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On fuzzification in Hilbert algebras

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Abstract

The paper is devoted to the fuzzification of subalgebras in Hilbert algebras. We show that any subalgebra can be realized as a level subalgebra of some fuzzy subalgebra and describe properties of such subalgebras.

Introduction

The concept of Hilbert algebra was introduced in early 50-ties by L.Henkin and T.Skolem for some investigations of implication in intuicionistic and other non-classical logics. In 60-ties, these algebras were studied especially by A.Horn and A.Diego from algebraic point of view. A.Diego proved (cf. [4]) that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by D.Busneag (cf. [1], [2]) and Y.B.Jun (cf. [5]) and recent of their filters forming deductive systems were recognized. I.Chajda and R.Halaš introduced in [3] the concept of ideals in Hilbert algebras and described connections between such ideals and congruences. Many interesting results on Hilbert algebras is contained in [6].

Our work in this paper is concerned with the fuzzyfication of subalgebras. We show that every subalgebra can be realized as a level subalgebra of some fuzzy subalgebra and we describe properties of such subalgebras.

Since there exist various modifications of the definition of Hilbert algebra, we use that of [1].

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Definition 1.1. A nonempty set X with a constant $\mathbf{1}$ and a binary operation $\star$ is called a *Hilbert algebra* if for all $x, y, z \in X$ the following axioms hold:

(I) $x \star (y \star x) = \mathbf{1}$,
(II) $(x \star (y \star z)) \star ((x \star y) \star (x \star z)) = \mathbf{1}$,
(III) $x \star y = \mathbf{1}$ *and* $y \star x = \mathbf{1}$ *imply* $x = y$.

Note that the binary operation on a Hilbert algebra is traditionally denoted by implication $\rightarrow$, but from the algebraic point of view this notion is not good. Therefore we use $\star$ instead of $\rightarrow$.

In a Hilbert algebra, the following conditions are valid (cf. for example [4]):

(1) $x \star x = \mathbf{1}$,
(2) $\mathbf{1} \star x = x$,
(3) $x \star \mathbf{1} = \mathbf{1}$.

It can be easily shown that in a Hilbert algebra X the relation $\leq$ defined by

$$x \leq y \iff x \star y = \mathbf{1}$$

is a partial order on X with $\mathbf{1}$ as the greatest element.

According to the general idea presented by L.A.Zadeh (cf. [8]) every function $\mu : X \rightarrow [0,1]$ is called *a fuzzy set* on X. The set $\mu_t = \{x \in X : \mu(x) \geq t\}$, where $t \in [0,1]$ is fixed, is called *a level subset of* μ. $Im(\mu)$ denotes the image set of μ.

2. Fuzzy subalgebras

Definition 2.1. A fuzzy set μ in a Hilbert algebra X is called a *fuzzy subalgebra* of X if

$$\mu(x \star y) \geq \min\{\mu(x),\, \mu(y)\} \quad \forall x, y \in X\,.$$

Example 2.2. Let $X = \{a, b, c, d, \mathbf{1}\}$ be the set with Cayley table (Table 1) and Hasse diagram (Diagram 1) as follows:

$\star$	a	b	c	d	$\mathbf{1}$
a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
b	a	$\mathbf{1}$	c	$\mathbf{1}$	$\mathbf{1}$
c	a	b	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
d	a	b	c	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	a	b	c	d	$\mathbf{1}$

Table 1

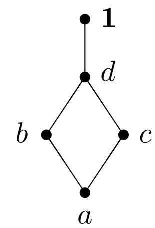

Diagram 1

It easy to see that $(X, \star, \mathbf{1})$ is a Hilbert algebra and $\mu : X \to [0,1]$ defined by $\mu(\mathbf{1}) = 0.8$ and $\mu(x) = 0.4$ for all $x \neq \mathbf{1}$ is a fuzzy set which is a fuzzy subalgebra of X. For this fuzzy subalgebra we have $\mu_t = X$ for $t \leq 0.4$, $\mu_t = \{\mathbf{1}\}$ for $0.4 < t \leq 0.8$ and $\mu_t = \emptyset$ for $t > 0.8$. $Im(\mu)$ contains only two elements: $t_1 = 0.8$ and $t_2 = 0.4$. □

Lemma 2.3. *If μ is a fuzzy subalgebra of a Hilbert algebra X, then $\mu(\mathbf{1}) \geq \mu(x)$ for every $x \in X$.*

Proof. Since $x \star x = \mathbf{1}$ for any $x \in X$, then

$\mu(\mathbf{1}) = \mu(x \star x) \geq \min\{\mu(x), \mu(x)\} = \mu(x)\,.$ □

Theorem 2.4. *A fuzzy set μ of a Hilbert algebra X is a fuzzy subalgebra iff for every $t \in [0,1]$, μ_t is either empty or a subalgebra of X.*

Proof. If μ is a fuzzy subalgebra of X and $\mu_t \neq \emptyset$, then for any $x, y \in \mu_t$

$$\mu(x \star y) \geq \min\{\mu(x), \mu(y)\} \geq t\,,$$

which implies $x \star y \in \mu_t$, and so μ_t is a subalgebra of X.

