Maximizing deviation method for interval-valued intuitionistic fuzzy multi-attribute decision making

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Abstract - The fuzzy multi-attribute decision making problems are investigated, in which the attribute values are given as interval-valued intuitionistic fuzzy numbers and the attribute weight information is incomplete. Some operational laws of interval-valued intuitionistic fuzzy numbers, score function and accuracy function of interval-valued intuitionistic fuzzy numbers are introduced. An optimization model based on the maximizing deviation method, by which the attribute weights can be determined, is established. We utilize the interval-valued intuitionistic weighted arithmetic averaging (IIWAA) operator to aggregate the interval-valued intuitionistic fuzzy information corresponding to each alternative, and then rank the alternatives and select the most desirable one(s) according to the score function and accuracy function. Finally, a practical example is provided to illustrate the developed approach and to demonstrate its effectiveness.

Index Terms - Multi-attribute decision making; interval-valued intuitionistic fuzzy number; interval-valued intuitionistic weighted arithmetic averaging (IIWAA) operator

I. INTRODUCTION

Atanassov [1] introduced the concept of intuitionistic fuzzy sets (IFSs) characterized by a membership function and a non-membership function, which is a generalization of Zadeh’s fuzzy sets [2]. Later, Atanassov and Gargov [3] further introduced the interval-valued intuitionistic fuzzy sets (IVIFSs), which are a further generalization of the IFSs. The fundamental characteristic of the IVIFS is that the values of its membership function and non-membership function are intervals rather than exact numbers. IVIFS has been proven to be highly useful to deal with uncertainty and vagueness, and a lot of work has been done to develop and enrich the IVIFS theory. In many complex decision making problems, the decision information provided by a decision maker is often imprecise or uncertain due to time pressure, lack of knowledge, and the DM’s limited expertise related with problem domain. Accordingly, IVIFS is a very suitable tool to be used to describe the imprecise or uncertain decision information. Some researchers have shown great interest in the IVIFS theory and applied it to the field of decision making. Atanassov [4] gave some operations and relations over IVIFSs, and studied their basic properties. Bustince and Burillo [5] defined the concepts of correlation and correlation coefficient of IVIFSs, and introduced two decomposition theorems of the correlation of IVIFSs. Hong [6] generalized the concepts of correlation and correlation coefficients of IVIFSs in a general probability space, and also derived some decomposition theorems of the correlation of IVIFSs in terms of the correlation of interval-valued fuzzy sets and the entropy of IFSs. Hung and Wu [7] proposed a method for calculating the correlation coefficient of IVIFSs by means of “centroid”. Mondal and Samanta [8] defined a topology of IVIFSs and studied some topological properties. Deschrijver and Kerre [9] established the relationships between intuitionistic fuzzy sets, L-fuzzy sets, interval-valued fuzzy sets and IVIFSs. Recently, Xu [10, 11] developed some aggregation operators, and gave an application of the operators to multi-attribute decision making with interval-valued intuitionistic fuzzy information, in which the information about attribute weights is completely known. However, the information about attribute weights is often incompletely known. The decision makers may provide the information about attribute weights with value ranges or order relation. Therefore, it is necessary to pay attention to this issue. The aim of this paper is to develop another method, based on the maximizing deviation method, to overcome this limitation. In this paper, we shall develop the maximizing deviation method for multiattribute decision making under interval-valued intuitionistic fuzzy environment, where the information about attribute weights is partially known or completely unknown.

The rest of this paper is organized as follows. Section II gives a review of basic concepts and operations related to interval-valued intuitionistic fuzzy sets. In Section III we introduce the MADM problem with interval-valued
Maximizing deviation method for interval-valued intuitionistic fuzzy multi-attribute decision making

Intuitionistic fuzzy information, in which the information about attribute weights is incompletely known. To determine the attribute weights, an optimization model based on the maximizing deviation method is established. A practical procedure based on maximizing deviation is developed in Section IV to handle the interval-valued intuitionistic fuzzy multi-attribute decision making. A practical example and a short conclusion are given in Section V and VI, respectively.

