Perfect matchings in regular graphs from eigenvalues

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Abstract

Let $G$ be a $k$-regular graph of even order. We find a best upper bound on the third largest adjacency eigenvalue $\lambda_3(G)$ that is sufficient to guarantee that $G$ has a perfect matching.

Key words: perfect matching, eigenvalues

1 Preliminaries

Throughout, $G$ denotes a simple graph of order $n$ (the number of vertices) and size $e$ (the number of edges). The eigenvalues of $G$ are the eigenvalues $\lambda_i$.

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of its adjacency matrix $A$, indexed so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The greatest eigenvalue, $\lambda_1$, is also called the spectral radius. If $G$ is $k$-regular, then it is easy to see that $\lambda_1 = k$ and also, $\lambda_2 < k$ if and only if $G$ is connected.

The eigenvalues of a graph are related to many of its properties and key parameters. The most studied eigenvalues have been the spectral radius $\lambda_1$ (in connection with the chromatic number, the independence number and the clique number of the graph [12,13,20,18]), $\lambda_2$ (in connection with the expansion property of the graph [14]) and $\lambda_n$ (in connection with the chromatic and the independence number of the graph [13] and the maximum cut [16]). We refer the reader to the monographs [5,9–11] as well as the recent surveys [14,16] for more details about eigenvalues of graphs and their applications.

In [3], Brouwer and Haemers give sufficient conditions for the existence of a perfect matching in a graph in terms of its Laplacian eigenvalues and, for a regular graph, give an improvement in terms of the third largest adjacency eigenvalue, $\lambda_3$. In this paper, we refine the result of [3] for regular graphs as well as some results from [6,8]. Our main result is the following theorem. There, $\theta$ is the greatest solution of $x^3 - x^2 - 6x + 2 = 0$.

**Theorem 1** If $G$ is a connected $k$-regular graph of even order such that

$$
\lambda_3 < \begin{cases} 
\theta = 2.85577 \ldots & \text{if } k = 3 \\
\frac{1}{2} \left( k - 2 + \sqrt{k^2 + 12} \right) & \text{if } k \text{ is even} \\
\frac{1}{2} \left( k - 3 + \sqrt{(k + 1)^2 + 16} \right) & \text{if } k \text{ is odd}
\end{cases} 
$$

(1)

then $G$ contains a perfect matching.

We also show that Theorem 1 is best possible by presenting for each $k \geq 3$, an example $G(k)$ of a $k$-regular graph of even order with no perfect matching and with $\lambda_3(G(k))$ equal to the upper bound in the theorem.

### 2 The proof of Theorem 1

Let $\mathcal{H}(k)$ denote the class of all connected irregular graphs with maximum degree $k$, odd order $n$, and size $e$ with $2e \geq kn - k + 2$. Suppose also that each graph in $\mathcal{H}(k)$ has at least 4 vertices of maximum degree $k$ if $k$ is odd and at least 3 if $k$ is even. In the course of the proof of Theorem 3.1 in [3], it is shown that if $G$ is a connected $k$-regular graph of even order with no perfect matching, then $(k \geq 3$ and) $G$ has three vertex disjoint induced subgraphs $H_1, H_2, H_3$ in $\mathcal{H}(k)$. Consequently, by interlacing,

$$
\lambda_3(G) \geq \lambda_3(H_1 \cup H_2 \cup H_3) \geq \min_i \lambda_1(H_i)
$$

(2)
For the sake of completeness, we present now the argument of Brouwer and Haemers. Let $G$ be a connected $k$-regular graph of even order without a perfect matching. By Tutte’s theorem [19, p.137], there is a set $S$ of vertices of $G$ such that the number $q$ of odd order components of $G \setminus S$ is greater than $s = |S|$. Because $G$ has even order, $s > 0$. Let $H_1, \ldots, H_q$ denote the odd components of $G \setminus S$. Denote by $n_i$ and $e_i$ the order and the size of $H_i$, respectively. It is easy to see that $\sum_{i=1}^q n_i + s$ is even. Since each $n_i$ is odd, it follows that $q + s$ is even and, because $q > s$, that $q \geq s + 2$.

