Bounded travelling wave solutions for a modified form of generalized Degasperis–Procesi equation

He Bin *, Rui Weiguo, Li Shaolin, Chen Chan
Department of Mathematics, Honghe University, Mengzi, Yunnan 661100, PR China

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ABSTRACT

In this paper, a modified form of generalized Degasperis–Procesi equation is studied by using the bifurcation theory and by the method of phase portraits analysis. Some bounded travelling waves (loop solitons, peakons, solitary waves and periodic waves) are found. Under different parameter conditions, exact parametric representations of the above travelling waves in explicit form and in implicit form are obtained.

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1. Introduction

In 1999, Degasperis and Procesi derived the following equation [1]:

\[ u_t u_{x x} + 4u u_x = 3u_x u_{xx} + uu_{xxx}, \]  

(1)

which is called Degasperis–Procesi (DP) equation.


Shen and Xu [8] investigated the generalized Degasperis–Procesi (gDP) equation

\[ u_t + c_0 u_x - u_{x x} + au u_x = 3u_x u_{xx} + uu_{xxx}, \]  

(2)

Rui et al. [9] obtained peakons and loop soliton solutions for a family of third-order dispersive partial differential equations

\[ u_t + c_0 u_x + \gamma u_{x x x} - \alpha^2 u_{x x} = (c_1 u^2 + c_2 u_x^2 + c_3 uu_{x x})_x. \]  

(3)

To study the property of solitary wave solutions, Wazwaz [10] suggested studying the following modified form of Degasperis–Procesi (mDP) equation:

\[ u_t + u_{x x t} + 4u^2 u_x = 3u_x u_{xx} + uu_{xxx}. \]  

(4)

* Corresponding author.
E-mail address: hbbhu@yahoo.com.cn (B. He).

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and he showed that mDP equation (4) has bell-shaped solitary wave solutions using tanh and sine–cosine methods. In Ref. [11], a new characteristic of solitary wave solutions, bell-shaped solitary wave and peakon coexisting for the same wave speed in mDP equation was found by Liu and Quyang, and the exact parametric representations of bell-shaped solitary wave and peakon were given.

In this paper, we study a modified form of the generalized Degasperis–Procesi (mgDP) equation

\[ u_t + c_0 u_x - u_{xxx} + uu_x = 3uu_x + uu_{xxx}. \]  

(5)

Setting

\[ u(x,t) = \phi(x - ct) = \phi(\xi), \]  

(6)

where \( c \) is the wave speed. Substituting (6) in Eq. (5), we obtain

\[ \left(c_0 - c\right)\phi' + c\phi''' + a\phi^2 \phi' - 3\phi\phi'' - \phi\phi''' = 0, \]  

(7)

where "'" is the derivative with respect to \( \xi \).

Integrating (7) once with respect to \( \xi \) and the integral constant taken as zero, we have

\[ \left(c - \phi\right)\phi'' + \frac{1}{3}a\phi^3 + (c_0 - c)\phi - (\phi')^2 = 0. \]  

(8)

Letting \( \phi' = y \), we obtain a planar integrable system

\[ \frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{1}{3}a\phi^3 + (c_0 - c)\phi - y^2 \]  

(9)

System (9) is a three-parameter planar dynamical system depending on the parameter set \( \left(c, c_0, a\right) \). Since the phase orbits defined by the vector field of system (9) determine all travelling wave solutions of Eq. (5), we should investigate the bifurcations of phase portraits of system (9) in \((\phi, y)\)-phase plane as the parameters \( c, c_0, a \) are changed. Since the physical model with bounded travelling wave solutions is meaningful, we pay attention only to the bounded travelling wave solutions of Eq. (5) in this paper.

Suppose that \( \phi(\xi) \) is a continuous solution of Eq. (5) for \( \xi \in (-\infty, \infty) \) and \( \lim_{\xi \to -\infty} \phi(\xi) = p, \lim_{\xi \to -\infty} \phi(\xi) = q \). Recall that (i) \( \phi(\xi) \) is called a solitary wave solution if \( p = q \); (ii) \( \phi(\xi) \) is called a kink or anti-kink solution if \( p \neq q \). Usually, a solitary wave solution of Eq. (5) corresponds to a homoclinic orbit of the system (9); a kink (or anti-kink) wave solution of Eq. (5) corresponds to a heteroclinic orbit (or the so-called connecting orbit) of system (9). Similarly, a periodic orbit of the system (9) corresponds to a periodic travelling wave solution of Eq. (5). Thus, to investigate all possible bifurcations of solitary waves and periodic waves of Eq. (5), we need to find all periodic annuli and homoclinic orbits of system (9), which depend on the system parameters [12,13].