Denote for $x, y \in X$ by t the set $\min\{\mu(x), \mu(y)\}$. Then by the assumption μ_t is a subalgebra of X, which implies $x \star y \in \mu_t$. Hence $\mu(x \star y) \geq t = \min\{\mu(x), \mu(y)\}$. Thus μ is a fuzzy subalgebra of X. □

Theorem 2.5. *Any subalgebra of a Hilbert algebra X can be realized as a level subalgebra of some fuzzy subalgebra of X.*

Proof. Let A be a subalgebra of a given Hilbert algebra X and let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} t & if \;\; x \in A, \\ 0 & if \;\; x \notin A, \end{cases}$$

where $t \in (0,1]$ is fixed. It is clear that $\mu_t = A$.

We prove that such defined μ is a fuzzy subalgebra of X. Let $x, y \in X$. If $x, y \in A$, then also $x \star y \in A$. Hence $\mu(x) = \mu(y) = \mu(x \star y) = t$ and $\mu(x \star y) \geq \min\{\mu(x), \mu(y)\}$. If $x, y \notin A$, then $\mu(x) = \mu(y) = 0$, and, consequently $\mu(x \star y) \geq \min\{\mu(x), \mu(y)\} = 0$. If exactly one of x, y belongs to A, then exactly one of $\mu(x)$ and $\mu(y)$ is equal to 0. Therefore $\min\{\mu(x), \mu(y)\} = 0$ and $\mu(x \star y) \geq 0$, which completes the proof. □

Note that in a finite Hilbert algebra the number of subalgebras is finite whereas the number of level subsets may be infinite. But not all these level subsets are distinct.

Theorem 2.6. *Two level subalgebras μ_s, μ_t $(s < t)$ of a fuzzy subalgebra μ are equal iff there no $x \in X$ such that $s \leq \mu(x) < t$.*

Proof. Let $\mu_s = \mu_t$ for some $s < t$. If there exists $x \in X$ such that $s \leq \mu(x) < t$, then μ_t is a proper subset of μ_s, which is a contradiction. Conversely assume that there is no $x \in X$ such that $s \leq \mu(x) < t$. If $x \in \mu_s$, then $\mu(x) \geq s$, and so $\mu(x) \geq t$, because $\mu(x)$ does not lie between s and t. Thus $x \in \mu_t$, which gives $\mu_s \subseteq \mu_t$. The converse inclusion is obvious since $s < t$. Therefore $\mu_s = \mu_t$. □

The set of level subalgebras of a given fuzzy Hilbert algebra X is totally ordered by inclusion. Since $\mu(x) \leq \mu(\mathbf{1})$ for all $x \in X$, then μ_{t_1}, where $t_1 = \mu(\mathbf{1})$, is the least level subalgebra. Hence we have the chain

$$\mu_{t_1} \subset \mu_{t_2} \subset ... \subset \mu_{t_p} = X,$$

where $t_1 > t_2 > ... > t_p$. In Example 2.2 this chain has only two elements: $\mu_{t_1} = \{\mathbf{1}\}$ and $\mu_{t_2} = X$.

Corollary 2.7. *Let μ be a fuzzy subalgebra of X. If $Im(\mu) = \{t_1, t_2, ..., t_n\}$, where $t_1 < t_2 < ... < t_n$, then the family of levels μ_{t_j}, $1 \leq j \leq n$, is the set of all the level subalgebras of μ.*

Proof. Consider some level subalgebras μ_s for $s \in [0, 1]$ and $s \notin Im(\mu)$, be a some level subalgebras. If $s < t_1$, then $\mu_{t_1} \subseteq \mu_s$. Since $\mu_{t_1} = X$, it follows that $\mu_s = X$ and $\mu_s = \mu_{t_1}$. If $t_j < s < t_{j+1}$, then there is no $x \in X$ such that $s \leq \mu(x) < \mu_{t_{j+1}}$. Thus, by Theorem 2.6, we have $\mu_s = \mu_{t_{j+1}}$. □

Note that two different fuzzy subalgebras may have an identical family of level subalgebras. For example, putting in Example 2.2 $\rho(\mathbf{1}) = 0.7$ and $\rho(x) = 0.3$ for all $x \neq \mathbf{1}$ we obtain a new fuzzy subalgebra which has the same family of levels as μ.

