Homogenization of a variational problem in three-dimension space

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Abstract

In this paper, we investigate the variational problem for a sequence of 3-dimensional domains with highly oscillating boundaries. Using the unfolding method and the averaging method, we obtain the result of the homogenization problem, that is, a sequence of solutions of Eq. (3.1) converges to the solution of Eq. (3.4) as the periodic length approaches zero. It is noteworthy that the convergence is in the strong sense.

1. Introduction

The periodic unfolding method was introduced in [6] by Cioranescu et al. for the study of classical periodic homogenization in the case of fixed domains and further described in [1–4,8,9,11]. This method was also applied to problems with holes and truss-like structures or in linearized elasticity.

The homogenization of periodic structures was carried out in the last 30 years for various kinds of problems involving differential equations [12–15] and systems, as well as integral energies. But most of these works all got the weak convergence. Recently, there was a break in [5,7], where the achievement of strong convergence was obtained. In [10], the unfolding method was applied to a linear elliptic equation in the oscillating boundary cases in two-dimension space, and the new result of strong convergence was obtained. The purpose of this paper is to generalize the work in [10], i.e., we apply the periodic unfolding method to a variational problem in the oscillating boundary cases in three-dimension space, and obtain the strong convergence result. The symbols used in this paper are the same as the ones to those in [10].

We will work on domains which are constructed as follows. Let \( \beta \in \mathbb{R} \) such that \( \beta \in (0, 1), 1/\epsilon = N \), where \( N \) is a positive integer. Define

\[
\Omega_k = \bigcup_{k=0}^{N-1} (k\epsilon, k\epsilon + \beta \epsilon) \times \bigcup_{l=0}^{N-1} (l\epsilon, l\epsilon + \beta \epsilon) \times (0, 1),
\]

\[
\Omega_\epsilon = (0, 1) \times (0, 1) \times (-1, 0), \quad \Omega_k \cap \Omega_\epsilon = \Gamma_z, \quad \Omega_c = \Omega_k \cup \Omega_\epsilon \cup \Gamma_z, \quad \Omega_A = (0, 1)^3, \quad \Omega = (0, 1) \times (0, 1) \times (-1, 1).
\]

2. The unfolding operator

A linear operator on \( L^1(\Omega_k) \) will be defined and used to interpret integrals over \( \epsilon \)-dependent domains as integrals over a fixed domain. This operator is called the unfolding operator.
We will use the following notations: Let \([\cdot] : \mathbb{R} \to \mathbb{Z}\) and \(\{\cdot\} : \mathbb{R} \to [0, 1)\) denote the functions which map every real number to its integer part and the fractional part.

**Definition 2.1 (The unfolding operator).** For every \(\varepsilon > 0\), \(u \in L^1(\Omega^\varepsilon_A)\), we define the unfolding operator \(T^\varepsilon : L^1(\Omega^\varepsilon_A) \to L^1(\Omega_A \times (0, \beta)^2)\) by setting

\[
T^\varepsilon(u)(x_1, x_2, x_3, x_4, x_5) = u\left(\frac{[x_1]}{\varepsilon} + \varepsilon x_4, \frac{[x_2]}{\varepsilon} + \varepsilon x_5, x_3\right)
\]

for every \((x_1, x_2, x_3) \in \Omega_A\) and \((x_4, x_5) \in (0, \beta)^2\).

If \(U\) is an open subset of \(\mathbb{R}^2\) containing \(\Omega_A^\varepsilon\) and \(u\) is a real-valued function on \(U\), \(T^\varepsilon u\) will mean \(T^\varepsilon\) acting on the restriction of \(u\) to \(\Omega_A^\varepsilon\). The following propositions state the properties of \(T^\varepsilon\) which will be used later. Most of them are straightforward and their proofs are omitted.

**Proposition 2.1.** \(T^\varepsilon\) is linear.

**Proposition 2.2.** Let \(u, v\) be functions: \(\Omega_A^\varepsilon \to \mathbb{R}\), then \(T^\varepsilon(uv) = T^\varepsilon u T^\varepsilon v\).

**Proposition 2.3.** Let \(x \in \Omega_A^\varepsilon\) and \(u : \Omega_A^\varepsilon \to \mathbb{R}\), then \(T^\varepsilon u(x_1, x_2, x_3, \frac{[x_4]}{\varepsilon}, \frac{[x_5]}{\varepsilon}) = u(x_1, x_2, x_3)\).

**Proposition 2.4.** Let \(u \in L^1(\Omega_A^\varepsilon)\), then

\[
\int_{\Omega_A^\varepsilon \times (0, \beta)^2} T^\varepsilon u \, dx = \int_{\Omega_A^\varepsilon} u \, dx.
\]

**Proof.** Suppose that \(u \in L^1(\Omega_A^\varepsilon)\). By Fubini’s theorem and the fact that \(T^\varepsilon u\) is piecewise constant in \(x_1\) and \(x_2\),

\[
\int_{\Omega_A^\varepsilon \times (0, \beta)^2} T^\varepsilon u \, dx = \int_{x_1=1}^{\beta} \int_{x_2=0}^{\beta} \int_{x_3=0}^{\beta} \int_{x_4=0}^{\beta} \int_{x_5=0}^{\beta} u\left(\frac{[x_1]}{\varepsilon} + \varepsilon x_4, \frac{[x_2]}{\varepsilon} + \varepsilon x_5, x_3\right) dx_1 dx_2 dx_3 dx_4 dx_5
\]

\[
= \int_{x_1=1}^{\beta} \int_{x_2=0}^{\beta} \int_{x_3=0}^{\beta} \int_{x_4=0}^{\beta} \int_{x_5=0}^{\beta} u(\varepsilon x_1 + \varepsilon x_4, \varepsilon x_2 + \varepsilon x_5, x_3) dx_1 dx_2 dx_3 dx_4 dx_5
\]

\[
= \int_{x_1=0}^{\beta} \int_{x_2=0}^{\beta} \int_{x_3=0}^{\beta} u(x_1, x_2, x_3) dx_1 dx_2 dx_3 = \int_{\Omega_A^\varepsilon} u \, dx.
\]

**Proposition 2.5.** Let \(u \in L^2(\Omega_A^\varepsilon)\), then \(T^\varepsilon u \in L^2(\Omega_A \times (0, \beta)^2)\). Moreover, \(T^\varepsilon u\) is a linear isometry between \(L^2(\Omega_A^\varepsilon)\) and \(L^2(\Omega_A \times (0, \beta)^2)\).

**Proof.** Suppose that \(u \in L^2(\Omega_A^\varepsilon)\), then \(|u|^2 \in L^1(\Omega_A^\varepsilon)\). By Proposition 2.2 and 2.4, we have:

\[
\int_{\Omega_A^\varepsilon \times (0, \beta)^2}|T^\varepsilon u|^2 \, dx = \int_{\Omega_A^\varepsilon \times (0, \beta)^2}|u|^2 \, dx = \int_{\Omega_A^\varepsilon}|u|^2 \, dx < \infty.
\]

By the previous calculation we can see that \(T^\varepsilon\) is a mapping preserving norm, that is \(\|T^\varepsilon u\|_{L^2(\Omega_A \times (0, \beta)^2)} = \|u\|_{L^2(\Omega_A^\varepsilon)}\). This Proposition and Proposition 2.1 imply that \(T^\varepsilon\) is a linear isometry between \(L^2(\Omega_A^\varepsilon)\) and \(L^2(\Omega_A \times (0, \beta)^2)\).

