Risk-sensitive dynamic pricing for a single perishable product

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ABSTRACT

We show that the monotone structures of dynamic pricing for a single perishable product under risk-neutrality are preserved under risk-sensitivity with the additive general utility and atemporal exponential utility functions. We also show that the optimal price is decreasing over the degree of risk-sensitivity under the exponential class of both additive and atemporal utility functions.

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1. Introduction

Traditional research on dynamic pricing considers risk-neutral decision makers to maximize the expected revenue. This approach falls short of taking account of revenue volatility and risk-sensitivity. Although risk-neutrality is often appropriate for repeated business activities where a single poor outcome has no severe impact on the financial condition of the firm, it may not hold for event organizers [1], firms with cash-flow problems [2], firms with short-term revenue targets [3], among others. These firms are sensitive to revenue variations and are willing to trade off lower expected revenue for downside protection against possible underperformances. Moreover, most managers in charge of dynamic pricing policies present some degree of risk aversion [4]. These findings call for the development of dynamic pricing strategies for risk-sensitive decision makers.

The literature on dynamic pricing strategies that incorporate risk-sensitivity is very limited and quite recent. Lim and Shanthikumar [5] establish an interesting equivalence between dynamic pricing for a single product with an atemporal exponential utility and robust dynamic pricing when studying uncertainty originated from forecast errors in revenue management. However, no structural properties of the optimal policy have been characterized. Levin et al. [1] study the risk-adjusted dynamic pricing and incorporate a risk factor to control the probability that total revenue falls below a minimum acceptable level. They show that the risk-adjusted optimal policy coincides with the risk-neutral optimal policy (Gallego and van Ryzin [6]) after the required revenue level has been reached, and the risk-adjusted optimal price may decrease whereas the risk-neutral optimal price increases immediately following a sale.

Capacity control and dynamic pricing are two main strategies in revenue management to balance constrained capacity and fluctuating demand. The literature on risk-sensitive capacity control is also very limited. Our work is closely related to the generic single-resource capacity control problem under risk-sensitivity studied in [2,3]. Barz [2] studies structural properties of the static and dynamic model under the additive general utility and atemporal exponential utility function, and raises a monotonicity conjecture that the optimal protection level is decreasing over the degree of risk-sensitivity for the decision maker with an atemporal exponential utility function. Feng and Xiao [3] study risk-sensitive capacity control for single perishable product with an atemporal exponential utility function. They show that the optimal control policy preserves the attractive properties under risk-neutrality such as nested active price set, monotonicity over time and inventory, and threshold-type control. They also show that the optimal price set will grow larger as the degree of risk-sensitivity goes up, indicating that the firm is lowering the threshold price. This is consistent with Barz’s monotonicity conjecture.

Different from [2,3] that fall into the capacity control framework with prices given, this work studies generic dynamic pricing strategies for a single perishable product under risk-sensitivity. Two main findings are made. First, we show that the intuitively appealing monotone structures with respect to inventory level and time under risk-neutrality, originally established in [6], are preserved under risk-sensitivity with the additive general utility and atemporal exponential utility functions. Second, we prove that the optimal price is decreasing over the degree of risk-sensitivity under the exponential class of both additive and atemporal utility functions.

In accordance with the monotonicity conjecture in [2] and results from [3], we show that a risk-sensitive decision maker...
will set a lower price to hedge against risks originated from uncertainties about consumer’s willingness-to-pay. In view of findings from [3], we conclude that a risk-sensitive decision maker will adopt a more conservative control policy under both capacity control and dynamic pricing to trade off lower expected revenue against possible unfavorable losses.

The rest of the paper is organized as follows. Section 2 presents the models and preliminaries. Sections 3 and 4 study structural properties of risk-sensitive dynamic pricing under the additive general utility and atemporal exponential utility functions respectively. In Section 5, numerical studies are used to illustrate the analytical results.