We notice that the right-hand side of the second equation in system (9) is generally not continuous when \( \phi = c \). In other words, on such straight lines in the phase plane \((\phi, y)\), the function \( \phi'' \) is not well defined. It implies that system (9) sometimes has non-smooth travelling wave solutions. This phenomenon has been considered before (see Refs. [12,13]).

The rest of the paper is organized as follows: in Section 2, we discuss the bifurcation phase portraits of system (9). In Section 3, we shall give some bounded travelling wave solutions of Eq. (5) in explicit form and in implicit form. A short conclusion will be given in Section 4.

2. Bifurcations of phase portraits of system (9)

In this section, we shall study all phase portraits and bifurcation sets of system (9) in the parameter space. Making transformation \( d\xi = (\phi - c) d\tau \), system (9) becomes a Hamiltonian system

\[ \frac{d\phi}{d\tau} = (\phi - c)y, \quad \frac{dy}{d\tau} = \frac{1}{3}a\phi^3 + (c_0 - c)\phi - y^2 \]  

(10)

with first integral

\[ H(\phi, y) = (\phi - c)^2 y^2 - \left(\frac{2}{15}a\phi^5 - \frac{1}{6}ac\phi^4 + \frac{2}{3}(c_0 - c)\phi^3 - (c_0 - c)c\phi^2\right) = h. \]  

(11)

Eq. (11) can be rewritten as

\[ y^2 = \frac{2}{15}a\phi^5 - \frac{1}{6}ac\phi^4 + \frac{2}{3}(c_0 - c)\phi^3 - (c_0 - c)c\phi^2 + h. \]  

\[ (\phi - c)^2. \]  

(12)

System (9) has the same first integral (11) and has the same phase orbits as system (10), except for the straight line \( \phi = c \). Denote \( \phi_0 = 0, \phi_\infty = c, \Lambda_1 = \frac{c_0 - c_0}{a}, \phi_{1,2} = \pm \sqrt{\Lambda_1}, \Lambda_2 = \frac{1}{4}a^2 + (c_0 - c)c, Y_\pm = \pm \sqrt{\Lambda_2}. \) Obviously, system (10) has one equilibrium point \( O(0,0) \) on the \( \phi \)-axis, when \( a \neq 0, c_0 = c \); system (10) has three equilibrium points \( O(\phi_0, 0), \Lambda_{1,2}(\phi_{1,2}, 0) \) on the \( \phi \)-axis, when \( \Lambda_1 > 0 \); system (10) has two equilibrium points \( S_\pm(\phi_\infty, Y_\pm) \) on the line \( \phi = \phi_\infty \), when \( \Lambda_2 > 0 \); system (10) has no equilibrium points on the line \( \phi = \phi_\infty \), when \( \Lambda_2 < 0 \).

Let $M(\phi, y)$ be the coefficient matrix of the linearized system of system (10) at an equilibrium point $(\phi, y)$. Then we have $\text{Trace}(M(\phi, y)) = 0$ and

$$J_0 = J(0, 0) = \det M(0, 0) = (c_0 - c)c,$$

$$J_1 = J(\phi_1, 0) = \det M(\phi_1, 0) = 2\left(\frac{\sqrt{3(c - c_0)}}{a} - c\right)(c - c_0),$$

$$J_2 = J(\phi_2, 0) = \det M(\phi_2, 0) = 2\left(\frac{\sqrt{3(c - c_0)}}{a} + c\right)(c - c_0),$$

and

$$J_s = J(\phi_s, y_s) = \det M(c, y_s) = -2y^2.$$

By the theory of planar dynamical systems, we know that for an equilibrium point of a planar integrable system, if $J < 0$, then the equilibrium point is a saddle point; if $J > 0$ and $\text{Trace}(M(\phi_1, y)) = 0$, then it is a center point; if $J = 0$ and the Poincaré index of the equilibrium point is zero, then it is a cusp.