For $i \in [q]$, denote by $t_i$ the number of edges with one endpoint in $H_i$ and the other in $S$. Because $G$ is connected, it follows that $t_i \geq 1$ for each $i \in [q]$. Also, since vertices in $H_i$ are adjacent only to vertices in $H_i$ or $S$, we deduce that $2e_i = kn_i - t_i = k(n_i - 1) + k - t_i$. Because $n_i$ is odd, it follows that $k - t_i$ is even. Thus, $t_i$ and $k$ have the same parity for each $i \in [q]$.

The sum of the degrees of the vertices in $S$ is at least the number of edges between $S$ and $\bigcup_{i=1}^q H_i$. Thus, $ks \geq \sum_{i=1}^q t_i$. Since $q \geq s + 2$, it follows that there are at least three $t_i$’s ($t_1, t_2, t_3$, say) such that $t_i < k$.

Suppose now that $i \in [3]$. Then, $t_i \leq k - 2$, so $n_i > 1$ and $2e_i = kn_i - t_i \geq kn_i - k + 2$. Also, $n_i(n_i - 1) \geq 2e_i \geq kn_i - k + 2$, so $n_i \geq k + 2/(n_i - 1)$. Hence, $n_i \geq k + 2 \geq t_i + 4$ if $k$ is odd and $n_i \geq k + 1 \geq t_i + 3$ if $k$ is even. Thus each of the odd components $H_i$, $i \in [3]$, has at least 4 vertices of degree $k$ if $k$ is odd, and at least 3 if $k$ is even. Thus $H_1, H_2, H_3 \in \mathcal{H}(k)$, as required.

Let $\rho(k) = \min \lambda_1(H)$ where the minimum is taken over all graphs $H$ in $\mathcal{H}(k)$. Then inequality (2) implies the following result.

**Lemma 2** Let $G$ be a connected $k$-regular graph with $k \geq 3$. If $G$ has even order and $\lambda_3(G) < \rho(k)$, then $G$ has a perfect matching.

To prove Theorem 1, it remains to determine $\rho(k)$. Our arguments will require frequent use of the interlacing inequality (3) below.

Suppose that $V = V_1 \cup V_2$ is a partition of the vertex set $V$ of a graph $G$ of order $n$ and size $e$. For $i = 1, 2$, let $G_i$ be the subgraph of $G$ induced by $V_i$, and let $n_i$ and $e_i$ be the order and size, respectively, of $G_i$. Also, let $G_{12}$ be the bipartite subgraph induced by the partition and let $e_{12}$ be the size of $G_{12}$. A theorem of Haemers [12] shows that the eigenvalues of the quotient matrix of the partition interlace the eigenvalues of the adjacency matrix of $G$ (see also Godsil and Royle [11, p.197]). Applying this result to the greatest eigenvalue of $G$, it turns out that

\[
\lambda_1(G) \geq \frac{e_1}{n_1} + \frac{e_2}{n_2} + \sqrt{\left(\frac{e_1}{n_1} - \frac{e_2}{n_2}\right)^2 + \frac{e_{12}^2}{n_1n_2}}, \tag{3}
\]
with equality if and only if the partition is *equitable* [11, p.195]; equivalently, if and only if $G_1$ and $G_2$ are regular, and $G_{12}$ is semiregular.

In the remainder of this section, we show that the parameters $\rho(k), k \geq 3$ are the spectral radii of the graphs $H(k)$ described in the next three paragraphs.

For even $k \geq 4$, let $H(k) = K_3 \lor \overline{M}_{k-2}$, the join of the complete graph of order 3 and the complement of a matching on $k - 2$ vertices. Then $H(k)$ has odd order $n = k + 1$ with 3 vertices of degree $k$ and $k - 2$ vertices of degree $k - 1$, so $2e = 3k + (k - 2)(k - 1) = kn - k + 2$. Thus, $H(k) \in \mathcal{H}(k)$.