II. PRELIMINARIES

Let us first review some basic concepts related to IFSs and IVIFSs

Definition 2.1[1]. Let \( X = \{x_1, x_2, \cdots, x_n\} \) be a finite universal set. An intuitionistic fuzzy Set (IFS) \( A \) in \( X \) is an object having the following form:

\[
A = \{< x, \mu_A(x), \nu_A(x) > | x \in X \}
\]

where the functions

\[
\mu_A : X \rightarrow [0,1], \; x \in X \rightarrow \mu_A(x) \in [0,1]
\]

and

\[
\nu_A : X \rightarrow [0,1], \; x \in X \rightarrow \nu_A(x) \in [0,1]
\]

with the condition

\[
0 \leq \mu_A(x) + \nu_A(x) \leq 1, \; \forall \; x \in X
\]

The numbers \( \mu_A(x) \) and \( \nu_A(x) \) denote the degree of membership and the degree of non-membership of the element \( x \in X \) in the set \( A \), respectively.

We call \( \pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \) the intuitionistic fuzzy index of the element \( x \) in the set \( A \). It is the degree of indeterminacy membership of the element \( x \) in the set \( A \).

It is obvious that for every \( x \in X \), \( 0 \leq \pi_A(x) \leq 1 \). However, sometime it is not appropriate to assume that the membership degrees for certain elements of \( A \) are exactly defined, but a value range can be given. In such cases, Atanassov and Gargov[3] defined the notion of interval-valued intuitionistic fuzzy set (IVIFS) as below:

Definition 2.2. Let \( X = \{x_1, x_2, \cdots, x_n\} \) be a given set, and \( D[0,1] \) be the set of all closed subintervals of the interval [0, 1], an interval-valued intuitionistic fuzzy set (IVIFS) \( \tilde{A} \) in \( X \) is an expression given by

\[
\tilde{A} = \{< x, \tilde{\mu}_A(x), \tilde{\nu}_A(x) > | x \in X \}
\]

where

\[
\tilde{\mu}_A : X \rightarrow D[0,1], \; \tilde{\nu}_A : X \rightarrow D[0,1]
\]

with the condition

\[
\sup(\tilde{\mu}_A(x)) + \sup(\tilde{\nu}_A(x)) \leq 1.
\]

Especially, if each of the intervals \( \tilde{\mu}_A(x) \) and \( \tilde{\nu}_A(x) \) contains exactly one element, i.e., if for every \( x \in X \),

\[
\mu_A(x) = \inf(\tilde{\mu}_A(x)) = \sup(\tilde{\mu}_A(x))
\]

\[
\nu_A(x) = \inf(\tilde{\nu}_A(x)) = \sup(\tilde{\nu}_A(x))
\]

Then, the given IVIFS \( \tilde{A} \) is transformed to an ordinary intuitionistic fuzzy set.

Based on IVIFS, Xu[12] defined the notion of interval-valued intuitionistic fuzzy number (IVIFN):

Definition 2.3. Let \( \tilde{A} = \{< x, \tilde{\mu}_A(x), \tilde{\nu}_A(x) > | x \in X \} \) be an IVIFS, then we call the pair \((\tilde{\mu}_A(x), \tilde{\nu}_A(x))\) an IVIFN.

For convenience, we denote the pair \((\tilde{\mu}_A(x), \tilde{\nu}_A(x))\) by \((|a, b|, [c, d])\), where \([a, b] \in D[0,1], [c, d] \in D[0,1]\) and \(a + b \leq 1\).

Distance between interval-valued intuitionistic fuzzy sets was introduced by Xu[13]. Here, we introduce a normalized Hamming distance, which will be employed in Section III.

