partial^2 h}{\partial u_k^2}(u) = \frac{\delta^2 + \delta \theta}{\gamma^2} C(u^\gamma) \delta \left( \theta \sum_{i=1}^{m} \psi(u_i) + 1 \right)^{-2} \left( u_k^{-\frac{\theta+\gamma}{\gamma}} \right)^2 \\
- \frac{\delta}{\gamma} \left( \theta \sum_{i=1}^{m} \psi(u_i) + 1 \right)^{-1} u_k^{-\frac{\theta+\gamma}{\gamma}-1}
\]

\[
\frac{\partial^2 h}{\partial u_j \partial u_k}(u) = \frac{\delta^2 + \delta \theta}{\gamma^2} C(u^\gamma) \delta \left( \theta \sum_{i=1}^{m} \psi(u_i) + 1 \right)^{-2} u_k^{-\frac{\theta+\gamma}{\gamma}} u_j^{-\frac{\theta+\gamma}{\gamma}}, j \neq k.
\]

We have to show that the Hessian is positive semi-definite for all \( u \in (0, 1]^m \), we will do this directly, by picking \( z \neq 0, z \in \mathbb{R}^m \) and forming \( z^T \nabla^2 h(u) z \). The special structure of the above second-derivatives is as follows \( \frac{\partial^2 h}{\partial u_k^2}(u) = \alpha(u) x_k^2 + \beta_k(u) \) and \( \frac{\partial^2 h}{\partial u_j \partial u_k}(u) = \alpha(u) x_j x_k \). Here we have abbreviated as follows for \( u \in (0, 1]^m, j = \)
1, ..., m:

\[
\alpha(u) = \frac{\delta^2 + \delta \theta}{\gamma^2} C(u^\gamma)^\delta (\theta \sum_{i=1}^{m} \psi(u_i^\gamma) + 1)^{-2},
\]

\[
x_j = u_j - \theta + \gamma
\]

\[
\beta_j(u) = -\frac{\delta \theta + \gamma}{\gamma} C(u^\gamma)^\delta (\theta \sum_{i=1}^{m} \psi(u_i^\gamma) + 1)^{-1} u_j - \gamma - 1.
\]

Now

\[
z^T \nabla^2 h(u) z = \sum_{j=1}^{m} \sum_{k=1}^{m} z_j \frac{\partial^2 h}{\partial u_j \partial u_k}(u) z_k
\]

\[
= \sum_{j,k \neq k} \alpha(u) z_j x_j x_k z_k + \sum_{k=1}^{m} \alpha(u) z_k x_k^2 + z_k^2 \beta_k(u)
\]

\[
= \alpha(u) \sum_{j=1}^{m} \sum_{k=1}^{m} z_j x_j x_k z_k + \sum_{k=1}^{m} z_k^2 \beta_k(u)
\]

\[
= \alpha(u) \left( \sum_{j=1}^{m} z_j x_j \right)^2 + \sum_{k=1}^{m} z_k^2 \beta_k(u).
\]

Substituting the above expressions, we obtain:

\[
z^T \nabla^2 h(u) z = C(u^\gamma)^\delta \frac{\delta^2 + \delta \theta}{\gamma^2} (\theta \sum_{i=1}^{m} \psi(u_i^\gamma) + 1)^{-2} (\sum_{i=1}^{m} z_i u_i^{-\frac{\delta \gamma + 1}{\gamma}})^2
\]

\[
- C(u^\gamma)^\delta \frac{\delta \theta + \gamma}{\gamma} (\theta \sum_{i=1}^{m} \psi(u_i^\gamma) + 1)^{-1} \sum_{i=1}^{m} z_i^2 u_i^{-\frac{\delta \gamma + 1}{\gamma} - 1}.
\] (24)