From (11), we have

$$h_0 = H(\phi_0, 0) = 0,$$

$$h_1 = H(\phi_1, 0) = \left(\frac{8\sqrt{3(c - c_0)}}{a} - 15c\right)(c - c_0)^2,$$

$$h_2 = H(\phi_2, 0) = -\left(\frac{8\sqrt{3(c - c_0)}}{a} + 15c\right)(c - c_0)^2,$$

and

$$h_s = H(\phi_s, y_s) = \frac{1}{30}ae^5 + \frac{1}{3}c^3c_0 - \frac{1}{3}c^4.$$

For a fixed $h$, the level curve $H(\phi, y) = h$ defined by (11) determines a set of invariant curves of system (10), which contains different branches of curves. As $h$ is varied, it defines different families of orbits of system (10) with different dynamical behaviors.

By using the property of equilibrium points and bifurcation theory, we can obtain the bifurcation lines as follows:

(I) For $c < 0$, there are three bifurcation lines as follows:

$$L_1 : \quad c_0 = c,$$

$$L_2 : \quad a_2(c_0) = \frac{3(c_0 - c)}{c^2},$$

and

$$L_3 : \quad a_3(c_0) = -\frac{10(c_0 - c)}{c^2}, \quad c_0 > c,$$

which divide $(c_0, a)$-parameter plane into 13 subregions: $A_1 = \{(c_0, a) | c_0 > c, a < a_2(c_0)\}; A_2 = \{(c_0, a) | c_0 > c, a = a_2(c_0)\}; A_3 = \{(c_0, a) | c_0 > c, a_2(c_0) < a < a_3(c_0)\}; A_4 = \{(c_0, a) | c_0 > c, a_2(c_0) < a < a_2(c_0)\}; A_5 = \{(c_0, a) | c_0 > c, a_2(c_0) < a \leq 0\}; A_6 = \{(c_0, a) | c_0 > c, 0 < a \leq a_2(c_0)\}; A_7 = \{(c_0, a) | c_0 = 0, 0 < a \leq a_2(c_0)\}; A_8 = \{(c_0, a) | c_0 = 0, 0 = a \}; A_9 = \{(c_0, a) | c_0; A_{10} = \{(c_0, a) | c_0 < c, a < 0 \}; A_{11} = \{(c_0, a) | c_0 < c, 0 < a < a_2(c_0)\}; A_{12} = \{(c_0, a) | c_0 < c, a = a_2(c_0)\}; \quad \text{and} \quad A_{13} = \{(c_0, a) | c_0 < c, a_2(c_0) < a \}.

(II) For $c > 0$, there are three bifurcation lines as follows:

$$L_1 : \quad c_0 = c,$$

$$L_2 : \quad a_2(c_0) = -\frac{3(c_0 - c)}{c^2},$$

and

$$L_4 : \quad a_4(c_0) = -\frac{10(c_0 - c)}{c^2}, \quad \quad c_0 < c,$$

which divide $(c_0, a)$-parameter plane into 13 subregions: $B_1 = \{(c_0, a) | c_0 < c, a < 0 \}; B_2 = \{(c_0, a) | c_0 < c, 0 < a \}; B_3 = \{(c_0, a) | c_0 < c, a_2(c_0) < a < a_4(c_0)\}; B_4 = \{(c_0, a) | c_0 < c, a_2(c_0) < a < a_2(c_0)\}; B_5 = \{(c_0, a) | c_0 < c, a = a_2(c_0)\}; B_6 = \{(c_0, a) | c_0 < c, a_4(c_0) < a \}; B_7 = \{(c_0, a) | c_0 = 0, 0 < a \}; B_8 = \{(c_0, a) | c_0 = 0, 0 = a \}; B_9 = \{(c_0, a) | c_0 = 0, 0 \}; B_{10} = \{(c_0, a) | c_0 > c, a < 0 \}; B_{11} = \{(c_0, a) | c_0 > c, a = a_2(c_0)\}; B_{12} = \{(c_0, a) | c_0 > c, a_2(c_0) < a < 0 \}; \quad \text{and} \quad B_{13} = \{(c_0, a) | c_0 > c, 0 \leq a \}.

The bifurcations of phase portraits of the system (10) shown is in Figs. 1 and 2.

3. Bounded travelling wave solutions of Eq. (5)

In this section, by using (9), (10) and (12) to do calculations, we shall give some bounded travelling wave solutions of Eq. (5) under the given parameter conditions shown in Section 2. We shall use the Jacobian elliptic functions $sn(x, m)$ and $dn(x, m)$ with the modulus $m$.