For odd $k \geq 5$, let $H(k) = C_4 \lor \overline{C}_{k-2}$, the join of a cycle on 4 vertices and the complement of a cycle on $k - 2$ vertices. Then $H(k)$ has odd order $n = k + 2$ with 4 vertices of degree $k$ and $k - 2$ vertices of degree $k - 1$, so $2e = 4k + (k - 2)(k - 1) = kn - k + 2$. Thus, $H(k) \in \mathcal{H}(k)$. Note that if $k - 2 \geq 7$, we may replace $\overline{C}_{k-2}$ by the complement of a union of cycles to obtain another graph in $\mathcal{H}(k)$. By (3), it has the same spectral radius as $H(k)$.

For $k = 3$, let $H(k)$ be the graph of order 5 obtained from the complete graph $K_4$ by subdividing one of its edges by a new vertex. Then $H(3)$ has degree sequence $2, 3, 3, 3, 3$ and so is in $\mathcal{H}(3)$.

Using the inequality $\sqrt{x^2 + a} \leq x + \frac{a}{2x}$ and the interlacing formula (3) on each graph $H(k)$, we find that for all even $k \geq 4$,

$$\rho(k) \leq \lambda_1(H(k)) = \frac{1}{2} \left( k - 2 + \sqrt{k^2 + 12} \right) < k - 1 + \frac{3}{k} \quad (4)$$

while, for all odd $k \geq 5$,

$$\rho(k) \leq \lambda_1(H(k)) = \frac{1}{2} \left( k - 3 + \sqrt{(k + 1)^2 + 16} \right) < k - 1 + \frac{4}{k + 1} \quad (5)$$

For $k = 3$, we have $\rho(3) \leq \lambda_1(H(3)) = \theta$ where $\theta = 2.85577 \ldots$ is the greatest root of the characteristic polynomial $x^3 - x^2 - 6x + 2$ of the quotient matrix of an equitable three part partition of $H(3)$.

To show that $\rho(k) = \lambda_1(H(k))$, we first prove that we may restrict our search to graphs on $k + 1$ or $k + 2$ vertices.

**Lemma 3** Let $H$ be a graph in $\mathcal{H}(k)$ with $\lambda_1(H) = \rho(k)$. Then $H$ has order $n$ and size $e$ where $n = k + 1$ if $k$ is even, $n = k + 2$ if $k$ is odd, and $2e = kn - k + 2$.

**Proof.** Suppose that $2e > kn - k + 2$. Then $2e \geq kn - k + 4$. Because the spectral radius of a graph is at least the average degree, $\lambda_1(H) \geq \frac{2e}{n} = k - \frac{k-4}{n}$. Noting
that the final upper bound in (5) is at least as great as that in (4), we have
\[
\lambda_1(H) - \rho(k) \geq \lambda_1(H) - \lambda_1(H(k)) > k - \frac{k - 4}{k + 1} - (k - 1) - \frac{4}{k + 1} > 0
\]
and so \( \lambda_1(H) \neq \rho(k) \). Thus, \( 2e = kn - k + 2 \).

Because \( H \) has odd order \( n \) with maximum degree \( k \), we have \( n \geq k + 1 \) if \( k \) is even and \( n \geq k + 2 \) if \( k \) is odd.

If \( k \) is even, \( H \) has at least 3 vertices of degree \( k \) and \( kn - k + 2 = 2e \leq 3k + (n - 3)(k - 1) \), so \( n \leq k + 1 \). Thus \( n = k + 1 \) if \( k \) is even.

Similarly, if \( k \) is odd, \( H \) has at least 4 vertices of degree \( k \) and \( kn - k + 2 = 2e \leq 4k + (n - 4)(k - 1) \), so \( n \leq k + 2 \). Thus \( n = k + 2 \) if \( k \) is odd.

The following lemma, together with Lemma 2, implies Theorem 1. (The case \( k = 2 \) in Theorem 1 is immediate.) The proof is rather long, but employs a variety of techniques that may be of interest.