**Proposition 2.6.** Let \(u \in H^1(\Omega_A^\varepsilon)\), then \(T^\varepsilon u \in L^2((0, 1) \times (0, 1); H^1((0, 1) \times (0, \beta) \times (0, \beta)))\). Furthermore, \(\frac{\partial}{\partial x_1} T^\varepsilon u = T^\varepsilon \frac{\partial u}{\partial x_1}\), \(\frac{\partial}{\partial x_2} T^\varepsilon u = T^\varepsilon \frac{\partial u}{\partial x_2}\), \(\frac{\partial}{\partial x_3} T^\varepsilon u = T^\varepsilon \frac{\partial u}{\partial x_3}\), \(\frac{\partial}{\partial x_4} T^\varepsilon u = T^\varepsilon \frac{\partial u}{\partial x_4}\), \(\frac{\partial}{\partial x_5} T^\varepsilon u = T^\varepsilon \frac{\partial u}{\partial x_5}\).
This Proposition and Proposition 2.4 imply the following result,
\[
\int_{x_1}^1 \int_{x_2}^1 \int_{x_3}^1 \int_{x_4}^\beta \int_{x_5}^\beta |T'u| dx_1 dx_2 dx_3 dx_4 dx_5 = \int_{x_1}^1 \int_{x_2}^1 \int_{x_3}^1 \int_{x_4}^\beta \int_{x_5}^\beta \left[ |T'u|^2 + \left| \frac{\partial}{\partial x_5} T'u \right|^2 \right] dx_1 dx_2 dx_3 dx_4 dx_5 
+ \int_{x_1}^1 \int_{x_2}^1 \int_{x_3}^1 \int_{x_4}^\beta \int_{x_5}^\beta \left| \frac{\partial}{\partial x_5} T'u \right| dx_1 dx_2 dx_3 dx_4 dx_5 
+ \varepsilon^2 \int_{x_1}^1 \int_{x_2}^1 \int_{x_3}^1 \int_{x_4}^\beta \int_{x_5}^\beta \left[ \frac{\partial^2 T'u}{\partial x_5^2} + \left| \frac{\partial^2}{\partial x_5^2} T'u \right|^2 \right] dx_1 dx_2 dx_3 dx_4 dx_5 
\leq \int_{\Omega_4 \times (0, \beta)^2} T^c \left( |u|^2 + \left| \frac{\partial u}{\partial x_1} \right|^2 + \left| \frac{\partial u}{\partial x_2} \right|^2 + \left| \frac{\partial u}{\partial x_3} \right|^2 \right) dx_1 dx_2 dx_3 dx_4 dx_5 
\leq \int_{\Omega_4} \left( |u|^2 + \left| \frac{\partial u}{\partial x_1} \right|^2 + \left| \frac{\partial u}{\partial x_2} \right|^2 + \left| \frac{\partial u}{\partial x_3} \right|^2 \right) dx_1 dx_2 dx_3 = \|u\|^2_{H^3(\Omega_4)} < \infty.
\]

Therefore, we have completed the proof of the Proposition. \(\square\)

2.1. Convergence properties

In this section some convergence results for sequences of \(\varepsilon\)–unfolding are presented. All limits are taken as \(\varepsilon \to 0\). First note that \(\varepsilon |x| \to x\) for any real number \(x\).

**Proposition 2.7.** Let \(u \in L^2(\Omega_4)\), then \(T'u \rightharpoonup u\) in \(L^2(\Omega_4 \times (0, \beta)^2)\).

**Proof.** Suppose first that \(u \in D(\Omega_4)\). We have

\[
\sup_{x \in \Omega_4} \|T'u(x,y) - u(x)\| = \omega_u(\varepsilon),
\]

where \(\omega_u\) is the modulus of continuity of \(u\). Hence \(T'u \rightharpoonup u\). The general case of \(u \in L^2(\Omega_4)\) follows by a density argument, by using Proposition 2.5. \(\square\)

**Proposition 2.8.** Let \(u_c \rightharpoonup u\) in \(L^2(\Omega_4)\), then \(T' u_c \rightharpoonup u\) in \(L^2(\Omega_4 \times (0, \beta)^2)\).

**Proof.** Suppose that \(u_c \rightharpoonup u\) in \(L^2(\Omega_4)\). By the triangle inequality, we have:

\[
\|T'u_c - u\|_{L^2(\Omega_4 \times (0, \beta)^2)} \leq \|T'u_c - T'u\|_{L^2(\Omega_4 \times (0, \beta)^2)} + \|T'u - u\|_{L^2(\Omega_4 \times (0, \beta)^2)}.
\]

By Proposition 2.7, the second term on the right hand side converges to zero. By Proposition 2.1 and 2.5,

\[
\|T'u_c - T'u\|_{L^2(\Omega_4 \times (0, \beta)^2)} = \|T(u_c - u)\|_{L^2(\Omega_4 \times (0, \beta)^2)} = \|u_c - u\|_{L^2(\Omega_4)} \leq \|u_c - u\|_{L^2(\Omega_4)} \to 0.
\]

Thus \(T'u_c \rightharpoonup u\) in \(L^2(\Omega_4 \times (0, \beta)^2)\). \(\square\)

For a function \(u\) defined in \(\Omega_4\), let \(\bar{u}\) denote the extension of \(u\) by zero outside \(\Omega_4\).

**Proposition 2.9.** Let \(u_c \in L^2(\Omega_4\varepsilon)\) for every \(\varepsilon > 0\) and \(T'u_c \rightharpoonup u\) weakly in \(L^2(\Omega_4 \times (0, \beta)^2)\), then \(\bar{u}_c \rightharpoonup \int_0^\beta \int_0^\beta u dx_4 dx_5\) weakly in \(L^2(\Omega_4)\).

**Proof.** Suppose that \(u_c \in L^2(\Omega_4\varepsilon)\) for every \(\varepsilon\) and \(T'u_c \rightharpoonup u\) weakly in \(L^2(\Omega_4 \times (0, \beta)^2)\) and thus \(\bar{u}_c\) is bounded in \(L^2(\Omega_4)\). Let \(\phi \in D(\Omega_4)\). Then, by Proposition 2.2, 2.7 and 2.8,

\[
\int_{\Omega_4} \bar{u}_c \phi dx = \int_{\Omega_4} u_c \phi dx = \int_{\Omega_4 \times (0, \beta)^2} T'(u_c, \phi) dx = \int_{\Omega_4 \times (0, \beta)^2} T'u_c \cdot T' \phi dx \rightharpoonup \int_{\Omega_4 \times (0, \beta)^2} u \phi dx
\]

\[
= \int_{\Omega_4} \int_0^\beta \int_0^\beta u dx_4 dx_5 \cdot \phi dx_1 dx_2 dx_3.
\]

Since \(\phi\) is arbitrary, \(\bar{u}_c \rightharpoonup \int_0^\beta \int_0^\beta u dx_4 dx_5\) weakly in \(L^2(\Omega_4)\). \(\square\)
**Proposition 2.10.** Let \( u_\varepsilon \in H^1(\Omega_\varepsilon) \) for every \( \varepsilon > 0 \) be such that \( T^*u_\varepsilon - u \) weakly in \( L^2((0,1)^2 \times (0,\beta)^2; H^1((0,1)), \) then \( \bar{u}_\varepsilon \rightarrow \int_0^\beta \int_0^\beta u \, dx \, dx_5 \) weakly in \( L^2((0,1)^2; \Omega_\varepsilon((0,1)), \) \)