3.1. Loop soliton solutions

3.1.1. Suppose that $c < 0$, $c_0 > c$, $a = 0$ (Fig. 1(1-6)), taking $h = h_0$, we have from (12) that

\[(1-13)\quad (c_0, a) \in A_{13}.
\]

Fig. 1. The phase portraits of the system (10) when $c < 0$. (1-1) $(c_0, a) \in A_1$; (1-2) $(c_0, a) \in A_2$; (1-3) $(c_0, a) \in A_3$; (1-4) $(c_0, a) \in A_4$; (1-5) $(c_0, a) \in A_5$; (1-6) $(c_0, a) \in A_6$; (1-7) $(c_0, a) \in A_7$; (1-8) $(c_0, a) \in A_8$; (1-9) $(c_0, a) \in A_9$; (1-10) $(c_0, a) \in A_{10}$; (1-11) $(c_0, a) \in A_{11}$; (1-12) $(c_0, a) \in A_{12}$; (1-13) $(c_0, a) \in A_{13}$.
where

\[ \phi_m = \frac{3}{2}. \]

Substituting (27) into the first expression of (10) and integrating it, we get

\[ \int_{\phi(0)}^{\phi} \frac{d\sigma}{\sqrt{\phi_m - \phi \sigma}} = \pm \int_0^r d\sigma, \]  

where \( \phi(0) \) and 0 are initial constants.
Choosing $\phi(0) = \phi_m$ and completing (28), we obtain one explicit function expressions for the loop soliton solution of Eq. (5)

$$
\begin{align*}
\phi(\tau) &= \phi_m \text{sech}^2(\omega_1 \tau), \\
\zeta(\tau) &= \frac{a m}{\phi_m} \tanh(\omega_1 \tau) - ct,
\end{align*}
$$

where $\tau$ is a parameter and $\omega_1 = -\sqrt{\frac{1}{8} \phi_m (c - c_0)}$. The profile of (29) is shown in Fig. 3(3-1).

3.1.2. Suppose that $c < 0, c_0 = c, a > 0$ (Fig. 1(1-7)), taking $h = h_0$, we have from (12) that

$$
y = \pm \sqrt{\frac{7 a}{2} m^2} \sqrt{\frac{\phi - \phi_m}{\phi - c}},
$$

where $\phi_m = \frac{x}{c}$. Substituting (30) into the first expression of (9) and integrating it, we get

$$
\int_{\phi(0)}^{\phi} \frac{(\sigma - c) d\sigma}{\sqrt{\frac{7 a}{2} m^2} \sigma^2 \sqrt{\sigma - \phi_m}} = \pm \int_0^\zeta d\sigma.
$$

Choosing $\phi(0) = \phi_m$ and completing (31), we obtain one implicit function expressions for the loop soliton solution of Eq. (5)

$$
Q \tanh^{-1} \left( \frac{\sqrt{\phi - \phi_m}}{\sqrt{-\phi_m}} \right) = \frac{c \sqrt{\phi - \phi_m}}{\phi_m \phi} + \frac{2}{15} a \zeta,
$$

where $Q = \frac{c}{\sqrt{\phi_m \phi}} - \frac{2}{\sqrt{-\phi_m}}$.

Let $\mu = \frac{c \sqrt{\phi - \phi_m}}{\phi_m \phi} + \sqrt{\frac{a}{15}} \zeta$, then (32) can be rewritten as

$$
\begin{align*}
\phi(\mu) &= \phi_m \text{sech}^2\left( \frac{\mu}{2} \right), \\
\zeta(\mu) &= \frac{1}{\mu_2} \left( \mu - \frac{c \sqrt{\phi - \phi_m}}{2 \phi m} \sinh\left( \frac{2 \mu}{\phi} \right) \right),
\end{align*}
$$

where $\omega_2 = \sqrt{\frac{a}{15}}. The profile of (33) is shown in Fig. 3(3-2).
3.1.3. Suppose that $c > 0, c_0 < c, a = 0$ (Fig. 2(2-1)), taking $h = h_0$, we have from (12) that

$$y = \pm \sqrt{\frac{2(c-c_0)}{3}} \phi \sqrt{\phi_M - \phi} \over \phi - c, \quad (34)$$

where $\phi_M = \frac{2}{3} c$. Substituting (34) into the first expression of (10) and integrating it, we get

$$\int_{\phi(0)}^{\phi} \frac{d\sigma}{\sqrt{\phi_M - \phi} \sqrt{\phi_M - \sigma}} = \pm \int_0^c d\xi. \quad (35)$$