Lemma 2.8. *Let μ be a fuzzy subalgebra with finite image. If $\mu_s = \mu_t$ for some $s, t \in Im(\mu)$, then $s = t$.*

Proof. Without lost of generality, let $s < t$. Since $s \in Im(\mu)$, then there exists $x \in X$ such that $\mu(x) = s < t$, hence $x \in \mu_s$ and $x \notin \mu_t$, a contradiction. □

Theorem 2.9. *Let μ and ρ be two fuzzy subalgebras of X with identical family of level subalgebras. If $Im(\mu) = \{t_1, ..., t_n\}$ and $Im(\rho) = \{s_1, ..., s_m\}$, where $t_1 > t_2 > ... > t_n$ and $s_1 > s_2 > ... > s_m$, then*

a) $m = n$,

b) $\mu_{t_j} = \rho_{s_j}$ for $j = 1, ..., n$,

c) if $\mu(x) = t_j$, then $\rho(x) = s_j$ for $x \in X$ and $j = 1, ..., n$.

Proof. (a) By Corollary 2.7 the only level subalgebras of μ and ρ are $\{\mu_{t_j}\}$ and $\{\rho_{s_j}\}$. Since, by the assumption, these families are identical, we have $m = n$.

(b) It follows directly from Corollary 2.7 and Theorem 2.6.

(c) Let $x_0 \in X$ be such that $\mu(x) = t_i$ and $\rho(x_0) = s_j$. From (b) and $\mu(x) = t_i$ follows $x_0 \in \rho_{s_j}$. Thus $\rho(x) \geq s_j$ and $s_j \geq s_i$, i.e. $\rho_{s_j} \subseteq \rho_{s_i}$. Since $x_0 \in \rho_{s_j} = \mu_{t_j}$, we obtain $t_i = \mu(x) \geq t_j$. This gives $\mu_{t_i} \subseteq \mu_{t_j}$, and, in the consequence (by (b)) $\rho_{s_i} = \mu_{t_i} \subseteq \mu_{t_j} = \rho_{s_j}$. Thus $\rho_{s_i} = \rho_{s_j}$. But, by Lemma 2.8, $s_i = s_j$. Therefore $\rho(x_0) = s_i$. □

Theorem 2.10. *Let μ and ρ be two fuzzy subalgebras of X with identical family of level subalgebras. Then $\mu = \rho$ iff $Im(\mu) = Im(\rho)$.*

Proof. Let $Im(\mu) = Im(\rho) = \{s_1, ..., s_n\}$ and $s_1 > ... > s_n$. Assume that $x_1, ..., x_n \in X$ are distinct and $\mu(x_j) = s_j$ for $1 \leq j \leq n$. By Theorem 2.9 also $\rho(x_j) = s_j$. Noticing that for any $x \in X$ there exists s_j such that $\mu(x) = s_j$, then $x \in \mu_{s_j} = \rho_{s_j}$. Hence $\rho(x) \geq s_j$, which gives $\rho(x) \geq \mu(x)$. Analogously we prove $\mu(x) \geq \rho(x)$. Thus $\mu(x) = \rho(x)$, which gives $\mu = \rho$. □

3. Fuzzy deductive systems

Definition 3.1. A nonempty subset D of a Hilbert algebra X is called a *deductive system* of X if it satisfies

(i) $\mathbf{1} \in D$,
(ii) $x \in D$ *and* $x \star y \in D$ *imply* $y \in D$.

Y.B.Jun proved (cf. [5]) that for all $x, y \in X$ the set

$$A(x, y) = \{z \in X : x \star (y \star z) = \mathbf{1}\}$$

is a deductive system. Moreover, any deductive system is the set-theoretic union of sets of the form $A(x, \mathbf{1})$ (cf. [5]). On the other hand,

$$[c, \mathbf{1}] = \{y \in X : c \leq y \leq \mathbf{1}\} = \{y \in X : \mathbf{1} \star (c \star y) = \mathbf{1}\} = A(\mathbf{1}, c)\,,$$

which proves that for every $c \in X$ the segment $[c, \mathbf{1}]$ is a deductive system and any deductive system of a Hilbert algebra is the set-theoretic union of some segments.