Definition 2.4. Let \( \tilde{\alpha}_1 = (|[a_1, b_1]|, [c_1, d_1]) \) and \( \tilde{\alpha}_2 = (|[a_2, b_2]|, [c_2, d_2]) \) be any two IVIFNs, then the normalized Hamming distance between \( \tilde{\alpha}_1 \) and \( \tilde{\alpha}_2 \) can be defined as:

\[
d(\tilde{\alpha}_1, \tilde{\alpha}_2) = \frac{1}{4}(|a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2| + |d_1 - d_2|)
\]

Definition 2.5. Let \( \tilde{\alpha} = ([a, b], [c, d]) \) be an IVIFN, a score function \( S \) of an interval-valued intuitionistic fuzzy number can be represented as follows [10, 12]:

\[
S(\tilde{\alpha}) = \frac{a - c + b - d}{2}
\]

where \( S(\tilde{\alpha}) \in [-1, 1] \).

Definition 2.6. Let \( \tilde{\alpha} = ([a, b], [c, d]) \) be an IVIFN, a variance function \( H \) of an interval-valued intuitionistic fuzzy number can be represented as follows [10, 12]:

\[
H(\tilde{\alpha}) = \frac{a + b + c + d}{2}
\]

where \( H(\tilde{\alpha}) \in [0, 1] \). The larger the value of \( H(\tilde{\alpha}) \), the more the degree of accuracy of the interval-valued intuitionistic fuzzy number \( \tilde{\alpha} \).

As presented above, the score function \( S \) and the accuracy function \( H \) are, respectively, defined as the difference and the sum of the membership function \( \tilde{\mu}_A(x) \) and the non-membership function \( \tilde{\nu}_A(x) \). Hong[14] showed that the relation between the score function \( S \) and the accuracy function \( H \) is similar to the relation between mean and variance in statistics. Based on the score function \( S \) and the accuracy function \( H \), an order relation between two interval-valued intuitionistic fuzzy numbers is given by Xu[12] as follows:

Definition 2.7. Let \( \tilde{\alpha}_1 \) and \( \tilde{\alpha}_2 \) be two interval-valued intuitionistic fuzzy numbers, then

if \( S(\tilde{\alpha}_1) < S(\tilde{\alpha}_2) \), then \( \tilde{\alpha}_1 < \tilde{\alpha}_2 \);

if \( S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2) \), then...
1) if \( H(\tilde{a}_i) = H(\tilde{a}_2) \), then \( \tilde{a}_1 = \tilde{a}_2 \);  
2) if \( H(\tilde{a}_1) < H(\tilde{a}_2) \), then \( \tilde{a}_1 < \tilde{a}_2 \); 

Definition 2.8. Let \( \tilde{a}_j = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}]) \) \((j=1, 2, \ldots, m)\) be a collection of interval-valued intuitionistic fuzzy numbers. The interval-valued intuitionistic fuzzy arithmetic aggregation (IIWAA) operator [10-12] is defined by

\[
IIWAA_{\omega} (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m) = \sum_{j=1}^{m} \omega_j \tilde{a}_j = \left( \prod_{j=1}^{m} (1-a_{ij})^{\omega_j}, \prod_{j=1}^{m} (1-b_{ij})^{\omega_j}, \prod_{j=1}^{m} c_{ij}^{\omega_j}, \prod_{j=1}^{m} d_{ij}^{\omega_j} \right)
\]

where \( \omega = (\omega_1, \omega_2, \ldots, \omega_m)^T \) be the weight vector of \( \tilde{a}_j \)(\( j=1, 2, \ldots, m \)), and \( \omega_j > 0, \sum_{j=1}^{m} \omega_j = 1 \).

