Abstract: This paper deals with the problem of delay-dependent robust $H_\infty$ control for discrete-time recurrent neural networks (DRNNs) with norm-bounded parameter uncertainties and interval time-varying delay. The activation functions are assumed to be globally Lipschitz continuous. For the robust stabilization problem, a state feedback controller is designed to ensure global robust stability of the closed-loop system about its equilibrium point for all admissible uncertainties, while for the robust $H_\infty$ control problem, attention is focused on the design of a state feedback controller such that in addition to the requirement of the global robust stability, a prescribed $H_\infty$ performance level for all delays to satisfy both the lower bound and upper bound of the interval time-varying delay is also required to be achieved. A linear matrix inequality approach is developed to solve these problems. It is shown that the desired state feedback controller can be constructed by solving certain LMIs. A numerical example is provided to demonstrate the effectiveness and applicability of the proposed results.

Keywords: Interval time-varying delay, Linear matrix inequality, Robust $H_\infty$ control, Stability, Discrete-time recurrent neural network

1. Introduction. In recent years, the study of delayed recurrent neural networks has attracted considerable attention due to the fact that delayed recurrent neural networks can be applied in patterns recognition, associative memories and optimization solvers, etc. Some of these applications have been reported on the existence, uniqueness, and global asymptotic or exponential stability of the equilibrium point for recurrent neural networks with constant delays or time-varying delays. Especially, the problem of stability analysis of delayed recurrent neural networks has been an important topic for researcher [1-6].

Up to now, most works on recurrent neural networks have been focused on acting in a continuous-time manner. However, when it comes to the implementation of continuous-time networks for the sake of computer-based simulation, experimentation or computation, generally speaking, it is usual to discretize the continuous-time networks. Accordingly, discrete-time neural networks have already been applied in a wide range of areas, such as image processing [7], time series analysis [8], quadratic optimization problems [9] and system identification [10], etc. In an ideal case, the discrete-time analogues should be produced in a way to reflect the dynamics of their continuous-time counterparts. Specifically, the discrete-time analogue should inherit the dynamical characteristics of the continuous-time networks under mild or no restriction on the discretization step-size, and also maintain functional similarity to the continuous-time system and any physical or biological reality that the continuous-time network has [11]. Unfortunately, as pointed out in [12],
the discretization cannot preserve the dynamics of the continuous-time counterpart even for a small sampling period. Therefore, there is a crucial need to study the dynamics of discrete-time neural networks.

Recently, the exponential stability analysis problem for discrete-time recurrent neural networks with time-varying delays was studied in [13], where a LMI approach [14] was developed to solve the problem. However, there might also be some uncertainties such as parameter perturbations, which can lead to oscillations and chaos because of the existence of modeling errors, external disturbance and parameter fluctuation. Thus it is necessary to study the robust stability of the discrete-time recurrent neural networks. In the case when interval time-varying delay and parameter uncertainties appear simultaneously, the global robust stability problem with dependence on both the lower bound and upper bound of the interval time-varying delay was solved in [15] via a LMI approach. One application of the existence, uniqueness, and globally exponential stability of the periodic solution for the discrete-time recurrent neural network can be found in [16], in which the sufficient conditions have been obtained in terms of LMIs with dependence on both the lower bound and upper bound of the interval time-varying delay.

Control of time-delay systems is a subject of both practical and theoretical importance which has received much attention in the past years. A great number of results on system analysis and control synthesis of such systems have been proposed [17-20]. On the other hand, the study of the control theory for stability of uncertain delayed stochastic neural network has been reported in [21], where memoryless stabilizing state feedback controllers were designed via an LMI approach. It is worth pointing out that all the above mentioned results in [21] were obtained in the context of continuous-time recurrent neural network. When discrete-time recurrent neural networks with interval time-varying delay and parameter uncertainties are concerned, no results on robust $H_{\infty}$ control for such systems have been available in the literature, so far, which motivates the present study.

