A Class of Second Order Difference Approximations for Solving Space Fractional Diffusion Equations

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Abstract

A class of second order approximations, called the weighted and shifted Grünwald difference operators, are proposed for Riemann-Liouville fractional derivatives, with their effective applications to numerically solving space fractional diffusion equations in one and two dimensions. The stability and convergence of our difference schemes for space fractional diffusion equations with constant coefficients in one and two dimensions are theoretically established. Several numerical examples are implemented to testify the efficiency of the numerical schemes and confirm the convergence order, and the numerical results for variable coefficients problem are also presented.

Keywords: Riemann-Liouville fractional derivative, Fractional diffusion equation, Weighted and shifted Grünwald difference operator.

AMS subject classifications: 26A33, 65L12, 65L20

1 Introduction

Fractional calculus is a fundamentally mathematical tool for describing some special phenomenons arising from engineering and science [15, 18, 22]. One of its most important applications is to describe the subdiffusion and superdiffusion process [5, 10, 16]. The suitable mathematical models are the diffusion equations with time and/or space fractional derivatives, where the classical first order derivative in time is replaced by the Caputo fractional derivative of order $\alpha \in (0, 1)$, and the second order derivative in space is essentially replaced by the Riemann-Liouville fractional derivative of order $\alpha \in (1, 2]$. The physical interpretation and practical applications of fractional diffusion equations have been discussed a lot with some common ideas [1, 9, 14]. Based on these, our main purpose of this paper is to study the higher accurate numerical solution of the space fractional diffusion equation by a novel finite difference approximation.

From the perspective of the numerical analysis, there are some fundamental difficulties in numerically approximating the fractional derivatives, because some good properties of classical approximating operators are lost. Over the last decades, the finite difference method has some developments in solving the fractional partial differential equations, e.g., [2, 12, 13, 27]. The Riemann-Liouville fractional derivative can be discretized by the standard Grünwald-Letnikov formula [18] with only the first order accuracy, but the difference scheme based on the Grünwald-Letnikov formula for time dependent problems is unstable [12]. To overcome this problem, Meerschaert and Tadjeran in [12] firstly proposed the shifted Grünwald-Letnikov formula to approximate fractional advection-dispersion flow equations. Recently, second order approximations to fractional derivatives are studied, Sousaa and Li presented a second order discretization for Riemann-Liouville fractional derivative and established an unconditionally stable weighted average finite difference method for one-dimensional fractional diffusion equation in [23], and the results in two-dimensional two-sided space fractional convection diffusion equation in finite domain can be seen in [6];
Ortigueira [17] gave the “fractional centred derivative” to approximate the Riesz fractional derivative with second order accuracy, and this method was used by Çelik and Duman in [2] to approximate fractional diffusion equation with the Riesz fractional derivative in a finite domain. In this paper, we propose a more general and flexible approach to approximate the Riemann-Liouville fractional derivative via combining the distinct shifted Grünwald-Letnikov formulae with their corresponding weights, and the weighted and shifted Grünwald-Letnikov formulae achieve second and higher order accuracy. A detailed algorithm shows that the weights are related to not only the shifted numbers but also the order of the fractional derivative, which implies the numerical algorithm is more related to the equation itself.

The paper is briefly summarized as follows. In Sec. 2, we propose a class of discrete operators to approximate the Riemann-Liouville fractional derivatives with high order truncating errors. In Sec. 3 and 4, one dimensional and two dimensional fractional diffusion equations are numerically solved by using the finite difference method based on the weighted and shifted Grünwald-Letnikov formulae, and the stability analysis of each case is presented. We prove that the finite difference solutions approximate the exact ones with $O(\tau^2 + h^2)$ in the discrete $L^2$ norm. Some numerical experiments are performed in Sec. 5 to verify the efficiency and accuracy of the methods. And the concluding remarks are given in the last Section.

### 2 High Order Approximations for Riemann-Liouville Fractional Derivatives

We begin with the definitions of the Riemann-Liouville fractional derivatives and the properties of their Fourier transform.

**Definition 1 ([18]).** The $\alpha (n - 1 < \alpha < n)$ order left and right Riemann-Liouville fractional derivatives of the function $u(x)$ on $[a, b]$ are defined as

1. **left Riemann-Liouville fractional derivative:**
   \[
   a D^\alpha_x u(x) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_a^x \frac{u(\xi)}{(x - \xi)^{\alpha-n+1}} d\xi;
   \]

2. **right Riemann-Liouville fractional derivative:**
   \[
   x D^\alpha_b u(x) = \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_x^b \frac{u(\xi)}{(\xi - x)^{\alpha-n+1}} d\xi.
   \]

**Property 1 ([8]).** Let $\alpha > 0$, $u \in C_0^\infty(\Omega)$, $\Omega \subset \mathbb{R}$. The Fourier transforms of the left and right Riemann-Liouville fractional derivatives satisfy

\[
\mathcal{F}(a D^\alpha_x u(x)) = (i\omega)^\alpha \hat{u}(\omega),
\]
\[
\mathcal{F}(x D^\alpha_b u(x)) = (-i\omega)^\alpha \hat{u}(\omega),
\]

where $\hat{u}(\omega)$ denotes the Fourier transform of $u$,

\[
\hat{u}(\omega) = \int_{\mathbb{R}} e^{-i\omega x} u(x) dx.
\]

In [12], the shifted Grünwald difference operator

\[
A_{h,p}^\alpha u(x) = \frac{1}{h^n} \sum_{k=0}^\infty g_k^{(\alpha)} u(x - (k - p)h),
\]

approximates the Riemann-Liouville fractional derivative uniformly with first order accuracy, i.e,

\[
A_{h,p}^\alpha u(x) = -D^\alpha_x u(x) + O(h),
\]
where $p$ is an integer and $g_k^{(α)} = (-1)^k \binom{α}{k}$. In fact, the coefficients $g_k^{(α)}$ in (2.1) are the coefficients of the power series of the function $(1 - z)^α$,

$$(1 - z)^α = \sum_{k=0}^{∞} (-1)^k \binom{α}{k} z^k = \sum_{k=0}^{∞} g_k^{(α)} z^k,$$

for all $|z| \leq 1$, and they can be evaluated recursively

$$g_0^{(α)} = 1, \quad g_1^{(α)} = 1 - \frac{α + 1}{k}, \quad k = 1, 2, \ldots .$$

(2.4)

**Lemma 1** ([12, 13, 18]). The coefficients in (2.1) satisfy the following properties for $1 < α \leq 2$,

$$\begin{cases} 
    g_0^{(α)} = 1, \quad g_1^{(α)} = -α < 0, \\
    1 \geq g_2^{(α)} \geq g_3^{(α)} \geq \ldots \geq 0, \\
    \sum_{k=0}^{∞} g_k^{(α)} = 0, \quad \sum_{k=0}^{m} g_k^{(α)} \leq 0, \quad m \geq 1.
\end{cases}$$

(2.5)

### 2.1 Second Order Approximations

Inspired by the shifted Grünwald difference operator (2.1) and multi-step method, we derive the following second order approximation for the Riemann-Liouville fractional derivatives.

**Theorem 1.** Let $u ∈ L^1(\mathbb{R})$, $-∞ < D_x^{α+2} u$ and its Fourier transform belong to $L^1(\mathbb{R})$, and define the weighted and shifted Grünwald difference (WSGD) operator by

$$L D_{h,p,q}^{α} u(x) = \frac{α - 2q}{2(p - q)} A_{h,p}^{α} u(x) + \frac{2p - α}{2(p - q)} A_{h,q}^{α} u(x),$$

then we have

$$L D_{h,p,q}^{α} u(x) = -∞ D_x^{α} u(x) + O(h^2)$$

(2.7)

uniformly for $x ∈ \mathbb{R}$, where $p, q$ are integers and $p \neq q$.

**Note.** The role of $p$ and $q$ is symmetric, i.e., $L D_{h,p,q}^{α} u(x) = L D_{h,q,p}^{α} u(x)$.

**Proof of Theorem 1.** By the definition of $A_{h,p}^{α}$ in (2.1), we can rewrite the WSGD operator as

$$L D_{h,p,q}^{α} u(x) = \frac{α - 2q}{2(p - q)} \frac{1}{h^α} \sum_{k=0}^{∞} g_k^{(α)} u(x - (k - p)h) + \frac{2p - α}{2(p - q)} \frac{1}{h^α} \sum_{k=0}^{∞} g_k^{(α)} u(x - (k - q)h).$$

(2.8)

Taking Fourier transform on (2.8), we obtain

$$\mathcal{F}[L D_{h,p,q}^{α} u]\!(ω) = \frac{1}{h^α} \sum_{k=0}^{∞} g_k^{(α)} \!\left( \frac{α - 2q}{2(p - q)} e^{-iω(k - p)h} + \frac{2p - α}{2(p - q)} e^{-iω(k - q)h} \right) \hat{u}(ω)$$

$$= \frac{1}{h^α} \left( \frac{α - 2q}{2(p - q)} (1 - e^{-iωh})^α e^{iωph} + \frac{2p - α}{2(p - q)} (1 - e^{-iωh})^α e^{iωqh} \right) \hat{u}(ω)$$

$$= (iω)^α \left( \frac{α - 2q}{2(p - q)} W_p(iωh) + \frac{2p - α}{2(p - q)} W_q(iωh) \right) \hat{u}(ω),$$

(2.9)
where
\[ W_r(z) = \left( 1 - \frac{e^{-z}}{z} \right)^{\alpha} e^{rz} = 1 + (r - \frac{\alpha}{2})z + O(z^2), \ r = p, q. \] (2.10)

Denoting \( \hat{\varphi}(\omega, h) = \mathcal{F}[r^\alpha_{h,p,q} u]|(\omega) - \mathcal{F}[r^\alpha_{-\infty} u]|(\omega) \), then from (2.9) and (2.10) there exists
\[ |\hat{\varphi}(\omega, h)| \leq C \lambda^2 |\omega|^\alpha |\hat{u}(\omega)|. \] (2.11)

With the condition \( \mathcal{F}[r^\alpha_{-\infty} u]|(\omega) \in L^1(\mathbb{R}) \), it yields
\[ |r^\alpha_{h,p,q} u - r^\alpha_{-\infty} u| = |\phi| \leq \frac{1}{2\pi} \int_\mathbb{R} |\hat{\varphi}(\omega, h)| \leq C \| \mathcal{F}[r^\alpha_{-\infty} u]|(\omega) \|_{L^1} h^2 = O(h^2). \] (2.12)

