An analytical solution to optimal focal distance in catadioptric imaging systems

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Abstract—Catadioptric imaging systems are important in many computer vision and robotics applications. This work addresses the issue of optimally setting the focal distance of a lens based camera in a catadioptric imaging system in order to acquire a best focused image. To this end, understanding the spatial distribution of virtual feature formed by mirror reflection is important. It is known that the virtual features of infinite range of scene depth are limited to a finite depth extent, which is named a caustic volume. In this work we further find that for a variety of quadric mirror based catadioptric systems, when the objects are located at a certain distance to the system, the corresponding virtual features can be considered to be located on the caustic volume boundary. We verify this property with real catadioptric images. Based on this property, an analytical solution is derived for the optimal focal distance setting, which can only be calculated by software simulation or numerical approaches in previous work. This solution is compared with a numerical solution of previous method and is also verified by a simulation of the optical process.

I. INTRODUCTION

Catadioptric imaging systems are highly desirable in many computer vision and robotics applications for their wide field of view. Such a system typically uses a curved mirror combined with a lens based camera [1], [2]. As in other image-forming optical devices, acquiring well focused images is important in catadioptric systems. It is well known that to capture a well focused image using a conventional lens based camera, the focal distance must be set to the depth of the objects of interest being imaged. However, in a catadioptric system, 3D objects in the world are first reflected by the curved mirror and form virtual features. These virtual features are then captured by the lens based camera. As the lens based camera aims at imaging virtual features of the objects rather than the objects themselves, the focal distance setting becomes substantially different. While most prior work have focused on the system design [1]-[6], calibration [7]-[9] and their applications [10]-[12], relatively few work paid attention to the study on how one should set the focal distance of the lens based camera in order to obtain an optimally well-focused image. Though the basic laws of geometric optics that describe the imaging process have been long established, this problem is still considered to be open and not completely understood [13].

Among the related previous work, an earlier one was performed by Baker and Nayar [2], where the authors studied the defocus blur caused by a curved mirror and presented the field curvature effects [14] using professional tools such as ZEMAX [15]. It was found that, for a scene of constant depth, significantly different focal distance settings are required to best focus on different object points, which is substantially different from that needed in a conventional camera. The authors also suggested using additional lens components to compensate for the field curvature effects. Without using additional lenses, Ishiguro [4] discussed the focus problem for low-cost catadioptric systems. The author stated that if the camera’s depth of field (DOF) could cover the zone where the virtual features of all object points reside, the image would appear clearly focused for all objects. In [16], the authors gave expressions for this virtual image zone using second order approximations and applied these expressions to analyze the image blur in omnidirectional stereo sensors. Recently, Swaminathan [13] explicitly derived expressions for the positions of the virtual features and found that the infinite range of scene depth is limited to a finite extent of virtual features which is named a caustic volume. Following this, Swaminathan gave a framework that calculates the optimal focal distance so that the focal plane passes the center of mass of the caustic volume and an optimal sharp image can be acquired. However, as stated in [13], in many cases analytical solutions are not available in this framework and they can only be calculated using numerical methods.

This work further explores the spatial distribution property of virtual feature within the caustic volume. We find that this distribution is non-uniform and the majority of virtual features are compressed in a narrow neighborhood of the caustic volume boundary (CVB). By examining the spatial distribution of virtual feature at physical scales for a variety of quadric mirror based catadioptric systems, we find that the great majority of object points which are at a distance to the system have their virtual features located on the caustic volume boundary. We verified this conclusion through experiments with real catadioptric images. Based on this conclusion, computation of the optimal focal distance can be simplified to consider only the caustic volume boundary. Following this, we derive an analytical solution for the optimal focal distance. Experiments show that this analytical solution is well consistent with the solution given in paper [13] which in many cases can only be computed using a numerical approach. The predicted optimal focal distance by our method is also well consistent with that given by a simulation using the professional optical design software ZEMAX [15].
II. SPATIAL DISTRIBUTION OF VIRTUAL FEATURE WITHIN THE CAUSTIC VOLUME

A catadioptric system studied in this work consists of a quadric mirror and a lens based camera with co-axis installation as shown in Fig. 1. A point \( P \) in space \( W \subset \mathbb{R}^3 \) is reflected by the mirror and forms its virtual feature in a virtual feature space \( V \subset \mathbb{R}^3 \). According to [1] and [14], the virtual feature of \( P \) can be modeled by a caustic. When we only consider the virtual feature corresponding to the principal incidence ray as in [13], the virtual feature of \( P \) can be regarded as a unique point. Therefore a one-to-one mapping can be established as \( M : W \rightarrow V \). As stated in [13], \( V \) is named a caustic volume. The caustic volume is between the mirror and a caustic volume boundary (CVB) (see Fig. 1).

