Danjon Limit: Bruin’s Method

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Abstract. Bruin (1977) devised a procedure to find out the visibility of the first crescent Moon. He applied various simplifications to his theory, not all of them acceptable. We rethink Bruin's method by making some corrections: we take into account the variation of the luminance of the Moon with the phase, we use the experimental results of Knoll et al. (1946) on threshold contrast, we apply Riccò's law, and we consider the atmospheric extinction coefficient to be variable. We use the theory to derive the Danjon limit.

1. Introduction

The crescent of the Moon is visible for the first time shortly after the conjunction, on the western horizon and after sunset. We want to know in advance when an observer with good vision, sees the crescent Moon from a geographical position and with good atmospheric conditions. The importance of this problem is that the first sight of the lunar crescent marks the beginning of the lunar month in the Islamic calendar.

This problem has been treated in two different ways: using empirical criteria and by physical procedures.

Bruin (1977) is considered as the first author who has dealt with the visibility of the crescent Moon by physical methods, but Samaha, Assad, and Mikhail (1969) previously devised a theory with physical criteria to find the first vision of the crescent Moon; however, Bruin's research, which appeared in a prestigious astronomical journal, has been the most widely publicized.

To determine the moment when the first vision of the Moon crescent occurs, we have to consider the following three data:

a) *The luminance of the twilight sky where the Moon is,* which depends on the place of observation; depression of the Sun below the horizon; the altitude of the Moon; azimuth difference between the Moon and the Sun and other unpredictable factors.

b) *The observed luminance of the Moon,* which depends on the phase angle; of the atmospheric extinction coefficient; the altitude of the Moon, and the libration.

c) *The limit of human vision* or minimum contrast (A.4) is required to see the Moon against the bright background of the twilight sky. This criterion depends on several factors such as the age of the observer; shape, orientation, and color of the illuminated object; prior knowledge of where to look; whether it is mono or binocular vision; whether there is magnification; whether artificial or natural pupil is used; the duration of the observation; the number of observers,...

In the critical zone, the visibility of an object is probabilistic (Segura, 2021c), that is to say, that, under the same conditions, sometimes the observer sees the object and sometimes not, with a certain probability. Finally, note that the results of the measurements are dependent on the methodology of the experiment, so the results of the measurements carried out to determine the threshold contrast are somewhat different from each other.

In this investigation, we consider that the distance from the Earth to the Moon and the Sun is the average. The altitude of the Moon is apparent, that is, corrected for parallax and refraction; the Sun's depression is without refraction, and the phase angle is topocentric, that is, it is the selenocentric angle between the center of the Sun and the position of the observer on the surface of the Earth (see appendix).

As the observation of the Moon is near the horizon, we identify the parallax of the Moon with the equatorial horizontal parallax π, to which we give the approximate value of one degree (A.12).

In this investigation, we will expose the Bruin method for the visibility of the crescent Moon; we will expose its defects and propose a modified method. Specifically, what interests us is the determination of the Danjon limit (1932 and 1936), the minimum angle between the centers of the Moon and the Sun at which the crescent can still be visible (Segura, 2021a, 2021b, 2021d).

2. Bruin's method simplifications

As we will see later, Bruin (1977) adopts simplifications in his theory of lunar visibility, some acceptable and others wrong.

a) He assumes that the luminance of the twilight sky B_S is uniform and only depends on the depression of the Sun below the horizon.

b) Bruin considers the observed luminance of the Moon uniform, that is, the same in all the illuminated points of the crescent, and assumes that it is 3,600 cd/ m^2 at the zenith and sea level.

c) Assume the observed luminance of the Moon independent of the phase angle χ; as we shall see, this simplification is manifestly erroneous and represents the most severe mistake of Bruin's method.

d) He gives a fixed value to the atmospheric extinction coefficient k (approximately 0.25).

e) At limit vision, he reduces the center of the crescent to a circle with a diameter equal to the maximum width of the crescent.

f) Determine contrast and threshold luminance by extrapolating Siedentopf (1941) measurements.

3. Bruin's method of crescent visibility

To apply Bruin's method and find out if the crescent will be seen at a specific time, we need to know: the depression of the Sun below the horizon d, the topocentric phase angle χ, the apparent altitude of the Moon h, and the atmospheric extinction coefficient k. With this data, we do the following:

a) We find the limit altitude $h_{\lim}$, the minimum altitude that the Moon must have at the moment considered to be visible.

b) If the altitude of the Moon h is greater than the limit altitude, the Moon will be visible, and it will not be otherwise.

The following procedure calculates the limit altitude:

a) We determine the luminance of the sky, knowing the depression of the Sun.

b) We find the maximum width of the lunar crescent $w_{\max}$ knowing the topocentric phase angle (A.11).

c) From the experimental data, we find the threshold contrast and the threshold luminance B_{th} for viewing the Moon.

d) From the law of atmospheric absorption (A.6) and (A.7), we find the limit altitude or altitude at which the Moon has a luminance equal to the threshold luminance.

Bruin developed a graphical method and plotted $h_{\lim} + d$ as a function of d for various phase angles. He used a single value for the atmospheric extinction coefficient k; a more thorough investigation requires that k be variable.

For each value of the phase angle, Bruin gives values to the depression d, finding by the method indicated above the limit altitude $h_{\lim}$. Thus he obtained several curves, each of them characterized by the value of the phase angle.

To determine if the Moon would be visible, he determined for the time of sunset of the day of interest, the apparent altitude of the Moon h, the geocentric depression of the Sun d, and the topocentric phase angle χ. Then draw the horizontal line $h + d$ on the graph. If this line intersects the curve characterized by the phase angle χ, the Moon will be visible; the cut-off points will indicate the beginning and end of the Moon's visibility period. We will not see the Moon if the horizontal line $h + d$ is below the graph (drawing 2).

4. Riccò's law for non resolvable images

Riccò's law proposed in 1877 applies to images of unresolvable size, that is to say, smaller in size than the resolving of the human eye (A.14), which we estimate to be 1 arc minute; then the image will be seen if its illuminance exceeds a threshold limit E_{th} that depends exclusively on the background luminance B_S (Segura, 2021c). Let us consider a circular image, with uniform luminance, diameter θ_0, and solid angle Ω_0, smaller than the resolution of the human eye; then the threshold contrast when the image is on a background of luminance B_S is

$$C_{th}\left(B_S,\theta_0\right)=\frac{B_{th}}{B_S}=\frac{E_{th}\left(B_S\right)}{\Omega_0 B_S},$$

for another image of size θ, solid angle Ω and not resolvable, its threshold contrast will be

$$C_{th}\left(B_S,\theta\right)=\frac{E_{th}\left(B_S\right)}{\Omega B_S} \quad \Rightarrow \quad C_{th}\left(B_S,\theta\right)\theta^2=C_{th}\left(B_S,\theta_0\right)\theta_0^2$$

we have applied (A.3) and (A.9); then Riccò's law for not resolvable images is

$$C_{th}\left(B_S,\theta\right)=\beta\left(B_S\right)\theta^{-2} \tag{1}$$

the coefficient β only depends on B_S

$$\beta\left(B_S\right)=C_{th}\left(B_S,\theta_0\right)\theta_0^2 \tag{2}$$

In table 1, we have calculated the coefficient $\beta\left(B_S\right)$ according to Knoll, Tousey, and Hulburt (1946) for a probability of vision of 100%. Table 2 represents coefficient $\beta\left(B_S\right)$ according to Blackwell (1946), which corresponds to a probability of vision of 50%. Finally, in table 3 are the results of Seidentopf (1941).