This paper deals with the problems of global robust stabilization and robust $H_{\infty}$ control for discrete-time uncertain recurrent neural network with interval time-varying delay. The parameter uncertainties are assumed to be time-varying norm-bounded, and the interval time-varying delay includes both lower and upper bounds of delay. The purpose of the robust stabilization is the design of a state feedback controller such that the resulting closed-loop system is globally robustly stable about its equilibrium point for all admissible uncertainties. For the robust $H_{\infty}$ control problem, the purpose is the design of a state feedback controller such that, in addition to the requirement of the global robust stability of the closed-loop system, a specified $H_{\infty}$ performance level is also required to be achieved. Based on certain LMIs, sufficient conditions for the solvability of these problems are obtained, which are dependent on the both the lower bound and upper bound of the interval time-varying delay. When these LMIs are feasible, a desired state feedback controller gain can be constructed. Finally, a numerical example is given to demonstrate the effectiveness.

Throughout this paper, the notation $X \geq Y$ ($X > Y$) for symmetric matrices $X$ and $Y$ indicates that the matrix $X - Y$ is positive and semi-definite (respectively, positive definite), $Z^T$ represents the transpose of matrix $Z$, and the vector norm $\|\cdot\|$ indicates the Euclidean vector norm, that is $\|W\| = \lambda_{\max}(W^T W)$, where $\lambda_{\max}(W)$ (respectively $\lambda_{\min}(W)$) denotes the operation of taking the maximum (respectively, minimum) eigenvalue of $W$. $l_2[0, \infty]$ refers to the space of square summable infinite vector sequences.

2. Problem Formulation. Consider the following discrete-time recurrent neural network with interval time-varying delay described by non-linear differential equations of the
form

\[ x(k+1) = (A + \Delta A(k))x(k) + (W + \Delta W(k))f(x(k)) + (W_1 + \Delta W_1(k))f(x(k) - \tau(k)) + u(k) + v(k), \]

(1)

\[ z(k) = Lx(k), \]

(2)

where \( x(k) = (x_1(k), x_2(k), \ldots, x_n(k))^T \) is the state vector, \( A = \text{diag}(a_1, a_2, \ldots, a_n) \) is a real constant diagonal with entries \( |a_i| < 1 \), \( W = [w_{ij}]_{n \times n} \) and \( W_1 = [(w_{ij})_1]_{n \times n} \) are the connection weight matrix and the delayed connection weight matrix, respectively.

The uncertain matrices \( \Delta A(k), \Delta W(k) \) and \( \Delta W_1(k) \) are unknown matrices representing time-varying parameter uncertainties, which are assumed to be of the form

\[ [\Delta A(k) \Delta W(k) \Delta W_1(k)] = MF(k) [N_1 \quad N_2 \quad N_3], \]

(4)

where \( M, N_1, N_2, \) and \( N_3 \) are known real constant matrices, and \( F(\cdot) : R \to R^{s \times j} \) is unknown time-varying matrix satisfying

\[ F^T(k)F(k) \leq I, \quad k \in N. \]

(5)

The uncertain matrices \( \Delta A(k), \Delta W(k) \) and \( \Delta W_1(k) \) are said to be admissible if both (4) and (5) hold.

In order to obtain our main results, the following assumption will be made throughout the paper.

**Assumption 2.1.** The activation function \( f(x) \) is nondecreasing, bounded and globally Lipschitz; that is

\[ 0 \leq \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leq \sigma_i, \quad \forall s_1 \neq s_2, \quad i = 1, 2, \ldots, n. \]

(6)

Throughout this paper, the following definitions will be used.

**Definition 2.1** (18, 22). The equilibrium point (trivial solution) 0 is said to be globally asymptotically stable if it is locally stable in the Lyapunov sense and is globally attractive, where global attractivity indicates that every trajectory tends to the equilibrium point as \( t \to \infty \).

**Definition 2.2.** The discrete-time uncertain recurrent neural network is said to be globally robustly stable about its equilibrium point with disturbance attenuation level \( \gamma \) if it is globally robustly stable and under zero initial conditions,

\[ ||z||_2 \leq \gamma ||v||_2, \]

(7)

For all nonzero \( v \in L_2([0, \infty); R^p) \) and all admissible uncertainties, where \( \gamma > 0 \) is a given scalar.

Clearly, the origin is the equilibrium point of recurrent neural network (1) without the controller and the disturbance input. In this paper, the problem is to develop techniques of robust stabilization to design a controller to control the state to converge to the origin and robust \( H_\infty \) control for discrete-time uncertain recurrent neural network with interval time-varying delay. Moreover, the following two robust control problems are given.
1. Robust stabilization problem: Determine a state feedback controller
\[ u(k) = Kx(k), \] (8)
for the system (1) such that the resulting closed-loop system is globally robustly stable about its equilibrium point.