2. The models

Consider a firm that has C units of a perishable product to sell over a finite horizon. The objective is to develop a dynamic pricing strategy to maximize the expected revenue under risk-neutrality or to maximize the expected utility under risk-sensitivity. Since the selling horizon for the perishable product is relatively short, discounting is generally not considered.

Let \( \mathcal{T}, \delta, A, q_t(\cdot|x, p), r_t(x, p) \) denote the underlying Markov decision process (MDP) for the dynamic pricing problem. The set of decision epochs is given by \( \mathcal{T} = \{T, T-1, \ldots, 1, 0\} \) with time going backward. There is at most one customer arrival with probability \( \lambda \) within each period \( t \). The state space is \( \mathcal{S} = \{x \in \mathbb{R} | 0 \leq x \leq C\} \) where \( x \) is the inventory level. Action space \( \mathcal{A} = \mathcal{P} \cap \mathcal{P}_p \) is the set of allowable prices, where \( \mathcal{P} \) is a compact set within \( (0, +\infty) \) and \( \mathcal{P}_p \) is a null price used to model out-of-stock situation (see [6]). Consumer’s willingness-to-pay for the product has a tail probability \( F(p) \) such that \( F(p_{\infty}) = 0 \) and \( F(p) \geq 0 \) for all \( p \in \mathcal{P} \). The dynamic pricing decision rule is a mapping \( p_t : \delta \rightarrow \mathcal{A} \). The transition law \( q_t : \delta \times \mathcal{A} \rightarrow \mathcal{S} \) is given by \( q_t(x|x-1|p) = \lambda F(p) \) and \( q_t(x|x|p) = 1 - \lambda F(p) \). One-period reward function is defined by \( r_t : \delta \times \mathcal{A} \rightarrow \mathbb{R} \). A \( T \)-stage Markovian pricing policy is given by \( \pi = (p_1, \ldots, p_T) \in \Pi \equiv \mathcal{A} \times \cdots \times \mathcal{A} \). The risk-neutral decision maker determines an optimal pricing policy \( \pi^* \) that maximizes expected revenue:

\[
J^{\pi^*}(C) = \sup_{\pi \in \Pi} J^{\pi}(C) = \max_{\pi \in \Pi} E^{\pi} \left[ \sum_{t=1}^{T} r_t(X_t, p_t(X_t)) + J^\pi_0(X_0) \right]_{X_T = C}. \tag{1}
\]

Two classes of utility functions are commonly used to study sequential decision problems under risk due to tractability [2], namely, additive utility function and atemporal utility function. Given any wealth profile \( \omega = (\omega_0, \omega_1, \ldots, \omega_T) \) over \( T + 1 \) periods, a utility function \( u \) is said to be additive if

\[
u(\omega) = \sum_{t=0}^{T} u_t(\omega_t), \]

and it is called atemporal if

\[
u(\omega) = u \left( \sum_{t=0}^{T} \omega_t \right). \]

The key difference between an additive utility and an atemporal utility lies in the decision maker’s time preference over reward streams. An additive utility deals with utility dynamically over time; while an atemporal utility is insensitive to the time aspect and evaluates the total rewards within the planning horizon. Technically, an additive utility is separable in time, hence simple to work analytically; while an atemporal utility is generally non-separable in time, thus more challenging analytically.

An additive utility maximizer determines an optimal pricing policy \( \pi_{\text{add}}^* \) that maximizes the following expected utility:

\[
J^{\pi_{\text{add}}^*}(C) = \sup_{\pi \in \Pi} J^{\pi_{\text{add}}}(C) = \max_{\pi \in \Pi} E^{\pi} \left[ \sum_{t=1}^{T} u_t r_t(X_t, p_t(X_t)) + u_0 \left( J^{\pi_{\text{add}}}_0(X_0) \right)_{X_T = C} \right]. \tag{2}
\]

An atemporal utility maximizer determines an optimal pricing policy \( \pi_{\text{amp}}^* \) that maximizes the following expected utility:

\[
J^{\pi_{\text{amp}}^*}(C) = \sup_{\pi \in \Pi} J^{\pi_{\text{amp}}}(C) = \max_{\pi \in \Pi} E^{\pi} \left[ u \left( \sum_{t=1}^{T} r_t(X_t, p_t(X_t)) + J^{\pi_{\text{amp}}}_0(X_0) \right)_{X_T = C} \right]. \tag{3}
\]

Next we will explore structural properties and comparative statics for risk-sensitive dynamic pricing. The following lemma derived from the envelope theorem [7] is used to study the comparative statics.