We have obtained Riccó's law assuming that for images smaller than the resolving of the

$\theta_0=1'$ $\quad \Omega_0=6.64572\cdot10^{-8}$ sr			
$\log B_s$ $\left(\text{cd/m}^2\right)$	$\log E_{th}\left(B_s\right)$ (lux)	$B_{th}\left(B_S,\theta_0\right)$	$\beta\left(B_S\right)$
-2*			44.99
-1.7*			29.01
-1.5*			21.55
-1	-7.1671	1.0241	10.24
-0.5	-6.9609	1.6465	5.21
0	-6.7255	2.8311	2.83
0.5	-6.4607	5.2090	1.65
1	-6.1667	10.251	1.03
1.5	-5.8433	21.585	0.68
2	-5.4906	48.625	0.49
2.5	-5.1086	117.18	0.37
3	-4.6972	302.18	0.30
3.5	-4.2566	833.41	0.26

Table 1.- From the data of Knoll et al. (1946), we have calculated the threshold illuminance E_{th} for the background luminance B_S . B_{th} is the threshold luminance obtained by $B_{th}=E_{th}\Omega_0$, Ω_0 is the solid angle of a circle with an angular diameter of 1 minute. In the last column is the coefficient $\beta\left(B_S\right)$ that appears in Riccó's law (1) and that we calculate by (2). The asterisks signify extrapolated values when fitting the curve with a sixth-order polynomial curve.

$\theta_0 = 0.595'$ $\Omega_0 = 2.35275 \cdot 10^{-8}$ sr			
$\log B_s$ $\left(\mathrm{cd/m^2}\right)$	$\log E_{th}\left(B_s\right)$ $\left(\mathrm{lux}\right)$	B_{th}	$\beta\left(B_S\right)$
-1	-7.5881	1.0974	4.0023
-0.5	-7.5061	1.3254	1.5995
0	-7.2237	2.5391	1.0246
0.5	-6.9202	5.1080	0.5864
1	-6.5735	11.3480	0.4408
1.5	-6.1820	27.9545	0.3170
2	-5.7706	72.08	0.2579
2.5	-5.3689	181.7677	0.2038
3	-4.9781	447	0.1752
3.5	-4.5060	1325.63	0.1484

Table 2.- Riccò's law coefficient β derived from Blackwell's data for an image of 0.595 arc minutes and a probability of vision of 50%. For a probability of 100%, we must multiply β by 2.

$\theta_0 = 1'$ $\Omega_0 = 6.64572 \cdot 10^{-8}$ sr			
$\log B_s$ $\left(\mathrm{cd/m^2}\right)$	$\log E_{th}\left(B_s\right)$ $\left(\mathrm{lux}\right)$	$B_{th}\left(B_S, \theta_0\right)$	$\beta\left(B_S\right)$
-1*	-7.4239	0.57	5.67
-0.5*	-7.1198	1.14	3.61
0	-6.7963	2.41	2.41
0.5	-6.5189	4.56	1.44
1	-6.2033	9.42	0.94
1.5	-5.9468	17.01	0.54
2	-5.6092	37.01	0.37
2.5	-5.3027	74.95	0.24
3*	-4.9908	153.71	0.15
3.5*	-4.6867	309.60	0.10

Table 3.- Riccò's law coefficient β derived from Seidentopf (1941). The asterisk means extrapolations

human eye, the illuminance that reaches the observer is the factor that determines the visibility of the image on a bright background (Segura, 2021c), that is, that there is a threshold illuminance and if the illuminance of the image is greater than the threshold, the image will be seen.

The opposite argument is also true. If we assume that Riccó's law is valid, that is, that the threshold contrast is inversely proportional to the square of the angular diameter of the image (supposedly circular), then there is a threshold illuminance, showing that for non-resolvable images, the factor that determines its visibility is the illuminance and not luminance.

Riccó's law is exact for non-resolvable images and approximately true for small images although larger than the resolving of the human eye, as confirmed by the experiment of Blackwell (1946).

5. The magnitude of the Moon

Photometric measurements of the Moon at large values of the phase angle are difficult since its observation has to be made at a low altitude above the horizon and therefore, is highly affected by atmospheric attenuation; also, the observation has to be done with twilight light, therefore the Moon's own illumination is added to the illumination of the sky, and finally, it must be added that the Moon is rarely observed, and for a very short time, with a phase angle greater than 170º because the brightness twilight sky masks the light emitted by the Moon.

Allen (1973, p. 144) gives the following formula for the magnitude of the Moon out of the atmosphere as a function of the geocentric phase angle χ'

$$m = -12.73 + 0.026|\chi'| + 4\cdot 10^{-9}\chi'^4 \tag{3}$$

χ' is in degrees. We cannot extend the formula (3) to more than 150º of the phase angle, since the luminance of the Moon at large phase angles is affected by the shielding and micro-shielding of the lunar surface caused by the inclination of the solar rays close to the horizon of the Moon. We check that (3) is unsuitable for use at large phase angles. In summary, (3) gives an illuminance greater than the real one for large phase angles.

