Blending Circular Quadrics with Parametric Patches

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Abstract

A method of blending circular quadrics with parametric patches is proposed in this paper. It needs $n$ rational bicubic Bézier patches and two S-patches to blend $n$ ($n > 2$) quadrics. The blend is $G^1$ continuous. Explicit formulae of control points of both Bézier patches and S-patches are derived from the corresponding $G^1$-continuity conditions. In addition, the shape can be intuitively modified by adjusting the free parameters of the blending surfaces.

1. Introduction

Surface blending is an important topic in geometric modelling. Simple and classical quadrics are also popularly used in engineering applications. The blending problem of quadrics thus deserves further research. For 2-way blending of circular quadrics, many methods were proposed. In those methods, many different blending surfaces are adopted, including parametric surfaces, implicit surfaces and subdivision surfaces etc. For $n$-way blending of circular quadrics, one may also find a few existing methods([1-6]). Some subdivision methods designed for general parametric base surfaces can be used to blend quadrics. But we know that a suitable initial frame of subdivision surface is often difficult to construct for blending problem. Additionally, subdivision methods can only obtain an approximate surface result. Wu and Zhou([3]) blended several quadrics with an algebraic surface. They transformed blending into solving a linear system. But such an algebraic surface exists only when all blending boundaries lie on a certain quadric. Chen et al.([4]) proposed to blend several quadrics with piecewise algebraic surfaces. A space partition is needed at first. But they haven’t presented an automatic and effective approach for such a space partition.

In this paper, we adopt a new framework to blend $n$ circular quadrics. The blending process includes two main steps. At first, we divide each boundary circle into two halves and then blend along two adjacent semicircles with rational bicubic Bézier patches. These blending patches enclose two holes. We then use S-patches to fill the holes. The whole blend is $G^1$ continuous and the shape of each blending patch can be intuitively adjusted by changing its free parameters.

2. Boundary division and 2-way blending

In this paper, circular quadrics refer to six primitives: cylinder, cone, sphere, one-sheet hyperboloid, paraboloid and plane (a degenerate quadratic surface). When there are $n(n > 2)$ quadrics (base surfaces), the blend belongs to $n$-way blending. Especially, when $n = 2$, it is 2-way blending.

Definition 2.1 Let $QS_1, QS_2, ..., QS_n$ for ($n > 2$) be the base surfaces, $P_{l1}, P_{l2}, ..., P_{ln}$ be their section planes, $C_1, C_2, ..., C_n$ be the intersection circles and $O_1, O_2, ..., O_n$ be the centers of these circles. Assuming that $O_1, O_2, ..., O_n$ form a spacial $n$-polygon, we need to construct a surface which meets the base surfaces with $G^1$ continuity along the respective intersection circles.

Let $O$ be the barycenter of $O_1, O_2, ..., O_n$. The plane through the axis of $QS_i$ and the point $O$ divides $C_i$ into two semicircles $SC_{i1}$ and $SC_{i2}$. Now we need to construct a rational bicubic patch $BP_i$ to blend $QS_i$ and $QS_{i+1}$ along $SC_{i1}$ and $SC_{i+11}$ ($i = 1, ..., n$ and $i+1 = 1$ when $i = n$). This is a 2-way blending problem in fact, we use a bicubic rational Bézier patch as the blending surface.

As illustrated in Fig. 1, $BP_i$ blends $QS_i$ and $QS_{i+1}$ along $SC_{i2}$ and $SC_{i+12}$ respectively. Assume that

$$BP_i(u, v) = \sum_{j_1=0}^{3} \sum_{j_2=0}^{3} \tilde{\omega}_{j_1,j_2} P_{j_1,j_2} B_{j_1}^3 (u) B_{j_2}^3 (v)/W, \quad (1)$$

where

$$W = \sum_{j_1=0}^{3} \sum_{j_2=0}^{3} \tilde{\omega}_{j_1,j_2} B_{j_1}^3 (u) B_{j_2}^3 (v), u, v \in [0, 1].$$

The concrete method of determining $BP_i$ has been discussed in paper [7].

3. Construction of S-patches

The theory of S-patches was presented by Loop and DeRose in 1989 ([8]). This class of patches refer to a
generalization of Bézier surfaces which allowed to be defined over any convex polygonal domain. They are popularly used to connect surfaces ([9][10][11]). We select S-patches to fill holes because that they have similar properties to surrounding Bézier surfaces and only one patch is needed to filling a hole. The boundaries of one hole are given by \( BP_i(u_i, 0), i = 1, \ldots, n \). Another hole is surrounded by \( BP_i(u_i, 1), i = 1, \ldots, n \). Two holes are filled using the same method, so we only introduce the method dealing with the latter. As indicated by Fig. 2, the regular \( n \)-polygon \( P \) is the definition domain. \( p_1, \ldots, p_n \) are its vertices, \( t_i = p_{i+1} - p_i \) and \( E(t_i) = (1 - t_i)p_i + t_iP_{i+1}, t_i \in [0, 1] \). According to the representation of surrounding patches \( BP_i \), constructed by 2-way blending method, it’s not difficult to obtain the following relations:

\[
\begin{align*}
(A0) \quad BP_{i-1}(1, 1) &= BP_i(0, 1); \\
(A1) \quad \frac{\partial BP_{i-1}}{\partial u_i}(1, 1) &= -\frac{\partial BP_i}{\partial u_i}(0, 1); \\
(A2) \quad \frac{\partial BP_{i-1}}{\partial v_i}(1, 1) &= -\frac{\partial BP_i}{\partial v_i}(0, 1).
\end{align*}
\]