2. Robust \( H_{\infty} \) control problem: Given a constant scalar \( \gamma > 0 \), determine a state feedback in the form of (8) such that the resulting closed-loop system is globally robustly stable with disturbance attenuation level \( \gamma \), thus (7) in fact gives an estimation of the deviation of the perturbed trajectory from the equilibrium point.

3. Main Results.

3.1. Robust stabilization. This subsection explores the global robust stabilization of the discrete-time recurrent uncertain neural network with interval time-varying delay given in (1). Specially, an LMI approach is employed to solve the problem in (1) is globally robustly stable about its equilibrium point for all admissible uncertainties \( \Delta A(k), \Delta W(k) \) and \( \Delta W_1(k) \). The analysis commences by using the LMI approach to develop some results which are essential to introduce the following Lemma 1 for the development of our main theorem.

**Lemma 3.1.** Let \( A, D, S, F \) and \( P \) be real matrices of appropriate dimensions with \( P > 0 \) and \( F \) satisfying \( F^T(k)F(k) \leq I \). Then the following statements hold

(a) For any \( \varepsilon > 0 \) and vectors \( x,y \in \mathbb{R}^n \)
\[ 2x^TD\varepsilon y \leq \varepsilon^{-1}x^TD^TDx + \varepsilon y^TS^TSy, \] (9)

(b) For vectors \( x,y \in \mathbb{R}^n \)
\[ 2x^TDsy \leq x^TD^TDx + y^TS^T \varepsilon^{-1}Sy. \] (10)

For any matrices \( E_i, S_i, T_i \quad (i = 1,2) \) and \( H_1 \) of appropriate dimensions, the null equations can be shown that
\[ \Phi_1 = 2[x^T(k)E_1 + x^T(k - \tau(k))E_2] \times [x(k) - x(k - \tau(k))] \]
\[ - \sum_{j=k-\tau(k)+1}^{k} (x(j) - x(j - 1))] = 0, \] (11)
\[ \Phi_2 = 2[x^T(k - \tau(k))S_1 + x^T(k - \tau_2)S_2] \times [x(k - \tau(k)) - x(k - \tau_2)] \]
\[ - \sum_{j=k-\tau_2+1}^{k-\tau(k)} (x(j) - x(j - 1))] = 0, \] (12)
\[ \Phi_3 = -2[x^T(k - \tau(k))T_1 + x^T(k - \tau_1)T_2] \times [x(k - \tau_1) - x(k - \tau(k))] \]
\[ - \sum_{j=k-\tau_1+1}^{k-\tau(k)} (x(j) - x(j - 1))] = 0, \] (13)
\[ \Phi_4 = -2[x^T(k + 1)H_1] \times [x(k + 1) - (A + \Delta A(k))x(k) - (W + \Delta W(k))f(x(k)) \]
\[ -(W_1 + \Delta W_1(k))f(x(k - \tau(k)))], \] (14)
\[ \Phi_5 = 2f^T(x(k))R_1f(x(k)) - 2f^T(x(k))R_1f(x(k)) + 2f^T(x(k - \tau(k)))R_2x(k - \tau(k)) \]
\[ - 2f^T(x(k - \tau(k)))R_2x(k - \tau(k)) = 0. \] (15)
To study the global robust stabilization problem of the discrete-time uncertain recurrent neural network with interval time-varying delay, the following theorem reveals that such conditions can be expressed in terms of LMIs.

**Theorem 3.1.** Under the Assumption 1, consider the system (1) with \( v(k) = 0 \), the equilibrium point of model (1) is globally robustly stabilizable if there exist matrices \( P > 0, Q_1 > 0, Q_2 > 0, Q_3 > 0, Z_1 > 0, Z_2 > 0, Z_3 > 0, \) a nonsingular \( H_1 \) and matrix \( Y \) and diagonal matrices \( R_1 > 0, R_2 > 0 \) and matrices \( E_i, S_i, T_i \) (\( i = 1, 2 \)) of appropriate dimensions and positive scalar \( \varepsilon \) such that the following LMI holds