III. SOME MODELS FOR DETERMINING ATTRIBUTE WEIGHTS

Consider an alternative set \( X = \{X_1, X_2, \ldots, X_n\} \), which consists of \( n \) non-inferior decision making alternatives, each alternative is assessed on \( m \) attributes. The multi-attribute decision making problem is to select a most preferred alternative from \( X \) based on the assessment of their attributes. To evaluate each alternative in the set \( X \), we use the interval-valued intuitionistic fuzzy number to represent the satisfaction and dissatisfaction degree to the fuzzy concept “excellence” expressed by the decision maker. Specifically, denote the set of all attributes \( A = \{A_1, A_2, \ldots, A_m\} \), let \( \tilde{R} = (\tilde{r}_{ij})_{n \times m} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])_{n \times m} \) be the interval-valued intuitionistic fuzzy decision matrix, where \([a_{ij}, b_{ij}]\) indicates the degree that the alternative \( X_i \) satisfies the attribute \( A_j \) to the fuzzy concept “excellence” expressed by the decision maker, while \([c_{ij}, d_{ij}]\) indicates the degree that the alternative \( X_i \) doesn’t satisfy the attribute \( A_j \) to the fuzzy concept “excellence”, expressed by the decision maker, and \([a_{ij}, b_{ij}] \in D(0, 1], [c_{ij}, d_{ij}] \in D(0, 1], b_{ij} + d_{ij} \leq 1 \), where \( i = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, m \). In other words, the evaluation of the alternative \( X_i \in X \) with respect to the attribute \( A_j \in A \) is an interval-valued intuitionistic fuzzy number \( \tilde{r}_{ij} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}]) \), the closed interval \([a_{ij}, b_{ij}]\) is the satisfaction degree and the closed interval \([c_{ij}, d_{ij}]\) is the non-satisfaction degree.

The information about attribute weights provided by the decision makers is usually incomplete because that the increasing complexity of the socio-economic environment makes it less and less possible for a single decision maker to consider all relevant aspects of a problem [15-16]. For convenience, we denote \( H \) by the set of the known information about attribute weights given by the decision makers, which can be constructed by the following forms [17,18], for \( i \neq j \):

1) A weak ranking: \( \{w_j \geq w_i\} \);  
2) A strict ranking: \( \{w_j - w_i \geq \alpha_i (> 0)\} \);  
3) A ranking with multiples: \( \{w_j \geq \alpha_i w_i\} \), \( 0 \leq \alpha_i \leq 1 \);  
4) An interval form: \( \{\alpha_i \leq w_j \leq \alpha_i + \varepsilon_i\} \), \( 0 \leq \alpha_i < \alpha_i + \varepsilon_i \leq 1 \);  
5) A ranking of differences: \( \{w_j - w_i \geq w_k - w_i\} \), for \( j \neq k \neq i \).

Let \( w = (w_1, w_2, \ldots, w_m)^T \in H \) be the weight vector of attributes, where \( w_j \geq 0, j = 1, 2, \ldots, m, \sum_{j=1}^{m} w_j = 1 \).

If the information about attribute weights is entirely known , i.e, the DM offer each attribute weight with crisp numerical value, then each attribute value can be weighted and all the weighted attribute values corresponding to each alternative can be aggregated into an overall one by using \( \tilde{r}_i = IIWAA_{\omega_i} (\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{im}) \). According to the values \( \tilde{r}_i \) of the alternatives \( X_i (i = 1, 2, \ldots, n) \), all these alternatives can be ranked and then the most desirable one(s) can be selected. The greater \( \tilde{r}_i \), the better the alternative \( X_i \) will be.

Because of the complexity of the objective things and the ambiguity of human thinking, people often do not know the information about attribute weights completely. In this situation, we can use the maximum deviation method which is proposed by Wang [19] to deal with MADM problems with numerical information. For a MADM problem, we need to compare the collective preference values to rank the alternatives. If the performance values of each alternative have little differences under an attribute, it shows that such an attribute plays a small role in the priority procedure. Contrarily, if some attribute makes the performance values among all the alternatives have obvious differences, such an attribute plays an important role in choosing the best alternative. From the perspective of ranking the alternatives, the attribute which makes larger deviations should be given a greater weight. Especially, if all available alternatives score about equally with respect to a given attribute, then the attribute is useless and should be assigned a very small weight.