**Lemma 1.** Let \( f : X \times \Theta \rightarrow \mathbb{R} \).

(a) If \( x^*(\theta) = \arg \max_x f(x, \theta) \), then \( f_{x\theta}(x^*(\theta), \theta) \cdot \frac{\partial x^*(\theta)}{\partial \theta} > 0 \).

(b) If \( x^*(\theta) = \arg \min_x f(x, \theta) \), then \( f_{x\theta}(x^*(\theta), \theta) \cdot \frac{\partial x^*(\theta)}{\partial \theta} < 0 \).

3. Additive general utility

Additive utility is frequently used in MDP [e.g. [2,8]] because of its time dynamics and analytical simplicity. For instance, firms with certain financial obligations to fulfill at certain time may have time-varying risk preferences which can be modeled by additive utility functions.

The Bellman equation of (2), dynamic pricing under the additive general utility, is given by

\[
j^{\pi_{\text{add}}}(x) = j^{\pi_{\text{add}}}_{t-1}(x) + \lambda \cdot \max_{p_t \in \mathcal{P}} \left\{ \tilde{F}(p) \left( u_t(p) - \Delta j^{\pi_{\text{add}}}_{t-1}(x) \right) \right\},
\]

\[
\forall x = 1, \ldots, C; \forall t = 1, \ldots, T, \tag{4}
\]

\[
j^{\pi_{\text{add}}}_0(x) = 0, \quad \forall x = 1, \ldots, C \quad \text{and} \quad j^{\pi_{\text{add}}}_{T}(0) = 0,
\]

\[
\forall t = 1, \ldots, T, \tag{5}
\]

where \( u_t(\cdot) \) is a utility function with \( u_t'(\cdot) \geq 0, u_t''(\cdot) \leq 0 \) and \( \Delta j^{\pi_{\text{add}}}_{t-1}(x) = j^{\pi_{\text{add}}}_{t-1}(x) - j^{\pi_{\text{add}}}_{t-1}(x-1) \) is the marginal opportunity cost in terms of utility. Let

\[
R^{\pi_{\text{add}}}(p, \Delta j^{\pi_{\text{add}}}_{t-1}(x)) = \tilde{F}(p) \left( u_t(p) - \Delta j^{\pi_{\text{add}}}_{t-1}(x) \right).
\]

The optimal price is given by

\[
p^*_{\text{add}}(\Delta j^{\pi_{\text{add}}}_{t-1}(x)) = \arg \max_{p_t \in \mathcal{P}} R^{\pi_{\text{add}}}(p, \Delta j^{\pi_{\text{add}}}_{t-1}(x)).
\]

The sufficient condition for the regularity of \( R^{\pi_{\text{add}}}(p, \Delta j^{\pi_{\text{add}}}_{t-1}(x)) \) is characterized in Lemma 2.

**Lemma 2.** If \( F(\cdot) \) is an increasing failure rate distribution, then \( R^{\pi_{\text{add}}}(p, \Delta j^{\pi_{\text{add}}}_{t-1}(x)) \) is quasiconcave on \( p \).

**Proof.** See Appendix. \( \square \)
Note that increasing failure rate (IFR) implies increasing generalized failure rate which is equivalent to increasing price elasticity (see [9]). IFR captures most common distributions such as uniform, normal, exponential and logistic. Distributions such as the Weibull, Beta, and Gamma are IFR distributions if their parameters fall into certain ranges.