**Lemma 4** For \( k \geq 3 \),
\[
\rho(k) = \rho(H(k)) = \begin{cases} 
\theta = 2.8558\ldots & \text{if } k = 3 \\
\frac{1}{2} \left( k - 2 + \sqrt{k^2 + 12} \right) & \text{for even } k \geq 4 \\
\frac{1}{2} \left( k - 3 + \sqrt{(k + 1)^2 + 16} \right) & \text{for odd } k \geq 5
\end{cases}
\]

**Proof.** Let \( \mathcal{H}(k) \) denote the set of all graphs in \( \mathcal{H}(k) \) that satisfy the additional conditions in Lemma 3. It is straightforward to check that the graphs in \( \mathcal{H}(k) \) are the graphs with maximum degree \( k \) obtained by deleting \( (k - 2)/2 \) edges from the complete graph \( K_{k+1} \) when \( k \geq 4 \) is even and by deleting \( k \) edges from \( K_{k+2} \) when \( k \geq 3 \) is odd.

If \( k \) is even and \( k \geq 4 \), then by Lemma 3, \( H \) has order \( n = k + 1 \) and at least 3 vertices of degree \( k = n - 1 \). Let \( G_1 \) be the subgraph of \( H \) induced by \( n_1 = 3 \) of the vertices of degree \( k \) and let \( G_2 \) be the subgraph induced by the remaining \( n_2 = n - 3 \) vertices. Because each vertex in \( G_1 \) is adjacent to all other vertices in \( H \), it follows that \( H \) has the same parameters \( n_1, n_2, e_1, e_2, e_{12} \) in (3) as \( H(k) \). Thus \( \lambda_1(H) \geq \lambda_1(H(k)) \) for even \( k \geq 4 \).

Suppose now that \( k \) is odd. By Lemma 3, \( H \) has order \( n = k + 2 \) and at least 4 vertices of degree \( k \). For \( k = 3 \), \( H(3) \) is the only graph in \( \mathcal{H}(3) \), so we may assume that \( k \geq 5 \). Let \( G_1 \) be the subgraph of \( H \) induced by \( n_1 = 4 \) of the vertices of degree \( k \) and let \( G_2 \) be the subgraph induced by the remaining \( n_2 = n - 4 = k - 2 \) vertices.

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Case 1. If $G_1$ may be chosen to be the complement of a perfect matching on 4 vertices, then $e_1 = 4$ and each vertex in $G_1$ is adjacent to each vertex in $G_2$. Thus $H$ has the same parameters $n_1, n_2, e_1, e_2, e_{12}$ as $H(k)$ in (3), and so $\lambda_1(H) \geq \lambda_1(H(k))$ in this case.

Case 2. If $G_1$ may be chosen to be complete, then $e_1 = 6$ and each vertex in $G_1$ is adjacent to all but one vertex in $G_2$, so $e_{12} = n_1 n_2 - 4 = 4(k-3)$. Also, because $2e = kn - k + 2 = k^2 + k + 2$, it follows that $e_2 = e - e_1 - e_{12} = \frac{1}{2}(k^2 - 7k + 14)$. Substitution in (3) gives

$$
\lambda_1(H) \geq \frac{1}{2} \left(3 + \left(k - 5 + \frac{4}{k-2}\right)\right) + \frac{1}{2} \sqrt{\left(k - 8 + \frac{4}{k-2}\right)^2 + \frac{16(k-3)^2}{k-2}}
$$

$$
= \frac{1}{2} \left(k - 2 + \frac{4}{k-2}\right) + \frac{1}{2} \sqrt{k^2 + 8 - \frac{32}{k-2} + \frac{16}{(k-2)^2}}
$$

$$
> \frac{1}{2} \left(k - 3 + \sqrt{(k+1)^2 + 16}\right)
$$

where the last inequality follows by a straightforward calculation. Thus, $\lambda_1(H) \geq \lambda_1(H(k))$ in this case as well.