**Proof.** Suppose that \( u_\varepsilon \) belongs to \( H^1(\Omega_\varepsilon) \) and that \( T^*u_\varepsilon \) converges weakly to \( u \) in \( L^2((0,1)^2 \times (0,\beta)^2; H^1((0,1)), \) Then by Proposition 2.9,
\[
\bar{u}_\varepsilon \rightarrow \int_0^\beta \int_0^\beta u \, dx \, dx_5
\]
and
\[
\frac{\partial \bar{u}_\varepsilon}{\partial x_3} = \frac{\partial u}{\partial x_3} \rightarrow \int_0^\beta \int_0^\beta \frac{\partial u}{\partial x_3} \, dx \, dx_5
\]
weakly in \( L^2(\Omega_\varepsilon) \). It follows that \( \bar{u}_\varepsilon \rightarrow \int_0^\beta \int_0^\beta u \, dx \, dx_5 \) weakly in \( L^2((0,1)^2; H^1((0,1)), \) \)

2.2. The boundary unfolding operator

As defined before, the two-dimensional common boundary between \( \Omega_\varepsilon \) and \( \Omega_\varepsilon \), is \( \Omega_\varepsilon \cap \Omega_\varepsilon = \Gamma_\varepsilon \) \( \times \bigcup_{k=0}^{N-1} (ke, ke + \beta e) \). We define an unfolding operator on \( \Gamma_\varepsilon \) as follows.

**Definition 2.2 (The boundary unfolding operator).** For every \( \varepsilon > 0, u \in L^1(\Gamma_\varepsilon), \) we define the boundary unfolding operator \( T_{x_1=0}^e : L^1(\Gamma_\varepsilon) \rightarrow L^1((0,1)^2 \times (0,\beta)^2) \) by setting
\[
T_{x_1=0}^e(u)(x_1, x_2, x_4, x_5) = u\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, \frac{x_4}{\varepsilon}, \frac{x_5}{\varepsilon}\right)
\]
for every \( (x_1, x_2, x_4, x_5) \in (0,1)^2 \times (0,\beta)^2 \).

The boundary unfolding operator \( T_{x_1=0}^e \) has the same properties as \( T^* \) and the proofs are very similar to those of \( T^* \), so we omit the proofs. Again, if \( U \) is an open subset of \( \Omega^2 \) such that \( \Gamma_\varepsilon \subset U \) and \( u : U \rightarrow \mathbb{R} \), then \( T_{x_1=0}^e u \) means that \( T_{x_1=0}^e \) acts on the restriction of \( u \) to \( \Gamma_\varepsilon \). The main properties of the boundary unfolding operator are listed in the following proposition.

**Proposition 2.11.** \( T_{x_1=0}^e \) has the following properties:

(a) \( T_{x_1=0}^e \) is linear.
(b) Let \( u, v \) be functions: \( \Gamma_\varepsilon \rightarrow \mathbb{R}, \) then \( T_{x_1=0}^e(uv) = T_{x_1=0}^e u \cdot T_{x_1=0}^e v. \)
(c) Let \( u \in L^1(\Gamma_\varepsilon), \) then \( \int_{(0,1)^2 \times (0,\beta)^2} T_{x_1=0}^e u \, dx = \int_{\Gamma_\varepsilon} u \, dx. \)
(d) Let \( u \in L^2(\Gamma_\varepsilon), \) then \( T_{x_1=0}^e u \in L^2((0,1)^2 \times (0,\beta)^2). \) Moreover, \( T_{x_1=0}^e u \) is a linear isometry between \( L^2(\Gamma_\varepsilon) \) and \( L^2((0,1)^2 \times (0,\beta)^2). \)
(e) Let \( u \in H^1(\Gamma_\varepsilon), \) then \( T_{x_1=0}^e u \in L^2((0,1)^2; H^1((0,\beta)^2)), \) and \( \frac{\partial}{\partial x_4} T^e u = \varepsilon T_{x_1=0}^e \frac{\partial u}{\partial x_1}, \frac{\partial}{\partial x_5} T^e u = \varepsilon T_{x_1=0}^e \frac{\partial u}{\partial x_2} \).
(f) Let \( u \in L^2(\Gamma_\varepsilon), \) then \( T_{x_1=0}^e u \rightarrow u \) in \( L^2((0,1)^2 \times (0,\beta)^2). \)
(g) Suppose that \( u_\varepsilon \rightarrow u \) in \( L^2((0,1)^2), \) then \( T_{x_1=0}^e u_\varepsilon \rightarrow u \) in \( L^2((0,1)^2 \times (0,\beta)^2). \)
(h) Suppose that \( u_\varepsilon \) is a sequence in \( L^2((0,1)^2) \) such that \( T_{x_1=0}^e u_\varepsilon \rightarrow u \) in \( L^2((0,1)^2 \times (0,\beta)^2), \) then \( \bar{u}_\varepsilon \rightarrow \int_0^\beta \int_0^\beta u \, dx \, dx_5 \) in \( L^2((0,1)^2 \times (0,1)), \)

3. Homogenization of a variational problem

We consider the following variational problem: \( L \in L^2(\Omega). \) Find \( u_\varepsilon \in H^1(\Omega_\varepsilon) \) such that
\[
\int_{\Omega_\varepsilon} (\nabla u_\varepsilon \cdot \nabla \varphi + u_\varepsilon \cdot \varphi) \, dx = \int_{\Omega_\varepsilon} f \varphi \, dx
\]
for all \( \varphi \in H^1(\Omega_\varepsilon). \)

A homogenized problem of (3.1) will be defined in the following sense. Considering a sequence \( u_\varepsilon \) of solutions to (3.1), a limit of \( u_\varepsilon \) extended to a fixed domain is identified. This limit is characterized as the solution of another variational problem which will be called the homogenized problem of (3.1).

Since the lower part \( \Omega_\varepsilon \) of the domain \( \Omega_\varepsilon \) does not depend on \( e, \) there is a natural division of the problem into two parts: \( \Omega_\varepsilon \) and \( \Omega_\varepsilon \). This gives rise to a transmission condition on the solutions on \( \Gamma_\varepsilon \). This condition will be expressed in terms of
traces of $H^1$-functions. The following notation will be used: with $u \in H^1$, $u|_A$ denotes the trace of $u$ restricted to $A$ where $A$ is a set for which $u|_A$ makes sense.

The limit problem will be formulated as a variational problem on the space $V$ defined by

$$V = \left\{ \psi \in L^2(\Omega) : \frac{\partial \psi}{\partial x_1} \in L^2(\Omega_b), \frac{\partial \psi}{\partial x_2} \in L^2(\Omega_b) \text{ and } \frac{\partial \psi}{\partial x_3} \in L^2(\Omega) \right\}.$$  

This is a Hilbert space when endowed with the norm defined by

$$\|\psi\|_V^2 = \|\psi\|_{L^2(\Omega)}^2 + \left\| \frac{\partial \psi}{\partial x_1} \right\|_{L^2(\Omega_b)}^2 + \left\| \frac{\partial \psi}{\partial x_2} \right\|_{L^2(\Omega_b)}^2 + \left\| \frac{\partial \psi}{\partial x_3} \right\|_{L^2(\Omega)}^2$$

and the corresponding inner product. Note that a function $\psi$ belongs to the space $V$ if and only if $\psi|_{\Omega_a} \in L^2((0,1);H^1(\Omega_b)), \psi|_{\Omega_b} \in H^1(\Omega_b)$, and the traces of $\psi|_{\Omega_a}$ and $\psi|_{\Omega_b}$ on $[x_3 = 0]$ coincide in the sense of the space $L^2((0,1) \times (0,1))$ (Indeed, both of the traces make sense for elements of the space $V$).