(Samaha, Asaad and Mikhail, 1969) and (Russell, 1916) proposes a law by which the Moon's magnitude depends on the logarithm of the cube of $180-\chi'$. Adjusting (3) to a law of this type we obtain *

$$m = 3.62548 - 2.33551\log\left(180-\chi'\right)^3 = 3.62548 - 2.5\log\left(180-\chi'\right)^{2.8026}, \tag{4}$$

(4) can be extended for high phase angle. We assume that (4) is the geocentric magnitude at the mean distance from Earth $\bar{r}$, hence χ' is the geocentric phase angle. From formula (4), we derive the Moon's luminance, assuming it is the same throughout the crescent. The surface of the lunar crescent of phase angle χ' seen in the direction of the Earth is

$$S = \frac{1}{2}\pi R^2\left(1+\cos\chi'\right)$$

which is the subtraction between the areas of a semi-circle and a semi-ellipse (Segura 2018, p.190), (Segura 2020b). Its solid angle is

$$\Omega = \frac{S}{\bar{r}^2} = \frac{\pi}{2}\frac{R^2}{\bar{r}^2}\left(1+\cos\chi'\right)$$

R is the radius of the Moon, and $\bar{r}$ is the mean distance from the Moon to the Earth. We define the magnitude by (A.5), then by (4), we find the geocentric illuminance at the mean distance from the Moon $\bar{E}$ expressed in lux

$$\log\bar{E} = -7.0422 + \log\left(180-\chi'\right)^{2.8026} \quad\Rightarrow\quad \bar{E} = 9.0742\cdot 10^{-8}\left(180-\chi'\right)^{2.8026}$$

and the luminance in cd/m^2 is by (A.3)

$$B = \frac{\bar{E}}{\Omega} = 9.0742\cdot 10^{-8}\,\frac{2\bar{r}^2}{\pi R^2\left(1+\cos\chi'\right)}\left(180-\chi'\right)^{2.8026}, \tag{5}$$

this formula is valid on the assumption that the luminance of the lunar crescent is uniform and before atmospheric absorption. Let us indicate that luminance is independent of the distance from the Moon since it is an intrinsic magnitude of the luminous object.

As we see the crescent near the horizon, the relationship between the topocentric χ and geocentric χ' phase angles is

$$\chi' \approx \chi - \pi \approx \chi - 1^{\circ}.$$

In table 4, we have applied (5), and we find the luminance of the Moon as a function of the topocentric phase angle, assuming that the luminance of the Moon is uniform. In Table 4, we see that the luminance of the Moon is highly dependent on the phase angle; therefore, it is unacceptable

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* We have adjusted formula (3) between the values of 90 and 150 degrees, thus avoid considering the opposition effect of the Moon. To obtain (4), we have first put the magnitude m of (3) as a function of $\log\left(180-\chi'\right)^3$, and we fit it to the straight line $m = a + b\log\left(180-\chi'\right)^3$, determining the parameters a and b. Sahama et al. (1969) adopt the formula $m = 4.5245 - 2.5\log\left(180-\chi'\right)^3$ instead of (4).

to assume, as Bruin does, that the luminance is the same regardless of the phase angle.

The luminance of the Moon is not uniform; it depends on the selenocentric geographic coordinates, especially on the lunar longitude. However, since the Moon crescent has a minimal width, it is acceptable to take the uniform luminance.

Schaefer (1991) used Hapke's lunar photometric theory to find the luminance of the Moon at large phase angles. However, as we have shown (Segura, 2021d), Hapke's theory gives an excessive darkening of the Moon when the phase angle is large.

6. Atmospheric absorption

The atmospheric absorption is characterized by two factors: the extinction coefficient *k* and the air-mass *X*, a measure of the distance traveled by light through the atmosphere. The extinction coefficient is highly variable, and we cannot know its value in advance. As we will see, this coefficient greatly affects the visibility of the crescent and is, therefore, the most uncertain factor in the theory of the visibility of the lunar crescent.

We are interested in calculating the minimum altitude at which the Moon must be to be visible; that is, by (A.6), we have to calculate the air-mass

$$X = \frac{2.5}{k} \log \frac{B_M(\chi)}{B_{th}(B_S)} \tag{6}$$

$B_M(\chi)$ is the luminance of the Moon according to the phase angle and outside the atmosphere, and B_{th} is the threshold luminance that depends exclusively on the luminance of the sky B_S (table 4). Finally, from (A.7) or table 6, we determine the apparent altitude of the Moon or limit altitude to see it.

7. Modified Bruin's method

Taking the ideas of Bruin, we develop a modified method, where we admit the following simplifications:

a) The sky luminance B_S only depends on the depression of the Sun below the horizon and we use Bruin's measurements (table 5).

b) The crescent has a uniform luminance.

c) We assume that at the limit of visibility, the visible zone of the Moon is a circular disk of diameter the maximum width of the crescent $w_{\max}$ (A.11).

d) Riccò's law (1) is valid.

χ (degrees)	B_S (cd/m^2)
90	860.1
100	748.6
110	647.2
120	553.7
130	465.9
140	381.9
150	299.6
160	216.3
170	127.7
175	78.3

Table 4.- Average luminance of the Moon according to formula (5) as a function of the topocentric phase angle.

d (degrees)	$\log B_S$ $\left(\text{cd/m}^2\right)$
0	3.102
1	2.932
2	2.659
3	2.296
4	1.773
5	1.227
6	0.705
7	0.182
8	-0.341
9	-0.818
10	-1.251
11	-1.6125
12	-1.9375
13*	-2.2042
14*	-2.4296

Table 5.- Luminance of the twilight sky as a function of the Sun depression without refraction, according to Bruin measurements. Asterisks mean extrapolated values.

e) We take the coefficient of Riccò's law (2) from the Knoll et al. experiment (table 1) from which we deduce the threshold contrast and the threshold luminance B_{th} .
f) The luminance of the Moon B_M depends on the phase angle by equation (5) and table 4.
g) We deduce the air-mass *X* of the limit altitude of the Moon from (6).
h) We calculate by (A.7) (table 6) the limit altitude.

The procedure to build the graphs that will allow us to know when the Moon will be visible is as follows:

a) For an extinction coefficient *k* and a topocentric phase angle χ, we give values to the depression of the Sun *d*.
b) Once the depression is known, we interpolate table 5 and calculate the luminance of the sky B_S .
c) Knowing the topocentric phase angle, we determine the maximum width of the crescent by (A.11).
d) Knowing the luminance of the sky, we determine the coefficient β from table 1.
e) By Riccò's law (1), we calculate the threshold contrast and the threshold luminance of the Moon B_{th} , where we take $\theta = w_{\max}$.
f) By (6), we calculate the air-mass $X\left(h_{\lim}\right)$ of the limit altitude of to see the Moon.
g) By (A.7) find the apparent altitude limit $h_{\lim}$.
h) We represent in the diagram the point $\left(d, h_{\lim} + d\right)$.
i) We take a new value for the depression and obtain a new point, and so on until we can draw the curve corresponding to the extinction coefficient *k* and the phase angle χ.
j) We obtain new curves with other values of the phase angle and the extinction coefficient.