Definition 3.2. A fuzzy set μ on a Hilbert algebra X is called a *fuzzy deductive system* of X if

(d_1) $\mu(\mathbf{1}) \geq \mu(x) \quad \forall x \in X$,
(d_2) $\mu(y) \geq \min\{\mu(x \star y),\, \mu(x)\} \quad \forall x, y \in X$.

Example 3.3. Let X be a Hilbert algebra as in Example 3.2 and let μ be defined by $\mu(a) = 0.5$, $\mu(c) = 0.8$ and $\mu(b) = \mu(d) = \mu(\mathbf{1}) = 0.9$. Then μ is a fuzzy deductive system. □

Proposition 3.4. *If* μ *is a fuzzy deductive system of a Hilbert algebra* X*, then*

(a) $\mu(x) \leq \mu(y)$ *for* $x \leq y$,
(b) $\mu(x) \leq \mu(y \star x)$ *for all* $x, y \in X$.

Proof. For $x \leq y$ we have $x \star y = \mathbf{1}$, hence

$$\mu(y) \geq \min\{\mu(x \star y),\, \mu(x)\} = \min\{\mu(\mathbf{1}),\, \mu(x)\} = \mu(x)\,,$$

proving (a). Applying (a) to (I) we obtain (b). □

Theorem 3.5. *A fuzzy deductive system is a fuzzy subalgebra.*

Proof. Let μ be a fuzzy deductive system of a Hilbert algebra X. Then $\mu(\mathbf{1}) \geq \mu(x)$ for all $x \in X$ and from (I) it follows $\mu(x \star (y \star x)) = \mu(\mathbf{1}) \geq \mu(y)$ for all $x, y \in X$. Thus

$$\mu(y \star x) \geq \min\{\mu(x \star (y \star x),\, \mu(x)\} = \mu(x) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in X$, ending the proof. $\square$

Theorem 3.6. *A fuzzy set μ on a Hilbert algebra X is a fuzzy deductive system iff for every $t \in [0,1]$, μ_t is either empty or it is a deductive system of X.*

Proof. Let μ_t be a deductive system of X for every $t \in [0,1]$. If $\mu(\mathbf{1}) \geq \mu(x)$ is not true, then there is $x_0 \in X$ such that $\mu(x_0) > \mu(\mathbf{1})$. But in this case for $a = (\mu(\mathbf{1}) + \mu(x_0))/\,2$ we have $\mu(x_0) > a > \mu(\mathbf{1})$, which implies $x_0 \in \mu_a$, i.e. $\mu_a \neq \emptyset$. Since μ_a is a deductive system, then $\mu(\mathbf{1}) \geq a$, a contradiction. Hence $\mu(\mathbf{1}) \geq \mu(x)$ for all $x \in X$.

Now, if (d_2) is false, then $\mu(y_0) < \min\{\mu(x_0 \star y_0),\, \mu(x_0)\}$ for some $x_0, y_0 \in X$. Let

$$b = \frac{1}{2}(\mu(y_0) + \min\{\mu(x_0 \star y_0),\, \mu(x_0)\}).$$

Then clearly $b \in [0,1]$ and

$$\mu(y_0) < b < \min\{\mu(x_0 \star y_0), \mu(x_0)\}\,,$$

which shows that $x_0 \star y_0 \in \mu_b$ and $x_0 \in \mu_b$. Hence $y_0 \in \mu_b$ because μ_b is a deductive system. Thus $\mu(y_0) \geq b$, also a contradiction. Therefore (d_2) is true and μ is a fuzzy deductive system.

The converse follows from Theorem 2.4 because any fuzzy deductive system is a subalgebra. $\square$

Similarly as Theorem 2.5 we prove

Theorem 3.7. *Any deductive system of a Hilbert algebra X can be realized as a level deductive system of some fuzzy deductive system of X.* $\square$

As a consequence of the above results and Theorem 2.6 we have

Theorem 3.8. *Two level deductive systems μ_s, μ_t $(s < t)$ of a fuzzy deductive system are equal iff there is no $x \in X$ such that $s \leq \mu(x) < t$.* $\square$

Corollary 3.9. *Let μ be a fuzzy deductive system of X. If $Im(\mu) = \{t_1, t_2, ..., t_n\}$, where $t_1 < t_2 < ... < t_n$, then the family of levels μ_{t_i}, $1 \leq i \leq n$, is the set of all the level deductive systems of μ.* $\square$

Also theorems 2.9 and 2.10 will be true if we replace "subalgebra" by "deductive system".

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