Recall that $\tilde{u}$ denotes the extension of $u$ by zero outside $\Omega_b^c$. Let $u^e_\epsilon = u|_{\Omega_b}, u^b_\epsilon = u|_{\Omega_b}, u_a = u|_{\Omega_a}$ and $u_b = u|_{\Omega_b}$.

**Theorem 3.1.** For every $\epsilon > 0$ there exists a unique solution $u_\epsilon \in H^1(\Omega_b)$ of (3.1) and a unique element $u \in V$ such that, as $\epsilon$ goes to 0,

$$\tilde{u}^e_\epsilon \rightharpoonup \beta^2 u_a \text{ weakly in } L^2((0,1) \times (0,1);H^1(\Omega_b));$$

$$\tilde{u}^b_\epsilon \rightharpoonup u_b \text{ weakly in } H^1(\Omega_b).$$

Moreover, $u$ is the unique solution of the following variational problem: Find $u \in V$ such that

$$\beta^2 \int_{\Omega_a} \left( \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_3^2} + u\psi \right) dx + \int_{\Omega_b} (\nabla u \cdot \nabla \psi + u \cdot \psi) dx = \beta^2 \int_{\Omega_a} f \psi dx + \int_{\Omega_b} f \psi dx$$

for all $\psi \in V$.

**Proof.** For every $\epsilon > 0$ the left hand side of (3.1) is an inner product on $H^1(\Omega_b)$. By Riesz representation theorem (or the Lax–Milgram theorem) implies that there exists a unique element $u_\epsilon \in H^1(\Omega_b)$ that satisfies (3.1) for all $\varphi \in H^1(\Omega_b)$. That is the first statement of the theorem.

Next, we show that the sequence $\|u_\epsilon\|_{H^1(\Omega_b)}$ is bounded. The fact that $u_\epsilon$ satisfies (3.1) and the Cauchy–Schwarz inequality yield

$$\|u_\epsilon\|_{H^1(\Omega_b)}^2 = \int_{\Omega_b} (|\nabla u_\epsilon|^2 + |u_\epsilon|^2) dx \leq \int_{\Omega_b} f u_\epsilon dx \leq \|f\|_{L^2(\Omega_b)} \|u_\epsilon\|_{H^1(\Omega_b)} \leq \|f\|_{L^2(\Omega_b)} \|u_\epsilon\|_{H^1(\Omega_b)},$$

from which follows

$$\|u_\epsilon\|_{H^1(\Omega_b)} \leq \|f\|_{L^2(\Omega_b)}.$$  

To estimate the norm of $T^e u_\epsilon$, we use Proposition 2.2, 2.4 and inequality (3.5):

$$\|T^e u_\epsilon\|_{L^2((0,1) \times (0,1);H^1(0,1) \times (0,\beta^2))} = \int_{\Omega_a \times (0,\beta^2)} \left( \|rac{\partial u_\epsilon}{\partial x_1}\|^2_1 + \|rac{\partial u_\epsilon}{\partial x_2}\|^2_1 + \|rac{\partial u_\epsilon}{\partial x_3}\|^2_1 + \|u_\epsilon\|^2_1 \right) \frac{dx_1 dx_2}{dx_3},$$

$$= \int_{\Omega_a \times (0,\beta^2)} \left( \|rac{\partial u_\epsilon}{\partial x_1}\|^2_1 + \|rac{\partial u_\epsilon}{\partial x_2}\|^2_1 + \|rac{\partial u_\epsilon}{\partial x_3}\|^2_1 + \|T^e u_\epsilon\|^2_1 \right) \frac{dx_1 dx_2}{dx_3},$$

$$= \int_{\Omega_a} \left( \|rac{\partial u_\epsilon}{\partial x_1}\|^2_1 + \|rac{\partial u_\epsilon}{\partial x_2}\|^2_1 + \|rac{\partial u_\epsilon}{\partial x_3}\|^2_1 + \|T^e u_\epsilon\|^2_1 \right) dx \leq \|u_\epsilon\|_{H^1(\Omega_b)} \leq \|f\|_{L^2(\Omega_b)}.$$
Since $T^*u^e_{\varepsilon} - u_0$ in $L^2((0,1) \times (0,1); H^1((0,1) \times (0,\beta)^2))$ we have $\frac{\partial}{\partial x_3} T^*u^e_{\varepsilon} - \frac{\partial u_0}{\partial x_3}$, $\frac{\partial}{\partial x_4} T^*u^e_{\varepsilon} - \frac{\partial u_0}{\partial x_4}$ and $\frac{\partial}{\partial x_5} T^*u^e_{\varepsilon} - \frac{\partial u_0}{\partial x_5}$ weakly in $L^2(\Omega_A \times (0,\beta)^2)$. It follows from Proposition 2.6 that

\begin{equation}
T^*\frac{\partial u^e_{\varepsilon}}{\partial x_3} - \frac{\partial u_0}{\partial x_3},
\end{equation}

\begin{equation} \varepsilon T^*\frac{\partial u^e_{\varepsilon}}{\partial x_4} - \frac{\partial u_0}{\partial x_4}, \end{equation}

\begin{equation} \varepsilon T^*\frac{\partial u^e_{\varepsilon}}{\partial x_5} - \frac{\partial u_0}{\partial x_5}, \end{equation}

weakly in $L^2(\Omega_A \times (0,\beta)^2)$. However, the fact that $T^*$ is a linear isometry and (3.5) yield

\begin{equation}
\left\| T^*\frac{\partial u^e_{\varepsilon}}{\partial x_1} \right\|_{L^2(\Omega_A \times (0,\beta)^2)} \leq \left\| \frac{\partial u^e_{\varepsilon}}{\partial x_1} \right\|_{L^2(\Gamma_A)} 
\end{equation}

\begin{equation}
\left\| T^*\frac{\partial u^e_{\varepsilon}}{\partial x_2} \right\|_{L^2(\Omega_A \times (0,\beta)^2)} \leq \left\| \frac{\partial u^e_{\varepsilon}}{\partial x_2} \right\|_{L^2(\Gamma_A)}.
\end{equation}

Hence $T^*\frac{\partial u^e_{\varepsilon}}{\partial x_1}$ and $T^*\frac{\partial u^e_{\varepsilon}}{\partial x_2}$ are bounded in $L^2(\Omega_A \times (0,\beta)^2)$. Therefore the left hand side of (3.7) and (3.8) converge to zero in $L^2(\Omega_A \times (0,\beta)^2)$, so $\frac{\partial u^e_{\varepsilon}}{\partial x_1} = 0$, and $\frac{\partial u^e_{\varepsilon}}{\partial x_2} = 0$.