As no priori information about scene structure is known, it is reasonable to assume that \( P \) is uniformly distributed in \( W \). Due to the distortion effect of the curved mirror, the spatial distribution of virtual feature becomes non-uniform. In this work, we explore optical geometry to explicitly describe the distortion effects of \( M \) and present the property for spatial distribution of virtual feature in \( V \).

For an object point \( P \), the location of its corresponding virtual feature can be explicitly calculated [13]. Therefore when \( P \) is uniformly distributed, the spatial distribution of its virtual feature can also be calculated and analytically studied. As a catadioptric system can be assumed to be rotationally symmetric, analysis can be performed in 2D as shown in Fig. 1. Here we follow the same notation system as in [13]. In Fig. 1, point \( C \) is the entrance pupil center of the lens. Two lines \( L_1 \) and \( L_2 \) define the field of view (FOV). The position of a point on the quadric mirror surface can be expressed as:

\[
S_i(t) = (t,[(e^2-1)t^2 + 2pt - p^2]^{1/2}) \in \mathbb{R}^2
\]

where \( e \) is the eccentricity of the quadric and \( p \) is the focal point of the quadric. \( S_i(t) \) is located on different positions on the mirror surface when \( t \) takes different values. For a particular \( S_i(t) \), an optical path can be determined by \( V_i(t) \) and \( V_j(t) \) as shown in Fig. 1, where \( V_i(t) \in \mathbb{R}^2 \) represents the principal incidence ray direction from \( C \) to \( S_i(t) \), and \( V_j(t) \in \mathbb{R}^2 \) represents the reflectance ray direction. Following this, the position of an object point along \( V_i(t) \) can be parameterized by:

\[
P(t,k) = S_i(t) + k \cdot V_i(t)
\]

where \( t \) determines the optical path and \( k \) is the object distance to \( S_i(t) \) along \( V_i(t) \). The corresponding virtual feature of \( P(t,k) \) is given by:

\[
S_i(t,k) = S_i(t) + \tilde{d}_i(t,k)V_i(t)
\]

where \( \tilde{d}_i(t,k) \) is defined as virtual feature depth and is the distance between \( S_i(t,k) \) and \( S_i(t) \). The dotted curve behind the mirror surface in Fig. 1 is the CVB and is given by:

\[
S_i(t) = S_i(t) + \tilde{d}_i(t)V_i(t)
\]

where \( \tilde{d}_i(t) \) is the caustic volume boundary depth. The explicit expressions for \( V_i(t) \), \( V_j(t) \), \( \tilde{d}_i(t,k) \) and \( \tilde{d}_i(t) \) are given in [13].

Consider a set of \( P(t,k) \) along \( V_i(t) \) with \( k \) sampled at equal interval. Then the spatial distribution of virtual feature is revealed from \( S_i(t,k) \). Since \( S_i(t,k) \) is determined by \( \tilde{d}_i(t,k) \), next we study the spatial distribution of \( \tilde{d}_i(t,k) \).

A. Spatial distribution of virtual feature depth

The distribution of \( \tilde{d}_i(t,k) \) with different rays is plotted in Fig. 2. Let \( p=50 \text{ mm} \), \( d=50 \text{ mm} \) and \( e=1.2 \). Here we consider three representative samples on the mirror whose horizontal distances to the optical axis are 15 mm, 30 mm, and 45 mm. Therefore the angles between \( V_i(t) \) and the optical axis are 11.34°, 20.5°, and 26.9°. In Fig. 2, from the below one, three curves of \( \tilde{d}_i(t,k) \) correspond to 11.34°, 20.5°, and 26.9° respectively. Fig. 2 shows that, as \( k \) increases, the virtual depth \( \tilde{d}_i(t,k) \) at each direction soon approaches the caustic boundary depth \( \tilde{d}_i(t) \), which is plotted as the horizontal dotted line. Note that logarithm coordinate is used for \( k \)-axis.