Now with our choice of \(\delta\), we obtain \(\delta^2 + \delta \theta \geq 0\) and similarly \(-\frac{\delta \gamma}{\gamma} \geq 0\). Together with \(u \in (0, 1]^m\), one can see that all expressions in (24) are positive. This therefore yields \(z^T \nabla^2 h(u) z \geq 0\), i.e., \(h\) is a convex function, as was to be shown. \(\square\)

**Remark 5.1:** Numeric evidence would indicate that the Clayton Copula is also \(\delta\)-\(\gamma\)-concave for \(\delta\) slightly bigger than \(-\theta\), but not for all \(\delta < 0\). Indeed, in dimension 2, picking \(\delta = -0.03\), \(\theta = 0.1\), \(\gamma = 0.5\) and evaluating the above Hessian at the point \(u = (0.96, 0.985)\), one obtains a negative eigenvalue.

**Remark 5.2:** We can also prove that the Clayton Copula is \(\delta\)-0 concave for \(\delta \leq -\theta\). Indeed setting \(h(u) = C(e^u)^\delta\) one obtains for \(u \in (-\infty, 0]^m\):

\[
\frac{\partial h}{\partial u_k}(u) = \delta C(e^u)^\delta (\theta \sum_{i=1}^{m} \psi(e^{u_i}) + 1)^{-1} e^{-\theta u_k}, k = 1, ..., m
\]
and for \( j, k = 1, \ldots, m \):
\[
\frac{\partial^2 h}{\partial u_i^2}(u) = (\delta^2 + \delta \theta) C(e^u)^\delta (\theta \sum_{i=1}^{m} \psi(e^{u_i}) + 1)^{-2} e^{-2\theta u_k} \\
- \theta \delta C(e^u)^\delta (\theta \sum_{i=1}^{m} \psi(e^{u_i}) + 1)^{-1} e^{-\theta u_k} \\
\frac{\partial^2 h}{\partial u_j \partial u_k}(u) = (\delta^2 + \delta \theta) C(e^u)^\delta (\theta \sum_{i=1}^{m} \psi(e^{u_i}) + 1)^{-2} e^{-\theta u_k} e^{-\theta u_j}, j \neq k.
\]

And the same results follow.

We have shown in Lemma 5.5 that the Clayton Copula is \( \delta-\gamma \)-concave for all \( \gamma > 0 \) and \( \delta \leq -\theta \). Since this Copula is not log-exp concave, the results of [9] could not be applied. We can however use our theorem to derive convexity of feasible sets \( M(p) \). As such we get the example:

**Example 5.6** Consider again the same setting as that of example 5.1, except that this time we use the Clayton Copula to link the components of \( \xi \) together. Since this Copula is \( \delta-\gamma \)-concave for any \( \gamma > 0 \) and \( \delta \leq -\theta \), we have to show that the mappings \( z \mapsto F(1/z) \) are \( \gamma \)-concave on some set \((0,(t^\#)^{-1})\). Since results hold in particular for \( \gamma < 1 \), concavity of those maps suffices. That is obtained whenever their densities are for instance 2-decreasing. Assuming that \( \xi \) follows an exponential distribution with parameter \( \lambda \), we obtain the very rough bound \( p^* = 1 - e^{-2} = 0.864 \). But we can do better, to this end we have to show that the mapping \( h(z) = F(1/z)^\gamma = (1 - \exp(-\frac{2}{z}))^\gamma \), is concave. A computation gives
\[
h'(z) = -\lambda \gamma z^{-2}(1 - \exp(-\lambda z^{-1}))^{-1} \exp(-\lambda z^{-1}) \\
h''(z) = h'(z)[-2z^{-1} - (\gamma - 1)\lambda z^{-2}(1 - e^{-\lambda z^{-1}})^{-1} e^{-\lambda z^{-1}} + \lambda z^{-2}].
\]

Applying Theorem 4.1, and picking for instance \( \gamma = \frac{1}{2} \), one obtains \( z^\# = 0.54807 \), giving the estimate \( p^* = 0.8387 \). With \( \gamma = 0.01 \), one obtains \( z^\# = 0.62537 \) and the estimate \( p^* = 0.7979 \). Again \( p^* \) does not depend on \( \lambda \).