For the attribute \( A_j \in A \), the deviation of alternative \( X_i \) to all the other alternatives can be represented by \( D_j(w) \) where
\[ D_j(w) = \sum_{k=1}^{n} d(\tilde{r}_{ij}, \tilde{r}_{kj}) w_j, (i = 1, 2, \cdots, n; j = 1, 2, \cdots, m) \]

Moreover, let
\[ D_j(w) = \sum_{j=1}^{m} D_j(w) = \sum_{j=1}^{m} \sum_{k=1}^{n} d(\tilde{r}_{ij}, \tilde{r}_{kj}) w_j, j = 1, 2, \cdots, m \]

Then \( D_j(w) \) represents the deviation value of all alternatives to other alternatives for the attribute \( A_j \); So we choose the weight vector \( w \) which can maximize all deviation values among all the alternatives for all the attributes.

Based on the above analysis, a non-linear programming model (M1) is constructed as follows:

\[
\begin{align*}
\text{max } D(w) &= \sum_{i=1}^{n} \sum_{j=1}^{m} D_j(w) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{n} d(\tilde{r}_{ij}, \tilde{r}_{kj}) w_j \\
\text{s.t. } w &\in H, \sum_{j=1}^{m} w_j = 1, w_j \geq 0, j = 1, 2, \cdots, m
\end{align*}
\]

By solving the model (M1), we get the optimal solution \( w = (w_1, w_2, \cdots, w_m)^T \), which can be used as the weight vector of attributes.

If the information about attribute weights is completely unknown, we can establish another programming model (M2):

\[
\begin{align*}
\text{max } D(w) &= \sum_{i=1}^{n} \sum_{j=1}^{m} D_j(w) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{n} d(\tilde{r}_{ij}, \tilde{r}_{kj}) w_j \\
\text{s.t. } w &\in H, \sum_{j=1}^{m} w_j = 1, w_j \geq 0, j = 1, 2, \cdots, m
\end{align*}
\]

To solve this model, we construct the Lagrange function:

\[
L(w, \lambda) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{n} w_j \left( |a_i - a_j| + |b_i - b_j| + |c_i - c_j| + |d_i - d_j| \right) + \lambda \left( \sum_{j=1}^{m} w_j^2 - 1 \right)
\]

where \( \lambda \) is the Lagrange multiplier.

Differentiating Eq. (5) with respect to \( w = (w_1, w_2, \cdots, w_m)^T \) and \( \lambda \), and setting these partial derivatives equal to zero, the following set of equations is obtained:

\[
\begin{align*}
\frac{\partial L}{\partial w_i} &= \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{n} \left( |a_i - a_j| + |b_i - b_j| + |c_i - c_j| + |d_i - d_j| \right) + \lambda w_i \\
\frac{\partial L}{\partial w_j} &= \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \left( |a_i - a_j| + |b_i - b_j| + |c_i - c_j| + |d_i - d_j| \right) + \lambda w_j \\
\frac{\partial L}{\partial \lambda} &= \sum_{j=1}^{m} w_j^2 - 1
\end{align*}
\]

By solving above equations, we get a simple and exact formula for determining the attribute weights as follows:

\[
w_j^* = \frac{1}{\sqrt{\sum_{i=1}^{n} \sum_{k=1}^{n} \left( |a_i - a_j| + |b_i - b_j| + |c_i - c_j| + |d_i - d_j| \right)^2}}
\]

By normalizing \( w_j^* (j = 1, 2, \cdots, m) \) be a unit, we have

\[
w_j^* = \frac{\sum_{i=1}^{n} \sum_{k=1}^{n} \left( |a_i - a_j| + |b_i - b_j| + |c_i - c_j| + |d_i - d_j| \right)}{\sqrt{\sum_{i=1}^{n} \sum_{k=1}^{n} \left( |a_i - a_j| + |b_i - b_j| + |c_i - c_j| + |d_i - d_j| \right)^2}}
\]

IV. AN APPROACH TO MULTI-ATTRIBUTE DECISION MAKING

Based on the above models, we develop a practical method for solving the MADM problems, in which the information about attribute weights is incompletely known or completely unknown, and the attribute values take the form of interval-valued intuitionistic fuzzy number. The method involves the following steps:

Step 1. Let \( \tilde{R} = (\tilde{r}_{ij})_{n \times m} \) be an interval-valued intuitionistic fuzzy decision matrix, where \( \tilde{r}_{ij} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}]) \), which is an attribute value, given by the DM(s), for the alternative \( X_i \in X \) with respect to the attribute \( A_j \in A \), \( w = (w_1, w_2, \cdots, w_m)^T \) be the weight vector of attributes, where \( w_j \in [0, 1] \); \( j = 1, 2, \cdots, m \), \( H \) is a set of the known weight information.