This observation indicates that as an object gets farther from the mirror surface, its virtual feature would fast approach the CVB. Next we examine this property quantitatively for a variety of catadioptric imaging systems.

B. Measurement of virtual feature spatial distribution

We construct a quantitative measurement to describe the spatial distribution of virtual feature. This measurement is denoted as \( K_{\text{mvf}} \) and it defines an object distance from the system, where all objects farther than this distance would have their virtual features located in a \( m\% \) neighborhood of the CVB. Next we introduce how \( K_{\text{mvf}} \) is derived in details.
Given a specific mirror surface point $S_t(t)$, an optical path can be determined by $V_i(t)$ and $V_f(t)$. Let the distance $k$ of a point along $V_i(t)$ take values from the interval $[0, \infty)$, then its virtual feature depth satisfies: $d_v(t,k) \in [0, \hat{d}_v(t))]$. Let $N_{m\%}(t)$ represent an interval $[\hat{d}_v(t)(1-m\%), \hat{d}_v(t)]$ along $V_i(t)$. Then in each direction $V_i(t)$, the object point whose virtual feature lies on the left boundary of $N_{m\%}(t)$ can be determined and denoted by $P(t,K(t;m\%))$. Following this, when $k> K(t;m\%)$, a point $P(t,k)$ will have its virtual feature located within $N_{m\%}(t)$. $K(t;m\%)$ can be obtained by solving the equation: $d_v(t,K(t;m\%)) = \hat{d}_v(t)(1-m\%)$ and it is:

$$K(t;m\%)=N_t / D_t$$

where $N_t = (1-m\%)t(2p+(1+e^2)t)(p(d+p)−(d−de^2 + p)t)$, $D_t = 2m\%p^3[d^2−p^2+2(d+p)t+e^2t^2]/2$.

In Fig. 2, the three triangular markers indicate the positions of $K(t;m\%)$ for $t=15$, $t=30$, and $t=45$ respectively.

For a given catadioptric system, the maximum value of $K(t;m\%)$ is defined as: $K_{m\%} = \max_{t\in\mathcal{T}}(K(t,m\%))$, where $R_t$ is the range of $t$. Therefore, the definition of $K_{m\%}$ has considered all the rays of interest. For an object point $P(t,k)$, as long as $k> K_{m\%}$, its virtual feature is within $N_{m\%}(t)$.

C. Property of virtual feature spatial distribution

Following its definition, $K_{m\%}$ is dependent on the configuration of the catadioptric system. Next we calculate and examine $K_{m\%}$ for a variety of quadric mirror based catadioptric systems at physical scales.

Here we consider two key parameters for the system configuration: the mirror surface eccentricity $e$ and the focal point $p$. Following this, $K_{m\%}$ is a function of $e$ and $p$. Let $e$ range from 0.8 to 1.2 and $p$ range from 3 mm to 60 mm. As $e$ takes different values, the mirror shapes are different. For a parabolic mirror, $e=1$ and for elliptic and hyperbolic mirrors, $e<1$ and $e>1$ respectively. In many applications the systems are expected to have a single viewpoint [2]. Thus let $d=p$. Since $S_t(t)$ and the light ray path are dependent on $t$, the effective field of view for the lens camera can be represented by specifying a range of $t$ as: $R_t=[t_{\min}, t_{\max}]$, where $t_{\min} = p/(1+e)$ and $\varphi(t_{\max}) = 20^\circ$. Here $\varphi(t)$ is the angle difference between the incidence ray direction $V_i(t)$ and the tangential direction of the mirror surface $T_i(t)$.

Following the above settings, catadioptric systems with different sizes are considered. The height from the mirror apex to the camera lens ranges from 5.9 mm to 114.7 mm and the diameter of the mirror ranges from 4.6 mm to 136.0 mm. These are consistent with the production list by ACCOWLE Co. LTD, which provides sensors for many omnidirectional vision applications. According to ACCOWLE’s production list [17], their sensors vary from 6.2 mm (Nano Type) to 104 mm (Extra-Large Type) in outer diameter.