\[
\begin{bmatrix}
\Pi & \tau_2 E & \tau_{21} S & \tau_{21} T & H M & \varepsilon N \\
\tau_2 E^T & -\tau_2 Z_1 & 0 & 0 & 0 & 0 \\
\tau_{21} S^T & 0 & -\tau_{21} (Z_1 + Z_2) & 0 & 0 & 0 \\
\tau_{21} T^T & 0 & 0 & -\tau_{21} Z_2 & 0 & 0 \\
M^T H^T & 0 & 0 & 0 & -\varepsilon I & 0 \\
\varepsilon N^T & 0 & 0 & 0 & 0 & -\varepsilon I \\
\end{bmatrix} < 0, \quad (16)
\]

where

\[
\Pi = \Pi(i, j), \quad i, j = 1, \ldots, 7,
\]
\[
\Pi_{11} = -P + (\tau_{21} + 1) Q_1 + Q_2 + Q_3 + \tau_2 Z_1 + \tau_{21} Z_2 + E_1 + E_1^T, \quad \Pi_{12} = E_2^T - E_1, \quad \Pi_{13} = 0,
\]
\[
\Pi_{14} = 0, \quad \Pi_{15} = \Sigma R_1, \quad \Pi_{16} = 0, \quad \Pi_{17} = -\tau_2 Z_1 - \tau_{21} Z_2 + A^T H_1^T + Y^T,
\]
\[
\Pi_{22} = -Q_1 - E_2 - E_2^T + S_1 + S_1^T + T_1 + T_1^T, \quad \Pi_{23} = -S_1 + S_1^T, \quad \Pi_{24} = -T_1 + T_1^T,
\]
\[
\Pi_{25} = 0, \quad \Pi_{26} = R_2, \quad \Pi_{27} = 0, \quad \Pi_{33} = -Q_3 - S_2 - S_2^T, \quad \Pi_{34} = 0, \quad \Pi_{35} = 0, \quad \Pi_{36} = 0, \quad \Pi_{37} = 0, \quad \Pi_{44} = -T_2 - T_2^T - Q_2, \quad \Pi_{45} = 0, \quad \Pi_{46} = 0, \quad \Pi_{47} = 0, \quad \Pi_{55} = -R_1 - R_1^T, \quad \Pi_{56} = 0, \quad \Pi_{57} = W_0^T H_1^T, \quad \Pi_{66} = -R_2 \Sigma^{-1} - (R_2 \Sigma^{-1})^T, \quad \Pi_{67} = W_1^T H_1^T,
\]
\[
\Pi_{77} = -H_1 - H_1^T + P + \tau_2 Z_1 + \tau_{21} Z_2, \quad E = [ E_1^T \quad E_2^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 ],
\]
\[
S = [ 0 \quad S_1^T \quad S_2^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 ], \quad T = [ 0 \quad T_1^T \quad 0 \quad T_2^T \quad 0 \quad 0 \quad 0 \quad 0 ],
\]
\[
H = [ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad H_1^T ], \quad N = [ \quad N_1 \quad 0 \quad 0 \quad 0 \quad N_2 \quad N_3 \quad 0 \quad 0 ],
\]
\[
\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n), \quad \tau_{21} = \tau_2 - \tau_1, \quad \text{in which } 0 \leq \tau_1 < \tau_2 \text{ is given satisfying (3). In this case, an appropriate delay-dependent global robust stabilizing state feedback controller can be chosen as}
\]

\[
u(k) = H_1^{-1} Y x(k). \quad (17)
\]

**Proof:** Consider the following Lyapunov-Krasovskii functional for the discrete-time system in (1) with (8) and \( v(k) = 0 \)

\[
V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k)
= x^T(k) P x(k) + \sum_{i=-\tau_2}^{k} \sum_{j=k+i+1}^{k} (x(j) - x(j - 1))^T Z_1(x(j) - x(j - 1))
+ \sum_{i=-\tau_1}^{k} \sum_{j=k+i+1}^{k} (x(j) - x(j - 1))^T Z_2(x(j) - x(j - 1)) + \sum_{j=k+\tau}^{k+1} x^T(j) Q_1 x(j)
+ \sum_{i=-\tau_1}^{k} \sum_{j=k+i+1}^{k} x^T(j) Q_1 x(j) + \sum_{j=k+\tau_2}^{k+1} x^T(j) Q_2 x(j) + \sum_{j=k+\tau_2}^{k+1} x^T(j) Q_3 x(j).
\]