**Remark 1.** For the right Riemann-Liouville fractional derivative, similar to Theorem 1, we can check that
\[ r^\alpha_{h,p,q} u(x) = \frac{\alpha - 2q}{2(p - q)} B^\alpha_{h,p} u(x) + \frac{2p - \alpha}{2(p - q)} B^\alpha_{h,q} f(x) = x^\alpha u(x) + O(h^2), \] (2.13)
uniformly for \( x \in \mathbb{R} \) under the conditions that \( u \in L^1(\mathbb{R}) \), \( x^\alpha u \) and its Fourier transform belong to \( L^1(\mathbb{R}) \), where \( p, q \) are integers and
\[ B^\alpha_{h,r} u(x) = \frac{1}{h^\alpha} \sum_{k=0}^\infty g_k(\alpha) u(x + (k - r)h). \] (2.14)

**Remark 2.** Considering a well defined function \( u(x) \) on the bounded interval \( [a, b] \), if \( u(a) = 0 \) or \( u(b) = 0 \), the function \( u(x) \) can be zero extended for \( x < a \) or \( x > b \). And then the \( \alpha \) order left and right Riemann-Liouville fractional derivatives of \( u(x) \) at each point \( x \) can be approximated by the WSGD operators with second order accuracy
\[ a^\alpha x^\alpha u(x) = \frac{\lambda_1}{h^\alpha} \sum_{k=0}^{[\frac{x - a}{h}]+p} g_k(\alpha) u(x - (k - p)h) + \frac{\lambda_2}{h^\alpha} \sum_{k=0}^{[\frac{x - a}{h}]+q} g_k(\alpha) u(x - (k - q)h) + O(h^2), \] (2.15)
\[ z^\alpha b^\alpha u(x) = \frac{\lambda_1}{h^\alpha} \sum_{k=0}^{[\frac{b - x}{h}]+p} g_k(\alpha) u(x + (k - p)h) + \frac{\lambda_2}{h^\alpha} \sum_{k=0}^{[\frac{b - x}{h}]+q} g_k(\alpha) u(x + (k - q)h) + O(h^2), \]
where \( \lambda_1 = \frac{\alpha - 2q}{2(p - q)} \), \( \lambda_2 = \frac{2p - \alpha}{2(p - q)} \).

**Remark 3.** The integers \( p, q \) are the numbers of the points located on the right/left hand of the point \( x \) used for evaluating the \( \alpha \) order left/right Riemann-Liouville fractional derivatives at \( x \), thus, when employing the difference method with (2.15) for approximating non-periodic fractional differential equations on bounded interval, \( p, q \) should be chosen satisfying \( |p| \leq 1, |q| \leq 1 \) to ensure that the nodes at which the values of \( u \) needed in (2.15) are within the bounded interval; otherwise, we need to use another way to discretize the fractional derivative when \( x \) is close to the right/left boundary. When \( (p, q) = (0, -1) \), the approximation method turns out to be unstable for time dependent problems. So two sets of \( (p, q) \) can be selected to establish the difference scheme for fractional diffusion equations, that is \( (1, 0), (1, -1) \), and the corresponding weights in (2.6) and (2.13) are \( (\frac{\alpha}{2}, \frac{\alpha}{2}) \) and \( (\frac{\alpha + 1}{2}, \frac{\alpha - 1}{2}) \). For \( \alpha = 2 \), the WSGD operator (2.6) is the centered difference approximation of second order derivative when \( (p, q) \) equals to \( (1, 0) \) or \( (1, -1) \); for \( \alpha = 1 \), \( (p, q) = (1, 0) \), the centered difference scheme for first order derivative is recovered.
The simplified forms of the discreted approximations (2.15) for Riemann-Liouville fractional derivatives with \((p, q) = (1, 0), (1, -1)\) are

\[
a D_x^\alpha u(x_i) = \frac{1}{h^\alpha} \sum_{k=0}^{i+1} w_k^{(\alpha)} u(x_{i-k+1}) + O(h^2),
\]

\[
x D_h^\alpha u(x_i) = \frac{1}{h^\alpha} \sum_{k=0}^{N-i+1} w_k^{(\alpha)} u(x_{i+k-1}) + O(h^2),
\]

where

\[
\begin{cases}
(p, q) = (1, 0), & w_0^{(\alpha)} = \frac{\alpha}{2} g_0^{(\alpha)}, w_k^{(\alpha)} = \frac{\alpha}{2} g_k^{(\alpha)} + \frac{2-\alpha}{2} g_{k-1}^{(\alpha)}, k \geq 1; \\
(p, q) = (1, -1), & w_0^{(\alpha)} = \frac{2 + \alpha}{4} g_0^{(\alpha)}, w_1^{(\alpha)} = \frac{2 + \alpha}{4} g_1^{(\alpha)}, \\
& w_k^{(\alpha)} = \frac{2 + \alpha}{4} g_k^{(\alpha)} + \frac{2-\alpha}{4} g_{k-2}^{(\alpha)}, k \geq 2.
\end{cases}
\]

With Lemma 1 and some calculations, we obtain the properties of the coefficients \(w_k^{(\alpha)}\) in (2.16) corresponding to \((p, q) = (1, 0), (1, -1)\) as follows.

**Lemma 2.** The coefficients in (2.16) satisfy the following properties for \(1 < \alpha \leq 2\),

1. if \((p, q) = (1, 0)\),

\[
\begin{align*}
& w_0^{(\alpha)} = \frac{\alpha}{2}, \quad w_1^{(\alpha)} = \frac{2 - \alpha - \alpha^2}{2} < 0, \quad w_2^{(\alpha)} = \frac{\alpha(\alpha^2 + \alpha - 4)}{4}, \\
& 1 \geq w_0^{(\alpha)} \geq w_3^{(\alpha)} \geq w_4^{(\alpha)} \geq \ldots \geq 0, \\
& \sum_{k=0}^{\infty} w_k^{(\alpha)} = 0, \quad \sum_{k=0}^{m} w_k^{(\alpha)} < 0, \quad m \geq 2;
\end{align*}
\]

2. if \((p, q) = (1, -1)\),

\[
\begin{align*}
& w_0^{(\alpha)} = \frac{2 + \alpha}{4}, \quad w_1^{(\alpha)} = -\frac{2\alpha + \alpha^2}{4} < 0, \\
& w_2^{(\alpha)} = \frac{\alpha^3 + \alpha^2 - 4\alpha + 4}{8} > 0, \quad w_3^{(\alpha)} = \frac{\alpha(2-\alpha)(\alpha^2 + \alpha - 8)}{6} < 0, \\
& 1 \geq w_0^{(\alpha)} \geq w_2^{(\alpha)} \geq w_4^{(\alpha)} \geq w_5^{(\alpha)} \geq \ldots \geq 0, \\
& \sum_{k=0}^{\infty} w_k^{(\alpha)} = 0, \quad \sum_{k=0}^{m} w_k^{(\alpha)} < 0, \quad m = 1 \text{ or } m \geq 3.
\end{align*}
\]

Next, we will explore the properties of the eigenvalues of the difference matrix of (2.16) on grid points \(\{x_k = a + kh, h = (b - a)/n, k = 1, 2, \ldots, n - 1\}\). In the following, we denote by \(H\) the symmetric (respectively, hermitian) part of \(A\) if \(A\) is real (respectively, complex) matrix.

**Lemma 3** ([20]). A real matrix \(A\) of order \(n\) is positive definite if and only if its symmetric part \(H = \frac{A + A^T}{2}\) is positive definite; \(H\) is positive definite if and only if the eigenvalues of \(H\) are positive.

**Lemma 4** ([20]). If \(A \in \mathbb{C}^{n \times n}\), let \(H = \frac{A + A^*}{2}\) be the hermitian part of \(A\), \(A^*\) the conjugate transpose of \(A\), then for any eigenvalue \(\lambda\) of \(A\), there exists

\[
\lambda_{\min}(H) \leq \text{Re}(\lambda) \leq \lambda_{\max}(H),
\]

where \(\text{Re}(\lambda)\) represents the real part of \(\lambda\), and \(\lambda_{\min}(H), \lambda_{\max}(H)\) are the minimum and maximum of the eigenvalues of \(H\).
Definition 2 ([4]). Let Toeplitz matrix $T_n$ be of the following form,

$$
T_n = \begin{pmatrix}
t_0 & t_{-1} & \cdots & t_{2-n} & t_{1-n} \\
t_1 & t_0 & t_{-1} & \cdots & t_{2-n} \\
\vdots & t_1 & t_0 & \ddots & \vdots \\
t_{n-2} & \cdots & \ddots & \ddots & t_{-1} \\
t_{n-1} & t_{n-2} & \cdots & t_1 & t_0
\end{pmatrix},
$$

if the diagonals $\{t_k\}_{k=-n+1}^{n-1}$ are the Fourier coefficients of a function $f$, i.e.,

$$
t_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx}dx,
$$

then the function $f$ is called the generating function of $T_n$.

Lemma 5 (Grenander-Szegö theorem [4, 3]). For the above Toeplitz matrix $T_n$, if $f$ is a $2\pi$-periodic continuous real-valued function defined on $[-\pi, \pi]$, denote $\lambda_{\min}(T_n)$ and $\lambda_{\max}(T_n)$ as the smallest and largest eigenvalues of $T_n$, respectively. Then we have

$$
f_{\min} \leq \lambda_{\min}(T_n) \leq \lambda_{\max}(T_n) \leq f_{\max},
$$

where $f_{\min}$, $f_{\max}$ denote the minimum and maximum values of $f(x)$. Moreover, if $f_{\min} < f_{\max}$, then all eigenvalues of $T_n$ satisfy

$$
f_{\min} < \lambda(T_n) < f_{\max},
$$

for all $n > 0$; and furthermore if $f_{\min} \geq 0$, then $T_n$ is positive definite.

Theorem 2. Let matrix $A$ be of the following form,

$$
A = \begin{pmatrix}
w_0^{(\alpha)} & w_1^{(\alpha)} & \cdots & w_{n-1}^{(\alpha)} \\
w_2^{(\alpha)} & w_1^{(\alpha)} & \cdots & w_0^{(\alpha)} \\
\vdots & w_2^{(\alpha)} & w_1^{(\alpha)} & \ddots \\
w_{n-2}^{(\alpha)} & \cdots & \ddots & w_0^{(\alpha)} \\
w_{n-1}^{(\alpha)} & w_{n-2}^{(\alpha)} & \cdots & w_1^{(\alpha)}
\end{pmatrix},
$$

(2.20)

where the diagonals $\{w_k^{(\alpha)}\}_{k=0}^{n-1}$ are the coefficients given in (2.16) corresponding to $(p, q) = (1, 0)$ or $(1, -1)$. Then we have that any eigenvalue $\lambda$ of $A$ satisfies

(1) $\text{Re}(\lambda) \equiv 0$, for $(p, q) = (1, 0)$, $\alpha = 1$,

(2) $\text{Re}(\lambda) < 0$, for $(p, q) = (1, 0)$, $1 < \alpha \leq 2$,

(3) $\text{Re}(\lambda) < 0$, for $(p, q) = (1, -1)$, $1 \leq \alpha \leq 2$.