We can also apply Theorem 4.2 to obtain \( p^* = 0.638 \) for \( \theta = 0.1 \). This time the result depends on \( \theta \) as can be shown empirically.

6. A Partial Characterization of the Gaussian Copula

So far we have only provided examples of Archimedean Copulas with the \( \delta-\gamma \)-concavity property. In this section we investigate the Gaussian Copula and provide a partial characterization of the \( \delta-\gamma \)-concavity properties of this Copula.

**Lemma 6.1:** Let \( R \) be an \( m \) times \( m \) correlation matrix. Assume furthermore that the inverse matrix is such that \( Q := R^{-1} - I \) has only positive components. Let \( \delta = \min_{i=1,\ldots,m} \max_{j=1,\ldots,m} Q_{ij} \) and define \( q = \max\{\log(\Phi(\frac{1}{2})),\log(\Phi(1))\} \), then the density of the Gaussian Copula is 0-0-concave for all \( u \in [q,0]^m \), where \( \Phi \) is the probability distribution function of a standard normal random variable.
Proof: Define the mapping \( f : (-\infty, 0]^m \rightarrow \mathbb{R}^m \) by setting \( f(u) = \varphi^{-1}(e^u) = (\varphi^{-1}(e^{u_1}), ..., \varphi^{-1}(e^{u_m})) \), where \( \varphi^{-1} \) is the inverse of the standard normal distribution function. We wish to show that the density of the Gaussian Copula wherein we substitute \( e^u \), is eventually log-concave, i.e., the mapping \( u \mapsto c_R(e^u) \) is concave. Now the density \( c_R \) of the Gaussian Copula is given by

\[
c_R(u) = \frac{1}{\sqrt{\det R}} \exp\left(-0.5 \varphi^{-1}(u)^T(R^{-1} - I)\varphi^{-1}(u)\right). \tag{25}
\]

This boils down to showing that the mapping \( u \mapsto h(u) := f(u)^T(R^{-1} - I)f(u) \) is convex.

We will begin by computing some derivatives. To this end, fix any \( u \in (-\infty, 0]^m \) completely arbitrarily. For convenience we will note \( Q := R^{-1} - I \). We begin by computing the first and second derivative of a component of the vector \( f(u) \), i.e.,

\[
\frac{df_i}{dv}(v) = \frac{d\varphi^{-1}(e^u)}{dv}(e^v)e^v = \sqrt{2\pi} \exp\left(\frac{1}{2} \varphi^{-1}(e^v)^2\right)e^v, \ i = 1, ..., m, \ \forall v \in (-\infty, 0]
\]

and

\[
\frac{d^2 \varphi^{-1}}{ds^2}(s) = \sqrt{2\pi} \exp\left(\frac{1}{2} \varphi^{-1}(s)^2\right)\frac{1}{2} \varphi^{-1}(s) \frac{d^2 \varphi^{-1}}{ds}(s)
\]

\[
= (\sqrt{2\pi} \exp\left(\frac{1}{2} \varphi^{-1}(s)^2\right))^2 \varphi^{-1}(s), \ \forall s \in (0, 1).
\]

Giving

\[
\frac{d^2 f_i}{dv^2}(v) = \frac{d^2 \varphi^{-1}}{ds^2}(e^v)e^v + \frac{df_i}{dv}(v)
\]

\[
= f_i(v)(\frac{df_i}{dv}(v))^2 + \frac{df_i}{dv}(v), \ i = 1, ..., m, \ \forall v \in (-\infty, 0].
\]