In Fig. 3, we plot the values of $K_{5\%}$. Both the increase of $e$ and $p$ lead to an increase of $K_{5\%}$. However, for the set of catadioptric systems presented, the maximum of $K_{5\%}$ still lies below 1 meter. In a variety of real robotic applications with catadioptric systems, the objects of interest usually lie a certain distance to the system. Therefore, the great majority of virtual features can be considered to be located on the caustic volume boundary. We refer to this property as the virtual feature spatial distribution (VFSD) property. The work in the following section is based on this VFSD property.

III. AN ANALYTICAL SOLUTION TO OPTIMAL FOCAL DISTANCE

The optimal focal distance is the focal distance of the lens based camera that enables the camera’s depth of field (DOF) to contain a maximum number of virtual features. Following this, in [13] the author took the center of mass of the caustic volume to be the optimal focal distance and proposed a framework to calculate its position with the following form:

$$f_o = \int_{t_{\min}+p/(1+e)}^{T} \int_{-p/(1+e)}^{\varphi(t_{\max})} S_y^{(t)}(t,k)\omega(t,k)J(t,k)dkdt$$

$$\int_{t_{\min}+p/(1+e)}^{T} \int_{-p/(1+e)}^{\varphi(t_{\max})} \omega(t,k)J(t,k)dkdt$$

where $J(t,k) = \begin{vmatrix} \frac{\partial S_x^{(t)}(t,k)}{\partial t} & \frac{\partial S_x^{(t)}(t,k)}{\partial k} \\ \frac{\partial S_y^{(t)}(t,k)}{\partial t} & \frac{\partial S_y^{(t)}(t,k)}{\partial k} \end{vmatrix}$

$S_y^{(t)}$ and $S_x^{(t)}$ refer to the $x$ and $y$ coordinates of a virtual feature point in the caustic volume (as shown in Fig. 1) and $\omega(t,k)$ is a uniformly distributed weight function. This theoretical result was verified by experiments with real images in [13]. However, as the author mentioned, in many cases analytical solutions to the integral are not possible.

In this work, we propose an analytical solution to compute the optimal focal distance of a catadioptric system. Following the VFSD property, the virtual features are located on the CVB. Therefore, to obtain the center of mass of the caustic volume, the integral can be performed only on the CVB instead of on the entire caustic volume. As the integral field now takes a much more concise form, obtaining an analytical solution becomes possible. By limiting the integral field in (3) to the CVB, we derive the optimal focal distance as follows:
The image projection of this section is an annulus. Therefore, the shapes of the best focused regions are a set of neighboring and concentric annuluses. Note that, this indicates that the shapes of the best focused image regions are independent of the 3D scene structures, which is very different from that in a conventional camera. Based on this, a model based method was developed to acquire an overall well focused catadioptric image in a previous work [19].

Here we use a catadioptric system consisting of a Canon Power-Shot S50 camera and a hyperbolic mirror. Guaranteed by the image-space telecentric feature [18], the scene contents in the multifocal images are identical. We took multifocal images in four scenes with the same setting. For each scene, three multifocal images were captured \{I_1, I_2, I_3\}. We consider a dense set of grid sample points in the image plane. For each sample point, we evaluate in which one of the three multifocal images is it the best focused. This is done by calculating a focus measurement score using higher order statistics (HOS) [20] in each image and finding the highest score. In Fig. 5, one image for each scene is presented in the first row, under which a subfigure displays the sample points.

A. Verification of the virtual feature spatial distribution property with real catadioptric images

We design an experiment to verify the VFSD property with multifocal catadioptric images. If the VFSD property holds, the shapes of the best focused image regions in the multifocal images are supposed to be a set of neighboring and concentric annuluses. This is explained as follows.

Let \( I = \{I_i\} \) be a set of \( K \) multifocal catadioptric images, which are taken at the same view point for the same scene but by setting the lens based camera to \( K \) different focal distance settings \( \{f(I_i)\} \), where \( f(I) \) is the focal distance of \( I \), and \( f(I_1) < f(I_2) < ... < f(I_K) \). Note that when a camera with image-space telecentric feature [18] is used, the scene contents in images of different \( f(I_i) \) are aligned. The depth of field (DOF) of \( I_i \) can be modeled by a space between two parallel planes perpendicular with the optical axis. Then the space is divided into \( K \) depth regions \( \{d(I_i)\} \), which are separated by \( K-1 \) parallel planes as shown in Fig. 4. The objects in \( d(I_i) \) is the best focused in \( I_i \). With the VFSD property, all the virtual features are on the CVB. Following this, the best focused region in \( I \) is formed by the section of CVB that lies in \( d(I_i) \). As the system is rotational symmetric,
the noise resulting from local texture-less regions, sensor internal reflections, and low SNR in dark areas.