Choosing $\phi(0) = \phi_M$ and completing (35), we obtain one explicit function expressions for the loop soliton solution of Eq. (5)

$$\left\{ \begin{array}{l}
\phi(\tau) = \phi_M \text{sech}^2(\omega_3 \tau), \\
\zeta(\tau) = \frac{a \sqrt{M}}{\theta M} \tanh(\omega_3 \tau) - ct,
\end{array} \right. \quad (36)$$

where $\tau$ is a parameter and $\omega_3 = -\sqrt{\frac{2}{3} \phi_M(c - c_0)}$. The profile of (36) is shown in Fig. 3(3-3).

3.1.4. Suppose that $c > 0, c_0 = c, a < 0$ (Fig. 2(2-9)), taking $h = h_0$, we have from (12) that

$$y = \pm \sqrt{\frac{-2 a^2}{3}} \phi \sqrt{\phi_M - \phi} \over \phi - c, \quad (37)$$

where $\phi_M = \frac{2}{3} c$. Substituting (37) into the first expression of (9) and integrating it, we get

$$\int_{\phi(0)}^{\phi} \frac{(\sigma - c) d\sigma}{\sqrt{\phi_M - \sigma} \sqrt{\phi_M - \phi}} = \pm \int_0^c d\xi. \quad (38)$$

Choosing $\phi(0) = \phi_M$ and completing (38), we obtain one implicit function expressions for the loop soliton solution of Eq. (5)

$$R \tanh^{-1} \left( \frac{\sqrt{\phi_M - \phi}}{\sqrt{\phi_M}} \right) = \sqrt{-\frac{2}{15} a^2} \phi - c \sqrt{\phi_M - \phi} \over \phi_M \phi, \quad (39)$$

where $R = \frac{1}{\sqrt{\phi_M}} (\frac{c}{\phi_M} - 2)$.

Let $\eta = \sqrt{-\frac{2}{15} a^2} \phi - c \sqrt{\phi_M - \phi} \over \phi_M \phi$, then (39) can be rewritten as

$$\left\{ \begin{array}{l}
\phi(\eta) = \phi_M \text{sech}^2 \left( \frac{\eta}{\phi_M} \right), \\
\zeta(\eta) = \frac{a \sqrt{M}}{\theta M} \left( \eta + c \sqrt{\phi_M} \sinh \left( \frac{\eta}{\phi_M} \right) \right),
\end{array} \right. \quad (40)$$

where $\omega_M = \sqrt{-\frac{c}{\phi_M}} a$. The profile of (40) is shown in Fig. 3(3-4).

3.2. Coexistence of solitary wave and peakon solutions

3.2.1. Suppose that $c < 0, c_0 > c, a = -\frac{10(c_0 - c)}{c^2}$ (Fig. 1(1-2)), taking $h = h_0$, we have from (12) that

$$y = \pm \frac{2}{c} \sqrt{\frac{c_0}{3}} \phi \sqrt{\phi_M - \phi}, \quad (41)$$

where $\phi_M = -\frac{2}{3} c$. Substituting (41) into the first expression of (9) and integrating it, we get

$$\int_{\phi(0)}^{\phi} \frac{d\sigma}{\sqrt{\phi_M - \sigma} \sqrt{\phi_M - \phi}} = \pm \int_0^c d\xi. \quad (42)$$

Choosing $\phi(0) = \phi_M$ and completing (42), we obtain one explicit function expression for the solitary wave solution of Eq. (5)

$$\phi(x - ct) = \frac{2 \phi_M}{1 + \cosh(\omega_5(x - ct))}, \quad (43)$$

where $\omega_5 = \sqrt{c(c - c_0)}$. The profile of (43) is shown in Fig. 4(4-1).