Suppose now that $G_1$ cannot be chosen to be as in Case 1 or Case 2 when $k$ is odd, $k \geq 5$. Note that $H \in \mathcal{H}(k)$ if and only if the complement $\overline{H}$ has order $n = k + 2$, size $\tau = k = n - 2$, and no isolated vertices. Let $\omega$ be the number of components of $\overline{H}$. Each component has at least 2 vertices, since $\overline{H}$ has no isolated vertices. Because a spanning forest of $\overline{H}$ accounts for $n - \omega$ edges, it follows that $\omega \geq n - \tau = 2$ and the remaining $\omega - 2$ edges are distributed among the $\omega$ components of $\overline{H}$. Thus, at least two of the components of $\overline{H}$ are trees. If three or more components are trees, we have Case 1 or Case 2. Thus, precisely two components of $\overline{H}$ are trees. If either one of the trees has 3 or more vertices of degree 1, we have Case 2. Thus, both trees have 2 vertices of degree 2 and so are paths. The remaining components (if any) must be cycles, or again we have Case 2. If both path components have order greater than 2, then again we have Case 2, while if both have order 2, then we have Case 1. It follows that if $G_1$ cannot be chosen to be in either Case 1 or Case 2 for odd $k \geq 5$, then we have the following case.

Case 3. The graph $H$ is the complement of a disjoint union of $K_2$, a path $P_m$ on $m \geq 3$ vertices and, if $n \geq m + 5$, a union $C$ of cycles.

Assume first that $m \geq 5$. Consider the graph $G$ on $k + 2$ vertices whose complement is the disjoint union of $K_2, K_2, C_{l-2}$ and $C$, where $C_{l-2}$ denotes a cycle on $l - 2 \geq 3$ vertices. By Case 1, we know that $\lambda_1(G) \geq \lambda_1(H(k))$.

Let $x$ be the positive eigenvector of norm 1 corresponding to $\lambda_1(G)$. By using an equitable partition of $G$ [12, p.195], it follows that the entries of $x$ are constant on the vertices of degree $k - 1$ in $G$ (corresponding to the vertices
on the cycles in $\overline{G}$) and constant and greatest on the 4 vertices of degree $k$ (corresponding to the endpoints of the two $K_2$’s in $\overline{G}$). Let 12 and 34 denote the two $K_2$’s in $\overline{G}$. This means $12 \notin E(G)$ and $34 \notin E(G)$. Let 56 be an edge of the cycle $C_{l-2}$ in $\overline{G}$. Similarly, this means $56 \notin E(G)$.

Note that the graph obtained from $G$ by adding edges 34 and 56 to $G$ and removing edges 35 and 46 from $G$ is isomorphic to $H$. Also,

$$\lambda_1(H) \geq x^\top A(H)x = x^\top A(G)x + x^\top (A(H) - A(G))x$$

$$= \lambda_1(G) + 2(x_3x_4 + x_5x_6 - x_3x_5 - x_4x_6)$$

$$= \lambda_1(G) + 2(x_3 - x_6)(x_4 - x_5)$$

$$> \lambda_1(G)$$

Since $\lambda_1(G) \geq \lambda_1(H(k))$, it follows that $\lambda_1(H) > \lambda_1(H(k))$ when $m \geq 5$.

Suppose now that $m = 3$. Partition the vertex set of $V(\overline{H})$ (and therefore of $V(H)$) into four parts: the two endpoints of $K_2$; the two endpoints of $P_3$; the internal vertex of $P_3$; and, the $k - 3$ vertices of $C$. This is an equitable partition of $H$ with quotient matrix

$$\begin{bmatrix}
0 & 2 & 1 & k-3 \\
2 & 1 & 0 & k-3 \\
2 & 0 & 0 & k-3 \\
2 & 2 & 1 & k-6
\end{bmatrix}$$