B. Comparison with a numerical solution of the previous method

We make a comparison between the solution \( f_a \) [13] and our solution \( f_b \). As \( f_a \) can only be numerically calculated, we use a numerical solution to obtain an approximation \( \hat{f}_a \). Then the comparison is conducted between \( \hat{f}_a \) and \( f_b \).

The numerical solution considers a set of object points \( \{ P(t(i),k(j)) \} \) uniformly distributed in the object space, where \( i \) and \( j \) are index numbers. Denote \( S_a \) (\( t(i),k(j) \)) as the \( x \) coordinate of the virtual feature of point \( P(t(i),k(j)) \). Then following (4), the weighted average of \( S \) is taken as the approximation of \( f_a \) and it is denoted as \( \hat{f}_a \):

\[
\hat{f}_a = \frac{\sum_{i=1}^{N_i} \sum_{j=1}^{N_j} S_a (t(i),k(j)) \omega(t(i),k(j))}{\sum_{i=1}^{N_i} \sum_{j=1}^{N_j} \omega(t(i),k(j))}
\]

(5)

Here \( \omega(t(i),k(j)) \) is the probability distribution of \( P(t(i),k(j)) \) and it is assumed to be uniform as no scene assumption is made. Thus we make \( \omega(t(i),k(j)) = 1, \forall i, j \).

Here \( \{ t(i) \}, i = 1,2,...,N_i \) and \( \{ k(j) \}, j = 1,2,...,N_j \) are selected so that \( P(t(i),k(j)) \) is uniformly distributed in the space. Let \( \{ t(i) \} \) be discrete samples of \( R_i = [t_{\text{min}}, t_{\text{max}}] \) at equal interval \( \Delta t \). Therefore \( t(1) = t_{\text{min}} \), \( t(i) = t_{\text{min}} + (i - 1) \cdot \Delta t \), \( t(N_i) = t_{\text{max}} \), where \( N_i = 1 + (t_{\text{max}} - t_{\text{min}}) / \Delta t \) is the number of sample points. Since the spatial resolution of a pixel is limited by the size of CCD, \( \Delta t \) can be assumed to be a fixed value and we take \( \Delta t = 0.1 \). The set of distances \( \{ k(j) \} \) is constructed to have an equal interval \( \Delta k \) so that \( k(1) = 0 \), \( k(i) = (i - 1) \cdot \Delta k \), \( k(N_k) = MaxK \), where \( MaxK \) is the largest object distance of interest and \( N_k = 1 + MaxK / \Delta k \).

Following (5), the accuracy of \( \hat{f}_a \) is mainly dependent on \( \Delta k \) and \( MaxK \). As \( \Delta k \) approaches 0 and \( MaxK \) approaches infinite, \( \hat{f}_a \) is supposed to approach the real value of \( f_a \). To examine the convergence property, we consider a hyperbola mirror based catadioptric system, where \( e = 1.1 \), \( d = 50 \) mm, \( p = 50 \) mm. Let \( \Delta k \) take values from set \{\( \Delta k \)\}, where \( \Delta k_0 = 10 \) mm, \( \Delta k_s = s \cdot 50 \) mm for \( s = 1,2,...,10 \). Let \( MaxK \) takes values from set \{\( MaxK \)\}, where \( MaxK_0 = 10000 \) mm, \( MaxK_q = q \cdot 50000 \) mm for \( q = 1,2,...,5 \). The values of \( \hat{f}_a \) are plotted in Fig. 6. Either a decrease of \( \Delta k \) or an increase of \( MaxK \) leads to an increase of \( \hat{f}_a \). At some point, \( \hat{f}_a \) would reach a stable region where the decrease of \( \Delta k \) by 50 mm or the increase of \( MaxK \) by 50000 mm makes \( \hat{f}_a \) change less than 0.1 mm. The boundary of this region is the convergence of \( \hat{f}_a \) and it is denoted as \( \hat{f}_a \).