Taking the difference of \( V(k) \) along the trajectories of (1) yields

\[
\Delta V_1(k) = x^T(k+1) P x(k+1) - x^T(k) P x(k), \quad (19)
\]
\[ \Delta V_2(k) = \sum_{i=-\tau_2}^{-1} \sum_{j=k+i+2}^{k} (x(j) - x(j-1))^T Z_1(x(j) - x(j-1)) \\
- \sum_{i=-\tau_2}^{-1} \sum_{j=k+i+1}^{k} (x(j) - x(j-1))^T Z_1(x(j) - x(j-1)) \leq \tau_2 (x(k+1) - x(k))^T Z_1(x(k+1) - x(k)) \]
(20) \\
- \sum_{j=k-\tau_2+1}^{k} (x(j) - x(j-1))^T Z_1(x(j) - x(j-1)) \\
- \sum_{j=k-\tau(k)+1}^{k} (x(j) - x(j-1))^T Z_1(x(j) - x(j-1)),

\[ \Delta V_3(k) = \sum_{i=-\tau_2}^{-1} \sum_{j=k+i+2}^{k+1} (x(j) - x(j-1))^T Z_2(x(j) - x(j-1)) \\
- \sum_{i=-\tau_2}^{-1} \sum_{j=k+i+1}^{k+1} (x(j) - x(j-1))^T Z_2(x(j) - x(j-1)) \leq (\tau_2 - \tau_1) (x(k+1) - x(k))^T Z_2(x(k+1) - x(k)) \\
- \sum_{j=k-\tau_2+1}^{k+1} (x(j) - x(j-1))^T Z_2(x(j) - x(j-1)) \\
- \sum_{j=k-\tau(k)+1}^{k+1} (x(j) - x(j-1))^T Z_2(x(j) - x(j-1)),
\]

\[ \Delta V_4(k) \leq x^T(k)((\tau_2 - \tau_1 + 1) Q_1 + Q_2 + Q_3)x(k) - x^T(k - \tau(k)) Q_1 x(k - \tau(k)) \\
-x^T(k - \tau_1) Q_2 x(k - \tau_1) - x^T(k - \tau_2) Q_3 x(k - \tau_2). \]
(22)

Defining the following new variables \\
\[ \eta(k) = [x^T(k), x^T(k - \tau(k)), x^T(k - \tau_2), f^T(x(k)), f^T(x(k - \tau(k))), x^T(k+1)]^T, \]
\[ E = [E_1^T, E_2^T, 0 0 0 0 0 0]^T, \quad S = [0 S_1^T S_2^T 0 0 0 0]^T, \]
\[ T = [0 T_1^T 0 T_2^T 0 0 0]^T, \quad H = [0 0 0 0 0 0 H_1^T]^T, \]
\[ N = [N_1 0 0 0 N_2 N_3 0]^T. \]

Combining (11)-(15) and (19)-(22), \( \Delta V(k) \) becomes \\
\[ \Delta V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + \Delta V_4(k) + \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 \]
\[ \leq x^T(k+1)Px(k+1) - x^T(k)Px(k) \]
\[ + \tau_2 (x(k+1) - x(k))^T Z_1(x(k+1) - x(k)) \]
\[ - \sum_{j=k-\tau_2+1}^{k} (x(j) - x(j-1))^T Z_1(x(j) - x(j-1)) \]
\[ - \sum_{j=k-\tau(k)+1}^{k} (x(j) - x(j-1))^T Z_1(x(j) - x(j-1)) \]
\[ + (\tau_2 - \tau_1) (x(k+1) - x(k))^T Z_2(x(k+1) - x(k)) \\
- \sum_{j=k-\tau_2+1}^{k} (x(j) - x(j-1))^T Z_2(x(j) - x(j-1)) \\
- \sum_{j=k-\tau(k)+1}^{k} (x(j) - x(j-1))^T Z_2(x(j) - x(j-1)) \\
+ x^T(k)((\tau_2 - \tau_1 + 1) Q_1 + Q_2 + Q_3)x(k) \]
Moreover,
\[-2\eta^T(k)E \sum_{j=k-\tau(k)+1}^{k} (x(j) - x(j-1)) \leq \tau_2 \eta^T(k)EZ_1^{-1}ET \eta(k) \] (24)
\[+ \sum_{j=k-\tau(k)+1}^{k-\tau_1} (x(j) - x(j-1))^T Z_1(x(j) - x(j-1)), \]
\[-2\eta^T(k)S \sum_{j=k-\tau_2+1}^{k-\tau(k)} (x(j) - x(j-1)) \leq (\tau_2 - \tau_1) \eta^T(k)SZ_1 + Z_2^{-1} ST \eta(k) \] (25)
\[+ \sum_{j=k-\tau_2+1}^{k-\tau_1} (x(j) - x(j-1))^T (Z_1 + Z_2)(x(j) - x(j-1)), \]
\[2\eta^T(k)T \sum_{j=k-\tau(k)+1}^{k-\tau_1} (x(j) - x(j-1)) \leq (\tau_2 - \tau_1) \eta^T(k)TZ_2^{-1}T \eta(k) \] (26)
\[+ \sum_{j=k-\tau_2+1}^{k-\tau_1} (x(j) - x(j-1))^T Z_2(x(j) - x(j-1)). \]