Because the partition is equitable, a positive eigenvector of the quotient lifts [11, p.198] to a positive eigenvector of $H$; that is, to a principal eigenvector. Thus the spectral radius of $H$ equals the spectral radius of the quotient matrix. The characteristic polynomial of the quotient matrix is

$$P(x) = x^4 - (k - 5)x^3 - (4k - 3)x^2 - (3k + 7)x + 2k$$

$$= (x^2 - (k - 3)x - 2k - 2)(x^2 + 2x - 1) - 2$$

Since $\lambda_1 = \lambda_1(H)$ is a root of $P(x)$ and $\lambda_1 > 1$,

$$\lambda_1^2 - (k - 3)\lambda_1 - 2k - 2 = \frac{2}{\lambda_1^2 + 2\lambda_1 - 1} > 0$$

Because the polynomial $x^2 - (k - 3)x - 2k - 2$ has roots $\lambda_1(H(k))$ and a negative number, it follows that $\lambda_1(H) = \lambda_1 > \lambda_1(H(k))$ when $m = 3$.

Suppose finally that $m = 4$. Partition the vertex set of $V(\overline{H})$ into four parts: the two endpoints of $K_2$; the two endpoints of $P_3$; the two internal vertices of $P_4$; and, the $k - 4$ vertices of $C$. This is an equitable partition of $H$ with
The quotient matrix
\[
\begin{bmatrix}
0 & 2 & 2 & k - 4 \\
2 & 1 & 1 & k - 4 \\
2 & 1 & 0 & k - 4 \\
2 & 2 & 2 & k - 7 \\
\end{bmatrix}
\]

The characteristic polynomial of the quotient matrix is
\[
Q(x) = x^4 - (k - 6)x^3 - (5k - 8)x^2 - (7k + 3)x - 2k - 4
= (x^2 - (k - 3)x - 2k - 2) (x^2 + 3x + 1) - 2
\]

Since \(\lambda_1 = \lambda_1(H)\) is a root of \(Q(x)\),
\[
\lambda_1^2 - (k - 3)\lambda_1 - 2k - 2 = \frac{2}{\lambda_1^2 + 3\lambda_1 + 1} > 0
\]

Because the polynomial \(x^2 - (k - 3)x - 2k - 2\) has roots \(\lambda_1(H)\) and a negative number, we have \(\lambda_1(H) = \lambda_1 > \lambda_1(H(k))\). This completes the proof of Lemma 3.

If \(G\) is a \(k\)-regular graph of even order with \(\lambda_2(G) \leq k - 1\), then \(G\) is connected and Theorem 1 implies that \(G\) has a perfect matching \(M\). Deleting \(M\) from \(G\) yields a \(k - 1\)-regular graph \(G - M\) with \(\lambda_2(G - M) \leq \lambda_2(G) + 1\) [15, p.181]. Also, if \(\lambda_2(G) \leq k - 3\) then \(\lambda_2(G - M) \leq k - 2\) so \(G - M\) will be connected and will also have a perfect matching by Theorem 1. Repeating this observation, we see that if \(t\) is a positive integer such that \(\lambda_2(G) \leq k - 2t + 1\), then \(G\) has \(t\) edge-disjoint perfect matchings. This implies the following corollary, first stated more generally in [3, Cor. 3.3] in terms of Laplacian eigenvalues.

**Corollary 5** A \(k\)-regular graph \(G\) of even order has at least \(\left\lfloor \frac{k - \lambda_2(G) + 1}{2} \right\rfloor\) edge-disjoint perfect matchings.

Following Brouwer and Haemers [3], let \(G(k)\) be the \(k\)-regular graph obtained by attaching the \(k - 2\) vertices of degree \(k - 1\) in each of \(k\) copies of \(H(k)\) to a set \(S\) of \(|S| = k - 2\) independent vertices. Then \(G(k) \setminus S\) has \(k > |S|\) copies of the odd order graph \(H(k)\) as its components and so, by Tutte’s theorem, \(G(k)\) has no perfect matching. In [3], it is mentioned that \(\lambda_2(G(k)) = \lambda_3(G(k)) = \lambda_1(H(k))\) for \(k\) even. A proof for all odd \(k \geq 3\) is given in the lemma below. This implies that the bounds in Theorem 1 are best possible.