For comparison, \( \hat{f}_a \) and \( f_a \) are computed for a variety of configurations. Let \( e \) range from 0.8 to 1.2 and \( p \) range from 3 mm to 60 mm. The difference \( \Delta f = f_a - \hat{f}_a \) is shown in Fig. 7. As \( e \) increases or \( p \) increases, \( \Delta f \) also increases. However, \( \Delta f \) is generally small with a maximum value of 0.031 mm. Notice that the accuracy of focal distance in a real camera is dependent on various factors such as the mechanical adjustment step, which might be as large as 1 cm. Following this, \( f_a \) and \( f_b \) are well consistent.

C. Simulation with ZEMAX

We compare our method with a simulation with ZEMAX [15]. ZEMAX was also used in [2] and [6] for catadioptric systems. Different from these, the overall focus performance is analyzed and object points in an area are considered.

We simulate a typical catadioptric system. As in Section II, set \( e = 1.1 \) and \( p = 20.00 \) mm. To satisfy the single viewpoint constraint, set \( d = p = 20.00 \) mm. The mirror radius is 50 mm.

![Fig. 8. The geometry used to simulate the catadioptric imaging process.](image-url)
The lens radius is 35 mm. Different F-stop numbers are used, which are F16, F11, F8, F5.6, and F4.

In the simulation, ZEMAX tracks any ray emitting from an arbitrary object point $G_i$ as shown in Fig. 8. The distance between the images of the two marginal rays determines the blur region span $B_i$. The average of blur region span is used to evaluate the focus performance, which is defined as the overall defocus measurement function $C(f_s)$ and it is:

$$C(f_s) = \sum G_i w(G_i) B_i(f_s) / \sum G_i w(G_i)$$ (6)

Here, $f_s$ is the focal distance. $G = \{ G_i \}$ is the set of points of interest, which are uniformly distributed in a circular space $R_i$ centered at the mirror’s focus $F_m$. $w(G_i)$ is a binary function, whose value is 1 only if $G_i$ is within the effective field of view. Therefore, the optimal focal distance is searched by minimizing $C(f_s)$. Here, the radius of $R_i$ is set to 200 meters. This includes outdoor scenarios. Experiments show that the minimum of $C(f_s)$ is not affected by the selection of a larger object space.

It is known [14] that the lens focus $F_{lens}$, the focal distance setting $f_b$, and the image plane distance $d_{image}$ should satisfy:

$$\frac{1}{F_{lens}} = \frac{1}{f_b} + \frac{1}{d_{image}} \cdot (7)$$

Let $d_{image}$ increase from 40 mm to 180 mm at an equal step of 10 mm. With (7), a set of different focal distance $\{ f_i \}$ is obtained. By finding the converging point of axially parallel incident rays, the focal length of the lens is 33.75 mm. Then the focal distance varies from 41.53 mm to 216.00 mm.

Fig. 9 shows $C(f_s)$ with different F numbers. It can be seen that, $C(f_s)$ increases with the lens aperture. For each lens aperture, $C(f_s)$ presents a global minimum, which is independent of the lens aperture and is marked by a square. At these positions, the image distance $d_{image} = 80$ mm. A search refines this to 81 mm. Therefore the optimal focal distance is $f_b = (d_{image} \cdot F_{lens}) / (d_{image} - F_{lens}) = 57.8571$ mm.

An analytical solution for the optimal focal distance is given by (4). Using the same parameters as those in the simulation, let $e = 1.1, p = 20, d = 20, t_{min} = p(1+e)$, and $t_{max} = 56.02$ (so that the mirror radius is 50 mm, which coincides with the setting in the simulation). Substituting these parameters to (4), the result is $f_b = 58.9765$ mm. It can be seen that $f_b$ and $f_s$ are well consistent. The optimal focal distance is also consistent with the simulation under other system configurations, where $e$ ranges from 0.8 to 1.2 and $p$ ranges from 3 mm to 60 mm, as those in Section II-C.

V. CONCLUSIONS

In this work we found that when an object point is located at a distance to the system, its virtual feature can be considered to be located on the caustic volume boundary. This property was verified with real catadioptric images and optical simulation. Based on this, an analytical solution to the optimal focal distance is derived for quadric mirror based catadioptric systems. This solution provides a fast and light-weighted tool to compute the optimal focal distance when designing lens and mirror parameters for a catadioptric imaging system and may further inspire related work in the future.

REFERENCES