Step 2. If the information about the attribute weights is partially known, then we solve the model (M1) to obtain the attribute weights. If the information about the attribute weights is completely unknown, then we can obtain the attribute weights by using Eq. (7).

Step 3. Utilize \( \tilde{r}_i = IWA_{\tilde{R}}(\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_m) \), we obtain the overall values \( \tilde{r}_i \) of the alternative \( X_i (i = 1, 2, \cdots, n) \).

Step 4. Calculate the scores \( S(\tilde{r}_i) \) \( (i = 1, 2, \cdots, n) \) to rank all the alternatives \( X_i \) \( (i = 1, 2, \cdots, n) \) and then to select the best one(s). If there is no difference between two scores \( S(\tilde{r}_i) \) and \( S(\tilde{r}_j) \), then we need to calculate the accuracy degrees \( H(\tilde{r}_i) \) and \( H(\tilde{r}_j) \) of the \( \tilde{r}_i \) and \( \tilde{r}_j \), respectively, and then rank the alternatives \( X_i \) and \( X_j \) in accordance with the accuracy degrees \( H(\tilde{r}_i) \) and \( H(\tilde{r}_j) \).

Step 5. Rank all the alternatives \( X_i (i = 1, 2, \cdots, n) \) and select the best one(s) in accordance with \( S(\tilde{r}_i) \) and \( H(\tilde{r}_i) \) \( (i = 1, 2, \cdots, n) \).

Step 6. End.

V. ILLUSTRATIVE EXAMPLE

In this section, we discuss a problem concerning with an investment decision making problem which is adapted from [20]. An investment company wants to invest a sum of money in the best option. There is a panel with four possible alternatives to invest the money: \( X_1 \) is a car company; \( X_2 \) is
a food company; \( X_3 \) is a computer company; and \( X_4 \) is an arms company. The investment company must take a decision according to the following four attributes: \( A_1 \) is the risk analysis; \( A_2 \) is the growth analysis; \( A_3 \) is the social-political impact analysis, and \( A_4 \) is the environmental impact analysis.

The four possible alternatives \( X_i \) (\( i=1,2,3,4 \)) are to be evaluated using the interval-valued intuitionistic fuzzy information by the decision maker under the above four attributes, as listed in the following fuzzy decision matrix.

\[
\vec{R} = \begin{bmatrix}
[[0.6,0.7], [0.3,0.4], [0.5,0.8], [0.5,0.6]],
[0.2,0.3] & [0.4,0.5] & [0.1,0.2] & [0.3,0.4] \\
[[0.4,0.5], [0.5,0.8], [0.3,0.6], [0.6,0.7]],
[0.4,0.5] & [0.1,0.2] & [0.3,0.4] & [0.1,0.3] \\
[[0.3,0.5], [0.1,0.3], [0.7,0.8], [0.5,0.7]],
[0.4,0.5] & [0.2,0.4] & [0.1,0.2] & [0.1,0.2] \\
[[0.2,0.4], [0.6,0.7], [0.5,0.6], [0.7,0.8]],
[0.4,0.5] & [0.2,0.3] & [0.2,0.3] & [0.1,0.2]
\end{bmatrix}
\]

Then, we utilize the approach developed to get the most desirable alternative(s).