**Lemma 6** Let \(G = G(k)\), \(H = H(k)\) be defined as above. Then, for \(k \geq 3\),
\[
\lambda_1(H) = \lambda_2(G) = \cdots = \lambda_k(G).
\]

**Proof.** Suppose that \(k\) is odd, \(k \geq 5\). If \(J\) denotes an all-ones matrix, then the
vertices of \( H(k) \) may be ordered so that it has partitioned adjacency matrix

\[
A(H) = \begin{bmatrix}
A(\overline{C}_{k-2}) & J \\
J^\top & A(C_4)
\end{bmatrix}
\]

The \( n = k^2 + 3k - 2 \) vertices of \( G \) may then be ordered so that \( G \) has adjacency matrix:

\[
A(G) = \begin{bmatrix}
O & C & C & \cdots & C \\
C^\top A(H) & O & \cdots & O \\
C^\top & O & A(H) & \cdots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C^\top & O & O & \cdots & A(H)
\end{bmatrix}
\]

where each \( O \) denotes a zero matrix of the appropriate size and \( C = [I \ O] \) where \( I \) is an identity matrix of order \( k - 2 \).

The eigenvectors of \( H(k) \) that are constant on each part of its two part equitable partition have eigenvalues given by the quotient matrix

\[
\begin{bmatrix}
k - 5 & 4 \\
k - 2 & 2
\end{bmatrix}
\]

These are \( \frac{1}{2} \left( k - 3 \pm \sqrt{(k + 1)^2 + 16} \right) \) where, as observed in (5), the positive eigenvalue is \( \lambda_1(H(k)) \). Let \( x \) and \( y \) be eigenvectors of \( H(k) \) associated with these two eigenvalues.

If \( u \) is a column eigenvector of \( A(H) \), consider the \( k - 1 \) column \( n \)-vectors

\[
[0^\top, \ldots, 0^\top, u^\top, -u^\top, 0^\top, \ldots, 0^\top]^\top
\]

where the zero vectors are compatible with the partition of \( A(G) \) and the first zero vector is always present. It is straightforward to check that these \( k - 1 \) vectors are linearly independent eigenvectors of \( A(G) \) with the same eigenvalue as \( u \). Thus, each eigenvalue of \( A(H) \) of multiplicity \( t \) yields an eigenvalue of \( A(G) \) of multiplicity at least \( t(k - 1) \). In particular, taking \( u = x \) and \( u = y \), we see that the two eigenvalues of \( A(H) \) above yield eigenvalues of \( A(G) \) of multiplicity at least \( k - 1 \) each. Thus, \( \lambda_1(H) \) is an eigenvalue of \( G \) with multiplicity at least \( k - 1 \).

Now consider the \((2k+1)\)-part equitable partition of \( G(k) \) obtained by extending the 2-part partitions of the \( k \) copies of \( H(k) \) in \( G(k) \). Let \( W \) be the space consisting of \( n \)-vectors that are constant on each part of the partition. Then \( \dim W = 2k + 1 \). Note that each of the \((2k - 1)\) independent eigenvectors of
$G(k)$ inherited from the eigenvectors $x$ and $y$ of $H(k)$ are in $W$. The natural 3-part equitable partition of $G(k)$ has quotient matrix

\[
\begin{bmatrix}
0 & k & 0 \\
1 & k - 4 & 3 \\
0 & k - 2 & 2
\end{bmatrix}
\]

with eigenvalues $k$ and $(-3 \pm \sqrt{17})/2$. Their corresponding eigenvectors lift to eigenvectors of $G(k)$ in $W$ with the same eigenvalues, and these 3 eigenvalues are different from those above. Thus the three lifted eigenvectors, together with the previous $2(k - 1)$ eigenvectors of $G(k)$ inherited from $H(k)$, form a basis for $W$.