Using Assumption 2.1 and noting that $R_1 > 0$ and $R_2 > 0$ are diagonal matrices, one has
\[2f^T(x(k)) R_1 f(x(k)) \leq 2f^T(x(k)) R_1 \Sigma x(k), \] (27)
\[-2f^T(x(k)) R_2 x(k) \leq -2f^T(x(k)) R_2 \Sigma^{-1} f(x(k)), \] (28)
where $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$.

Following from Lemma 3.1 (a) results in
\[2\eta^T(k)H (\Delta A(k)x(k) + \Delta W(k)f(x(k))) + \Delta W_1(k)f(x(k) - \tau(k))) \]
\[= 2\eta^T(k)HM F(k) N T \eta(k) \leq \epsilon^{-1} \eta^T(k)HM M^T H^T \eta(k) + \epsilon \eta^T(k)N N^T \eta(k), \] (29)
where $N = [N_1 0 0 0 N_2 N_3 0]^T$.

Substituting (24)-(28) into (23), it is not difficult to deduce that
\[\Delta V(k) \leq \eta^T(k)[\Pi + \tau_2 E Z_1^{-1} E^T + \tau_2 S(Z_1 + Z_2)^{-1} S^T + \tau_2 S Z_2^{-1} S^T \]
\[+ \tau_1 T Z_2^{-1} T^T + \epsilon^{-1} HM M^T H^T + \epsilon N N^T] \eta(k), \] (30)
where
\[\Pi = \Pi(i, j), \quad i, j = 1, \ldots, 7, \]
\[\Pi_{11} = -P + (\tau_2 + 1) Q_1 + Q_2 + Q_3 + \tau_2 Z_1 + \tau_2 Z_2 + E_1 + E_1^T, \]
\[\Pi_{12} = E_2^T - E_1, \quad \Pi_{13} = 0, \]
Theorem 3.2. Lyapunov-Krasovaskii stability theorem [22], state feedback controller can be obtained by solving the (16). Furthermore by using the uncertain recurrent neural network with interval time-varying delay in (1) is globally asymptotically robust stability about its equilibrium point for all admissible uncertainties. This completes the proof of Theorem 3.1.

3.2. Robust $H_{\infty}$ control. This subsection will focus on the design of a state feedback controller such that the resulting closed-loop system is globally robustly stable with disturbance attenuation level $\gamma > 0$ for estimation of the deviation of the perturbed trajectory from the equilibrium point. For presentation convenience, define the following variables

$$
\eta(k) = [x^T(k) x^T(k-\tau(k)) x^T(k-\tau_2) x^T(k-\tau_1) f^T(x(k)) f^T(x(k-\tau_2(k))) x^T(k+1) v^T(k)]^T,
$$

where $
E = [ E_1^T E_2^T 0 0 0 0 ]^T,$

$$
S = [ 0 S_1^T S_2^T 0 0 0 0 ]^T,
$$

$$
T = [ T_1^T 0 T_2^T 0 0 0 ]^T,
$$

$$
H = [ 0 0 0 0 0 0 H_1^T ]^T,
$$

$$
N = [ N_1 0 0 0 N_2 N_3 0 ]^T.
$$

The following delay-dependent global robust $H_{\infty}$ performance analysis result is explored.