To facilitate the characterization of structural properties, we let $K(\Delta) = \max_p \{ F(p)(u(p) - \Delta) \}$ and present a lemma summarizing properties of $K(\Delta)$ (also see [10]). For simplicity of exposition, we use $\uparrow$ to indicate increasing and $\downarrow$ to indicate decreasing in the weak sense.

**Lemma 3.** $K(\Delta)$ has the following properties:
(a) $K(\Delta) \downarrow \Delta$;
(b) If $\Delta_1 \leq \Delta_2$ then $K(\Delta_1) - K(\Delta_2) \leq \Delta_2 - \Delta_1$.

**Lemma 3** follows immediately as $F(p)(u(p) - \Delta)$ is decreasing in $\Delta$ while $F(p)(u(p) - \Delta) + \Delta$ is increasing in $\Delta$ for any $p$. Structural properties are summarized below.

**Proposition 1.** The problem (4), namely risk-averse dynamic pricing under the additive general utility, has the following structural properties:
(a) $\Delta_{t+1}^{\text{add}}(x) \downarrow x \uparrow t$;
(b) $p^*_{\text{add}}(\Delta_{t+1}^{\text{add}}(x)) \uparrow \Delta_{t+1}^{\text{add}}(x) \downarrow x \uparrow t$.

**Proof.** (a) We prove by induction on $t$. It is trivial when $t = 0$.
Assume $\Delta_{t-1}^{\text{add}}(x) \leq \Delta_{t-1}^{\text{add}}(x-1)$, which is true because
$$\Delta_{t}^{\text{add}}(x) - \Delta_{t}^{\text{add}}(x-1) = \Delta_{t-1}^{\text{add}}(x) - \Delta_{t-1}^{\text{add}}(x-1)$$
$$+ \lambda \left[ K(\Delta_{t-1}^{\text{add}}(x)) - K(\Delta_{t-1}^{\text{add}}(x-1)) \right]$$
$$\leq \Delta_{t-1}^{\text{add}}(x) - \Delta_{t-1}^{\text{add}}(x-1)$$
$$+ \lambda \left[ K(\Delta_{t-1}^{\text{add}}(x)) - K(\Delta_{t-1}^{\text{add}}(x-1)) \right]$$
$$\leq (1 - \lambda) \left( \Delta_{t-1}^{\text{add}}(x) - \Delta_{t-1}^{\text{add}}(x-1) \right)$$
$$\leq 0,$$
where the three inequalities follow by Lemma 3(a), Lemma 3(b) and $\Delta_{t-1}^{\text{add}}(x) \leq \Delta_{t-1}^{\text{add}}(x-1)$ respectively.

For the time monotonicity, it follows because
$$\Delta_{t}^{\text{add}}(x) = \Delta_{t-1}^{\text{add}}(x) + \lambda \left[ K(\Delta_{t-1}^{\text{add}}(x)) - K(\Delta_{t-1}^{\text{add}}(x-1)) \right]$$
$$\geq \Delta_{t-1}^{\text{add}}(x).$$
(b) The property that $p^*_{\text{add}}(\Delta_{t-1}^{\text{add}}(x))$ is increasing in $\Delta_{t-1}^{\text{add}}(x)$ follows from the supermodularity of the function $R$ in $p$ and $\Delta_{t-1}^{\text{add}}(x)$, i.e.,
$$\frac{\partial^2 R(p, \Delta_{t-1}^{\text{add}}(x))}{\partial p \partial \Delta_{t-1}^{\text{add}}(x)} = f(p) > 0.$$ As $\Delta_{t}^{\text{add}}(x) \downarrow x \uparrow t$, it follows that $p^*_{\text{add}}(\Delta_{t-1}^{\text{add}}(x)) \downarrow x \uparrow t$. □