Case 1: The information about the attribute weights is partly known and the known weight information is given as follows:

\[ H = \{0.15 \leq w_1 \leq 0.3, 0.15 \leq w_2 \leq 0.25, \]
\[ 0.25 \leq w_3 \leq 0.4, 0.3 \leq w_4 \leq 0.45, 2.5w_1 \leq w_3 \} \]

Step 1. Utilize the model (M1) to establish the following single-objective programming model:

\[
\begin{align*}
\max D(w) &= 1.7w_1 + 2.7w_2 + 1.7w_3 + 1.3w_4 \\
\text{s.t.} & \sum_{i=1}^{4} w_j = 1, w_j \geq 0, j = 1,2,3,4
\end{align*}
\]

Solving this model, we get the weight vector of attributes:

\[
w = (0.15,0.175,0.375,0.3)^T
\]

Step 2. Utilize \( \vec{r}_i = \text{IIWAA}_w(\vec{r}_{i1}, \vec{r}_{i2}, \ldots, \vec{r}_{im}) \), we obtain the overall values \( \vec{r}_i \) of the alternatives \( X_i \) (\( i=1,2,3,4 \)).

\[
\vec{r}_1 = ([0.4871,0.6829],[0.1966,0.3072])
\]
\[
\vec{r}_2 = ([0.4548,0.6639],[0.1859,0.3361])
\]
\[
\vec{r}_3 = ([0.5188,0.6773],[0.1390,0.2591])
\]
\[
\vec{r}_4 = ([0.5573,0.6717],[0.1803,0.2868])
\]

Step 3. Calculate the scores \( S(\vec{r}_i) \) (\( i=1,2,3,4 \)):

\[
S(\vec{r}_1) = 0.3331, S(\vec{r}_2) = 0.2984, S(\vec{r}_3) = 0.3990, S(\vec{r}_4) = 0.3810.
\]

Step 4. Rank all the alternatives \( X_i \) (\( i=1,2,3,4 \)) in accordance with the scores \( S(\vec{r}_i) \) of \( \vec{r}_i \) (\( i=1,2,3,4 \)):

\[ X_3 \succ X_4 \succ X_1 \succ X_2 \]

Case 2: If the information about the attribute weights is completely unknown, we utilize another approach developed to get the most desirable alternative(s).

Step 1. Utilize the Eq. (7) to get the weight vector of attributes:

\[
w = (0.2297,0.3649,0.2297,0.1757)^T
\]

Step 2. Utilize \( \vec{r}_i = \text{IIWAA}_w(\vec{r}_{i1}, \vec{r}_{i2}, \ldots, \vec{r}_{im}) \), we obtain the overall values \( \vec{r}_i \) of the alternatives \( X_i \) (\( i=1,2,3,4 \)).

\[
\vec{r}_1 = ([0.4629,0.6298],[0.2359,0.3464])
\]
\[
\vec{r}_2 = ([0.4584,0.6892],[0.1770,0.3108])
\]
\[
\vec{r}_3 = ([0.4047,0.5813],[0.1771,0.3179])
\]
\[
\vec{r}_4 = ([0.5306,0.6500],[0.2076,0.3142])
\]

Step 3. Calculate the scores \( S(\vec{r}_i) \) of \( \vec{r}_i \) (\( i=1,2,3,4 \)):

\[
S(\vec{r}_1) = 0.2552, S(\vec{r}_2) = 0.3299, S(\vec{r}_3) = 0.2455, S(\vec{r}_4) = 0.3294.
\]

Step 4. Rank all the alternatives \( X_i \) (\( i=1,2,3,4 \)) in accordance with the scores \( S(\vec{r}_i) \):

\[ X_3 \succ X_4 \succ X_1 \succ X_2 \]

VI. CONCLUDING REMARKS

In this paper, we have investigated the multi-attribute decision making problems under interval-valued intuitionistic fuzzy environment where the information about attribute weights is incomplete, we have applied the maximizing deviation method to the interval-valued intuitionistic fuzzy multi-attribute decision-making, and established some optimization models to determine the attribute weights. Then we have utilized the obtained attribute weights and the interval-valued intuitionistic fuzzy weighted arithmetic averaging (IIWAA) operator to find the ranking of the alternatives and to select the most desirable one. All these procedures have been illustrated by a numerical example concerning with an investment company, searching the best option. In the future, we will consider the situations where the decision maker can provide and modify his preference information about attribute weights and attribute values gradually in the process of decision making.

REFERENCES