In table 7, we show an example of the calculations for a topocentric angle of 167º and atmospheric extinction coefficient 0.25. Until the depression of the Sun of 5º, the Moon is not observable since the luminance of the sky is very intense. For depression of 5º, there is a limit

Air-masss	
h (º)	Kasten-Young
0	37.9196
1	26.3106
2	19.4332
3	15.1477
4	12.302
5	10.3058
6	8.8415
7	7.7281
8	6.8565
9	6.1577
10	5.5860

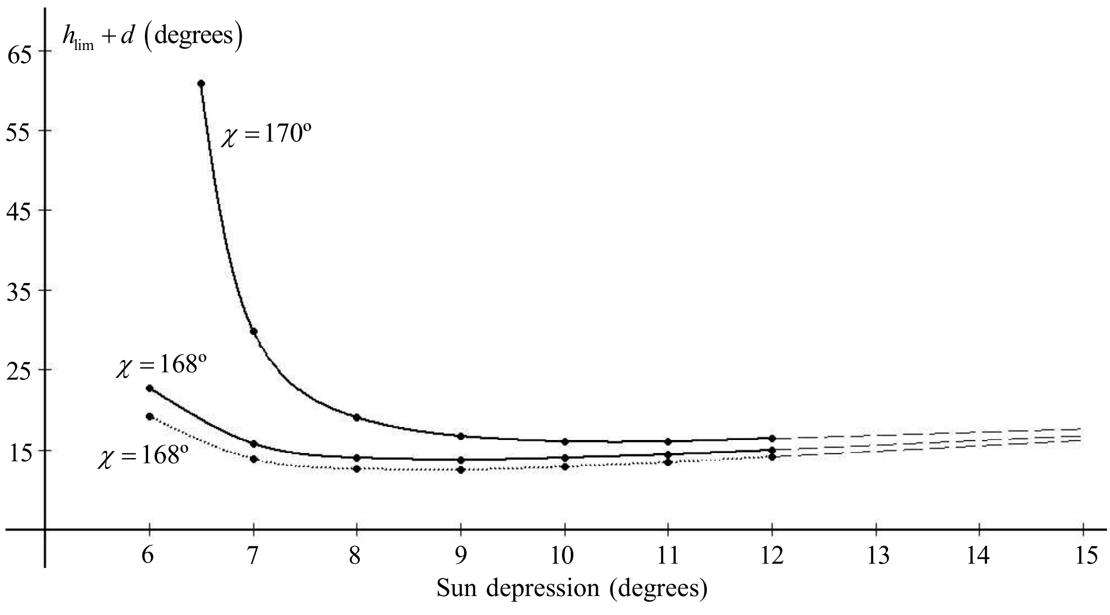
Table 6.- Air-mass as a function of apparent altitude, according to the Kasten-Young formula (A.7). *h* is the apparent altitude of the Moon, that is, corrected for parallax and refraction.

altitude of the Moon, although this does not mean that it will be visible; for this, the altitude of the Moon for the corresponding depression of the Sun must be greater than the limit altitude.

In drawing 1, we have drawn some graphs that help us determine if the Moon will be visible. The graphs are characterized by the topocentric phase angle and the atmospheric extinction coefficient. In drawing 1, we show the effect of extinction. The right end of the graphs, which we represent with a broken line, has been obtained by linear extrapolation.

We have found drawing 1 with the measurements made by Knoll et al. that give the minimum of vision with a probability of 100%. However, we can modify the results for another probability.

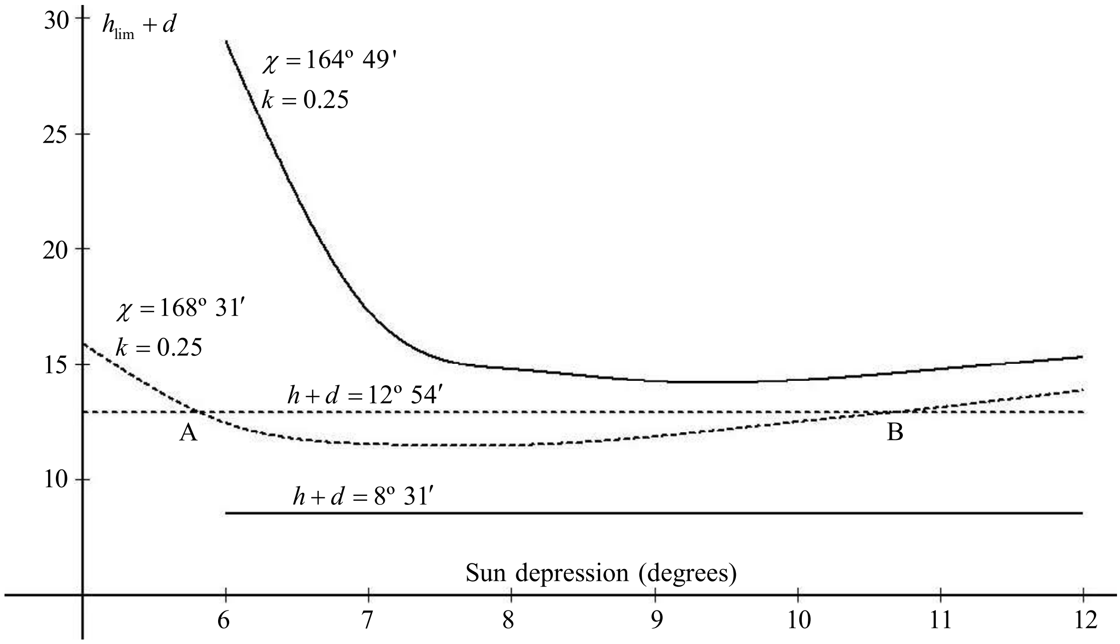
Let $C_{th}(p)$ the threshold contrast to see an image with a probability of vision p and $C_{th}(50\%)$ the threshold contrast to see the same image under the same conditions with a probability



Drawing 1.- Some of the curves obtained by the modified Bruin's method. The curves with solid lines correspond to an extinction constant of 0.25. The dotted curve is for $k = 0.2$. The dashed lines are the linear extrapolations.

$\chi = 167^o$ $k = 0.25$ $B_M = 145.08$ $\omega_{max} = 0.398'$ Probability = 100%							
d	$\log B_S$	B_S	$\beta(B_S)$	B_{th}	X	h_{lim}	$d + h_{lim}$
0	3.102	1264.70	0.2918	-	-	-	-
1	2.932	855.07	0.3095	-	-	-	-
2	2.659	456.04	0.3477	-	-	-	-
3	2.296	197.70	0.4190	-	-	-	-
4	1.773	59.29	0.5763	-	-	-	-
5	1.227	16.87	0.8711	92.77	1.942	30.94	35.94
6	0.705	5.07	1.3958	44.68	5.115	10.99	16.99
7	0.182	1.52	2.4005	23.03	7.992	6.76	13.76
8	-0.341	0.456	4.4532	12.82	10.537	4.88	12.88
9	-0.818	0.152	8.4091	8.07	12.548	3.91	12.91
10	-1.251	0.0561	15.918	5.64	14.105	3.37	13.37
11	-1.613	0.0244	25.508	3.93	15.673	2.88	13.88
12	-1.938	0.0115	41.200	2.99	16.858	2.60	14.60

Table 7.- Calculation of the limit altitude for a topocentric phase angle of 167º and an atmospheric extinction coeffecient of 0.25. We determine B_S by table 5. We calculate $\beta(B_S)$ by table 2. We find B_{th} by Riccó's law (1). We calculate the air mass by (6) and h_{lim} by table 6.