**Proposition 1** shows the monotone structures with respect to inventory level and remaining time for risk-sensitive dynamic pricing with the additive general utility, which includes risk-neutrality as a special case with $u_i(p) = p$. It is worth highlighting that the proof of Proposition 1 is different from that of Theorem 1 in Callegaro and van Ryzin [6] that uses a continuous-time MDP formulation. It is also different from the product usually adopted in dynamic pricing literature through discrete-time formulation, which follows the approach of Proposition 4.3 in Ross [11] that is an induction proof on $x + t$ (e.g. Lemma 5 in [12]). We prove the concavity of $\Delta_{t}^{\text{add}}(x)$ in $x$ by induction on $t$ and other properties follow immediately. This approach is much simpler.

### 3.1. Additive exponential utility

An exponential utility function has many appealing properties and is the most widely used risk-sensitive utility function (see [2]). Furthermore, a carefully selected exponential utility function can be a good approximation for a general utility function in most cases.

Consider the exponential utility function $u_i(\omega) = 1 - e^{-\gamma \omega}$.

Under this specification, the degree of the decision maker's risk-sensitivity to the reward variation is completely captured by one parameter, namely the absolute risk-aversion coefficient,
$$\gamma_i(\omega) = \frac{-u_i''(\omega)}{u_i'(\omega)} = \gamma_i,$$
where $\gamma_i > 0$ corresponds to risk aversion. The Bellman equation of risk-sensitive dynamic pricing under the additive exponential utility function is given by
$$J_{t}^{\text{add}}(x) = J_{t-1}^{\text{add}}(x) + \lambda \max_{p \in P} \left\{ \tilde{F}(p) \left( 1 - e^{-\gamma p} - \Delta_{t-1}^{\text{add}}(x) \right) \right\}.$$ (6)

The monotonicity property of the optimal price in the degree of risk-sensitivity is stated in the following result.

**Proposition 2.** The optimal price $p^*_{\text{add}}(\gamma_i, \Delta_{t-1}^{\text{add}}(x))$ of problem (6), risk-sensitive dynamic pricing under the additive exponential utility function, is decreasing in $\gamma_i$.

**Proof.** See Appendix. □

Barz [2] shows that the optimal protection level is monotone on the degree of risk aversion with an additive utility function.

### 4. Atemporal exponential utility

When the planning horizon is short and the time elapsing between two decisions is negligible, it is realistic to assume an atemporal utility if the decision maker is concerned on the total revenue gained with no time preference. However, using an atemporal utility function in the context of MDP is more challenging analytically than an additive utility function as it requires an enlarged state space to track the wealth level accumulated up to the decision period, which is computationally intractable. An atemporal exponential utility nevertheless has the unique advantage of not increasing the dimension of the state space for the underlying MDP; hence it is commonly used in the literature (see [2,3]).

Risk-sensitive dynamic pricing problem (3) under the atemporal exponential utility function $u_i(\omega) = 1 - e^{-\gamma \omega}$ can be rewritten as follows,
$$\max_{\pi \in \Pi} E_{\pi}^{\omega} \left[ \sum_{t=0}^{T} r_i(X_t, p_i(X_t)) \right]_{X_T = C}$$
$$= 1 - \min_{\pi \in \Pi} E_{\pi}^{\omega} \left[ e^{-\gamma \sum_{t=0}^{T} r_i(X_t, p_i(X_t))} \right]_{X_T = C}.$$ (7)

Let
$$J_i^{\text{amp}}(C) = \min_{\pi \in \Pi} E_{\pi}^{\omega} \left[ e^{-\gamma \sum_{t=0}^{T} r_i(X_t, p_i(X_t))} \right]_{X_T = C}$$ (8)
and the Bellman equation of (8) is given by
$$J_i^{\text{amp}}(x) = \min_{p} \left\{ \lambda \tilde{F}(p) \cdot e^{-\gamma p} J_{i-1}^{\text{amp}}(x - 1) $$
$$+ (1 - \lambda \tilde{F}(p)) J_{i-1}^{\text{amp}}(x) \right\}.$$ (9)
Note that a similar derivation of (9) in the discrete-time formulation can be found in [13] and for the continuous-time formulation in [3,5]. The certainty equivalent of $J_t^{\text{atmp}}(x)$ is given by