By Proposition 2.10, the weak convergence of $T^*u^e_{\varepsilon}$ implies $\overline{u^e_{\varepsilon}} - \int_0^1 \int_0^1 u_0 dx_4 dx_5 = \beta^2 u_0$ and $\int_0^1 \int_0^1 \frac{\partial u_0}{\partial x_4} dx_4 dx_5 = \beta^2 \frac{\partial u_0}{\partial x_5}$, so $\overline{u^e_{\varepsilon}} = \beta^2 u_0$ weakly in $L^2((0,1)^2; H^1((0,1)))$. Thus (3.2) is established.

Since $T^*\frac{\partial u^e_{\varepsilon}}{\partial x_1}$ and $T^*\frac{\partial u^e_{\varepsilon}}{\partial x_2}$ are bounded in $L^2(\Omega_A \times (0,\beta)^2)$, by weak compactness, there are elements $P$, $Q \in L^2(\Omega_A \times (0,\beta)^2)$ such that, up to a subsequence, $T^*\frac{\partial u^e_{\varepsilon}}{\partial x_1}$ and $T^*\frac{\partial u^e_{\varepsilon}}{\partial x_2}$ converge weakly to $P$, $Q$ in $L^2(\Omega_A \times (0,\beta)^2)$.

By estimate (3.5), $u^e_{\varepsilon}$ is bounded in $H^1(\Omega_B)$. Thus, up to a subsequence again, $u^e_{\varepsilon}$ converges weakly to some $u_0$ in $H^1(\Omega_B)$. From now on, we denote by a subsequence of $\varepsilon = 1/N$ such that $T^*u^e_{\varepsilon} - u_0$, $T^*\frac{\partial u^e_{\varepsilon}}{\partial x_1} - P$, $T^*\frac{\partial u^e_{\varepsilon}}{\partial x_2} - Q$ and $u^e_{\varepsilon} - u_0$ weakly in $L^2((0,1) \times (0,1); H^1((0,1) \times (0,\beta)^2))$, $L^2(\Omega_A \times (0,\beta)^2)$, $L^2(\Omega_A \times (0,\beta)^2)$ and $H^1(\Omega_B)$, respectively. This is possible since every subsequence of a weakly convergent sequence converges to the same limit.

We are to show that $u$ defined by

\begin{equation}
\begin{cases}
u \in \Omega_A, & \text{if } x \in \Omega_A; \\
u \in \Omega_B, & \text{if } x \in \Omega_B,
\end{cases}
\end{equation}

belongs to $V$ and satisfies (3.2) and (3.3). We already know that $u_0$ belongs to $L^2((0,1) \times (0,1); H^1((0,1)))$, $u_0$ belongs to $H^1(\Omega_B)$, and (3.2) and (3.3) hold. Left to show is that $u \in V$. We do this by showing that the traces of $u_0$ and $u_0$ are equal on $\{x_3 = 0\}$, that is on the set $(0,1) \times (0,1) \times \{x_3 = 0\}$.

Since $u^e_{\varepsilon}$ and $u^e_{\varepsilon}$ are the restrictions of $u_0$ on $\Omega_A$ and $\Omega_B$, respectively, the traces of $u^e_{\varepsilon}$ and $u^e_{\varepsilon}$ exist and must be equal on $\Gamma_\varepsilon$, that is $u^e_{\varepsilon}|_{x_3 = 0} = u^e_{\varepsilon}|_{x_3 = 0}$. The condition $u^e_{\varepsilon}|_{x_3 = 0} = u^e_{\varepsilon}|_{x_3 = 0}$ implies $T^*$ converges to $T^*$ weakly in $L^2((0,1) \times (0,1); (0,\beta)^2)$. By noting that $T^*$ converges to $T^*$ weakly in $L^2((0,1) \times (0,1); (0,\beta)^2)$, the continuity of the trace operator $T^*$ implies strong convergence of the trace $u^e_{\varepsilon}|_{x_3 = 0}$ to $u_0|_{x_3 = 0}$ in $L^2((0,1) \times (0,1))$. Thus by Proposition 2.11, $u_0$ belongs to $V$.

Now we turn to the proof of the variational problem (3.4) satisfied by $u$. Consider the first test functions of the form

$$
\phi^e = \phi^e(x) = \varepsilon \phi(x) \psi \left( \frac{x_1}{l}, \frac{x_2}{l} \right),
$$

where $\phi \in D(\Omega_A)$ and $\psi \in C^\infty((0,1) \times (0,1))$. These $\phi^e$ are piecewise continuous in $x_1$ and $x_2$ with jump possibilities at $\{x_1 = k\epsilon\} \cup \{x_2 = k\epsilon\}$ for $k = 1, 2, \ldots, N - 1$. Note that no jumps of $\phi^e$ occur on $\Omega_{\varepsilon_k}$. Recall that $T^*$ denotes $T^*$ operating on the restriction of $u$ to $\Omega_{\varepsilon_k}$. 

\end{document}
The \( \varepsilon \)-unfolding of \( \varphi^e \) is \( T \varepsilon \varphi^e = \varepsilon \varphi(\varepsilon x_1, \varepsilon x_2, \varepsilon x_3, x_4, x_5) \psi(x_4, x_5) \). By Proposition 2.6 we find

\[
T^e \frac{\partial \varphi^e}{\partial x_1} = \varepsilon \varphi_1 \left( \frac{\varepsilon x_1}{\varepsilon} + \varepsilon x_4, \frac{\varepsilon x_2}{\varepsilon} + \varepsilon x_5, x_3 \right) \psi(x_4, x_5) + \varphi \left( \frac{\varepsilon x_1}{\varepsilon} + \varepsilon x_4, \frac{\varepsilon x_2}{\varepsilon} + \varepsilon x_5, x_3 \right) \frac{\partial \psi}{\partial x_4}(x_4, x_5),
\]

\[
T^e \frac{\partial \varphi^e}{\partial x_2} = \varepsilon \varphi_2 \left( \frac{\varepsilon x_1}{\varepsilon} + \varepsilon x_4, \frac{\varepsilon x_2}{\varepsilon} + \varepsilon x_5, x_3 \right) \psi(x_4, x_5) + \varphi \left( \frac{\varepsilon x_1}{\varepsilon} + \varepsilon x_4, \frac{\varepsilon x_2}{\varepsilon} + \varepsilon x_5, x_3 \right) \frac{\partial \psi}{\partial x_5}(x_4, x_5),
\]

\[
T^e \frac{\partial \varphi^e}{\partial x_3} = \varepsilon \varphi_3 \left( \frac{\varepsilon x_1}{\varepsilon} + \varepsilon x_4, \frac{\varepsilon x_2}{\varepsilon} + \varepsilon x_5, x_3 \right) \psi(x_4, x_5),
\]

where \( \varphi_1 \), \( \varphi_2 \), and \( \varphi_3 \) denote the partial derivatives of \( \varphi \) with respect to the first, second and the third variables. So \( T^e \varphi^e \) and \( T^e \varphi^e \) converge strongly to zero in \( L^1(\Omega) \times (0, \beta)^2 \), and \( T^e \varphi^e \to \varphi(x_1, x_2, x_3) \frac{\partial \psi}{\partial x_4}(x_4, x_5) \). \( T^e \varphi^e \to \varphi(x_1, x_2, x_3) \frac{\partial \psi}{\partial x_5}(x_4, x_5) \) in the same space. Therefore, as \( \varepsilon \to 0 \),