Theorem 3.2. Under the Assumption 1, consider the system (1) is globally robustly stabilizable with disturbance attenuation level $\gamma > 0$ if there exist matrices $P > 0, Q_1 > 0, Q_2 > 0, Q_3 > 0, Z_1 > 0, Z_2 > 0, Z_3 > 0$, a nonsingular $H_1$ and matrix $Y$ and diagonal matrices $R_1 > 0, R_2 > 0$ and matrices $E_i, S_i$, and $T_i$ $(i = 1, 2)$ of appropriate dimensions and positive scalar $\varepsilon$ such that the following LMI holds

$$
\begin{bmatrix}
\Pi & \tau_2 E & \tau_2 S & \tau_2 T & H M & \varepsilon N \\
\tau_2 E^T & -\tau_2 Z_1 & 0 & 0 & 0 & 0 \\
\tau_2 S^T & 0 & -\tau_2 (Z_1 + Z_2) & 0 & 0 & 0 \\
\tau_2 T^T & 0 & 0 & -\tau_2 Z_2 & 0 & 0 \\
M^T H^T & 0 & 0 & 0 & -\varepsilon I & 0 \\
\varepsilon N^T & 0 & 0 & 0 & 0 & -\varepsilon I
\end{bmatrix} < 0,
$$

where $\Pi = \Pi(i, j), \ i, j = 1, \cdots, 8,$

$\Pi_{11} = -P + (\tau_2 + 1) Q_1 + Q_2 + Q_3 + L^T L + \tau_2 Z_1 + \tau_2 Z_2 + E_1 + E_1^T,$

$\Pi_{12} = E_2 - E_1,$

$\Pi_{13} = 0, \Pi_{14} = 0, \Pi_{15} = \Sigma^T R_1, \Pi_{16} = 0, \Pi_{17} = -\tau_2 Z_1 - \tau_2 Z_2 + A^T H_1^T + Y^T,$

$\Pi_{18} = 0, \Pi_{22} = -Q_1 - E_2 - E_2^T + S_1 + S_2^T + T_1 + T_1^T,$

$\Pi_{23} = -S_1 + S_2^T, \Pi_{24} = -T_1 + T_2^T,$

$\Pi_{25} = 0, \Pi_{26} = R_1^T, \Pi_{27} = 0, \Pi_{28} = 0, \Pi_{33} = -Q_3 - S_1 - S_2^T, \Pi_{34} = 0, \Pi_{35} = 0,$

$\Pi_{36} = 0, \Pi_{37} = 0, \Pi_{38} = 0, \Pi_{44} = -T_2 - T_2^T - Q_2, \Pi_{45} = 0, \Pi_{46} = 0, \Pi_{47} = 0,$

$\Pi_{48} = 0, \Pi_{55} = -R_1 - R_1^T, \Pi_{56} = 0, \Pi_{57} = W_0^T H_1^T, \Pi_{58} = 0,$

$\Pi_{66} = -R_2 \Sigma^{-1} -(R_2 \Sigma^{-1})^T, \Pi_{67} = W_1^T H_1^T, \Pi_{68} = 0,$

$\Pi_{77} = -H_1 - H_1^T + P + \tau_2 Z_1 + \tau_2 Z_2, \Pi_{78} = H_1, \Pi_{88} = -\gamma^2 I, \tau_2 = \tau_2 - \tau_1,$ when $0 \leq \Delta V(k) < 0$ holds if (16) is satisfied.
\( \tau_1 < \tau_2 \) is given satisfying in (3). In this case, an appropriate delay-dependent global robust stabilizing state feedback controller can be chosen as

\[
u(k) = H_1^{-1} Y x(k).
\]

**Proof:** Define

\[
J_N = \sum_{k=0}^{N} \|z(k)\|^2 - \gamma^2 \|v(k)\|^2,
\]

where the scalar \( N > 0 \) is an integer. Noting the zero initial condition and (30) yields

\[
J_N = \sum_{k=0}^{N} \|z(k)\|^2 - \gamma^2 \|v(k)\|^2 + \Delta V(k) - V_{N+1}
\leq \sum_{k=0}^{N} \eta^T(k) \theta \eta(k),
\]

where \( \theta = \Pi + \tau_2 E Z_1^{-1} E^T + \tau_{21} S (Z_1 + Z_2)^{-1} S^T + \tau_{21} S Z_2^{-1} S^T + \tau_{21} T Z_2^{-1} T^T + \varepsilon^{-1} H M M^T H^T + \varepsilon N N^T. \)

Now, by the Schur complement formula, it follows from (37) that \( \theta < 0 \), which together with (40) ensures that \( \|z\|_2 \leq \gamma \|v\|_2 \) holds under the zero initial condition. This completes the proof.