Drawing 2.- Prediction of lunar visibility by the modified Bruin method. The curve with a solid line corresponds to the time of sunset on March 14, 2021. The continuous horizontal line is the altitude of the Moon plus the depression of the Sun (which we take as a positive value) for the same moment. Since the horizontal line does not intersect the curve, the Moon will not be visible.

The discontinuous curve is the one that corresponds to the time of sunset on February 12, 2021. The discontinuous horizontal line is the altitude of the Moon plus the depression of the Sun for the same moment. As the horizontal line intersects the curve, the Moon will be seen. The vision of the crescent begins approximately when the Sun is 5º 48' below the horizon (point A) and will end when the solar depression is 10º 40' (point B). The two curves correspond to observations at the geographical position 36º 1' N, 5º 22' W, and an atmospheric extinction constant of 0.25.

of 50%, according to Blackwell

$$C_{th}(p)=\xi(p)C_{th}(50\%) \quad \Rightarrow \quad B_{th}(p)=\xi(p)B_{th}(50\%).$$

In drawing 3, we show the function $\xi(p)$ according to Blackwell's results. The horizontal axis is the function ξ, and the vertical axis is the probability of vision p expressed in units.

To find the threshold luminance for a probability p from the luminance found in Knoll et al., that is, with a probability of 100%, we apply

$$B_{th}(p)=\frac{\xi(p)}{\xi(100\%)}B_{th}(100\%)=\frac{1}{2}\xi(p)B_{th}(100\%)$$

From drawing 3 we find $\xi(100\%)=2$ and deduce the value of $\xi(p)$.

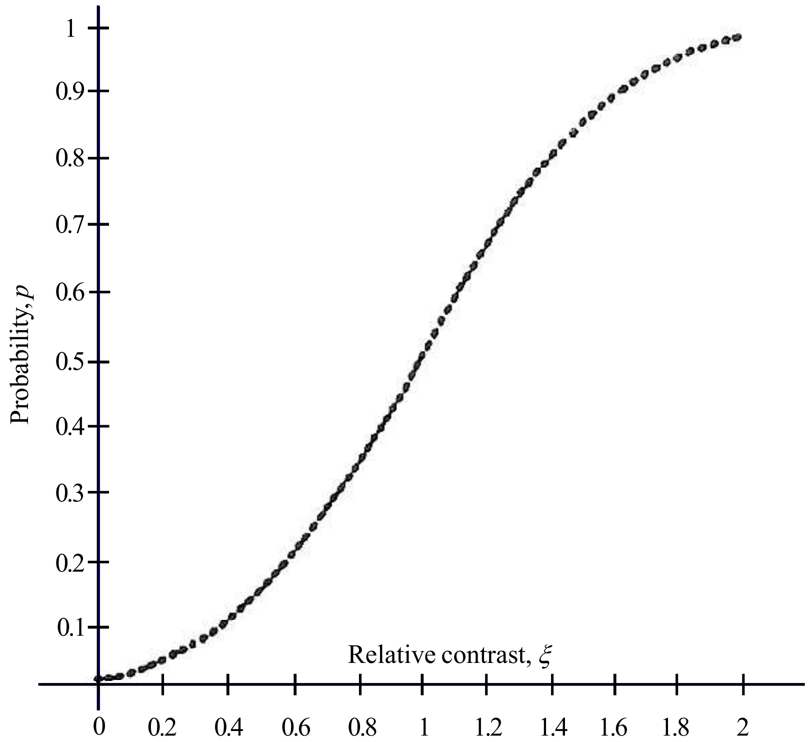
In drawing 4, we compare the curves for the probabilities 100%, 50%, and 20% for the topocentric phase angle 168º and the atmospheric extinction constant 0.2. The graph corresponding to the lowest probability of vision is the one with the lowest limit altitude, which indicates that we can see the Moon at the most inferior distance from the Sun; we can see it with more anticipation.

8. Visibility of the crescent Moon by Bruin's modified theory

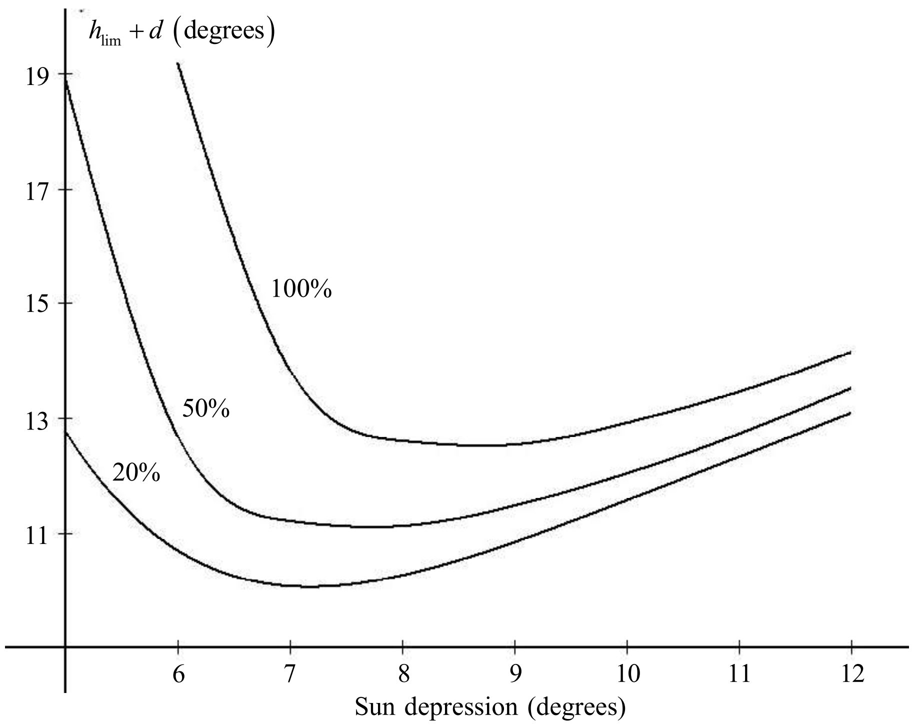
To find out if the lunar crescent will be visible we calculate the topocentric phase angle and the apparent altitude of the Moon h when the depression of the Sun is 0. We assume that during the time between sunset and vision of the Moon, the angle $h+d$ does not vary.

In drawing 2, we analyze the visibility of the Moon on March 14, 2021, and February 12, 2021, at a location with geographic coordinates of 36º 1' N and 5º 22' W and assuming that the atmospheric extinction constant is 0.25. We determine the phase angle of the Moon at sunset of the days considered and represent the visibility curves for the two moments considered and the straight lines $h+d$ for the two days.