$$\tilde{J}_t^{\text{atmp}}(x) = -\frac{1}{\gamma} \ln J_t^{\text{atmp}}(x).$$

We can rewrite (9) in terms of $\tilde{J}_t^{\text{atmp}}(x)$:

$$\tilde{J}_t^{\text{atmp}}(x) = \tilde{J}_{t-1}^{\text{atmp}}(x) - 1 - \ln \left[ 1 + \min_\gamma \frac{(e^{\gamma - \lambda x} - 1)}{\gamma} \right].$$

where $\Delta_{t-1}^{\text{atmp}}(x) = J_t^{\text{atmp}}(x) - J_t^{\text{atmp}}(x-1)$ is the opportunity cost in terms of certainty equivalent. Let

$$R^{\text{atmp}}(p, \gamma, \Delta_{t-1}^{\text{atmp}}(x)) = \tilde{f}(p) \cdot (e^{-\gamma(p - \Delta_{t-1}^{\text{atmp}}(x))} - 1).$$

The optimal price is given by

$$p^{\ast}_{\text{atmp}}(\gamma, \Delta_{t-1}^{\text{atmp}}(x)) = \arg\min_\gamma R^{\text{atmp}}(p, \gamma, \Delta_{t-1}^{\text{atmp}}(x)).$$

The sufficient condition for the regularity of $R^{\text{atmp}}(p, \gamma, \Delta_{t-1}^{\text{atmp}}(x))$ is characterized below.

**Lemma 4.** If $F(\cdot)$ is an increasing failure rate distribution, then $R^{\text{atmp}}(p, \gamma, \Delta_{t-1}^{\text{atmp}}(x))$ is quasiconvex on $p$. 

Fig. 1. Monotonicity of the optimal price under additive and atemporal exponential utility.
To facilitate the proofs of structural properties, define
\[ \tilde{K}(\Delta) = \min_p \{ F(p)(e^{-\gamma p(\Delta - \Delta_1)} - 1) \} \]
and
\[ G(\Delta) = \ln \left[ 1 + \lambda \tilde{K}(\Delta) \right], \]
and their properties are summarized in the following lemma.

**Lemma 5.** \( \tilde{K}(\Delta) \) and \( G(\Delta) \) have the following properties:
(a) \( \tilde{K}(\Delta) \uparrow \Delta \);
(b) \( G(\Delta) \uparrow \Delta \);
(c) If \( \Delta_1 \leq \Delta_2 \) then \( G(\Delta_2) - G(\Delta_1) \leq \gamma (\Delta_2 - \Delta_1) \).

**Proof.** It is evident that parts (a) and (b) are trivial.
(c) Let \( \tilde{G}(\Delta) = G(\Delta) - \gamma \Delta = \ln(1 + \min_p[\lambda \tilde{F}(p)(e^{-\gamma p(\Delta - \Delta_1)} - 1)]) \).

Structural properties are summarized in the following proposition.

**Proposition 3.** The problem (11), which is an equivalent problem of (7) on risk-sensitive dynamic pricing under the atemporal exponential utility function \( u(\omega) = 1 - e^{-\gamma \omega} \), has the following structural properties:
(a) \( \Delta_j^{\text{tamp}}(x) \downarrow x \uparrow t \)
(b) \( p^*_\omega(\gamma, \Delta_j^{\text{tamp}}(x)) \uparrow \Delta_j^{\text{tamp}}(x) \downarrow x \uparrow t \)