**Remark 3.1.** It is noted that in the stochastic context, state feedback controllers are designed in [21] for continuous-time stochastic recurrent neural network with time-varying delays. In this paper, however, state feedback controllers are designed in the discrete-time deterministic recurrent neural network context. It is worth noting that discrete-time deterministic recurrent neural network with interval time-varying delay and continuous-time stochastic recurrent neural network with time-varying delays have different properties, and needs to be dealt with separately. The results given in Theorem 2 provide an LMI approach to the design of state feedback controllers for discrete-time deterministic recurrent neural network with interval time-varying, which is new and presents a contribution to discrete-time uncertain recurrent neural network and control theory.


**Example 4.1.** Consider the following discrete-time uncertain recurrent neural network

\[
x(k+1) = (A + \Delta A(k))x(k) + (W + \Delta W(k))f(x(k))
+ (W_1 + \Delta W_1(k))f(x(k - \tau(k))) + u(k) + v(k),
\]

where

\[
A = \begin{bmatrix}
-1.5427 & 0 \\
0 & 0.8427 
\end{bmatrix},
W = \begin{bmatrix}
0 & 0.125 \\
0.125 & 0 
\end{bmatrix},
W_1 = \begin{bmatrix}
-0.1 & 0 \\
0 & 0.1 
\end{bmatrix},
M = \begin{bmatrix}
0 & 0.1 \\
0.1 & 0.2 
\end{bmatrix},
\]

\[
N_1 = \begin{bmatrix}
0.2 & 0.1 \\
0.2 & 0.1 
\end{bmatrix},
N_2 = \begin{bmatrix}
0.1 & 0.3 \\
0 & 0.1 
\end{bmatrix},
N_3 = \begin{bmatrix}
0.1 & 0.1 \\
0.1 & 0.1 
\end{bmatrix},
L = [0.2, 0.1].
\]

Furthermore, the activation functions in this example are assumed to satisfy Assumption 1 with \( \sigma_1 = 0.5633, \sigma_2 = 0.0478 \). By the Matlab LMI Control Toolbox, the compute upper bounds, \( \tau_2 \) which guarantee the global robust stability of system (1) for given lower bounds, \( \tau_1 = 2, \) are \( \tau_2 = 12 \). The noise attenuation level \( \gamma = 1.8 \). The purpose is the design of a state feedback controller such that the resulting closed-loop system is globally robustly stable with disturbance attenuation level \( \gamma \) for estimation of the deviation of the perturbed trajectory from the equilibrium point.

The feasible solutions for LMI (16) can be found as follows.
\[
P = \begin{bmatrix} 6.1464 & 0.0690 \\ 0.0690 & 5.9476 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.2173 & -0.0006 \\ -0.0006 & 0.1449 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.9238 & -0.0050 \\ -0.0050 & 0.9402 \end{bmatrix},
\]
\[
Q_3 = \begin{bmatrix} 0.5452 & -0.0036 \\ -0.0036 & 0.5544 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} 0.0793 & -0.0001 \\ -0.0001 & 0.0875 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0.1051 & 0.0000 \\ 0.0000 & 0.1135 \end{bmatrix},
\]
\[
R_1 = \begin{bmatrix} 1.4427 & 0 \\ 0 & 1.7928 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.4348 & 0 \\ 0 & 0.2024 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0.3397 & 0.0442 \\ 0.0432 & 0.2730 \end{bmatrix},
\]
\[
Y = \begin{bmatrix} 2.4220 & -0.0521 \\ -0.0248 & 1.8124 \end{bmatrix}, \quad \varepsilon = 1.0470.
\]

Therefore, by Theorem 2, it is easy to see that the global robust \(H_\infty\) control problem is solvable, and the desired state feedback controller can be constructed as
\[
u(k) = H_1^{-1}Yx(k) = \begin{bmatrix} 6.9743 & 0.6960 \\ -1.1943 & 6.5285 \end{bmatrix}x(k).
\]

5. **Conclusions.** This study has investigated the problem of globally robust stability for a discrete-time recurrent neural network with time-varying norm-bounded parameter uncertainties and interval time-varying delay. A sufficient condition to design the global robust stability state feedback controllers and guarantee a prescribed level on disturbance attenuation, which takes into account the range for the time delay, has been derived using the Lyapunov-Krasovskii functional and the LMI approach. A numerical example has been presented to demonstrate the effectiveness of the proposed approach.

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