\[
\int_{\Omega_3} (\nabla u^e \cdot \nabla \varphi^e + u^e \varphi^e) \, dx = \int_{\Omega_3} (\nabla u^e \cdot \nabla \psi + u^e \psi) \, dx
\]

\[
= \int_{\Omega_3 \times (0, \beta)^2} (T^e \frac{\partial \psi}{\partial x_1} + T^e \frac{\partial \psi}{\partial x_2} + T^e \frac{\partial \psi}{\partial x_3} + T^e \frac{\partial \psi}{\partial x_4} + T^e \frac{\partial \psi}{\partial x_5}) \, dx
\]

\[
- \int_{\Omega_3 \times (0, \beta)^2} (P \varphi(x_1, x_2, x_3) \frac{\partial \psi}{\partial x_4}(x_4, x_5) + Q \varphi(x_1, x_2, x_3) \frac{\partial \psi}{\partial x_5}(x_4, x_5)) \, dx
\]

and \( \int_{\Omega_3} \varphi^e \, dx = \int_{\Omega_3} f \, dx = \int_{\Omega_3 \times (0, \beta)^2} T^e \varphi^e \, dx \to 0 \). Since \( u^e \) satisfies \( (3.1) \) and \( \varphi \) is constant in \( x_4 \) and \( x_5 \), we must have

\[
\int_{\Omega_3 \times (0, \beta)^2} P(x_1, x_2, x_3, x_4, x_5) \varphi(x_1, x_2, x_3) \frac{\partial \psi}{\partial x_4}(x_4, x_5) + Q(x_1, x_2, x_3, x_4, x_5) \varphi(x_1, x_2, x_3) \frac{\partial \psi}{\partial x_5}(x_4, x_5) \, dx = 0,
\]

which implies that

\[
\int_0^\beta \int_0^\beta P(x_1, x_2, x_3, x_4, x_5) \frac{\partial \psi}{\partial x_4}(x_4, x_5) + Q(x_1, x_2, x_3, x_4, x_5) \frac{\partial \psi}{\partial x_5}(x_4, x_5) \, dx \, dx_5 = 0
\]

for almost every \( (x_1, x_2, x_3) \in \Omega_3 \). Since \( \varphi \in C^\infty((0,1) \times (0,1)) \). We must have \( P = Q = 0 \).

Now take \( \Psi \in C^\infty(\overline{\Omega}) \) as a test function. On the left hand side of \( (3.1) \), we have

\[
\int_{\Omega_3} (\nabla u^e \cdot \nabla \varphi^e + u^e \varphi^e) \, dx = \int_{\Omega_3} (\nabla u^e \cdot \nabla \psi + u^e \psi) \, dx + \int_{\Omega_3} (\nabla u^e \cdot \nabla \varphi^e + u^e \varphi^e) \, dx.
\]

In the limit as \( \varepsilon \to 0 \), by weak–strong convergence,

\[
\int_{\Omega_3} (\nabla u^e \cdot \nabla \varphi^e + u^e \cdot \varphi^e) \, dx = \int_{\Omega_3 \times (0, \beta)^2} (T^e \frac{\partial \varphi^e}{\partial x_1} + T^e \frac{\partial \varphi^e}{\partial x_2} + T^e \frac{\partial \varphi^e}{\partial x_3} + T^e \frac{\partial \varphi^e}{\partial x_4} + T^e \frac{\partial \varphi^e}{\partial x_5}) \, dx
\]

\[
- \int_{\Omega_3 \times (0, \beta)^2} (P \frac{\partial \varphi^e}{\partial x_4}(x_4, x_5) + Q \frac{\partial \varphi^e}{\partial x_5}(x_4, x_5)) \, dx = \beta^2 \int_{\Omega_3} \frac{\partial \varphi^e}{\partial x_4}(x_4, x_5) \, dx
\]

and \( \int_{\Omega_3} (\nabla u^e \cdot \nabla \varphi^e + u^e \varphi^e) \, dx \to \int_{\Omega_3} (\nabla u \cdot \nabla \varphi + u \varphi) \, dx \). Thus

\[
\int_{\Omega_3} (\nabla u \cdot \nabla \varphi + u \varphi) \, dx \to \beta^2 \int_{\Omega_3} \frac{\partial \varphi}{\partial x_4}(x_4, x_5) \, dx + \int_{\Omega_3} (\nabla u \cdot \nabla \psi + u \psi) \, dx.
\]

On the right hand side of \( (3.1) \)

\[
\int_{\Omega_3} f \Psi \, dx = \int_{\Omega_3} f \Psi \, dx + \int_{\Omega_3} f \Psi \, dx = \int_{\Omega_3 \times (0, \beta)^2} T^e (f \psi) \, dx \to \int_{\Omega_3} f \Psi \, dx \to \int_{\Omega_3 \times (0, \beta)^2} f \psi \, dx + \int_{\Omega_3} f \Psi \, dx
\]

\[
= \beta^2 \int_{\Omega_3} f \Psi \, dx + \int_{\Omega_3} f \Psi \, dx.
\]

In view of \( (3.1) \) this means

\[
\beta^2 \int_{\Omega_3} \frac{\partial \varphi}{\partial x_4}(x_4, x_5) \, dx + \int_{\Omega_3} (\nabla u \cdot \nabla \psi + u \psi) \, dx = \beta^2 \int_{\Omega_3} f \Psi \, dx + \int_{\Omega_3} f \Psi \, dx
\]

and \( (3.4) \) is established for all \( \Psi \in C^\infty(\overline{\Omega}) \) and by density for all \( \Psi \in V \).

Regard the left hand side of \( (3.4) \) as a bilinear form on \( V \) and denote it by \( a(u, v) \). Then \( a \) is clearly coercive and continuous since for \( u, v \in V \), \( a(u, v) \geq \beta \|u\|_V^2 \) and \( |a(u, v)| \leq \|u\|_V \|v\|_V \). So the uniqueness of the solution of problem \( (3.4) \) follows from the Lax–Milgram theorem. Thus every weakly convergent subsequence of \( (3.2) \) converges to the same limit and every weakly convergent subsequence of \( (3.3) \) converges to the same limit, so the whole sequence converges weakly to those limits. \( \square \)
Actually one can prove a strong result for the unfolded sequence in $\Omega_x^k$ and for the original sequence in $\Omega_x$. First of all, let us introduce the following lemma which will be used in the strong result.

**Lemma 3.2.** Let $m$ be a fixed integer, $x^k_n$, $k = 1, \ldots, m$ be $m$ bounded sequences of real numbers and $x^k$, $k = 1, \ldots, m$ be $m$ reals. Suppose that $\sum \limits_n x^k_n \to \sum \limits_k x^k$ and for every $k = 1, \ldots, m$, $\lim \limits_{n \to \infty} x^k_n \geq x^k$. Then $\lim \limits_{n \to \infty} x^k_n = x^k$ for every $k = 1, \ldots, m$.

**Proof.** By the property of lower limit, we have, for $a_n$, $b_n \in \mathbb{R}$, $n \in \mathbb{Z}$

$$\lim \limits_{n \to \infty} (a_n + b_n) \geq \lim \limits_{n \to \infty} a_n + \lim \limits_{n \to \infty} b_n,$$

thus

$$\sum \limits_k x^k = \lim \limits_{n \to \infty} \sum \limits_k x^k_n = \lim \limits_{n \to \infty} \sum \limits_k x^k_n \geq \sum \limits_k \lim \limits_{n \to \infty} x^k_n \geq \sum \limits_k x^k,$$

and also, $\lim \limits_{n \to \infty} x^k_n \geq x^k$ for every $k = 1, 2, \ldots, m$, so, $\lim \limits_{n \to \infty} x^k_n = x^k$, $\forall$ $k = 1, \ldots, m$.