In drawing 2, we see that for March 14, 2021, the horizontal line $h+d$ does not cut the



Drawing 3.- Average probability curve in Blackwell's experience. Relative contrast 1 corresponds to a probability of 50%. If another probability is desired, the curve determines the coefficient ξ by which the threshold contrast for a probability of 50% must be multiplied. For example, a probability of 90% corresponds to a threshold contrast of 1.62, which is the factor by which to multiply the contrast for a probability of 50%, to find the threshold contrast for the 90% probability. The maximum of the curve corresponds to a 98% probability. Curve reproduced from Blackwell's work.



Drawing 4.- Lunar visibility curves for a 168° topocentric phase angle, extinction constant 0.2 and various vision probabilities. When the probability of vision is small, the Moon can be seen at a smaller angular distance from the Sun; therefore, we can see it with more anticipation.

curve, indicating that the Moon will not be seen that afternoon. However, for February 12, 2021, the horizontal line $h+d$ cuts the visibility curve; therefore, the Moon will be seen that day. Cutting points of the $h+d$ line with curve indicate the depression of the Sun at the beginning and end of the vision of the crescent.

9. Danjon limit

We call the Danjon limit the minimum topocentric angular distance (or arc-light) between the centers of the Moon and the Sun in which it is still possible to see the Moon; that is, at a smaller angular distance, it is impossible to see the Moon. The smallest arc-light is found when the azimuth difference between the Moon and the Sun is zero. Then

$$a_{LT} = h' + d$$

h' is the topocentric altitude of the Moon without refraction. By (A.12)

$$a_{LT} = 180 - \chi \quad \Rightarrow \quad 180 - \chi = h' + d \quad \Rightarrow \quad h + d = 180 - \chi + R_R \qquad (7)$$

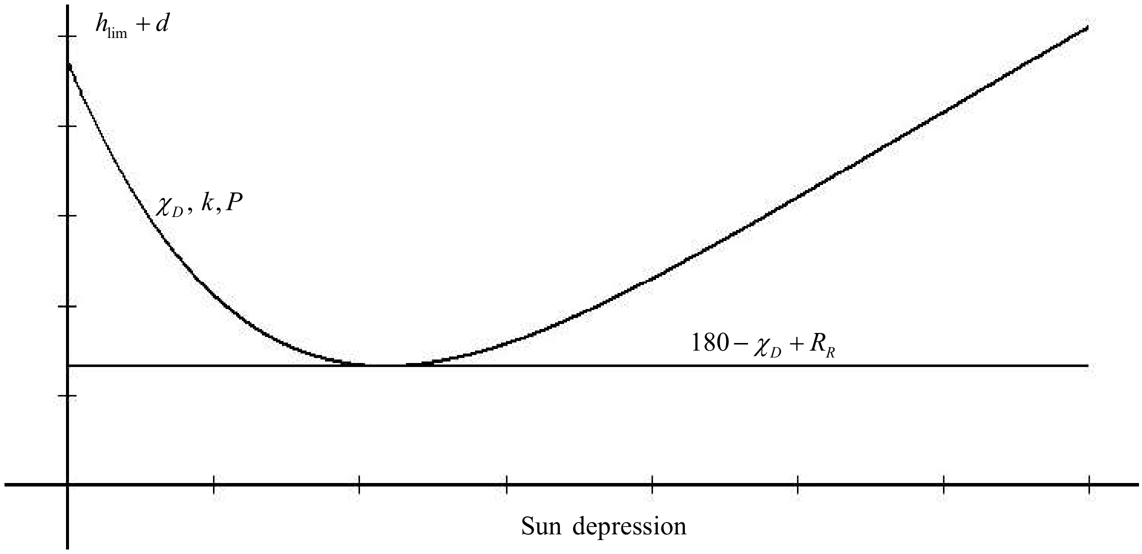
R_R is the angle of refraction calculated by (A.10) and h is the apparent altitude (with refraction and parallax) of the Moon.

To find the Danjon limit, we have to determine the topocentric phase angle that has a visibility curve such that its minimum or lower point is tangent to the horizontal line $180 - \chi + R_R$. R_R is the angle of refraction for the limiting height of the minimum of the visibility curve. Drawing 5 shows the conditions that the phase angle corresponding to the Danjon limit must meet.

It is important to note that the Danjon limit is a topocentric angle. It depends on the atmospheric extinction coefficient and the probability of vision; therefore, it is not a fixed value, not even for the same place of observation.

To set conditions for calculating the Danjon limit, we assume exceptional atmospheric conditions and consider an extinction coefficient of 0.15 and a probability of vision of 20%. Making calculations like those in table 7 for various phase angles, we find that the conditions required in drawing 5 occur when the topocentric phase angle is 169.9°, that is, the Danjon limit in Bruin's theory according to the requirements above is 10.1°, a little higher than that found by other authors.

If we assume other atmospheric conditions and another probability of vision, the obtained



Drawing 5.- The phase angle of the Danjon limit for an extinction coefficient *k* and a probability *P* is the topocentric phase angle χ_D for which the minimum of the visibility curve is tangent to the horizontal line $180 - \chi_D + R_R$. The Danjon limit is the topocentric arc-light $a_{LT} = 180 - \chi_D$.

Danjon limit will be different.

10. Conclusions

In 1977 Bruin proposed a method to determine the visibility of the crescent Moon by comparing the luminance of the twilight sky with the luminance of the Moon. Bruin proposed some simplifications, such as assuming that the Moon always has the same luminance; we have corrected it in our modified method proposal. Bruin did not consider the variation of the atmospheric extinction coefficient nor the probability of vision.

Like Bruin, we associate a curve to each phase angle which depends on the coefficient of vision and the probability of vision. Using a graphical method, we determine from these graphs whether the Moon will be visible or not.

Finally, applying the modified Bruin method, we have determined a value for the Danjon limit, which we have calculated for an extinction coefficient of 0.15 and a vision probability of 20%. The topocentric arc-light of the Danjon limit that we have found is approximately 10.1º, a little higher than the value found by other authors.

11. Appendix

* *Luminance.* Luminance *B* is the luminous flux (or luminous power) emitted per unit area perpendicular to the direction of emission and per unit solid angle

$$B = \frac{d^2\phi}{dS\cos\theta d\Omega} \quad \text{(A.1)}$$

its unit is $\mathrm{lm \cdot sr^{-1} \cdot m^{-2}}$ or $\mathrm{cd \cdot m^{-2}}$; θ is the angle between the direction in which the luminance is measured and the normal to the luminous surface element *dS*.

* *Iluminance.* Illuminance *E* is the luminous flux that reaches the unit area normal to the direction of incidence and has the unit $\mathrm{lm/m^2}$ or lux

$$E = \frac{d\phi}{dS'\cos\theta'} \quad \text{(A.2)}$$

dS' is the surface on which the light falls and θ' is the angle between the normal to the surface dS' and the incidence direction.