Moreover, when $1 < \alpha \leq 2$, matrix $A$ is negative definite, and the real parts of the eigenvalues $\lambda$ of matrix $c_1A + c_2A^T$ are less than 0, where $c_1, c_2 \geq 0, c_1^2 + c_2^2 \neq 0$.

Proof. We consider the symmetric part of matrix $A$, denoted as $H = \frac{A + A^T}{2}$. The generating functions of $A$ and $A^T$ are

$$
f_A(x) = \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{i(k-1)x}, \quad f_{A^T}(x) = \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{-i(k-1)x},
$$

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respectively. Then \( f(\alpha; x) = \frac{f_A(x) + f_{A^T}(x)}{2} \) is the generating function of \( H \), and \( f(\alpha; x) \) is a periodic continuous real-valued function on \([−\pi, \pi]\) since \( f_A(x) \) and \( f_{A^T}(x) \) are mutually conjugated.

Case \((p, q) = (1, 0)\): with the corresponding coefficients \( w_k^{(\alpha)} \) given by (2.17), then

\[
\begin{align*}
\hat{f}(\alpha; x) & = \frac{1}{2} \left( \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{i(k-1)x} + \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{-i(k-1)x} \right) \\
& = \frac{1}{2} \left( \alpha \frac{\alpha}{2} e^{-ix} \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{ikx} + \frac{2 - \alpha}{2} \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{-ikx} + \frac{\alpha}{2} e^{ix} \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{-ikx} + \frac{2 - \alpha}{2} \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{ikx} \right) \\
& = \frac{\alpha}{4} \left( (e^{ix}(1 - e^{ix})^{\alpha} + e^{ix}(1 - e^{-ix})^{\alpha}) + \frac{2 - \alpha}{4} (1 - e^{ix})^{\alpha} + (1 - e^{-ix})^{\alpha} \right).
\end{align*}
\]

Next we check \( f(\alpha; x) \leq 0 \) for \( 1 < \alpha \leq 2 \). Since \( f(\alpha; x) \) is a real-valued and even function, we just consider its principal value on \([0, \pi]\). By the formula

\[
e^{i\theta} - e^{i\phi} = 2i \sin \left(\frac{\theta - \phi}{2}\right) e^{i(\theta+\phi)/2},
\]

we obtain

\[
\begin{align*}
f(\alpha; x) & = (2 \sin \left(\frac{x}{2}\right))^{\alpha} \left( \frac{\alpha}{2} \cos \left(\frac{\alpha}{2} (x - \pi) - x \right) + \frac{2 - \alpha}{2} \cos \left(\frac{\alpha}{2} (x - \pi) \right) \right). 
\end{align*}
\]

It is easy to prove that \( f(\alpha; x) \) decreases with respect to \( \alpha \), then \( f(\alpha; x) \leq f(1; x) \equiv 0 \); by Lemma 4 and 5, \( \text{Re}(\lambda) \equiv 0 \) for \( \alpha = 1 \), and \( f(\alpha; x) \) is not identically zero for \( 1 < \alpha \leq 2 \), then we get \( \text{Re}(\lambda) < 0 \).

Case \((p, q) = (1, -1)\): the corresponding generating function \( f(\alpha; x) \) of \( \frac{A + A^T}{2} \) can be calculated in the following form with coefficients \( w_k^{(\alpha)} \) given by (2.17),

\[
\begin{align*}
f(\alpha; x) & = \frac{1}{2} \left( \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{i(k-1)x} + \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{-i(k-1)x} \right) \\
& = \frac{2 + \alpha}{8} \left( e^{-ix} \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{ikx} + e^{ix} \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{-ikx} \right) + \frac{2 - \alpha}{8} \left( e^{ix} \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{ikx} + e^{-ix} \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{-ikx} \right) \\
& = \frac{2 + \alpha}{8} \left( e^{-ix}(1 - e^{ix})^{\alpha} + e^{ix}(1 - e^{-ix})^{\alpha} \right) + \frac{2 - \alpha}{8} \left( e^{ix}(1 - e^{ix})^{\alpha} + e^{-ix}(1 - e^{-ix})^{\alpha} \right).
\end{align*}
\]

Next we check \( f(\alpha; x) \leq 0 \) for \( 1 < \alpha \leq 2 \). Since \( f(\alpha; x) \) is a real-valued and even function, we just consider its principal value on \([0, \pi]\). By simple calculation, we obtain

\[
\begin{align*}
f(\alpha; x) & = (2 \sin \left(\frac{x}{2}\right))^{\alpha} \left( \frac{\alpha}{2} \sin \left(\frac{\alpha}{2} (x - \pi) \right) \sin(x) + \cos \left(\frac{\alpha}{2} (x - \pi) \right) \cos(x) \right) \\
& = (2 \sin \left(\frac{x}{2}\right))^{\alpha} \left( \frac{\alpha}{2} \sin \left(\frac{\alpha}{2} (x - \pi) \right) \sin(x) + \cos \left(\frac{\alpha}{2} (x - \pi) \right) \cos(x) \right). 
\end{align*}
\]

We can also check that \( f(\alpha; x) \) decreases with respect to \( \alpha \), then \( f(\alpha; x) \leq f(1; x) = -2 \sin^4 \left(\frac{x}{2}\right) \leq 0 \), then by Lemma 4 and 5, we get \( \text{Re}(\lambda) < 0 \) for \( 1 \leq \alpha \leq 2 \).

From the above discussions and Lemma 5, we know, for \( 1 < \alpha \leq 2 \), the matrix \( \frac{1}{2}(A + A^T) \) is negative definite, which implies matrix \( A \) is negative definite by Lemma 3. And the symmetric part of matrix \( c_1 A + c_2 A^T \) is \( \frac{c_1 + c_2}{2}(A + A^T) \), thus we obtain \( \text{Re}(\lambda(c_1 A + c_2 A^T)) < 0 \) for \( 1 < \alpha \leq 2 \).

Remark 4. For the case \((p, q) = (1, 0)\) and \(1 < \alpha \leq 2\), we can check that the symmetric part \( H \) of matrix \( A \) in (2.20) is strictly diagonally dominant by using Lemma 2, and the elements of the main diagonal of \( H \) are negative, then the eigenvalues of \( H \) are less than zero by the Gershgorin circle theorem ([20],P188), therefore, with Lemma 3 and 4, we can also get \( \text{Re}(\lambda(A)) < 0 \), and \( A \) is negative positive.

Remark 5. By the same approach described in Theorem 2, we can verify that the generating function of the symmetric part of difference matrix for \((p, q) = (0, -1)\) is not identically negative when \( 1 < \alpha \leq 2 \), which leads to the instability of the difference method to fractional diffusion equations for the same reason in the stability analysis in the sequel.
2.2 Third Order Approximations

Similar to the second order approximations for Riemann-Liouville fractional derivatives, we give a combination of three shifted Grünwald difference operators

\[ L G_{h,p,q,r}^\alpha u(x) = \lambda_1 A_{h,p}^\alpha u(x) + \lambda_2 A_{h,q}^\alpha u(x) + \lambda_3 A_{h,r}^\alpha u(x), \]  

(2.23)

where \( p, q, r \) are integers and mutually non-equal, and

\[ \lambda_1 = \frac{12pq - (6p + 6r + 1)\alpha + 3\alpha^2}{12(qr - pq - pr + p^2)}, \]
\[ \lambda_2 = \frac{12pr - (6p + 6r + 1)\alpha + 3\alpha^2}{12(pr - pq - qr + q^2)}, \]
\[ \lambda_3 = \frac{12pq - (6p + 6r + 1)\alpha + 3\alpha^2}{12(pq - pr - qr + r^2)}. \]

(2.24)

Assuming \( u \in L^1(\mathbb{R}) \), and taking Fourier transform on (2.23), we get

\[ \mathcal{F}[L G_{h,p,q,r}^\alpha u](\omega) = (i\omega)^\alpha (\lambda_1 W_p(i\omega h) + \lambda_2 W_q(i\omega h) + \lambda_3 W_r(i\omega h)) \hat{u}(\omega) = (i\omega)^\alpha \left(1 + C(i\omega h)^3\right) \hat{u}(\omega), \]

(2.25)

where \( W_s(z) \) is defined in (2.10). If \( -\infty D_x^{\alpha+3} u \) and its Fourier transform belong to \( L^1(\mathbb{R}) \), then we have

\[ |L G_{h,p,q,r}^\alpha u - -\infty D_x^{\alpha} u| \leq \frac{1}{2\pi} \int_\mathbb{R} |\mathcal{F}[L G_{h,p,q,r}^\alpha u - -\infty D_x^{\alpha} u]| \leq C \|\mathcal{F}[-\infty D_x^{\alpha+3} u](\omega)\|_{L^1} h^3 = O(h^3). \]

(2.26)

The above results can be stated in the following theorem.

**Theorem 3.** Let \( u \in L^1(\mathbb{R}) \), \( -\infty D_x^{\alpha+3} u \) and its Fourier transform belong to \( L^1(\mathbb{R}) \), and the following 3-WSGD operator (2.23) satisfies

\[ L G_{h,p,q,r}^\alpha u(x) = -\infty D_x^{\alpha} u(x) + O(h^3), \]

uniformly for \( x \in \mathbb{R} \).

If \( u \in L^1(\mathbb{R}) \), \( -\infty D_x^{\alpha+3} u \) and its Fourier transform belong to \( L^1(\mathbb{R}) \), we also have

\[ R G_{h,p,q,r}^\alpha u(x) = \lambda_1 B_{h,p}^\alpha u(x) + \lambda_2 B_{h,q}^\alpha u(x) + \lambda_3 B_{h,r}^\alpha u(x) = -\infty D_x^{\alpha} u + O(h^3), \]

(2.28)

uniformly for \( x \in \mathbb{R} \), where the operator \( B_{h,s}^\alpha \) is given by (2.14), and \( \lambda_i, i = 1, 2, 3 \) are the same as (2.24).

