Using Modules in Teaching Complex Analysis

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July 26, 2016

Abstract

Educational modules can play an important part in revitalizing the teaching and learning of complex analysis. At the Westmont College workshop on the subject in June 2014, time was spent generating ideas and creating structures for module proposals. Sharing some of those ideas and giving a few example modules is the main purpose of this paper. The author has started to construct *Mathematica*-based educational modules for complex analysis, one of which is available in the online repository of supplemental materials for this article. The author has also begun work on a website called *The Complex Moduli Project* to make modules in complex analysis available worldwide, and invites others to join him in this endeavor.

Keywords: Educational module, complex analysis, *Mathematica*

1 Introduction: What is an Educational Module?

The term *educational module* is flexible and can be envisioned in various ways in the teaching of mathematics. It can refer to a single class period group activity or a semester-long individual research project. It can be focused on developing conceptual understanding, improving problem-solving skills, learning how to do calculations, constructing rigorous proofs, setting up and using mathematical models, or using technology in effective ways. It can also involve demonstration of learning through essays, posters, papers, presentations, quizzes, and tests. For those who are thinking about "flipping" or "inverting" their classrooms, modules can be the main source of content for in-class work, though evidence of effectiveness is still mixed at this point [1, 5]

Are there any unifying features that can be described for educational modules? The following quote from an *eHow* webpage emphasizes that an educational module focuses on a single topic, includes background material, is activity-based, and builds toward projects.

> "In education, the term 'module' refers to an instructional unit that focuses on a particular topic. Although the details and activities vary according to the specific context, such as course and student level, most educational modules include information about the topic, focus on student-centered learning activities and culminate in a project for students to demonstrate understanding" [3]

At a *Reference* webpage, the emphasis is on the root meaning of the word, that a module is always a subunit of a whole.

> "In education, a 'module' is a fractional part of a student's education experience. In an entire degree program, each class represents a module focused on a given subject. In a single class, a module is a chapter, class meeting or lecture on a specific topic." [4]

In this paper, we will list topics in complex analysis that might lend themselves well to a module structure as described on *eHow*, give suggestions for module construction, and give outlines of a couple example modules. We also want to encourage reader participation in constructing a website to house complex analysis modules and provide a forum for interaction among authors and instructors.

2 Complex Analysis Module Topics and a Computer Tool

At the Revitalizing Complex Analysis Workshop at Westmont College in June 2014, time was spent by the participants on generating ideas for module topics. Then a small working group, which included the author along with Paul Zorn (St. Olaf, Northfield, MN) and Tamas Forgacs (California State University, Fresno), spent an afternoon fleshing out a few of those ideas into a structure with more details. We chose to initially split the topics into foundational topics and specialized topics. A full list of these topics can be requested from the author, but the foundational topics included: complex arithmetic and geometry, mapping and iterative behavior of specific basic functions, the Cauchy-Riemann equations, geometric and physical interpretations of complex integration, applications of Cauchy's theorem and the residue theorem, and tricks for calculating Taylor and Laurent series. Specialized topics included: infinite products and the Basel problem, quaternions and computer graphics, the prime number theorem, the Dirichlet problem, complex functions of several variables, linear operators on $\mathbb{R}[x]$ and $\mathbb{C}[x]$, Riemann surfaces, and various transforms (e.g., Fourier, Laplace, Hilbert, and Z). It was also suggested to emphasize, when possible, real-world applications of the module topics.

The author has an interest in using the computer algebra system *Mathematica* as a tool to create modules on complex analysis. *Mathematica*'s dynamic features can be used to make interactive plots that can solidify understanding both as they are constructed and as they are used. Of special interest is the use of *Mathematica* to animate the diagrams shown in *Visual Complex Analysis*, by Tristan Needham [7], and ultimately in making modules where students learn how to do this. Examples are given in the online repository of supplemental materials for this article.[1] More extensive examples can be found in [6], along with instructional videos at the author's *YouTube* channel.[2]

3 Example Module Outlines

The module outlines provided in what follows are flexible and open to revision, but the basic overarching framework the small working group suggested consisted of outlines typically of the form: 1) topic title, 2) short description of the topic, 3) longer description of some details of the topic along with basic examples, sample exercises to consider, questions to discuss and debate, and possible applications, 4) prerequisite knowledge, 5) the role played by complex analysis and/or the importance of the topic in complex analysis, 6) timeframe, and 7) resources. Items can be added, deleted, or rearranged by module constructors as desired.

3.1 Example Module Outline: Complex Arithmetic and Geometry

3.1.1 Description

In this module, students work to understand how complex numbers are added, subtracted, multiplied, and divided, and how to geometrically interpret these operations. The relative pros and cons of rectangular and polar coordinate systems can be considered. Euler's formula, application to trigonometric identities, and construction of roots can be highlights.