The \( G^0 \)-continuity condition requires that \( SP(E_i(t_i)) = BP_i(t_i, 1), t_i \in [0, 1], i = 1, \ldots, n \). Again, the sufficient and necessary conditions of \( G^1 \)-continuity between \( SP \) and \( BP_i(i = 1, \ldots, n) \) require:

\[
\begin{align*}
(B0) \quad D_{t_i} SP(E_i(t_i)) &= \frac{\partial BP_i}{\partial u_i}(t_i, 1), t_i \in [0, 1]; \\
(B1) \quad \exists \mu, \nu : [0, 1] \to R, s.t. \\
D_{t_i} SP(E_i(t_i)) &= \mu(t_i) \frac{\partial BP_i}{\partial u_i}(t_i, 1) + \nu(t_i) \frac{\partial BP_i}{\partial v_i}(t_i, 1).
\end{align*}
\]

In order to construct appropriate \( \mu \) and \( \nu \), we impose two endpoint conditions and one twice differentiable condition on \((B1)\). Then we get

\[
\begin{align*}
(C0) \quad -\frac{\partial BP_{i-1}}{\partial u_i}(1, 1) &= \mu(0) \frac{\partial BP_i}{\partial u_i}(0, 1) + \nu(0) \frac{\partial BP_i}{\partial v_i}(0, 1); \\
(C1) \quad \frac{\partial BP_i}{\partial u_i}(0, 1) &= (\mu(1) + 2 \cos \frac{\pi}{n}) \frac{\partial BP_i}{\partial u_i}(1, 1) + (\mu(1) + \nu(1) - 1) \frac{\partial BP_i}{\partial v_i}(1, 1); \\
(C2) \quad -\frac{\partial BP_i}{\partial v_i}(1, 1) &= \mu(0) \frac{\partial BP_i}{\partial u_i}(0, 1) + \nu(0) \frac{\partial BP_i}{\partial v_i}(0, 1) - \nu(0) \frac{\partial BP_{i-1}}{\partial u_i}(1, 1) + \mu(0) \frac{\partial BP_{i-1}}{\partial v_i}(1, 1) + (\mu(1) + \nu(1) - 1) \frac{\partial BP_i}{\partial v_i}(1, 1).
\end{align*}
\]

Based on \((A1)\) and \((A2)\), \( \mu(0) = 1, \mu(1) = -1 - 2 \cos \frac{\pi}{n}, \nu(0) = 1 \) and \( \nu(0) = 0, \nu(1) = 0, \nu'(0) = \nu'(1) \) are sufficient for the above constraints. Choosing \( \mu \) to be a quadratic polynomial, it can be uniquely determined using the above corresponding conditions, i.e. \( \mu(t) = (-2 - 2c)t + 1, \) where \( c = \cos \frac{\pi}{n} \). If \( \nu \) is also chosen to be quadratic polynomial, we’ll get \( \nu(t) \equiv 0 \), which conflicts with \( G^1 \)-continuity. So the minimal degree of \( \nu \) is three and we take \( \nu(t) = \delta_B(-2t^3 + 3t^2 - t) \), where \( \delta_B \) is a free constant factor. From the relation \((B1)\), we can conclude that the depth of S-patch \( SP \), which meets surrounding patches \( BP_i(i = 1, \ldots, n) \) with \( G^1 \)-continuity, must be at least 7.

Let \( SP \) be represented as

\[
SP(p) = \sum_{i=1}^{n} c_iB_i^2(l_i(p), \ldots, l_n(p)),
\]

where \( (l_1(p), \ldots, l_n(p)) \) is the barycenter coordinate of \( p \) in \( P \). Again, let blending boundaries be

\[
BP_i(u_i, 1) = \sum_{j=0}^{3} b_{i,j+1}B_j^3(u_i).
\]

We can easily get

\[
\begin{align*}
\frac{\partial BP_i}{\partial u_i}(t_i, 1) &= 3 \sum_{j=0}^{2} (b_{i,j+2} - b_{i,j+1})B_j^3(t_i), \\
\frac{\partial BP_i}{\partial v_i}(t_i, 1) &= 3 \sum_{j=0}^{3} (b_{i,j+1} - b_{i,j+1})B_j^3(t_i).
\end{align*}
\]

where \( P_{i,j} \) are the row adjacent to \( b_{i,j}(j = 1, 2, 3, 4) \) of control points of \( BP_i \). \( b_{i,j} \) are control points of boundary curves \( BP_i(u_i, 1) \) as illustrated in Fig. 2 (right).