As stated in Remark 3, the 3-WSGD operator can be utilized for approximating Riemann-Liouville fractional differential equations on bounded domain by finite difference method when choosing \( (p, q, r) = (1, 0, -1) \), then the corresponding weight coefficients in (2.24) are \( \lambda_1 = \frac{7}{24}\alpha + \frac{3}{8}\alpha^2 \), \( \lambda_2 = 1 + \frac{1}{12}\alpha - \frac{1}{4}\alpha^2 \), \( \lambda_3 = -\frac{7}{24}\alpha + \frac{1}{8}\alpha^2 \). For function \( u(x) \) satisfying \( u(a) = u(b) = 0 \) on grid points \( \{x_k = a + kh, h = \frac{b-a}{N}\} \).
\( (b - a)/n, k = 1, \ldots, n - 1 \), the approximation matrix of (2.23) with \((p, q, r) = (1, 0, -1)\) is

\[
G = \lambda_1 \begin{pmatrix}
g_1^{(a)} & g_0^{(a)} \\
g_2^{(a)} & g_1^{(a)} \\
\vdots & \vdots \\
g_{n-2}^{(a)} & g_{n-2}^{(a)} \\
g_{n-1}^{(a)} & g_{n-1}^{(a)}
\end{pmatrix} + \lambda_2 \begin{pmatrix}
g_0^{(a)} & g_1^{(a)} \\
g_1^{(a)} & g_0^{(a)} \\
\vdots & \vdots \\
g_{n-3}^{(a)} & g_{n-3}^{(a)} \\
g_{n-2}^{(a)} & g_{n-2}^{(a)}
\end{pmatrix} + \lambda_3 \begin{pmatrix}
g_0^{(a)} & 0 \\
\vdots & \vdots \\
g_4^{(a)} & 0 \\
g_{n-3}^{(a)} & g_{n-3}^{(a)} \\
g_{n-4}^{(a)} & g_{n-4}^{(a)}
\end{pmatrix}.
\]

**Example 1.** We utilize the approximation (2.23) for simulating the steady state fractional diffusion problem

\[-D_x^\alpha u(x) = -\frac{\Gamma(3 + \alpha)}{2}x^2, \quad x \in (0, 1),\]

with \(u(0) = 0\), \(u(1) = 1\), and \(1 < \alpha < 2\). The exact solution is \(u(x) = x^{2+\alpha}\).

The 3-WSGD operator with \((p, q, r) = (1, 0, -1)\) is utilized for computing the solution of Example 1, the numerical results are given in Table 1, from which the order and accuracy of the 3-WSGD operator is verified. As we do in the above, the generating function of the symmetric part \(\mathcal{G}_{\frac{G+GT}{2}}\) of the Topelitz matrix

<table>
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<tr>
<th>(N)</th>
<th>(|u^n - U^n|_\infty)</th>
<th>rate</th>
<th>(|u^n - U^n|)</th>
<th>rate</th>
<th>(|u^n - U^n|_\infty)</th>
<th>rate</th>
<th>(|u^n - U^n|)</th>
<th>rate</th>
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<td>-</td>
<td>5.92003E-04</td>
<td>-</td>
<td>3.20333E-04</td>
<td>-</td>
<td>1.59788E-04</td>
<td>-</td>
</tr>
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<td>7.51799E-05</td>
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<td>2.29262E-05</td>
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<td>1.04858E-05</td>
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<td>1.18999E-06</td>
<td>2.99</td>
<td>1.07818E-07</td>
<td>3.88</td>
<td>4.24776E-08</td>
<td>3.98</td>
</tr>
<tr>
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<tr>
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<td>1.86501E-08</td>
<td>3.00</td>
<td>4.89318E-10</td>
<td>3.89</td>
<td>1.67325E-10</td>
<td>4.00</td>
</tr>
</tbody>
</table>

\(G\) is

\[f(\alpha; x) = \left(\frac{5}{48} + \frac{1}{16} \alpha^2\right) \left(e^{-ix}(1 - e^{ix})^\alpha + e^{ix}(1 - e^{-ix})^\alpha\right)\]

\(+ \left(\frac{1}{2} + \frac{1}{24} \alpha - \frac{1}{8} \alpha^2\right) \left((1 - e^{ix})^\alpha + (1 - e^{-ix})^\alpha\right)\]

\(+ \left(-\frac{7}{48} \alpha + \frac{1}{16} \alpha^2\right) \left(e^{ix}(1 - e^{ix})^\alpha + e^{-ix}(1 - e^{-ix})^\alpha\right),\]

\(x \in [-\pi, \pi]\). As matrix \(\mathcal{G}_{\frac{G+GT}{2}}\) is symmetric, thus \(f(\alpha; x)\) is a real-valued and even function, so we consider it on \([0, \pi]\) and get

\[f(\alpha; x) = \left(2 \sin\left(\frac{x}{2}\right)^2\right) \left(\frac{5}{48} \alpha + \frac{1}{16} \alpha^2\right) \cos\left(\frac{\alpha}{2}(x - \pi) - x\right) + \left(\frac{1}{2} + \frac{1}{24} \alpha - \frac{1}{8} \alpha^2\right) \cos\left(\frac{\alpha}{2}(x - \pi)\right)\]

\(+ \left(-\frac{7}{48} \alpha + \frac{1}{16} \alpha^2\right) \cos\left(\frac{\alpha}{2}(x - \pi) + x\right)\).
We can check that $f(\alpha; x)$ is not identically positive or negative for $1 \leq \alpha \leq 2$, and the real parts of the eigenvalues of matrix $G$ are not always negative, so the finite difference scheme using (2.23) or (2.28) for time dependent fractional problems will not be unconditionally stable.

3 One Dimensional Space Fractional Diffusion Equation

In this section, we consider the following two-sided one dimensional space fractional diffusion equation

\[
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= K_1 a D_x^\alpha u(x,t) + K_2 x D_x^\alpha u(x,t) + f(x,t), \quad (x,t) \in (a,b) \times (0,T], \\
u(x,0) &= u_0(x), \quad x \in [a,b], \\
u(a,t) = \phi_a(t), \quad u(b,t) = \phi_b(t), \quad t \in [0,T],
\end{aligned}
\]

where both $a D_x^\alpha$ and $x D_x^\alpha$ are Riemann-Liouville fractional operators with $1 < \alpha \leq 2$. The diffusion coefficients $K_1$ and $K_2$ are nonnegative constants with $K_1^2 + K_2^2 \neq 0$. And if $K_1 \neq 0$, then $\phi_a(t) \equiv 0$; if $K_2 \neq 0$, then $\phi_b(t) \equiv 0$. Next we will discretize the problem (3.1) by the second order accurate WSGD formula (2.16). In the analysis of the numerical method that follows, we assume that (3.1) has a unique and sufficiently smooth solution.

3.1 CN-WSGD scheme

We partition the interval $[a,b]$ into a uniform mesh with the space step $h = (b - a)/N$ and the time step $\tau = T/M$, where $N, M$ being two positive integers. And the set of grid points are denoted by $x_i = ih$ and $t_n = n\tau$ for $1 \leq i \leq N$ and $0 \leq n \leq M$. Let $t_{n+1/2} = (t_n + t_{n+1})/2$ for $0 \leq n \leq M - 1$, and we use the following notations

\[
u_i^n = u(x_i, t_n), \quad f_i^{n+1/2} = f(x_i, t_{n+1/2}), \quad \delta_t u_i^n = (u_i^{n+1} - u_i^n)/\tau.
\]

Using the Crank-Nicolson technique for the time discretization of (3.1) leads to

\[
\delta_t u_i^n - \frac{1}{2} \left( K_1 (a D_x^\alpha u_i^n) + K_1 (a D_x^\alpha u_i^{n+1}) + K_2 (x D_x^\alpha u_i^n) + K_2 (x D_x^\alpha u_i^{n+1}) \right) = f_i^{n+1/2} + O(\tau^2).
\]

In space discretization, we choose the WSGD operators $L D_{h,p,q}^\alpha u(x,t)$ and $R D_{h,p,q}^\alpha u(x,t)$ to approximate the Riemann-Liouville fractional derivatives $a D_x^\alpha u(x,t)$ and $x D_x^\alpha u(x,t)$ with second order accuracy, respectively, and $(p, q) = (1, 0)$ or $(1, -1)$. This implies that

\[
\delta_t u_i^n - \frac{1}{2} \left( K_1 L D_{h,p,q}^\alpha u_i^n + K_1 L D_{h,p,q}^\alpha u_i^{n+1} + K_2 R D_{h,p,q}^\alpha u_i^n + K_2 R D_{h,p,q}^\alpha u_i^{n+1} \right) = f_i^{n+1/2} + \varepsilon_i^n,
\]

where

\[
|\varepsilon_i^n| \leq \hat{c}(\tau^2 + h^2).
\]

Multiplying (3.2) by $\tau$ and separating the time layers, we have

\[
u_i^{n+1} = \frac{K_1 \tau}{2} L D_{h,p,q}^\alpha u_i^{n+1} - \frac{K_2 \tau}{2} R D_{h,p,q}^\alpha u_i^{n+1}
\]

\[
u_i^n = \frac{K_1 \tau}{2} L D_{h,p,q}^\alpha u_i^n + \frac{K_2 \tau}{2} R D_{h,p,q}^\alpha u_i^n + \tau f_i^{n+1/2} + O(\tau^3 + \tau h^2).
\]
Substituting \( L D^\alpha_{h,p,q} u, R D^\alpha_{h,p,q} u \) by (2.16), we obtain that
\[
\begin{aligned}
\quad & u^{n+1}_i - \frac{K_1 \tau}{2h^\alpha} \sum_{k=0}^{i+1} w^{(\alpha)}_k u^{n+1}_{i-k+1} - \frac{K_2 \tau}{2h^\alpha} \sum_{k=0}^{N-i+1} w^{(\alpha)}_k u^{n+1}_{i+k-1} \\
= & u^n_i + \frac{K_1 \tau}{2h^\alpha} \sum_{k=0}^{i+1} w^{(\alpha)}_k u^n_{i-k+1} + \frac{K_2 \tau}{2h^\alpha} \sum_{k=0}^{N-i+1} w^{(\alpha)}_k u^n_{i+k-1} + \tau f^{n+1/2}_i + O(\tau^3 + \tau h^2).
\end{aligned}
\tag{3.5}
\]

Denoting \( U^n_i \) as the numerical approximation of \( u^n_i \), we derive the CN-WSGD scheme for (3.1)
\[
\begin{aligned}
\quad & U^{n+1}_i - \frac{K_1 \tau}{2h^\alpha} \sum_{k=0}^{i+1} w^{(\alpha)}_k U^{n+1}_{i-k+1} - \frac{K_2 \tau}{2h^\alpha} \sum_{k=0}^{N-i+1} w^{(\alpha)}_k U^{n+1}_{i+k-1} \\
= & U^n_i + \frac{K_1 \tau}{2h^\alpha} \sum_{k=0}^{i+1} w^{(\alpha)}_k U^n_{i-k+1} + \frac{K_2 \tau}{2h^\alpha} \sum_{k=0}^{N-i+1} w^{(\alpha)}_k U^n_{i+k-1} + \tau f^{n+1/2}_i.
\end{aligned}
\tag{3.6}
\]