Now we can compute the derivatives of the mapping \( h \),

\[
\frac{\partial h}{\partial u_k}(u) = 2 \frac{df_k}{du}(u_k) \sum_{j=1}^m Q_{kj} f_j(u_j), \ k = 1, ..., m
\]

and the second derivatives are:

\[
\frac{\partial^2 h}{\partial u_k^2}(u) = 2 \frac{d^2 f_k}{du^2}(u_k)Q_{kk} f_k(u_k) + 2 \frac{df_k}{du}(u_k)Q_{kk} \frac{df_k}{du}(u_k)
\]

\[
+ 2 \sum_{j=1, j \neq k}^m \frac{d^2 f_k}{du^2}(u_k)Q_{kj} f_j(u_j), \ k = 1, ..., m
\]

\[
\frac{\partial^2 h}{\partial u_j \partial u_k}(u) = 2 \frac{df_k}{du}(u_k)Q_{kj} \frac{df_j}{du}(u_j), \ j, k = 1, ..., m, \ j \neq k
\]

Substituting in the above expression the previously found identity for the second
derivative of $f_i$, we obtain

$$
\frac{\partial^2 h}{\partial u_k^2}(u) = 2 \frac{df_k}{du}(u_k)Q_{kk}\frac{df_k}{du}(u_k) + 2 \sum_{j=1}^{m} \frac{d^2 f_k}{du^2}(u_k)Q_{kj}f_j(u_j)
$$

$$
= 2 \frac{df_k}{du}(u_k)Q_{kk}\frac{df_k}{du}(u_k)
$$

$$
+ 2 \frac{d^2 f_k}{du^2}(u_k)^2[(f_k(u_k) + (\frac{df_k}{du}(u_k))^{-1})\sum_{j=1}^{m} Q_{kj}f_j(u_j)], \quad k = 1, \ldots, m
$$

where we have used that $\frac{df_k}{du}(u_k) > 0$ for all $u \in (-\infty, 0]^m$. Now picking $z \in \mathbb{R}^m$ and forming $z^T \nabla^2 h(u)z$, one obtains the following

$$
z^T \nabla^2 h(u)z = 2 \sum_{i,j=1}^{m} z_i \frac{df_i}{du}(u_k)Q_{ij} \frac{df_j}{du}(u_j)z_j
$$

$$
+ 2 \sum_{k=1}^{m} z_k \frac{d^2 f_k}{du^2}(u_k)^2[(f_k(u_k) + (\frac{df_k}{du}(u_k))^{-1})\sum_{j=1}^{m} Q_{kj}f_j(u_j)],
$$

which holds for all $u \in (-\infty, 0]^m$ as it was chosen arbitrarily.

If we define the vector $x(u) \in \mathbb{R}^m$ as $x(u) = (z_1 \frac{df_1}{du}(u_1), \ldots, z_m \frac{df_m}{du}(u_m))$, and the matrix $\bar{Q}(u)$ as

$$
\bar{Q}(u)_{ij} = \begin{cases} 
Q_{ii} + (f_i(u_i) + (\frac{df_i}{du}(u_i))^{-1})\sum_{j=1}^{m} Q_{ij}f_j(u_j) & \text{if } i, j = 1, \ldots, m, i \neq j \\
\bar{Q}_{ii} & \text{if } i = 1, \ldots, m
\end{cases}
$$

It is clear that $z^T \nabla^2 h(u)z = x(u)^T \bar{Q}(u)x(u)$ and $z^T \nabla^2 h(u)z \geq 0$ if and only if $\bar{Q}(u)$ is positive semi-definite. Defining the vector $\alpha(u) = (\alpha_1(u), \ldots, \alpha_m(u)) \in \mathbb{R}^m$ as

$$
\alpha_i(u) = (f_i(u_i) + (\frac{df_i}{du}(u_i))^{-1})\sum_{j=1}^{m} Q_{ij}f_j(u_j), \quad i = 1, \ldots, m
$$

it is clear that $\bar{Q}(u) = Q + \text{diag } \alpha(u)$, where $\text{diag } \alpha(u)$ is the diagonal matrix with elements of the vector $\alpha(u)$. It therefore follows that the eigenvalues of $\bar{Q}(u)$ are those of $R^{-1}$ to which we add $\alpha(u)-1$. Making sure that $\alpha_i(u)-1 \geq 0, \quad \forall i = 1, \ldots, m$ is therefore sufficient for $Q(u)$ to be positive semi-definite.