3.1.2 More Detailed Description: Ordering of Topics

By analogy with linear polynomials, students can conjecture how to add and subtract complex numbers, treating i as an indeterminate. The group properties of addition can then be tested and subtraction can be defined. Using rectangular coordinates, vectors, and prior knowledge, students can then discover, understand, and justify the parallelogram rule and they can be led to the triangle inequality. Related arithmetic conjectures and confirmations about multiplication can lead them to recognize that $\mathbb{C}$ is a field and division can then be defined. At this point, geometrically interpreting multiplication and division is the real prize, and will probably take a lot of conjecturing and testing, especially for those completely unfamiliar with the topic. Ultimately, the students should summarize their findings in polar coordinates. The instructor can then decide how to go about introducing and/or getting the students to use the complex

[1]The supplemental materials include four *Mathematica* notebooks that serve as a module on complex arithmetic and the complex plane, as well as some stand-alone *Mathematica* code to generate pictures found in [7].

[2]A playlist on complex arithmetic along with *Mathematica* instructions can be found at https://www.youtube.com/watch?v=wWBmwxzlTqU&list=PLmU0FIlJY-Mlju-7wtPuyX-JirE02rG8C

exponential and Euler's formula. Applications to trigonometric identities and construction of roots can be done if more time is allotted and guidance given.

3.1.3 Prerequisites

1. Field properties of $\mathbb{R}$
2. Rectangular and polar coordinates
3. Vectors and the parallelogram rule
4. A willingness to explore what happens when facts about real numbers are assumed to hold for complex numbers (e.g., properties of exponents)
5. (If desired) Taylor series facts

3.1.4 Timeframe

Two to Four 50-Minute Units: 10-20 minute introduction, 70-150 minutes for student exploration and summary of results, 20-30 minutes for wrap-up.

As the students explore, conjecture, and justify, instructors can have them share their hypotheses, calculations, and reasoning with the rest of the class.

3.1.5 Importance to Complex Analysis

Complex arithmetic in the complex plane is the foundation of everything.

3.1.6 Resources

Any standard text in complex analysis. The author enjoys using [8]. Online applets to test conjectures. Students with knowledge of *Mathematica* or another computer algebra system can confirm their conjectures symbolically and visually.

3.2 Example Module Outline: The Squaring Map $z \mapsto z^2$

3.2.1 Description

Viewing a complex function as a mapping is a conceptual leap for students. Though linear maps should be considered first, the squaring map $z \mapsto z^2$ is a natural second example. This simple function has much more subtlety to its behavior than students can possibly imagine, and it can ultimately lead to a study of chaotic behavior, Julia sets, and the Mandelbrot set.

3.2.2 More Detailed Description: Ordering of Topics

First, consider something the students might know already: squaring a complex number squares its modulus and doubles its argument. Next, consider the most basic behavior of $z \mapsto z^2$ under iteration. It has two fixed points at $z = 0$ and $z = 1$. For a seed $z_0 \in \mathbb{C}$, the points on the orbit tend to zero if $|z_0| < 1$, to infinity if $|z_0| > 1$, and have a constant modulus if $|z_0| = 1$. It's probably best to next consider how it maps simple geometric shapes, like disks and rectangles. The local behavior of the mapping can be studied if students are familiar with the geometric interpretation the complex derivative. If students are comfortable using the Jacobian determinant of the corresponding real mapping $\mathbb{R}^2 \to \mathbb{R}^2$ to compute double integrals with a change of variable, then consideration can be given to how the mapping affects areas. Finally, if there's time and if the instructor and students have patience and desire, the chaotic behavior of the mapping on the unit circle can be explored and the ties to the mapping $z \mapsto z^2 + c$, Julia sets, and the Mandelbrot set can be made.

3.2.3 Prerequisites

1. Complex arithmetic, the complex plane, rectangular and polar forms
2. Basic ideas about iteration
3. Parameterizations of circles and lines
4. Change of variables for double integrals using the Jacobian determinant
5. A willingness to explore

3.2.4 Timeframe

One to Four 50-Minute Units: 10-30 minute introduction, 30-130 minutes for student exploration and summary of results, 10-40 minutes for wrap-up.

3.2.5 Importance to Complex Analysis

The idea of a mapping is central to complex analysis, though we aren't getting at the true heart of the subject beyond basic ideas and the complex derivative here. This topic can spur interest and appreciation of unexpected and beautiful features of the subject, and lead to a greater desire to study it in more depth.

3.2.6 Resources

Any standard text in complex analysis. Online applets to test conjectures. Students with knowledge of *Mathematica* or another computer algebra system can confirm their conjectures symbolically and visually.