For the convenience of implementation, using the matrix form of the grid functions
\[
U^n = \begin{pmatrix} U^n_1, U^n_2, \cdots, U^n_{N-1} \end{pmatrix}^T, \quad F^n = \begin{pmatrix} f^{n+1/2}_1, f^{n+1/2}_2, \cdots, f^{n+1/2}_{N-1} \end{pmatrix}^T,
\]
makes the finite difference scheme (3.6) be described as
\[
\begin{pmatrix} I - \frac{\tau}{2h^\alpha}(K_1 A + K_2 A^T) \end{pmatrix} U^{n+1} = \begin{pmatrix} I + \frac{\tau}{2h^\alpha}(K_1 A + K_2 A^T) \end{pmatrix} U^n + \tau F^n + H^n,
\tag{3.7}
\]
where \( A \) is given by (2.20) and
\[
H^n = \frac{\tau}{2h^\alpha} \begin{bmatrix} K_1 w^{(\alpha)}_2 + K_2 w^{(\alpha)}_0 \\ K_1 w^{(\alpha)}_3 \\ \vdots \\ K_1 w^{(\alpha)}_{N-1} \\ K_1 w^{(\alpha)}_N \end{bmatrix} \begin{bmatrix} U^n_0 + U^{n+1}_0 \\ U^n_1 \\ \vdots \\ U^n_{N-1} \end{bmatrix}
\begin{bmatrix} K_2 w^{(\alpha)}_N \\ K_2 w^{(\alpha)}_{N-1} \\ \vdots \\ K_2 w^{(\alpha)}_1 \\ K_1 w^{(\alpha)}_0 + K_2 w^{(\alpha)}_2 \end{bmatrix}.
\tag{3.8}
\]

### 3.2 Stability and Convergence

Now we consider the stability and convergence analysis for the CN-WSGD scheme (3.7). Define
\[
V_h = \{ v : v = \{ v_i \} \text{ is a grid function in } \{ x_i = ih \}_{i=1}^{N-1} \text{ and } v_0 = v_N = 0 \}.
\]

For any \( v = \{ v_i \} \in V_h \), we define its pointwise maximum norm
\[
\| v \|_\infty = \max_{1 \leq i \leq N-1} | v_i | \tag{3.9}
\]
and the following discrete norm
\[
\| v \| = \sqrt{h \sum_{i=1}^{N-1} v_i^2}.
\]

**Theorem 4.** The finite difference scheme (3.6) is unconditionally stable.
Proof. Denoting $B = \frac{\tau}{2h^\alpha}(K_1 A + K_2 A^T)$. The matrix form of the difference approximation for problem (3.1) can be rewritten as

$$(I - B)U^{n+1} = (I + B)U^n + \tau F^n + H^n. \quad (3.10)$$

If denote $\lambda$ as an eigenvalue of matrix $B$, then $\frac{1 + \lambda}{1 - \lambda}$ is the eigenvalue of matrix $(I - B)^{-1}(I + B)$. The result of Theorem 2 shows that the eigenvalues of matrix $\frac{B + B^T}{2} = \frac{\tau(K_1 + K_2)}{4h^\alpha}(A + A^T)$ are negative, thus $\text{Re}(\lambda) < 0$, which implies that $\frac{1 + \lambda}{1 - \lambda} < 1$. Therefore, the spectral radius of matrix $(I - B)^{-1}(I + B)$ is less than one, then the discreted scheme (3.6) is unconditionally stable. \hfill \Box

Remark 6. Considering the $\theta$ weighted scheme for the time discretization of (3.1), then the iterative matrix of the full discrete scheme is

$$(I - \theta B)^{-1}(I + (1 - \theta)B), \quad (3.11)$$

if $\lambda$ is an eigenvalue of matrix $B$, then the eigenvalue of (3.11) is $\frac{1 + (1 - \theta)\lambda}{1 - \theta\lambda}$. As $\text{Re}(\lambda) < 0$, it is easy to check that

$$\left| \frac{1 + (1 - \theta)\lambda}{1 - \theta\lambda} \right| < 1 \quad (3.12)$$

for $\frac{1}{2} \leq \theta \leq 1$. Then the $\theta$ weighted WSGD scheme for (3.1) is unconditionally stable when $\frac{1}{2} \leq \theta \leq 1$.

Before verifying the unconditional convergence of the scheme (3.6), we first present the following several auxiliary lemmas.

Lemma 6 ([19]). Assume that $\{k_n\}$ and $\{p_n\}$ are nonnegative sequences, and the sequence $\{\phi_n\}$ satisfies

$$\phi_0 \leq g_0, \quad \phi_n \leq g_0 + \sum_{l=0}^{n-1} p_l + \sum_{l=0}^{n-1} k_l \phi_l, \quad n \geq 1,$$

where $g_0 \geq 0$. Then the sequence $\{\phi_n\}$ satisfies

$$\phi_n \leq \left(g_0 + \sum_{l=0}^{n-1} p_l\right) \exp\left(\sum_{l=0}^{n-1} k_l\right), \quad n \geq 1. \quad (3.13)$$

Theorem 5. Let $u^n_i$ be the exact solution of problem (3.1), and $U^n_i$ the solution of the finite difference scheme (3.6), then for all $1 \leq n \leq M$, we have

$$\|u^n - U^n\| \leq c(\tau^2 + h^2), \quad (3.14)$$

where $c$ denotes a positive constant and $\| \cdot \|$ stands for the discrete $L^2$-norm.

Proof. Let $e^n_i = u^n_i - U^n_i$, and from (3.5) and (3.6) we have

$$(e^{n+1} - e^n) - \frac{K_1 \tau}{2h^\alpha} A(e^{n+1} + e^n) - \frac{K_2 \tau}{2h^\alpha} A^T(e^{n+1} + e^n) = \tau e^n, \quad (3.15)$$

where

$$e^n = \begin{pmatrix} u^n_1 - U^n_1, u^n_2 - U^n_2, \ldots, u^n_{N-1} - U^n_{N-1} \end{pmatrix}^T, \quad e^n = \begin{pmatrix} e^n_1, e^n_2, \ldots, e^n_{N-1} \end{pmatrix}^T.$$
Multiplying (3.15) by \( h \), and acting \((e^{n+1} + e^n)^T\) on both sides, we obtain that
\[
h(e^{n+1} + e^n)^T I(e^{n+1} - e^n) - \frac{K_1}{2h^{a-1}}(e^{n+1} + e^n)^T A(e^{n+1} + e^n) \\
- \frac{K_2}{2h^{a-1}}(e^{n+1} + e^n)^T A^T(e^{n+1} + e^n) = \tau h(e^{n+1} + e^n)^T e^n.
\]

By Theorem 2, \( A \) and its transpose \( A^T \) both being the negative definite matrices, we get
\[
(e^{n+1} + e^n)^T A(e^{n+1} + e^n) < 0, \quad (e^{n+1} + e^n)^T A^T(e^{n+1} + e^n) < 0.
\]  
(3.16)

And it yields that
\[
\frac{1}{\tau} h \sum_{i=1}^{N-1} ((e_i^{n+1})^2 - (e_i^n)^2) \leq \tau h \sum_{i=1}^{N-1} (e_i^{n+1} + e_i^n) e_i^n.
\]  
(3.17)

Summing up for all \( 0 \leq k \leq n - 1 \), we have
\[
\frac{N-1}{2} \sum_{i=1}^{N-1} (e_i^n)^2 \leq \tau h \sum_{i=1}^{N-1} (e_i^{n+1} + e_i^n) e_i^n = \tau h \sum_{i=1}^{N-1} (e_i^{n+1} + e_i^n) e_i^{n-1} + \tau h \sum_{i=1}^{N-1} e_i^n e_i^{n-1} \\
\leq \tau h \sum_{i=1}^{N-1} (e_i^n)^2 + \frac{\tau h}{2} \sum_{i=1}^{N-1} \sum_{k=1}^{n-1} (e_i^{k-1} + e_i^k)^2 + \frac{h}{2} \sum_{i=1}^{N-1} (\tau e_i^{n-1})^2.
\]  
(3.18)

By noting that \( e_i^n \leq c(\tau^2 + h^2) \), and utilizing the discrete Gronwall’s inequality, we obtain that
\[
\|e^n\|^2 \leq \tau \sum_{k=1}^{n-1} \|e^k\|^2 + (c(\tau^2 + h^2))^2 \leq \exp(T)(c(\tau^2 + h^2))^2 \leq c((\tau^2 + h^2)^2),
\]  
(3.19)

which is the result that we need.

4 Two Dimensional Space Fractional Diffusion Equation

We next consider the following two-sided space fractional diffusion equation in two dimensions
\[
\left\{ \begin{array}{l}
\frac{\partial u(x,y,t)}{\partial t} = \left( K^+_1 \cdot D^\alpha_x u(x,y,t) + K^-_2 \cdot D^\beta_y u(x,y,t) \right) \\
+ \left( K^-_1 \cdot D^\alpha_y u(x,y,t) + K^+_2 \cdot D^\beta_x u(x,y,t) \right) + f(x,y,t), \\
(x,y,t) \in \Omega \times [0,T], \\
u(x,y,0) = u_0(x,y), \\
u(x,y,t) = \varphi(x,y,t), \\
(x,y) \in \Omega, \\
(x,y,t) \in \partial \Omega \times [0,T],
\end{array} \right.
\]  
(4.1)

where \( \Omega = (a,b) \times (c,d) \), \( \alpha D^\alpha_x \), \( \beta D^\beta_y \) and \( \alpha D^\alpha_y \), \( \beta D^\beta_x \) are Riemann-Liouville fractional operators with \( 1 < \alpha, \beta \leq 2 \). The diffusion coefficients satisfy \( K^+_i, K^-_i \geq 0 \), \( i = 1, 2 \), \((K^+_1)^2 + (K^-_1)^2 \neq 0 \) and \((K^+_2)^2 + (K^-_2)^2 \neq 0 \). And the boundary function \( \varphi \) satisfies, if \( K^+_1 \neq 0 \), then \( \varphi(a,y,t) = 0 \); if \( K^-_1 \neq 0 \), then \( \varphi(b,y,t) = 0 \); if \( K^-_2 \neq 0 \), then \( \varphi(x,c,t) = 0 \); if \( K^+_2 \neq 0 \), then \( \varphi(x,d,t) = 0 \). We assume that (4.1) has a unique and sufficiently smooth solution.
4.1 CN-WSGD scheme

Now we establish the Crank-Nicolson difference scheme by using WSGD formula (2.16) for problem (4.1). We partition the domain \( \Omega \) into a uniform mesh with the space steps \( h_x = (b - a)/N_1, h_y = (d - c)/N_2 \) and the time step \( \tau = T/M \), where \( N_1, N_2, M \) being two positive integers. And the set of grid points are denoted by \( x_i = ih_x, y_j = jh_y \) and \( t_n = n\tau \) for \( 1 \leq i \leq N_1, 1 \leq j \leq N_2 \) and \( 0 \leq n \leq M \). Let \( t_{n+1/2} = (t_n + t_{n+1})/2 \) for \( 0 \leq n \leq M - 1 \), and we use the following notations

\[
\delta_t u_{i,j}^n = (u_{i,j}^{n+1} - u_{i,j}^n)/\tau.
\]