On the other hand, we only to prove $\lim \limits_{n \to \infty} x^k_n = x^k$, $k = 1, \ldots, m$.

If it is not true, suppose that there exists $k_0$, such that

$$\lim \limits_{n \to \infty} x^k_n < x^k,$$

then

$$\sum \limits_k x^k = \lim \limits_{n \to \infty} \sum \limits_k x^k_n \leq \lim \limits_{n \to \infty} \sum \limits_k x^k_n \leq \sum \limits_k \lim \limits_{n \to \infty} x^k_n + \lim \limits_{n \to \infty} x^k_0 < \sum \limits_k x^k + x^k_0 = \sum \limits_k x^k.$$

Which is a contradiction. So, $\lim \limits_{n \to \infty} x^k_n = x^k$, for every $k = 1, \ldots, m$.

Thus, we have proved the lemma. $\square$

Using the lemma, we can have the following theorem.

**Theorem 3.3.** Let $u_n$ be the sequence of solutions to (3.1) and $u \in V$ the solution of the variational problem (3.4) in Theorem 3.1. Then the sequence $u_n^k$ converges strongly to $u$ in $H^1(\Omega_x^k)$ and the sequence $T^k(u_n^k)$ converges strongly to $u$ in $L^2((0, 1) \times (0, 1), H^1((0, 1) \times (0, b)^2))$.

**Proof.** Let $u_n$ be the sequence of solutions to (3.1) and $u \in V$ the solution of the variational problem (3.4) in Theorem 3.1. Let

$$x^1 = \|T^k u_n\|_{L^2(\Omega_x^k \times \Omega_y^k)}^2, \quad x^4 = \|P\|_{L^2(\Omega_x^k \times \Omega_y^k)} = 0;$$

$$x^2 = \|T^k u_n\|_{L^2(\Omega_x^k \times \Omega_y^k)}^2, \quad x^5 = \|Q\|_{L^2(\Omega_x^k \times \Omega_y^k)} = 0;$$

$$x^3 = \|T^k u_n\|_{L^2(\Omega_x^k \times \Omega_y^k)}^2, \quad x^6 = \|\partial u_n\|_{L^2(\Omega_x^k \times \Omega_y^k)}^2;$$

$$x^4 = \|T^k u_n\|_{L^2(\Omega_x^k \times \Omega_y^k)}^2, \quad x^7 = \|u_n\|_{L^2(\Omega_x^k \times \Omega_y^k)}^2;$$

$$x^5 = \|\partial u_n\|_{L^2(\Omega_x^k)}^2, \quad x^8 = \|u_n\|_{L^2(\Omega_x^k)}^2;$$

where $P$ and $Q$ are the weak limits of $T^k \frac{\partial u_n}{\partial t}$ and $T^k \frac{\partial u_n}{\partial x}$ in $L^2(\Omega_x \times (0, b)^2)$, and so on. Then by weak lower semi-continuity of the norms, $\lim \limits_{n \to \infty} x^k_n \geq x^k$ for $k = 1, \ldots, 8$.

From the proof of Theorem 3.1 we know that

$$\int_{\Omega} (|\nabla u_n|^2 + |u_n|^2) dx = \int_{\Omega_x \times \Omega_y^k} T^k f u_n dx + \int_{\Omega} f u_n^3 dx \to \beta^2 \int_{\Omega_x} f u_n dx + \int_{\Omega_x} f u_n dx.$$
Using $u$ itself as a test function in the limit problem (3.4) yields

$$
\beta^2 \int_{\Omega_a} f_u u \, dx + \int_{\Omega_b} f_u u \, dx = \beta^2 \int_{\Omega_a} \left( \frac{\partial u_a}{\partial x_1} \right)^2 + \left| u_a \right|^2 \, dx + \int_{\Omega_b} (|\nabla u_b|^2 + |u_b|^2) \, dx = \sum_k x_k^2,
$$

since $x_1^2 = x_2^2 = 0$. Thus $\int_{\Omega_a} (|\nabla u_a|^2 + |u_a|^2) \, dx \to \sum_k x_k^2$. On the other hand, we have

$$
\int_{\Omega_a} (|\nabla u_a|^2 + |u_a|^2) \, dx = \int_{\Omega_{a,\epsilon} \setminus (0,0)} \left( T^e \frac{\partial u^\epsilon_a}{\partial x_1} \right)^2 + \left| T^e \frac{\partial u^\epsilon_a}{\partial x_2} \right|^2 + \left| T^e \frac{\partial u^\epsilon_a}{\partial x_3} \right|^2 + \left| T^e u^\epsilon_a \right|^2 \, dx + \int_{\Omega_b} (|\nabla u^\epsilon_b|^2 + |u^\epsilon_b|^2) \, dx = \sum_k x_k^2,
$$

Hence

$$
\sum_k x_k^\epsilon \to \sum_k x_k^a.
$$

This and the lemma imply $x_k^\epsilon \to x_k$ for $k = 1, \ldots, 8$. The weak convergence and the convergence of the norms imply that these sequences converge strongly, that is in

$$
L^2(\Omega_a \times (0, \beta)^2), \quad L^2(\Omega_b),
$$

$$
\frac{\partial}{\partial x_1} u_a^\epsilon \to 0, \quad \frac{\partial u_a^\epsilon}{\partial x_2} \to 0, \quad \frac{\partial u_a^\epsilon}{\partial x_3} \to 0, \quad \frac{\partial u_a^\epsilon}{\partial x_4} \to 0,
$$

$$
T^e \frac{\partial u^\epsilon_a}{\partial x_1} \to 0, \quad T^e \frac{\partial u^\epsilon_a}{\partial x_2} \to 0, \quad T^e \frac{\partial u^\epsilon_a}{\partial x_3} \to 0, \quad T^e \frac{\partial u^\epsilon_a}{\partial x_4} \to 0.
$$

The partial derivatives of $T^e u^\epsilon_a$ with respect to $x_4$ and $x_5$ converge strongly to $\frac{\partial u_a^\epsilon}{\partial x_4} = \frac{\partial u_a^\epsilon}{\partial x_5} = 0$ which may be seen as follows

$$
\left\| \frac{\partial}{\partial x_4} T^e u^\epsilon_a \right\|_{L^2((0,1) \times (0,1))} \to 0,
$$

$$
\left\| \frac{\partial}{\partial x_5} T^e u^\epsilon_a \right\|_{L^2((0,1) \times (0,1))} \to 0.
$$

It follows that $T^e u^\epsilon_a \to u_a$ in $L^2((0,1) \times (0,1) ; H^1((0,1) \times (0,1)^2))$ and $u^\epsilon_b \to u_b$ in $H^1(\Omega_b)$.

We cannot have strong convergence when extending by zero unless the limit is zero. However, there exists a way of extending $u^\epsilon_a$ such that the extended sequence converges strongly to the element $u_a$ in $L^2(\Omega_a)$, where $u$ is the element in $V$ identified by Theorem 3.1. The extension is constructed as follows.