Choosing $\phi(0) = \phi$, and completing (42), we obtain one explicit function expression for the peakon solution of Eq. (5)
3.2.2. Suppose that $c > 0$, $c_0 < c$, $a = \frac{10(c_0 - c)}{c^2}$ (Fig. 2(2-5)), taking $h = h_0$, we have from (12) that

$$y = \pm \frac{2}{c} \sqrt{\frac{c - c_0}{3}} \phi \sqrt{\phi - \phi_m}. \quad (45)$$

where $\phi_m = -\frac{4}{3}c$. Substituting (45) into the first expression of (9) and integrating it, we get

$$\int_{\phi(0)}^{\phi} \frac{d\sigma}{\sqrt{\sigma - \phi_m}} = \pm \int_{0}^{\psi} d\sigma. \quad (46)$$

Choosing $\phi(0) = \phi_m$ and completing (46), we obtain one explicit function expression for the solitary wave solution of Eq. (5)

$$\phi(x - ct) = \frac{2\phi_m}{1 + \cosh(\omega_0(x - ct))}. \quad (47)$$

where $\omega_0 = \frac{1}{c} \sqrt{c(c - c_0)}$. The profile of (47) is shown in Fig. 4(4-3).

Choosing $\phi(0) = \phi_s$ and completing (46), we obtain one explicit function expression for the peakon solution of Eq. (5)

$$\phi(x - ct) = \frac{-4\phi_s\phi_m}{(\sqrt{\phi_s - \phi_m} - \sqrt{-\phi_m})^2 e^{\omega_0(x - ct)} + (\sqrt{\phi_s - \phi_m} + \sqrt{-\phi_m})^2 e^{-\omega_0(x - ct)} - 2\phi_s}. \quad (48)$$

The profile of (48) is shown in Fig. 4(4-4).

Fig. 4. The profiles of solitary wave and peakon of Eq. (5). (4-1) $c = -1$, $c_0 = -0.5$, $a = -5$. (4-2) $c = -1$, $c_0 = -0.5$, $a = -5$. (4-3) $c = 1$, $c_0 = 0.5$, $a = 5$. (4-4) $c = 1$, $c_0 = 0.5$, $a = 5$. 

(4-1) $c = -1$, $c_0 = -0.5$, $a = -5$. (4-2) $c = -1$, $c_0 = -0.5$, $a = -5$. (4-3) $c = 1$, $c_0 = 0.5$, $a = 5$. (4-4) $c = 1$, $c_0 = 0.5$, $a = 5$. 

$\phi(x - ct) = \frac{4\phi_s\phi_m}{(\sqrt{\phi_M - \phi_s} - \sqrt{\phi_M})^2 e^{\omega_0(x - ct)} + (\sqrt{\phi_M - \phi_s} + \sqrt{\phi_M})^2 e^{-\omega_0(x - ct)} + 2\phi_s}. \quad (44)$

The profile of (44) is shown in Fig. 4(4-2).

The profile of (47) is shown in Fig. 4(4-3).
3.3. Solitary wave solutions

3.3.1. Suppose that $c < 0, c_0 > c, -\frac{3(c_0 - c)}{2c^2} < a = -\frac{64(c_0 - c)}{75c^2} < 0$ (Fig. 1(1-5)), taking $h = h_0$, we have from (12) that

$$y = \pm \frac{8}{15c} \sqrt{\frac{2(c_0 - c)}{5}} (\phi - \phi_2) \phi \sqrt{\phi_M - \phi} \phi_3,$$

(49)

where $\phi_M = -\frac{2}{3} c$. Substituting (49) into the first expression of (9) and integrating it, we get

$$\int_{\phi(0)}^{\phi} (\sigma - \phi_3) d\sigma = \pm \int_0^c d\sigma.$$

(50)

Choosing the $\phi(0) = \phi_M$ and completing (50), we obtain one implicit function expression for the solitary wave solution of Eq. (5)

$$\left(\sqrt{\phi_M - \phi} - \sqrt{\phi_M - \phi_2}\right)^{\frac{\phi_3}{\phi_2}} = \left(\sqrt{\phi_M - \phi_2} - \sqrt{\phi_M - \phi_3}\right)^{\frac{\phi_2}{\phi_3}} e^{\frac{\phi_1}{\phi_2} |x - c|},$$

(51)

where $\omega_0 = \frac{8}{15c} \sqrt{\frac{2(c_0 - c)}{5}}$. The profile of (51) is shown in Fig. 5(5-1).