Discretizing (4.1) in time direction leads to

\[
\delta_t u_{i,j}^n = \frac{1}{2} \left( K_1^{+}(aD_x^\alpha u)_{i,j}^{n+1} + K_2^{+}(x D_x^\alpha u)_{i,j}^{n+1} + K_1^{-}(c D_y^\beta u)_{i,j}^{n+1} + K_2^{-}(y D_d^\beta u)_{i,j}^{n+1} + K_1^{+}(aD_x^\alpha u)_{i,j}^n + K_2^{+}(x D_x^\alpha u)_{i,j}^n + K_1^{-}(c D_y^\beta u)_{i,j}^n + K_2^{-}(y D_d^\beta u)_{i,j}^n \right) + f_{i,j}^{n+1/2} + O(\tau^2). \tag{4.2}
\]

In space discretization, we choose the WSGD operators \( L D_{h_x,p,q}^\alpha u \), \( R D_{h_x,p,q}^\alpha u \) and \( L D_{h_y,p,q}^\beta u \), \( R D_{h_y,p,q}^\beta u \) to respectively approximate the fractional diffusion terms \( a D_x^\alpha u \), \( x D_x^\alpha u \) and \( c D_y^\beta u \), \( y D_d^\beta u \). And multiplying (4.2) by \( \tau \) and separating the time layers, we have that

\[
\left( 1 - \frac{K_1^{+} \tau}{2} L D_{h_x,p,q}^\alpha - \frac{K_2^{+} \tau}{2} R D_{h_x,p,q}^\alpha - \frac{K_1^{-} \tau}{2} L D_{h_y,p,q}^\beta - \frac{K_2^{-} \tau}{2} R D_{h_y,p,q}^\beta \right) u_{i,j}^{n+1} + \tau f_{i,j}^{n+1/2} + \tau \varepsilon_{i,j}^n, \tag{4.3}
\]

where \( |\varepsilon_{i,j}^n| \leq \bar{c}(\tau^2 + h^2) \) denotes the truncation error. And we denote

\[
\delta_x^\alpha = K_1^{+} L D_{h_x,p,q}^\alpha + K_2^{+} R D_{h_x,p,q}^\alpha, \quad \delta_y^\beta = K_1^{-} L D_{h_y,p,q}^\beta + K_2^{-} R D_{h_y,p,q}^\beta.
\]

Using the Taylor expansion, we have

\[
\frac{\tau^2}{4} \delta_x^\alpha \delta_y^\beta (u_{i,j}^{n+1} - u_{i,j}^n) = \frac{\tau^3}{4} \left( (K_1^{+} aD_x^\alpha + K_2^{+} x D_x^\alpha)(K_1^{-} cD_y^\beta + K_2^{-} y D_d^\beta)u_t \right)_{i,j}^{n+1/2} + O(\tau^5 + \tau^3 h^2). \tag{4.4}
\]

Adding formula (4.4) to the right-hand side of (4.3) and making the factorization leads to

\[
\left( 1 - \frac{\tau}{2} \delta_x^\alpha \right) \left( 1 - \frac{\tau}{2} \delta_y^\beta \right) U_{i,j}^{n+1} = \left( 1 + \frac{\tau}{2} \delta_x^\alpha \right) \left( 1 + \frac{\tau}{2} \delta_y^\beta \right) U_{i,j}^n + \tau f_{i,j}^{n+1/2} + \tau \varepsilon_{i,j}^n + O(\tau^5 + \tau^3 h^2). \tag{4.5}
\]

Denoting by \( U_{i,j}^n \) the numerical approximation to \( u_{i,j}^n \), we obtain the finite difference approximation for problem (4.1)

\[
\left( 1 - \frac{\tau}{2} \delta_x^\alpha \right) \left( 1 - \frac{\tau}{2} \delta_y^\beta \right) U_{i,j}^{n+1} = \left( 1 + \frac{\tau}{2} \delta_x^\alpha \right) \left( 1 + \frac{\tau}{2} \delta_y^\beta \right) U_{i,j}^n + \tau f_{i,j}^{n+1/2}. \tag{4.6}
\]

For efficiently solving (4.6), the following techniques can be used. Peaceman-Rachford ADI scheme [24]:

\[
\begin{align*}
(1 - \frac{\tau}{2} \delta_x^\alpha) V_{i,j}^n & = (1 + \frac{\tau}{2} \delta_y^\beta) U_{i,j}^n + \frac{\tau}{2} f_{i,j}^{n+1/2}, \tag{4.7a} \\
(1 - \frac{\tau}{2} \delta_y^\beta) V_{i,j}^{n+1} & = (1 + \frac{\tau}{2} \delta_x^\alpha) V_{i,j}^n + \frac{\tau}{2} f_{i,j}^{n+1/2}. \tag{4.7b}
\end{align*}
\]
Douglas ADI scheme [7]:
\[
\begin{align*}
(1 - \frac{\tau}{2} \delta_x^n) V_{i,j}^n &= \left(1 + \frac{\tau}{2} \delta_x^n + \tau \delta_y^n\right) U_{i,j}^n + \tau f_{i,j}^{n+1/2}, \\
(1 - \frac{\tau}{2} \delta_y^n) U_{i,j}^{n+1} &= V_{i,j}^n - \frac{\tau}{2} \delta_y^n U_{i,j}^n.
\end{align*}
\]
(4.8a)

D’Yakonov ADI scheme [24]:
\[
\begin{align*}
(1 - \frac{\tau}{2} \delta_x^n) V_{i,j}^n &= \left(1 + \frac{\tau}{2} \delta_x^n\right) \left(1 + \frac{\tau}{2} \delta_y^n\right) U_{i,j}^n + \tau f_{i,j}^{n+1/2}, \\
(1 - \frac{\tau}{2} \delta_y^n) U_{i,j}^{n+1} &= V_{i,j}^n.
\end{align*}
\]
(4.9a)

A simple calculation shows that
\[
\frac{\tau^3}{4} \delta_x^n \delta_y^n f_{i,j}^{n+1/2} = \frac{\tau^3}{4} (K_x^n a D_x^n + K_y^n a D_y^n) (K_x^n c D_x^n + K_y^n c D_y^n) f_{i,j}^{n+1/2} + O(\tau^3 h^2).
\]
(4.10)

Then from (4.5) and (4.10), it yields that
\[
\left(1 - \frac{\tau}{2} \delta_x^n\right) \left(1 - \frac{\tau}{2} \delta_y^n\right) u_{i,j}^{n+1} = \left(1 + \frac{\tau}{2} \delta_x^n\right) \left(1 + \frac{\tau}{2} \delta_y^n\right) u_{i,j}^n + \tau f_{i,j}^{n+1/2} + \frac{\tau^3}{4} \delta_x^n \delta_y^n f_{i,j}^{n+1/2} + \tau \varepsilon_{i,j}^n.
\]
(4.11)

where
\[
\varepsilon_{i,j}^n = \varepsilon_{i,j}^n - \frac{\tau^2}{4} (K_x^n a D_x^n + K_y^n a D_y^n) (K_x^n c D_x^n + K_y^n c D_y^n) f_{i,j}^{n+1/2} + O(\tau^4 + \tau^2 h^2).
\]
(4.12)

Eliminating the truncating error and denoting \( U_{i,j}^n \) as the numerical approximation of \( u_{i,j}^n \), we have
\[
\left(1 - \frac{\tau}{2} \delta_x^n\right) \left(1 - \frac{\tau}{2} \delta_y^n\right) U_{i,j}^{n+1} = \left(1 + \frac{\tau}{2} \delta_x^n\right) \left(1 + \frac{\tau}{2} \delta_y^n\right) U_{i,j}^n + \tau f_{i,j}^{n+1/2} + \frac{\tau^3}{4} \delta_x^n \delta_y^n f_{i,j}^{n+1/2}.
\]
(4.13)

Introducing the intermediate variable \( V_{i,j}^n \), we obtain the locally one-dimensional (LOD) scheme mentioned in [21, 28],
\[
\begin{align*}
(1 - \frac{\tau}{2} \delta_x^n) V_{i,j}^n &= \left(1 + \frac{\tau}{2} \delta_x^n\right) U_{i,j}^n + \frac{\tau}{2} \left(1 + \frac{\tau}{2} \delta_x^n\right) f_{i,j}^{n+1/2}, \\
(1 - \frac{\tau}{2} \delta_y^n) V_{i,j}^{n+1} &= \left(1 + \frac{\tau}{2} \delta_y^n\right) V_{i,j}^n + \frac{\tau}{2} \left(1 - \frac{\tau}{2} \delta_y^n\right) f_{i,j}^{n+1/2}.
\end{align*}
\]
(4.14a)

4.2 Stability and Convergence

Now we consider the stability and convergence analysis for the CN-WSGD scheme (4.6). Define the sets of the index of the interior and boundary mesh grid points in domain \([a, b] \times [c, d]\), respectively, as
\[
\Lambda_h = \{(i, j) : 1 \leq i \leq N_x - 1, 1 \leq j \leq N_y - 1\}, \\
\partial \Lambda_h = \{(i, j) : i = 0, N_x; 0 \leq j \leq N_y\} \cup \{(i, j) : 0 \leq i \leq N_x; j = 0, N_y\}.
\]

For any \( v = \{v_i\} \in V_h \), we define its pointwise maximum norm and discrete \( L^2 \) norm, respectively, as follows
\[
\| v \|_\infty = \max_{(i,j) \in \Lambda_h} |v_{i,j}|, \quad \| v \| = \sqrt{h^2 \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} v_{i,j}^2},
\]
(4.15)

where
\[
V_h = \{ v : v = \{v_{i,j}\} \text{ is a grid function in } \Lambda_h \text{ and } v_{i,j} = 0 \text{ on } \partial \Lambda_h \}.
\]

In the following, we list some properties of Kronecker products of matrices.
Lemma 7 ([11]). Let $A \in \mathbb{R}^{n \times n}$ have eigenvalues $\{\lambda_i\}_{i=1}^n$, and $B \in \mathbb{R}^{m \times m}$ have eigenvalues $\{\mu_j\}_{j=1}^m$. Then the $mn$ eigenvalues of $A \otimes B$, which represents the kronecker product of matrix $A$ and $B$, are
\[ \lambda_1\mu_1, \ldots, \lambda_1\mu_m, \lambda_2\mu_1, \ldots, \lambda_2\mu_m, \ldots, \lambda_n\mu_1, \ldots, \lambda_n\mu_m. \]

Lemma 8 ([11]). Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{r \times s}$, $C \in \mathbb{R}^{n \times p}$, $D \in \mathbb{R}^{s \times t}$. Then
\[ (A \otimes B)(C \otimes D) = AC \otimes BD \quad (\in \mathbb{R}^{mr \times pt}). \]
Moreover, if $A, B \in \mathbb{R}^{n \times n}$, $I$ is a unit matrix of order $n$, then matrices $I \otimes A$ and $B \otimes I$ commute.