* *Relationship between luminance and illuminance.* There is a relationship between luminance and illuminance. If $d\Omega$ is the solid angle of the surface dS' on which the light falls observed from the light source, *r* is the distance between emitting and receiving surfaces, then it is satisfied

$$d\Omega = \frac{dS'\cos\theta'}{r^2},$$

the solid angle $d\Omega'$ of the emitting surface element dS seen from the surface on which the light falls is

$$d\Omega' = \frac{dS\cos\theta}{r^2}$$

by (A.1) we find

$$B = \frac{d^2\phi}{dS\cos\theta d\Omega} = \frac{d^2\phi}{dS\cos\theta\frac{dS'\cos\theta'}{r^2}} = \frac{d^2\phi}{d\Omega' dS'\cos\theta'} = \frac{dE}{d\Omega'} \quad\Rightarrow\quad dE = Bd\Omega'$$

$d\Omega'$ is the solid angle of an element of the emitting surface as measured by the observer. In the special case that the surface has uniform luminance (the same over the entire surface), then

$$E = B\Omega'. \tag{A.3}$$

* *Contrast.* We define the contrast of a image of luminance B that is on a background of luminance B_S as

$$C = \frac{B - B_S}{B_S}$$

it is a dimensionless quantity that, for our purposes, is always a positive number. The observed luminance B of the Moon is the sum of the luminance of the Moon B_M, after going through the atmosphere and the luminance of the twilight sky B_S

$$C = \frac{B - B_S}{B_S} = \frac{\left(B_M + B_S\right) - B_S}{B_S} = \frac{B_M}{B_S}. \tag{A.4}$$

* *Stellar magnitude.* We define stellar magnitude so that an increase of 5 of its units corresponds to an increase of 100 times its illuminance. We take as reference that an illuminance of 1 lux has a magnitude of -13.98; therefore, the visual magnitude m is determined by (Allen 1973, p.201)

$$m = -13.98 - 2.5\log E. \tag{A.5}$$

the unit of E is lux.

* *Atmospheric extinction.* When light passes through the atmosphere, it undergoes a weakening called extinction, caused by three factors: Rayleigh scattering by molecules, scattering by aerosols, and molecular absorption, mainly ozone. The attenuation of light rays entering the atmosphere follows the Beer-Lambert law

$$E = E'10^{-0.4kX} \quad\Rightarrow\quad B = B'10^{-0.4kX} \quad\Rightarrow\quad X = \frac{2.5}{k}\log\frac{B'}{B} \tag{A.6}$$

E and B are the illuminance and luminance observed, E' and B' illuminance and luminance outside the atmosphere, X es el air-mass, a measure of the distance traveled by light in the atmosphere, k is a constant called the extinction coefficient expressed in magnitudes per air-mass.

As a consequence of atmospheric extinction, the stellar magnitude m after passing through the atmosphere is

$$m = -13.98 - 2.5\log E = -13.98 - 2.5\log E' + kX = m' + kX$$

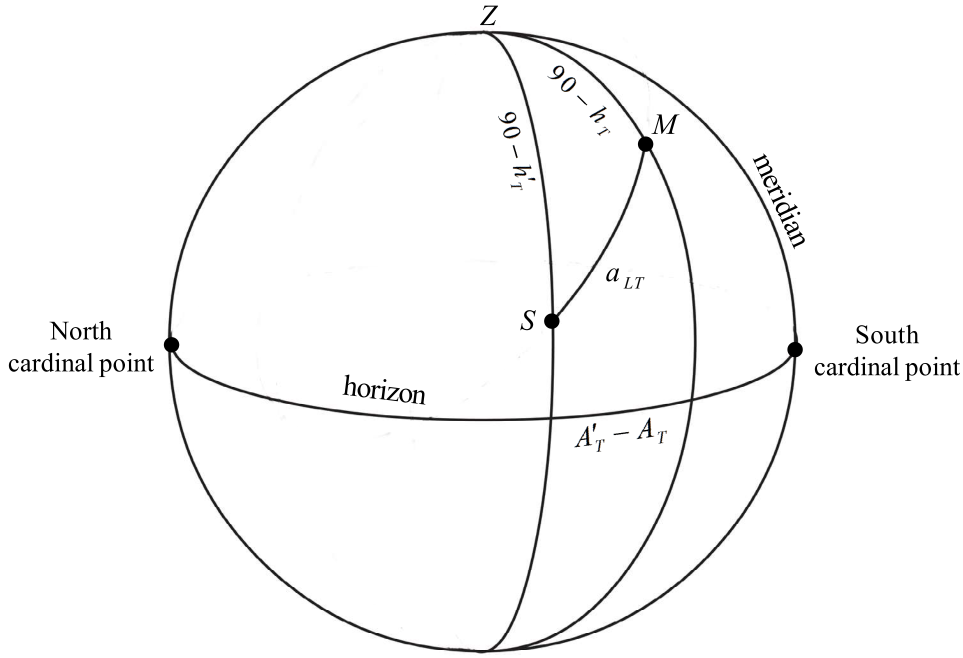
m' is the magnitude of the Moon before atmospheric attenuation.

There are several formulas for air-mass X that conform to realistic atmospheric models; among them is the formula of Kasten and Young (1989)

$$X = \frac{1}{\cos z + 0.50572\left(6.07995^\circ + 90^\circ - z\right)^{-1.6364}} \tag{A.7}$$

z is the apparent zenith distance in degrees (that is, topocentric and with refraction).

* *Phase angle.* The geocentric phase angle χ is the selenocentric angle between the centers of the Sun and the Earth. Note that when the Moon is in conjunction is not $\chi = 180^\circ$, because there is ecliptic latitude of the Moon. The topocentric phase angle χ_T is the selenocentric angle between



Drawing A.1.- In the celestial sphere, we have drawn the horizon, which is the horizontal circle. In the center of the sphere is the observer. *S* and *M* is the Sun and the Moon, *Z* is the zenith of the observation site. We measure azimuth in a retrograde direction from the meridian to the point where the horizon intersects the star's vertical. The arc of the great circle between the Sun and the Moon is the arc-light.We measure azimuth from the south. From triangle *SZM*, we calculate the arc-light by the cosine theorem. The angle at vertex *Z* is the azimuth difference between the Sun and the Moon.

the observer's position on the Earth's surface and the center of the Sun.

* *Arc-ligth.* The geocentric arc-light a_L is the angle measured from the center of the Earth between the Sun and the Moon centers. The topocentric arc-light a_{LT} is the angle measured from the observation point on the Earth's surface between the Moon and Sun's centers.