The remaining eigenvectors in a basis of eigenvectors for $G(k)$ may be chosen orthogonal to the vectors in $W$; equivalently, they may be chosen to be orthogonal to the characteristic vectors of the parts of the $(2k + 1)$-part partition because the characteristic vectors are also a basis for $W$. Consequently, they will be (some of the) eigenvectors of the matrix $A(\hat{G})$ obtained from $A(G)$ by replacing each all-ones block in each diagonal block $A(H)$ by an all-zeros matrix. But $A(\hat{G})$ is the adjacency matrix of a graph $\hat{G}$ with $k + 1$ connected components, one of which is the graph $G'$ obtained by attaching $k$ copies of $C_{k - 2}$ to a set $S$ of $k - 2$ independent vertices by perfect matchings. Each of the remaining $k$ components is a copy of $C_4$. It follows that the greatest eigenvalue of $\hat{G}$ is that of the component $G'$. Because $G'$ has a two part equitable partition with quotient matrix

\[
\begin{bmatrix}
0 & k \\
1 & k - 5
\end{bmatrix},
\]

its greatest eigenvalue is $\frac{1}{2} \left(k - 5 + \sqrt{k^2 - 6k + 25}\right)$ and this is easily seen to be less than $k - 1 < \lambda_1(H(k))$.

This completes the proof for the case where $k$ is odd, $k \geq 5$. The case of $k$ even, $k \geq 4$, is similar and the case $k = 3$ is easily handled.

\[\square\]

### 3 Comments and examples

There are a number of conditions in the literature that guarantee the existence of perfect matchings in regular graphs of even order. Unfortunately, Theorem 1 appears to be applicable to only some of these. Still it is interesting to note
that many special graphs can be quickly shown to have perfect matchings. For example, if $G$ is a $k$-regular Ramanujan graph of even order $k \geq 6$, then
\[ \lambda_2(G) \leq 2\sqrt{k-1} < k - 1 \]
and so $G$ has at least $\lfloor (k - 2\sqrt{k-1} + 1)/2 \rfloor$ edge-disjoint perfect matchings by Corollary 5. (See also Brandt, Broersma, Diestel and Kriesell [2, Cor. 5.3].)

It is easily checked that strongly regular graphs of even order satisfy the conditions of Theorem 1 and so have perfect matchings. But this is a limited case. For it is noted in [3] that every $k$-regular graph $G$ of even order with diameter at most 3 must have a perfect matching, otherwise there would be three odd components $H_i \in \mathcal{H}(k)$ in some $G \setminus S$, giving a contradiction.

From Theorem 1, it follows that a $k$-regular connected graph of even order with $\lambda_3(G) \leq k - 1$ has a perfect matching. Thus, any $k$-regular connected graph $G$ of even order for which $\lambda_2(G)$ or $\lambda_3(G)$ is an integer must have a perfect matching. This includes, for example, some special distance regular graphs such as the Johnson graphs, the Hamming graphs, and so, the Odd graphs and, in particular, the Petersen graph. But, in fact, it is shown in [3] that every distance regular graph of even order has a perfect matching. It has yet to be determined whether or not every distance regular graph of even order satisfies the conditions of Theorem 1.

If $G$ is a vertex transitive graph and $\lambda$ is a simple eigenvalue, then a result of Chan and Godsil [4] shows that $\lambda$ must be an integer. Thus, if $G$ is a connected vertex transitive graph of even order and either $\lambda_2(G)$ or $\lambda_3(G)$ is simple, then Theorem 1 implies that $G$ has a perfect matching. But this is also a limited case, because the Gallai-Edmonds structure theorem [17, p.94] implies that every vertex transitive graph of even order has a perfect matching. In particular, an abelian Cayley graph of degree $k$ and even order $n$ is vertex transitive and so has a perfect matching, but Theorem 1 can rarely be applied because, for fixed $k$, the spectral gap $k - \lambda_3$ approaches 0 as $n$ increases (see [7], for example).

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References