Lemma 9 ([11]). For all $A$ and $B$, $(A \otimes B)^T = A^T \otimes B^T$.

Lemma 10 ([29]). Let $A, B$ to be two positive semidefinite matrices, symbolized $A \geq 0$ and $B \geq 0$. Then $A \otimes B \geq 0$.

Theorem 6. The difference scheme (4.6) is unconditionally stable for $1 < \alpha, \beta \leq 2$.

Proof. We represent the discrete functions $U^n_{i,j}$ and $f^{n+1/2}_{i,j}$ into vector forms with
\[ U^n = (u^n_{1,1}, u^n_{2,1}, \ldots, u^n_{1,n-1}, u^n_{2,n-1}, \ldots, u^n_{1,1}, u^n_{2,2}, \ldots, u^n_{1,n-1}, u^n_{2,n-1}, \ldots, u^n_{1,n-1,1}, u^n_{2,n-1,1}, \ldots, u^n_{1,n-1,n-1})^T, \]
\[ F^{n+1/2} = (f^{n+1/2}_{1,1}, f^{n+1/2}_{2,1}, \ldots, f^{n+1/2}_{1,n-1,1}, f^{n+1/2}_{2,n-1,1}, \ldots, f^{n+1/2}_{1,n-1,1}, f^{n+1/2}_{2,n-1,1}, \ldots, f^{n+1/2}_{1,n-1,n-1})^T, \]
and denote
\[ D_x = \frac{K_1^+}{2h^\alpha} I \otimes A_\alpha + \frac{K_2^+}{2h^\alpha} I \otimes A_\alpha^T, \quad D_y = \frac{K_1^-}{2h^\beta} A_\beta \otimes I + \frac{K_2^-}{2h^\beta} A_\beta^T \otimes I, \]
where the symbol $\otimes$ denotes the Kronecker product, $I$ is the unit matrix, and matrices $A_\alpha$ and $A_\beta$ are defined in (2.20) corresponding to $\alpha, \beta$, respectively. Therefore, the difference scheme (4.6) can be expressed as
\[ (I - D_x)(I - D_y)U^{n+1} = (I + D_x)(I + D_y)U^n + \tau F^{n+1/2}, \]
then the relationship between the error $e^{n+1}$ in $U^{n+1}$ and the error $e^n$ in $U^n$ is given by
\[ e^{n+1} = (I - D_y)^{-1}(I - D_x)^{-1}(I + D_x)(I + D_y)e^n. \]
Using Lemma 8, we can check that $D_x$ and $D_y$ commute, i.e.,
\[ D_xD_y = D_yD_x = \frac{\tau^2}{4h^\alpha + h^\beta}(K_1^- A_\beta + K_2^- A_\beta^T) \otimes (K_1^+ A_\alpha + K_2^+ A_\alpha^T). \]
Thus (4.19) can be rewritten as
\[ e^n = ((I - D_y)^{-1}(I + D_y))^n ((I - D_x)^{-1}(I + D_x))^n e^0. \]
We can also calculate the symmetric part of $D_x$ by Lemma 9 as
\[ \frac{D_x + D_x^T}{2} = (K_1^+ + K_2^+)/(2h^\alpha) I \otimes \left( A_\alpha + A_\alpha^T \right)/(2h^\alpha), \quad \frac{D_y + D_y^T}{2} = (K_1^- + K_2^-)/(2h^\beta) I \otimes \left( A_\beta + A_\beta^T \right)/(2h^\beta). \]
And from Theorem 2, the eigenvalues of $A_\alpha + A_\alpha^T$ and $A_\beta + A_\beta^T$ are all negative when $1 < \alpha, \beta \leq 2$. Defining $\lambda_\alpha$ and $\lambda_\beta$ as an eigenvalue of matrices $D_x$ and $D_y$, respectively, then it yields from the consequences of Lemma 4 and 7 that the real parts of $\lambda_\alpha$ and $\lambda_\beta$ are both less than zero. Since $(1 + \lambda_\alpha)/(1 - \lambda_\alpha)$ and $(1 + \lambda_\beta)/(1 - \lambda_\beta)$ are eigenvalues of matrices $(I - D_x)^{-1}(I + D_x)$ and $(I - D_y)^{-1}(I + D_y)$, respectively, thus the spectral radius of each matrix is less than 1, which follows that $((I - D_x)^{-1}(I + D_x))^n$ and $((I - D_y)^{-1}(I + D_y))^n$ converge to zero matrix (see Theorem 1.5 in [20]). Therefore the difference scheme (4.6) is unconditionally stable. \[ \square \]
Remark 7. For the similar reason described in Remark 6 and the proof of Theorem 6, we conclude that the WSGD scheme with $\theta$ weighted scheme for the time discretization for (4.1) is unconditionally stable when $\frac{1}{2} \leq \theta \leq 1$.

Theorem 7. Let $u_{i,j}^n$ be the exact solution of (4.1) with $1 < \alpha, \beta \leq 2$, and $U_{i,j}^n$ the solution of the difference scheme (4.6), then for all $1 \leq n \leq M$, we have

$$
\|u^n - U^n\| \leq c(\tau^2 + h^2),
$$

where $c$ denotes the positive constant and $\|\cdot\|$ stands for the discrete $L^2$-norm.

Proof. Let $e_{i,j}^n = u_{i,j}^n - U_{i,j}^n$, subtracting (4.5) from (4.6) leads to

$$(I - D_x)(I - D_y)e^{n+1} = (I + D_x)(I + D_y)e^n + \tau \mathcal{E}^n,$$

where $D_x$ and $D_y$ are given in (4.17) and

$$
e = (e_{1,1}, e_{2,1}, \cdots, e_{N-1,1}, e_{1,2}, e_{2,2}, \cdots, e_{N-1,2}, \cdots, e_{1,N-1}, e_{2,N-1}, \cdots, e_{N-1,N-1})^T,$$
$$
\mathcal{E} = (e_{1,1}, e_{2,1}, \cdots, e_{N-1,1}, e_{1,2}, e_{2,2}, \cdots, e_{N-1,2}, \cdots, e_{1,N-1}, e_{2,N-1}, \cdots, e_{N-1,N-1})^T.
$$

Separating the time layer, we rewrite (4.23) as

$$(I + D_x D_y)(e^{n+1} - e^n) - (D_x + D_y)(e^{n+1} + e^n) = \tau \mathcal{E}^n,$$

Then by Lemma 9, the symmetric part of $D_x$ is

$$
\frac{D_x + D_x^T}{2} = \frac{(K_1^+ + K_2^+)}{2h^\alpha} I \otimes \left( A_\alpha + A_\alpha^T \right),
$$

thus $D_x$ is negative definite from Lemma 7 and Theorem 2, in the similar way, we can obtain that $D_y$ is negative definite. Using Lemma 8, we calculate that

$$
D_x D_y = D_y D_x = \frac{\tau^2}{4h^{\alpha+\beta}} (K_1^- A_\beta + K_2^- A_\beta^T) \otimes (K_1^+ A_\alpha + K_2^+ A_\alpha^T). \tag{4.25}
$$

Again, by using Lemma 4, Theorem 2 and Lemma 10, it is deduced that $D_x D_y$ is positive definite. Then multiplying row vector $(e^{n+1} + e^n)^T$ on both sides of (4.24), we have

$$
h^2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} (e_{i,j}^n)^2 \leq \tau h^2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \sum_{k=0}^{n-1} (e_{i,j}^{k+1} + e_{i,j}^k) e_{i,j}^k
$$

$$
= \tau h^2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \sum_{k=1}^{n-1} (e_{i,j}^{k+1} + e_{i,j}^k) + \tau h^2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} (e_{i,j}^{n+1} + e_{i,j}^n)
$$

$$
\leq \tau h^2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \sum_{k=1}^{n-1} (e_{i,j}^{k+1} + e_{i,j}^{k-1} + e_{i,j}^k)^2 + \frac{h^2}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} (e_{i,j}^n)^2 + (\tau e_{i,j}^{n-1})^2.
$$

It shows that

$$
\|e^n\|^2 \leq \tau \sum_{k=1}^{n-1} \|e^k\|^2 + \tau \sum_{k=1}^{n-1} \|e^{k-1}\|^2 + \tau^2 \|e^{n-1}\|^2. \tag{4.26}
$$

By noticing that $e_{i,j}^n \leq \tilde{e}(\tau^2 + h^2)$, and utilizing the discrete Gronwall’s inequality, we obtain that

$$
\|e^n\|^2 \leq \exp(T)\left( \tilde{e}(\tau^2 + h^2) \right) \leq c \left( (\tau^2 + h^2)^2 \right). \tag{4.27}
$$

Then we finish the argumentation by deriving that $\|e^n\| \leq c(\tau^2 + h^2)$. \hfill \Box

The convergence result for scheme (4.13) can also be obtained by the similar way as above.
5 Numerical Examples

5.1 One Dimensional Case

Example 2. Consider the following problem
\[ \frac{\partial u(x,t)}{\partial t} = 0 D_\alpha^\alpha u(x,t) - e^{-t} (x^{1+\alpha} + \Gamma(2 + \alpha) x), \quad (x,t) \in (0,1) \times (0,1), \]  
with the boundary conditions
\[ u(0,t) = 0, \quad u(1,t) = e^{-t}, \quad t \in [0,1], \]
and initial value
\[ u(x,0) = x^{1+\alpha}, \quad x \in [0,1]. \]
Then the exact solution of (5.1) is \( u(x,t) = e^{-t} x^{1+\alpha}. \)