3.3.2. Suppose that $c > 0, c_0 < c, 0 < a = -\frac{64(c_0 - c)}{75c^2} < -\frac{3(c_0 - c)}{2c^2}$ (Fig. 2(2-2)), taking $h = h_0$, we have from (12) that

$$y = \pm \frac{8}{15c} \sqrt{\frac{2(c_0 - c)}{5}} (\phi - \phi_1) \phi \sqrt{\phi_M - \phi} \phi_3,$$

(52)

where $\phi_M = -\frac{2}{3} c$. Substituting (52) into the first expression of (9) and integrating it, we get

$$\int_{\phi(0)}^{\phi} (\sigma - \phi_3) d\sigma = \pm \int_0^c d\sigma.$$

(53)

Choosing $\phi(0) = \phi_m$ and completing (53), we obtain one implicit function expression for the solitary wave solution of Eq. (5)

$$\left(\sqrt{\phi_m - \phi} - \sqrt{\phi_m - \phi_m}\right)^{\frac{\phi_3}{\phi_2}} = \left(\sqrt{\phi_m - \phi_2} - \sqrt{\phi_m - \phi_m}\right)^{\frac{\phi_2}{\phi_3}} e^{\frac{\phi_1}{\phi_2} |x - c|},$$

(54)

where $\omega_h = -\frac{8}{15c} \sqrt{\frac{2(c_0 - c)}{5}}$. The profile of (54) is shown in Fig. 5(5-2).

3.4. Periodic wave solutions

3.4.1. Suppose that $c < 0, c_0 > c, a = -\frac{3(c_0 - c)}{2c^2}$ (Fig. 1(1-4)), taking $h = h_0$, we have from (12) that

$$y = \pm \frac{8}{5c^2} \sqrt{\frac{2(c_0 - c)}{5}} \phi \phi (\phi_M - \phi) (\phi - \phi_3),$$

(55)

$$\frac{\phi_3}{\phi_2} \phi_3 \phi_2.$$

(5-1) $c = -1, c_0 = -0.5, a = -\frac{32}{75}$. (5-2) $c = 1, c_0 = 0, a = \frac{64}{75}$.

Fig. 5. The profiles of solitary wave of Eq. (5). (5-1) $c = -1, c_0 = -0.5, a = -\frac{32}{75}$. (5-2) $c = 1, c_0 = 0, a = \frac{64}{75}$. 

where \( \phi_{M,m} = -\left(\frac{7}{8} \pm \frac{\sqrt{105}}{24}\right) c \). Substituting (55) into the first expression of (9) and integrating it, we get

\[
\int_{\phi(0)}^{\phi(x)} \frac{d\sigma}{\sqrt{2(c-\sigma)(\phi_M - \sigma)(\phi - \phi_M)}} = \pm \int_{0}^{c} d\sigma. \tag{56}
\]

Choosing \( \phi(0) = \phi_m \) and completing (56), we obtain one explicit function expression of Jacobin elliptic function type for the periodic wave solution of Eq. (5)

\[
\phi(x - ct) = \phi_m - \phi_s \alpha m_1 \sin^2 \left(\omega(t)(x - ct), m_1 \right), \tag{57}
\]

where \( \omega = \sqrt{\frac{2(c-\phi_m)(\phi_m-\phi_s)}{5c^2}} \), \( m_1 = \sqrt{\frac{2(c-\phi_m)(\phi_m-\phi_s)}{5c^2}} \). The profile of (57) is shown in Fig. 6(6-1).

3.4.2. Suppose that \( c < 0, c_0 < c, a = -\frac{\sqrt{5}c_0}{c} \) (Fig. 1(1-12)), taking \( h = h_s \), we have from (12) that

\[
y = \pm \sqrt{\frac{2(c-c_0)}{5c^2}} \sqrt{(\phi_M - \phi)(\phi - \phi_M)}(\phi - \phi_s), \tag{58}
\]

where \( \phi_{M,m} = -\left(\frac{7}{8} \pm \frac{\sqrt{105}}{24}\right) c \). Substituting (58) into the first expression of (9) and integrating it, we get

\[
\int_{\phi(0)}^{\phi(x)} \frac{d\sigma}{\sqrt{2(c-\sigma)(\phi_M - \sigma)(\phi - \phi_M)}} = \pm \int_{0}^{c} d\sigma. \tag{59}
\]

Choosing the \( \phi(0) = \phi_s \) and completing (59), we obtain one explicit function expression of Jacobin elliptic function type for the periodic wave solution of Eq. (5)

\[
\phi(x - ct) = \phi_s + \left(\phi_m - \phi_s \right) \sin^2 \left(\omega(t)(x - ct), m_2 \right), \tag{60}
\]

where \( \omega = \sqrt{\frac{2(c_0-\phi_s)(\phi_m-\phi_s)}{5(c-c_0)}} \), \( m_2 = \sqrt{\frac{2(c_0-\phi_s)(\phi_m-\phi_s)}{5(c-c_0)}} \). The profile of (60) is shown in Fig. 6(6-2).