For $\nu \in L^1(\Omega^c_a)$ and $x \in \Omega_a \setminus \Omega^c_a$, let $m^c(\nu)(x)$ denote the average value of $\nu$ over the closest cog to the left of $x_1, x_2$ and at the height $x_3$:

$$
m^c(\nu)(x) = \begin{cases} \frac{1}{a_2} \int_{a_2}^{a_2 + \beta} \int_{b_2 - a_1}^{b_2} \nu(s, t, x_3) \, ds \, dt, & \text{if } u(s, t) = \nu(s, t, x_3) \text{ is integrable;} \\ 0, & \text{otherwise.} \end{cases}
$$

An equivalent definition is

$$
m^c(\nu)(x) = \frac{1}{\beta} \int_0^\beta \int_0^\beta T^e \nu(x_1, x_2, x_3, x_4, x_5) \, dx_4 \, dx_5.
$$

**Definition 3.1.** For $\nu \in L^1(\Omega^c_a)$, let $m^c(\nu)$ denote the extension of $\nu$ to $\Omega_a$ by $m^c(\nu)$ for all $x \in \Omega_a \setminus \Omega^c_a$:

$$
m^c(\nu)(x) = \begin{cases} \nu(x), & \text{if } x \in \Omega^c_a; \\ m^c(\nu)(x), & \text{if } x \in \Omega_a \setminus \Omega^c_a. \end{cases}
$$

**Proposition 3.1.** Suppose the sequence $\nu_\epsilon$ is such that, as $\epsilon$ goes to 0, $T^\epsilon(\nu_\epsilon)$ converges strongly in the space $L^2(\Omega_a \times (0, \beta)^2)$ to some $\nu_0$ which does not depend upon $x_4$ and $x_5$. Then, the sequence $m^c(\nu_\epsilon)$ converges strongly to $\nu_0$ in $L^2(\Omega_a)$ and the same holds true for the sequence $m^c(\nu_\epsilon)$.

**Proof.** As the complement of $\Omega^c_a$ is considered, a corresponding unfolding operator $T^c_\epsilon$ will be used, which is defined exactly as $T^\epsilon$. For $u$ defined on $\Omega_a$, $T^c_\epsilon u$ denotes $T^c_\epsilon$ operating on the restriction of $u$ to $\Omega_a \setminus \Omega^c_a$. Only the domains are different in the properties of $T^\epsilon$ and $T^c_\epsilon$.
From the formula of \( m^2(\nu_c) \) as the average of \( T^2(\nu_c) \) with respect to \( x_4 \) and \( x_5 \), it follows that \( m^2(\nu_c) \) converges strongly in \( L^2(\Omega) \) to \( \frac{1}{\mathcal{P}} \int_0^\mathcal{P} \int_0^\mathcal{P} \nu_0 \, dx \, dx_5 \). But the latter is just \( \nu_0 \) by hypothesis.

From the sequence \( \tilde{\nu}_\varepsilon \), we write

\[
\| \tilde{\nu}_\varepsilon - \nu_0 \|_{L^2(\Omega)}^2 = \| \nu_c - \nu_0 \|_{L^2(\Omega)}^2 + \| m(\nu_c) - \nu_0 \|_{L^2(\Omega)}^2 = \| T^2(\nu_c) - T^2(\nu_0) \|_{L^2(\Omega \times [0,\mathcal{P}^2])}^2 + \| T^2 m(\nu_c) - T^2 m(\nu_0) \|_{L^2(\Omega \times [0,\mathcal{P}^2])}^2 + \nu_0 - T^2 \nu_0 \|_{L^2(\Omega \times [0,\mathcal{P}^2])}^2 + \| \nu_0 - T^2 \nu_0 \|_{L^2(\Omega \times [0,\mathcal{P}^2])}^2.
\]

By hypothesis, \( \| T^2(\nu_c) - \nu_0 \|_{L^2(\Omega \times [0,\mathcal{P}^2])} \) goes to zero, and by Proposition 2.7, \( \| m^2(\nu_c) - \nu_0 \|_{L^2(\Omega \times [0,\mathcal{P}^2])} \) goes to zero. Consequently,

\[
\lim_{\varepsilon \to 0} \| \tilde{\nu}_\varepsilon - \nu_0 \|_{L^2(\Omega)} = \lim_{\varepsilon \to 0} \| T^2 m(\nu_c) - \nu_0 \|_{L^2(\Omega \times [0,\mathcal{P}^2])}.
\]

But \( T^2 m(\nu_c) = \frac{1}{N} \int_0^N T^2(\nu_c) \, dx_4 \, dx_5 = m^2(\nu_c) \), so that

\[
\| T^2 m(\nu_c) - \nu_0 \|_{L^2(\Omega \times [0,\mathcal{P}^2])} = \| m^2(\nu_c) - \nu_0 \|_{L^2(\Omega \times [0,\mathcal{P}^2])} \leq \| m(\nu_c) - \nu_0 \|_{L^2(\Omega)}.
\]

As seen above, the latter goes to zero, thus completes the proof.

**Table 1**

Values of the problem numerical convergence orders.

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>0.25</th>
<th>0.2</th>
<th>0.1</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_c )</td>
<td>0.06101192</td>
<td>0.05973446</td>
<td>0.057210494</td>
<td>0.055953777</td>
</tr>
<tr>
<td>( \beta_c )</td>
<td>0.00426555</td>
<td>0.003864344</td>
<td>0.003135465</td>
<td>0.0028275</td>
</tr>
</tbody>
</table>
Theorem 3.4. Let $u_\epsilon$ be the sequence of solutions to (3.1) and $u \in V$ the solution of the variational problem (3.4) in Theorem 3.1. Then both sequences $m^\epsilon(u_\epsilon^m)$ and $u_\epsilon^m$ converge to $u_\alpha$ strongly in $L^2((0, 1) \times (0, 1); H^1_\alpha(0, 1))$.

Proof. By Theorem 3.3, it is enough to apply Proposition 2.1 to $u_\epsilon^m$ and $\left.\frac{\partial}{\partial x_3}\left(u_\epsilon^m\right)\right|_{x_3}$, and note that by Proposition 2.6, $T^\epsilon\left(\left.\frac{\partial}{\partial x_3}\left(u_\epsilon^m\right)\right|_{x_3}\right)$ converges strongly to $\left.\frac{\partial}{\partial x_3}\left(u_\alpha\right)\right|_{x_3}$ which does not depend upon the local variables $x_4$ and $x_5$. Applying Proposition 3.1 to $u_\epsilon^m$ and $\left.\frac{\partial}{\partial x_3}\left(u_\epsilon^m\right)\right|_{x_3}$, we can obtain $m^\epsilon(u_\epsilon^m)$ and $u_\epsilon^m$ converge strongly to $u_\alpha$ in $L^2((0, 1) \times (0, 1); H^1_\alpha(0, 1))$. □

In order to verify the correctness of the theorem, we present some numerical calculations by the software comsol to show the iteration process by setting $f = x_3^1$, $\beta = 0.6$ (see Fig. 1 and Table 1).

Let $\varepsilon_c = \|u_\epsilon - u\|^2_{L^2((0, 1) \times (0, 1); H^1_\alpha(0, 1))}$, $\beta_c = \|u_\epsilon^m - u_\alpha\|_m^2\left(u_\alpha\right)$.

References