Example 3. Consider the following problem
\[ \frac{\partial u(x,t)}{\partial t} = 0 D_\alpha^\alpha u(x,t) - D_\alpha^\alpha u(x,t) + f(x,t), \quad (x,t) \in (0,1) \times (0,1), \]  
\[ u(0,t) = u(1,t) = 0, \quad t \in [0,1], \]
\[ u(x,0) = x^3(1-x)^3, \quad x \in [0,1], \]  
\[ (p,q) = (1,0) \quad (p,q) = (1,-1) \]
\[ \alpha \quad N \quad \|u^n - U^n\|_\infty \quad \text{rate} \quad \|u^n - U^n\|_\infty \quad \text{rate} \quad \|u^n - U^n\|_\infty \quad \text{rate} \]
\begin{align*}
32 & 1.54190E-05 & 2.11 & 8.91288E-06 & 2.02 & 2.28231E-04 & 1.99 & 1.69497E-05 & 2.54 & \\
64 & 3.59204E-06 & 2.10 & 2.20864E-06 & 2.01 & 5.54535E-05 & 2.04 & 3.18905E-06 & 2.41 & \\
128 & 8.38779E-07 & 2.10 & 5.50064E-07 & 2.01 & 1.3272E-05 & 2.07 & 6.62381E-07 & 2.27 & \\
256 & 2.07953E-07 & 2.01 & 1.37309E-07 & 2.00 & 3.12360E-06 & 2.08 & 1.49541E-07 & 2.15 & \\
512 & 5.19919E-08 & 2.00 & 3.43071E-08 & 2.00 & 7.33195E-07 & 2.09 & 3.5944E-07 & 2.07 & \\
1.5 & \quad 16 & 6.17157E-05 & - & 8.80121E-06 & - & 3.88221E-04 & - & 3.91200E-05 & - \\
32 & 1.25568E-05 & 2.30 & 2.30799E-06 & 1.93 & 7.85748E-05 & 2.30 & 5.04830E-06 & 2.95 & \\
64 & 2.47412E-06 & 2.34 & 6.07043E-07 & 1.93 & 1.54572E-05 & 2.35 & 7.43659E-07 & 2.76 & \\
128 & 4.76404E-07 & 2.38 & 1.56527E-07 & 1.96 & 2.97507E-06 & 2.38 & 1.49956E-07 & 2.31 & \\
256 & 9.01282E-08 & 2.40 & 3.97926E-08 & 1.98 & 5.62846E-07 & 2.40 & 3.72282E-08 & 2.01 & \\
512 & 1.93161E-08 & 2.22 & 1.00351E-08 & 1.99 & 1.05033E-07 & 2.42 & 9.60334E-09 & 1.95 & \\
1.9 & \quad 16 & 1.63058E-05 & - & 2.27814E-06 & - & 6.02603E-05 & - & 7.78084E-06 & - \\
64 & 4.93027E-07 & 2.34 & 1.81207E-07 & 1.84 & 1.35841E-06 & 2.76 & 1.49823E-07 & 2.59 & \\
128 & 9.01282E-08 & 2.38 & 4.81995E-08 & 1.91 & 1.94436E-07 & 2.80 & 4.02142E-08 & 1.90 & \\
256 & 3.23580E-08 & 1.98 & 1.24022E-08 & 1.96 & 3.06615E-08 & 1.99 & 1.12278E-08 & 1.84 & \\
512 & 8.15631E-09 & 1.99 & 3.14892E-09 & 1.98 & 7.93775E-09 & 1.95 & 2.99226E-09 & 1.91 & \\
\end{align*}
with the source term
\[
f(x, t) = -e^{-t} \left( x^3(1 - x)^3 + \frac{\Gamma(4)}{\Gamma(4 - \alpha)} (x^3 - (1 - x)^3) - 3 \frac{\Gamma(5)}{\Gamma(5 - \alpha)} (x^4 - (1 - x)^4) 
+ 3 \frac{\Gamma(6)}{\Gamma(6 - \alpha)} (x^5 - (1 - x)^5) - \frac{\Gamma(7)}{\Gamma(7 - \alpha)} (x^6 - (1 - x)^6) \right). 
\]

By simple evaluation, the exact solution of (5.2) is \( u(x, t) = e^{-t} x^3(1 - x)^3 \).

Table 3: The maximum and \( L^2 \) errors and their convergence rates to Example 3 approximated by the CN-WSGD scheme at \( t = 1 \) for different \( \alpha \) with \( \tau = h \).

| \( \alpha \) | \( N \) | \( ||u^n - U^n||_{\infty} \) | rate | \( ||u^n - U^n|| \) | rate | \( ||u^n - U^n||_{\infty} \) | rate | \( ||u^n - U^n|| \) | rate |
|---|---|---|---|---|---|---|---|---|---|
| 1.1 | 16 | 1.21351E-04 | - | 6.87244E-05 | - | 1.04202E-04 | - | 5.49761E-05 | - |
| | 32 | 3.10400E-05 | 1.97 | 1.75798E-05 | 1.97 | 4.32767E-05 | 1.27 | 2.00595E-05 | 1.45 |
| | 64 | 7.93983E-06 | 1.97 | 4.47207E-06 | 1.97 | 1.48399E-05 | 1.54 | 7.42486E-06 | 1.43 |
| | 128 | 2.01674E-06 | 1.98 | 1.12995E-06 | 1.98 | 4.19788E-06 | 1.82 | 2.23601E-06 | 1.73 |
| | 256 | 5.08051E-07 | 1.99 | 2.84150E-07 | 1.99 | 1.10967E-06 | 1.92 | 6.11319E-07 | 1.87 |
| | 512 | 1.27511E-07 | 1.99 | 7.12580E-08 | 2.00 | 2.84897E-07 | 1.96 | 1.59692E-07 | 1.94 |
| 1.5 | 16 | 2.03009E-04 | - | 5.46438E-05 | - | 2.99388E-04 | - | 8.57787E-05 | - |
| | 32 | 4.52559E-05 | 2.17 | 1.37190E-05 | 1.99 | 7.90624E-05 | 1.92 | 2.31127E-05 | 1.89 |
| | 64 | 1.31225E-05 | 2.00 | 3.45401E-06 | 1.99 | 2.01438E-05 | 1.97 | 6.01008E-06 | 1.94 |
| | 128 | 2.83579E-06 | 2.00 | 8.67756E-07 | 1.99 | 5.08147E-06 | 1.99 | 1.53528E-06 | 1.97 |
| | 256 | 7.09655E-07 | 2.00 | 2.17555E-07 | 1.99 | 1.27542E-06 | 1.99 | 3.88274E-07 | 1.98 |
| | 512 | 1.77509E-07 | 2.00 | 5.44715E-08 | 2.00 | 3.19447E-07 | 2.00 | 9.76542E-08 | 1.99 |
| 1.9 | 16 | 2.02959E-04 | - | 3.60448E-05 | - | 2.35899E-04 | - | 4.37067E-05 | - |
| | 32 | 4.57927E-05 | 2.15 | 8.97441E-06 | 2.01 | 5.44882E-05 | 2.11 | 1.15060E-05 | 1.98 |
| | 64 | 9.36312E-06 | 2.29 | 2.23928E-06 | 2.00 | 1.13848E-05 | 2.26 | 2.77301E-06 | 1.99 |
| | 128 | 2.83579E-06 | 2.00 | 8.67756E-07 | 1.99 | 5.08147E-06 | 2.11 | 1.15060E-05 | 1.98 |
| | 256 | 5.08051E-07 | 2.00 | 2.17555E-07 | 1.99 | 1.27542E-06 | 2.00 | 3.88274E-07 | 1.98 |
| | 512 | 1.27160E-07 | 2.00 | 3.49898E-08 | 2.00 | 1.58420E-07 | 2.00 | 6.11319E-07 | 1.87 |

Example 4. Consider the following variable coefficients problem
\[
\frac{\partial u(x, t)}{\partial t} = x^\alpha_0 D_x^\alpha u(x, t) + (1 - x)^\alpha_0 D_x^\alpha u(x, t) + f(x, t), \quad (x, t) \in (0, 1) \times (0, 1],
\]
\[
u(0, t) = u(1, t) = 0, \quad t \in [0, 1],
\]
\[
u(x, 0) = x^3(1 - x)^3, \quad x \in [0, 1],
\]
with the source term
\[
f(x, t) = -e^{-t} \left( x^3(1 - x)^3 + \frac{\Gamma(4)}{\Gamma(4 - \alpha)} (x^3 - (1 - x)^3) - 3 \frac{\Gamma(5)}{\Gamma(5 - \alpha)} (x^4 - (1 - x)^4) 
+ 3 \frac{\Gamma(6)}{\Gamma(6 - \alpha)} (x^5 - (1 - x)^5) - \frac{\Gamma(7)}{\Gamma(7 - \alpha)} (x^6 - (1 - x)^6) \right). 
\]

By simple evaluation, the exact solution of (5.3) is \( u(x, t) = e^{-t} x^3(1 - x)^3 \).
Table 4: The maximum and $L^2$ errors and their convergence rates to Example 4 approximated by the CN-WSGD scheme at $t=1$ for different $\alpha$ with $\tau=h$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$N$</th>
<th>$(p, q) = (1, 0)$</th>
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<th>$(p, q) = (1, -1)$</th>
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<td></td>
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5.2 Two Dimensional Case

Example 5. The following fractional diffusion problem

$$\frac{\partial u(x, y, t)}{\partial t} = 0D_x^{1.2}u(x, y, t) + xD_y^{1.2}u(x, y, t) + 0D_y^{1.8}u(x, y, t) + yD_x^{1.8}u(x, y, t) + f(x, y, t)$$

is considered in the domain $\Omega = (0, 1)^2$ and $t>0$ with boundary conditions $u(x, y, t)|_{\partial \Omega} = 0$ and the initial condition $u(x, y, 0) = x^3(1-x)^3y^3(1-y)^3$, where the source term

$$f(x, y, t) = -e^{-t}\left[\left(x^3(1-x)^3y^3(1-y)^3\right) + \left(\frac{\Gamma(4)}{\Gamma(2.8)}(x^{1.8} + (1-x)^{1.8}) - \frac{3\Gamma(5)}{\Gamma(3.8)}(x^{2.8} + (1-x)^{2.8})\right)
+ \frac{3\Gamma(6)}{\Gamma(4.8)}(x^{3.8} + (1-x)^{3.8}) - \frac{\Gamma(7)}{\Gamma(5.8)}(x^{4.8} + (1-x)^{4.8})y^3(1-y)^3\right].$$

Then the exact solution of the fractional partial differential equation is $u(x, y, t) = e^{-t}x^3(1-x)^3y^3(1-y)^3$.

We use four numerical schemes: LOD (4.14), PR-ADI (4.7), Douglas-ADI (4.8) and D’yakonov-ADI (4.9), to simulate Example 5, the maximum and $L^2$ errors and their convergence rates to Example 5 approximated at $t=1$ are listed in Table 5, where $N = N_1 = N_2$, and $p$, $q$ are the shifted numbers of the WSGD operators. From the numerical results, three ADI schemes obtain more accurate solution than the LOD scheme, and it also reflects that the three ADI schemes are equivalent in two dimensional case.
Table 5: The maximum and $L^2$ errors and their convergence rates to Example 5 approximated at $t = 1$ with $\tau = h$.

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6 Conclusion

The paper provides the novel second order approximations for fractional derivatives, called the weighted and shifted Grünwald difference operator; it also suggests a direction to gain higher order discretization and compact schemes of fractional derivatives. The discretizations are used to solve one and two dimensional space fractional diffusion equations; several numerical schemes are designed, their effectiveness are theoretically proved and numerically verified.

7Acknowledgements

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References