In order to show this, let $i = 1, \ldots, m$ be arbitrary. Since the mapping $f$ is strictly increasing, $u_i \geq q$ implies $f_i(u_i) \geq f_i(q) \geq \frac{1}{\delta} > 0$. It was observed above that $\frac{df_i}{du}(u_i) > 0$ for all $u_i \in (-\infty, 0]$. Altogether this provides the estimate:

$$
\alpha_i(u) \geq \frac{1}{\delta} \sum_{j=1}^{m} Q_{ij}f_j(u_j).
$$

According to the definition of $\delta$ there exists an index $j^*$ such that $\sum_{j=1}^{m} Q_{ij^*}f_j(u_j) \geq \delta f_j(q)$, since $Q_{ij} \geq 0$ for all $j = 1, \ldots, m$ and $f_j(u_j) \geq 0$ for all $u_j \geq q$. The choice of $q$ also implies $f_j(u_j) \geq 1, \quad \forall j = 1, \ldots, m$. By combining the above estimates we
have obtained that $\alpha_i(u) \geq \frac{1}{\delta} \sum_{j=1}^{m} Q_{ij} f_j(u_j) \geq 1$. Since $i = 1, ..., m$ was arbitrary the result follows. \qed

**Example 6.2** Let $R$ be the correlation matrix in dimension $\mathbb{R}^2$ having $-0.9$ on the off-diagonal. Then $\delta = 4.74$ and $q = \max \{-0.54, -0.17\}$. In fact for all 2-dimensional matrices with off-diagonal element $\rho \leq 1 - \sqrt{5}/2$, this same bound holds.

### 7. A Potential Application

In some engineering problems in energy one easily stumbles across mixed laws in "columns". In particular, offer demand equilibrium constraints in unit commitment require that we commit a production schedule producing enough energy in most situations. However, uncertainty is only discovered later. This uncertainty consists of load uncertainty and uncertainty on renewable generation such as wind power. Wishing to produce enough electricity in most situations for all time steps simultaneously would then result in a constraint of the type

$$ p \leq P[\xi + \eta \leq g(x)], \quad (26) $$

where $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the mapping associating with a decision vector $x \in \mathbb{R}^n$ its actual production level and $\xi \in \mathbb{R}^m$ and $-\eta \in \mathbb{R}^m$ are two random vectors, modeling for instance load uncertainty and wind generation respectively. Now to show that Copulae can be used to obtain a convex model requires an additional result. Theorem 4.2.3. of [13] (dating back to [3–5], [18, proof in dimension 1]) indicates that the convolution of log-concave densities is again a log-concave density. As such, picking each $\xi_i, \eta_i$ individually following a log-concave density, it follows that their sum follows a log-concave density, i.e., $F_i(z) = P[\xi_i + \eta_i \leq z]$ is a log-concave function. If we now use a $\delta$-$\gamma$-concave Copula with $\gamma \leq 0$ and concave functions $g_i(x)$, the set defined by (26) will then turn out to be eventually convex. Since in practice $g_i$ is often linear, as a sum of production levels, it will be concave. Hence we can come up with a tractable model for such a mixed law setting.

### 8. Conclusions and Perspectives

In this paper we have provided a new insight in the eventual convexity results obtained in [9]. We have shown that those results can be extended to a larger class of Copulae and that sharper probability thresholds can be obtained. We have also provided a potential application showing how probabilistic modelling with Copulae could arise in practice. In future work such an example will be explored from a numerical point of view. Other research perspectives consist of examining if the provided probability thresholds are tight. The $\delta$-$\gamma$-concavity properties of more Copulae will be investigated equally. In particular it can be conjectured that many Archimedean Copulae exhibit $\delta$-$\gamma$-concavity properties.

**References**

REFERENCES