* *Allen's formula*. The bolometric magnitude of the Moon, that is, measured for all wavelengths, in the absence of atmosphere and at the mean distance from the Earth, is calculated by (Allen 1973, p.144)

$$m = -12.73 + 0.026|\chi'| + 4 \cdot 10^{-9} \chi'^4$$

this formula cannot be extrapolated to large phase angles because they do not consider the attenuation of the illuminance of the Moon due to the inclination with which the solar rays reach its surface.

Following a suggestion from Russell (1916) we obtain a formula for the magnitude of the Moon that depends on the cube of $180-\chi'$ (Segura, 2021b)

$$m = 3.62548 - 2.33551\log(180-\chi')^3 = 3.62548 - 2.5\log(180-\chi')^{2.8026}. \qquad \text{(A.8)}$$

* *Solid angle of a circular surface.* When projecting a circular image of angular diameter θ onto a sphere of radius *r*, it forms a spherical shell, whose area is

$$S = 2\pi r^2\left[1-\cos\left(\frac{\theta}{2}\right)\right]$$

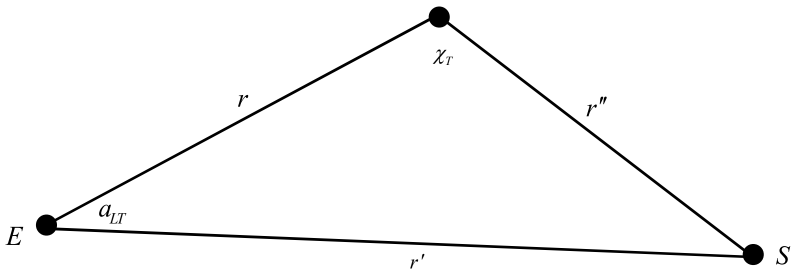
then the solid angle is

$$\Omega = \frac{S}{r^2} = 2\pi\left[1-\cos\left(\frac{\theta}{2}\right)\right].$$

If θ is very small

$$\cos\left(\frac{\theta}{2}\right) \approx 1 - \frac{1}{2}\left(\frac{\theta}{2}\right)^2$$

expressed θ in radians, therefore



Drawing A.2.- Positions of the observer on the Earth *E*, Moon *M*, and Sun *S*. χ_T is the phase angle and a_{LT} the light-arc, both topocentric,

$$\Omega = \pi\left(\frac{\theta}{2}\right)^2 \quad \Rightarrow \quad E = \pi\left(\frac{\theta}{2}\right)^2 B. \tag{A.9}$$

As the total luminance of the image is the sum of the luminance of the image *B* and the background luminance B_S, then using (A.2) and (A.3), the illuminance of the image is

$$E = \pi\left(\frac{\theta}{2}\right)^2 B = \pi\left(\frac{\theta}{2}\right)^2 B_S C,$$

E is the illuminance caused exclusively by the image, B_S is the background luminance, and *C* is the contrast, θ is in radians.

If θ is not very small, then the illuminance as a function of luminance for a circular image is

$$E = 2\pi B_S C\left[1-\cos(\theta/2)\right].$$

* *Refraction.* The angle of refraction is

$$R_R = z - z_0$$

z is the geometric zenith distance (without refraction), and z_0 is the apparent zenith distance (with refraction), R_R is measured in arc minutes. Bennett (1982) (Meeus, 1991, p. 102) obtained an empirical formula that gives with a very good approximation the angle of refraction for all the values of the apparent altitude of a star above the horizon

$$R_R = \frac{1}{\tan\left(h_0 + \dfrac{7.31}{h_0 + 4.4}\right)} \tag{A.10}$$

valid for normal atmospheric pressure and 10 ºC, h_0 being expressed in degrees. In (A.16), the angle of refraction is in minutes and the apparent altitude in degrees.

When we know the geometric altitude *h* (not including refraction) and not the apparent altitude h_0, we continue to use (A.10) to calculate the angle of refraction, using successive approximations.

* *Width of the crescent*. The width ω of a zone of the crescent Moon of position angle ψ when the phase angle is χ is

$$w(\chi,\psi) = \frac{R}{r}\left(1 + \frac{\cos\chi}{\sqrt{\sin^2\psi\cos^2\chi + \cos^2\psi}}\right),$$

R is the radius of the Moon, *r* the Earth-Moon distance, ω is expressed in radians and ψ is the angle of position. The maximum width of the Moon corresponds to $\psi = 0$

$$w_{\max}(\chi) = \frac{R}{r}(1+\cos\chi). \tag{A.11}$$

* *Parallax:* To calculate the topocentric arc-light, we apply the cosine theorem to the spherical triangle of drawing A.1

$$\cos a_{LT} = \cos(A'_T - A_T)\cos h'_T \cos h_T + \sin h'_T \sin h_T.$$

A'_T, h'_T is azimuth and altitude of the Sun and A_T, h_T those of the Moon, all of them topocentric. We neglect the parallax of the Sun and identify its topocentric altitude with the geocentric one; furthermore, since the parallax in azimuth is minimal, we also neglect it; in other words, we will only correct for parallax the altitude of the Moon and its distance from the place of observation.

Applying the sine theorem to the triangle in drawing A.2

$$\frac{\sin a_{LT}}{r''}=\frac{\sin \chi_T}{r'} \quad \Rightarrow \quad \chi_T=\sin^{-1}\left(\frac{r'\sin a_{LT}}{r''}\right),$$

r' is the distance from the Sun to the observation point, and r'' the distance between the Sun and the Moon's centers. Since $r' \sim r''$, then by (A.13) $\chi_T \approx 180 - a_{LT}$.

Applying the cosine theorem to the triangle in drawing A.2

$$r''=\sqrt{r_T^2+r'^2-2r_T r'\cos a_{TL}}$$

r_T is the topocentric distance from the center of the Moon.

If the azimuth difference between the Sun and the Moon is zero, then (A.12) reduces to

$$\cos a_{LT}=\cos\left(h_T-h'\right) \quad \Rightarrow \quad a_{LT}=h_T-h' \approx 180-\chi_T. \tag{A.12}$$

When the Moon is very close to the horizon then

$$\sin\pi \approx \pi; \quad \cos h_T \approx 1; \quad \sin h_T \approx h_T$$

and from equations (A.11) it follows that

$$h_T \approx h-\pi \tag{A.13}$$

π is the equatorial horizontal parallax of the Moon, which when it is at the mean distance from the Earth is 57' 2.6''.

* *Resolving power:* Suppose a point image observed at a great distance through a circular diaphragm, which could be the pupil of the eye. When light passes through the diaphragm, the diffraction phenomenon occurs, the image observed through the diaphragm is a central circle surrounded by circular rings. The central image is called the Airy disk and has the angular diameter

$$\theta_r=2.44\frac{\lambda}{\varphi}, \tag{A.14}$$

λ is the wavelength and φ is the diameter of the diaphragm or pupil.

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