3.4.3. Suppose that \( c > 0, c_0 < c, a = -\frac{\sqrt{5}c_0}{c} \) (see Fig. 2(2-3)), taking \( h = h_s \), we have from (12) that

\[
(6-1) \ c = -1, \ c_0 = 0, \ a = -3. \quad (6-2) \ c = -1, \ c_0 = -2, \ a = 3. \quad (6-3) \ c = 1, \ c_0 = 0.5, \ a = 1.5. \quad (6-4) \ c = 1, \ c_0 = 2, \ a = -3.
\]

Fig. 6. The profiles of periodic wave of Eq. (5). (6-1) \( c = -1, c_0 = 0, a = -3. \) (6-2) \( c = -1, c_0 = -2, a = 3. \) (6-3) \( c = 1, c_0 = 0.5, a = 1.5. \) (6-4) \( c = 1, c_0 = 2, a = -3. \)


\[ y = \pm \sqrt{\frac{2(c - c_0)}{5c^2}} (\phi_x - \phi)(\phi_M - \phi)(\phi - \phi_m), \]  

(61)

where \( \phi_{d,m} = \left( -\frac{c}{c_0} \pm \frac{10c}{24} \right) c \). Substituting (61) into the first expression of (9) and integrating it, we get

\[ \int_{\phi(0)}^{\phi} \frac{2(c - c_0)}{5c^2} (\phi_x - \phi)(\phi_M - \phi)(\phi - \phi_m) \, \mathrm{d}\sigma = \pm \int_{0}^{c} \, \mathrm{d}\sigma. \]  

(62)

Choosing \( \phi(0) = \phi_m \) and completing (62), we obtain one explicit function expression of Jacobin elliptic function type for the periodic wave solution of Eq. (5)

\[ \phi(x - ct) = \phi_m + (\phi_M - \phi_m) m_2^2 \left( \omega_{11}(x - ct), m_3 \right), \]  

(63)

where \( \omega_{11} = \sqrt{\frac{(a - c)(a - c)}{a^2 - c}} m_3 \). The profile of (63) is shown in Fig. 6.(6-3).

3.4.4. Suppose that \( c > 0, c_0 > c, a = \frac{(a - c)}{c} \) (Fig. 2(2-11)), taking \( h = h_i \), we have from (12) that

\[ y = \pm \sqrt{\frac{2(c_0 - c)}{5c^2}} (\phi_x - \phi)(\phi_M - \phi)(\phi - \phi_m), \]  

(64)

where \( \phi_{d,m} = \left( -\frac{c}{c_0} \pm \frac{10c}{24} \right) c \). Substituting (64) into the first expression of (9) and integrating it, we get

\[ \int_{\phi(0)}^{\phi} \frac{2(c_0 - c)}{5c^2} (\phi_x - \phi)(\phi_M - \phi)(\phi - \phi_m) \, \mathrm{d}\sigma = \pm \int_{0}^{c} \, \mathrm{d}\sigma. \]  

(65)

Choosing \( \phi(0) = \phi_M \) and completing (65), we obtain one explicit function expression of Jacobin elliptic function type for the periodic wave solution of Eq. (5)

\[ \phi(x - ct) = \frac{\phi_M - \phi_m}{m_2^2} \left( \omega_{12}(x - ct), m_4 \right), \]  

(66)

where \( \omega_{12} = \sqrt{\frac{(a - c)(a - c)}{a^2 - c}} m_4 = \sqrt{\frac{(a - c)(a - c)}{a^2 - c}} \). The profile of (66) is shown in Fig. 6.(6-4).

4. Conclusion

In this paper, we consider the bifurcation behavior of mgDP equation, some bounded travelling wave solutions are obtained. At the same time, we have shown that the loop soliton exists in mgDP equation ((29), (33), (36) and (40)) and we have shown the following property of solitary waves for mgDP equation: at the same wave speed, the solitary wave and the peakon coexist in mgDP equation ((43),(44),(47) and (48)).

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References