3.3 Example Module Outline: Convergence of Infinite Products

3.3.1 Description

In this module, students work to formulate a definition of convergence for infinite products, formulate and prove basic convergence criteria, and apply their criteria to typical examples. This can then lead to an instructor-led consideration of the "Basel problem", i.e. Euler's derivation of $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ via expansion of $\sin(z) = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\pi^2}\right)$. It can also lead to consideration of special functions such as the gamma function $\Gamma(z)$ and the Riemann zeta function $\zeta(z)$.

3.3.2 More Detailed Description: Definition and Tests of Convergence, Examples

Hopefully students can initially work toward construction of some form of the following definition, though they should have comfort with sequences and infinite sums before considering infinite products.

Definition: Let $\{a_n\}$ be a sequence of nonzero complex numbers and let $p_n = \prod_{k=1}^{n} a_k$. The expression $\prod_{k=1}^{\infty} a_k$ is called an **infinite product** and the sequence $\{p_n\}$ is called the corresponding **sequence of partial products**. The infinite product $\prod_{k=1}^{\infty} a_k$ is said to **converge** if the sequence $\{p_n\}$ converges to a nonzero number and the infinite product is said to **diverge** otherwise.

Questions to Discuss and Debate: Why not allow convergence to zero? If we allowed convergence to zero, would we want to put extra conditions on the factors? What are some reasons we might want to allow convergence to zero?

Another goal should be to have the students work toward the derivation of convergence/divergence tests, and possibly their proofs.

Basic Convergence/Divergence Tests:

1. If the sequence $\{a_n\}$ does not converge to 1, then $\prod_{k=1}^{\infty} a_k$ diverges.

2. $\prod_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} \text{Log}(a_k)$ converges, where $\text{Log}(z)$ is the principal branch of the logarithm. (In conjunction with the questions to discuss and debate above, there can be consideration of what happens if we allow $a_k = 0$ for some values of k).

3. If $\{r_k\}$ is a sequence of nonnegative real numbers, then $\prod_{k=1}^{\infty}(1 + r_k)$ converges if and only if $\sum_{k=1}^{\infty} r_k$ converges. (Use convenient inequalities, one of which is based on $e^x > 1 + x$ for all $x > 0$, and the monotone convergence theorem for real sequences to prove this).

Students will probably need some help to start thinking of examples. Instructors may want to suggest these at the beginning of the module.

Basic Examples to Consider:

1. $\prod_{k=2}^{\infty}\left(1 - \frac{1}{k^2}\right)$

   Expressing the factors as $1 - \frac{1}{k^2} = \frac{(k-1)(k+1)}{k^2}$ leads to a simple formula for p_n and a simple limit calculation to get the value $\frac{1}{2}$ for the infinite product.

2. $\prod_{k=1}^{\infty}\left(1 + z^{2^{k-1}}\right)$

   Use induction to prove p_n is a finite geometric series. For what values of z does this converge?

3. $\prod_{k=1}^{\infty}\left(1 + \frac{1}{k^p}\right)$

   For real p, converges if $p > 1$ and diverges if $p \leq 1$

3.3.3 Prerequisites

1. Definition of convergence of a sequence of complex numbers

2. Familiarity with convergence criteria for infinite sums

3. Familiarity with geometric series

4. Familiarity with the complex logarithm

3.3.4 Timeframe

One to Two 50-Minute Units: 10-20 minute introduction, 30-60 minutes for student exploration (of simplest examples and basic definition), 10-20 minutes for wrap-up. More time would probably be needed for construction and proofs of convergence/divergence tests. Consider making exercises out of harder examples.

As the students explore, conjecture, and prove, the instructor can write well-known convergence tests for infinite series on the board, as well as the formula for a finite geometric series and for infinite geometric series whose common ratio has modulus less than 1.

3.3.5 Importance to Complex Analysis

Discussion of applications to the Basel problem, the gamma function, Stirling's formula, the Riemann zeta function, and the prime number theorem can be hinted at or explored as exercises or in class if more time is available.

3.3.6 Resources

A text with convergence criteria for infinite series and possibly infinite products is helpful. A computer algebra system, such as *Mathematica*, to test conjectures should be considered.

4 The Complex Moduli Project

In addition to doing work creating *Mathematica*-based modules, the author would like to work with others in writing complex analysis educational modules in other formats and making them available for download to instructors and students worldwide. We could proof-read and critique each others' work and generate new ideas together. Towards this end, the author has started work on a website, titled *The Complex Moduli Project*, to host this effort. The name is intended to tie together in a mildly humorous way the idea of an educational module with the notion of the modulus of a complex number ("moduli" being the *actual* plural form of "modulus" and the *plausible (though inaccruate)* plural form of "module").

Presently the project can be found at a *Google Sites* webpage [2]. Please contact the author if you are interested in joining in this endeavor and especially if you also have skills at website development.

5 References

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