Substituting equations \((2)\) and \((3)\) into the relation formula \((B0)\), we obtain

\[
\begin{align*}
&c_{\ell 0} = b_{i,1} \\
&c_{\ell 0} + 2\ell \ell_{\ell+1} = \frac{1}{2} b_{i,1} + \frac{3}{2} b_{i,2} \\
&c_{\ell 0} + 2\ell \ell_{\ell+1} = \frac{1}{2} b_{i,1} + \frac{3}{2} b_{i,2} + \frac{1}{2} b_{i,3} \\
&c_{\ell 0} + 3\ell \ell_{\ell+1} = \frac{1}{2} b_{i,1} + \frac{3}{2} b_{i,2} + \frac{1}{2} b_{i,3} + \frac{1}{2} b_{i,4} \\
&c_{\ell 0} + 3\ell \ell_{\ell+1} = \frac{1}{2} b_{i,1} + \frac{3}{2} b_{i,2} + \frac{1}{2} b_{i,3} + \frac{1}{2} b_{i,4} \\
&c_{\ell 0} + 6\ell \ell_{\ell+1} = \frac{1}{2} b_{i,1} + \frac{3}{2} b_{i,2} + \frac{1}{2} b_{i,3} + \frac{1}{2} b_{i,4}
\end{align*}
\]

Further, we substitute \( u(t) = (-2 - 2c)t + 1, v(t) = \delta_B(-2t^3 + 3t^2 - t) \) and equations \((2)\), \((3)\) and \((4)\) into
In the following, we give the procedure for determining the control net $C$ of $SP$.

- Let $\tilde{C}$ be initial control net of a depth 3 S-patch. Its boundary control points are $c_{i-1} = b_{i-1}I, c_{i} = b_{i}I, c_{i+1} = b_{i+1}I$. Its interior control points are computed by $c_i = (i_1, i_2, \ldots, i_n) * \tilde{I}^{-1} * (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n)^T$, where $\tilde{v}_i = \delta_i(b_{i+1}I - b_iI), \delta_i$ is a free parameter, $\tilde{I} = (a_{ij})_{n \times n}, a_{ij} = 1$ only when $i = 1, \ldots, n$ and $j = i - 1, i + 1$, otherwise $a_{ij} = 0$;
- Elevate $\tilde{c}$ to depth 7 to get a new control net $C$;
- Reset those control points of $C$ marked with circle symbol in Fig. 3 by equations (5) and (6);
- Reset control points marked with star symbol by the following procedure.

For $i = 1$ to $n$ and for $j = 0$ to $6$ do

$$v_n = c_{i-1} + (7-j-1)\tilde{c}_i + j\tilde{c}_{i+1}; v_1 = c_i + (7-j)\tilde{c}_i + j\tilde{c}_{i+1}; v_2 = c_i + (7-j+1)\tilde{c}_i + j\tilde{c}_{i+1};$$

for $k = 2$ to $n - 2$, do

$$v_n = c_{i-1} + (7-j-1)\tilde{c}_i + j\tilde{c}_{i+1}; v_1 = c_i + (7-j)\tilde{c}_i + j\tilde{c}_{i+1}; v_2 = c_i + (7-j+1)\tilde{c}_i + j\tilde{c}_{i+1};$$

end $k$

end $i, j$.

4. Adjustment of free parameters and examples

For each S-patch used to fill a $n$-sided hole, we set two parameters $\delta_B$ and $\delta_I$ to adjust its shape. $\delta_B$ is called a boundary parameter. We often take $\delta_B \in (0, 20]$. Changing its value, we can adjust the shape of S-patch near each blending boundary. $\delta_I$ is called an interior parameter. We often take $\delta_I \in (0, 2]$. We can adjust the interior shape of the S-patch intuitively through changing the value of $\delta_I$. Figs. 4 and 5 illustrate the blending of three-cylinders and five-conics, respectively. In Fig. 4, the two cylinders are perpendicular to each other. From Fig. 5, we can see how the two free parameters affect the shape of an S-patch. The two figures have the same boundary parameter and different interior parameters. $\delta_I$ equals to 1.1 in the first figure and 1.5 in the second.

In Fig. 6, the axes of cylinders and paraboloid are agonic. The whole blending surface is smooth seen from different perspectives.

5. Conclusion

For $n$-way blending problem of quadrics, those existing methods are either with stern conditions, or without an automatic algorithm, or only suitable for special number of quadrics. Furthermore, their blending surfaces belong to subdivision surfaces or algebraic surfaces. We adopt parametric surfaces, which are more convenient for modelling. The whole blending surface is $G^1$-continuous.
Furthermore, each blending patch can be adjusted intuitively through changing some free parameters. It can be used to blend any number of quadrics. Because in the first step of blending, all computations upon base surfaces can be processed under their standard forms, modelling by this method becomes very convenient and effective. Furthermore, this method can be easily extended to blend other base surfaces, for example elliptical quadrics and toruses.

Acknowledgment

This work was partially supported by the National Science Foundation of China (Grant No. 60773179), the State Key Basic Research Program 973 of China (Grant No. G2004CB318000), and the Research Grants Council of the Hong Kong Special Administrative Region, China (Grant No. CityU 1184/06E).

References