**Proof.** See Appendix. \( \square \)

**5. Numerical examples**

In this section, we use a simple numerical example to illustrate the analytical results developed in this study. Given an initial inventory level \( C = 10 \) and a selling time horizon \( T = 15 \). Assume that the probability of a customer arriving during each period of time is \( \lambda = 0.78 \). We consider additive exponential utility functions \( u_i(\omega) = 1 - e^{-\gamma \omega}, \omega = 0, \ldots, T \) and atemporal exponential utility function \( u(\omega) = 1 - e^{-\gamma \omega} \) respectively. Consumer’s willingness-to-pay is assumed to have a Weibull distribution with scale 100 and shape 2, which is an increasing failure rate distribution. Note that a Weibull distribution is flexible with a variety of behavior of reservation prices, such as normal, exponential, a heavier left tail, or a heavier right tail, etc., thus it is frequently used to model the reservation distribution (e.g. [12]).

From Fig. 1 we observe that under both utility functions, the optimal price is decreasing in inventory level \( x \), increasing in time \( t \) and decreasing in the degree of risk-sensitivity \( \gamma \). The risk-sensitive decision maker will set a lower price to hedge against the risks originated from the uncertainty about consumer’s willingness-to-pay.

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### Appendix. Proofs

**Proof of Lemma 2.** The first-order condition (FOC) for (5) is given by

\[
\frac{\partial R^{\text{add}}(p, \Delta_j^{\text{add}}(x))}{\partial p} = \tilde{F}(p) \left[ u'_i(p) - (u_i(p) - \Delta_j^{\text{add}}(x)) \cdot \frac{f(p)}{\tilde{F}(p)} \right] = 0.
\]

The second partial derivative of \( R^{\text{add}}(p, \Delta_j^{\text{add}}(x)) \) evaluated at the FOC point is

\[
\frac{\partial^2 R^{\text{add}}(p, \Delta_j^{\text{add}}(x))}{\partial p^2} = \tilde{F}(p) \left[ u''_i(p) - u'_i(p) \cdot \frac{f(p)}{\tilde{F}(p)} \right] < 0,
\]

where the third term is nonnegative since \( u_i(p; \Delta_j^{\text{add}}(x)) - \Delta_j^{\text{add}}(x) \geq 0 \) from (A.1) and \( \frac{\partial}{\partial p} \left\{ \frac{f(p)}{\tilde{F}(p)} \right\} > 0 \) due to IFR of \( F(\cdot) \). Therefore, \( R^{\text{add}}(p, \Delta_j^{\text{add}}(x)) \) is quasiconcave on \( p \) and (A.1) determines the unique optimal solution \( p^*_\omega(\gamma, \Delta_j^{\text{add}}(x)) \).

**Proof of Proposition 2.** Let \( R^{\text{add}}(p, \gamma_t, \Delta_j^{\text{add}}(x)) = \tilde{F}(p) \left( 1 - e^{-\gamma_t p} - \Delta_j^{\text{add}}(x) \right) \), and \( p^*_\omega(\gamma_t, \Delta_j^{\text{add}}(x)) = \arg\max_{p \in [0,1]} R^{\text{add}}(p, \gamma_t, \Delta_j^{\text{add}}(x)) \).

By Proposition 1(b), we have

\[
\frac{\partial R^{\text{add}}(p, \gamma_t, 0)}{\partial p} = \left[ -f(p)(1 - e^{-\gamma_t p}) + \tilde{F}(p) \gamma_t e^{-\gamma_t p} \right]_{p=p^*_\omega(\gamma_t, \Delta_j^{\text{add}}(x))} \leq 0.
\]

Thus it is true for

\[
\frac{\partial^2 R^{\text{add}}(p, \gamma_t, \Delta_j^{\text{add}}(x))}{\partial p \partial \gamma_t} \bigg|_{p=p^*_\omega(\gamma_t, \Delta_j^{\text{add}}(x))} = \left[ (1 - \gamma_t p) \tilde{F}(p) - f(p)p \right] e^{-\gamma_t p} \leq \left[ 1 - e^{-\gamma_t p} - \gamma_t p \right] \tilde{F}(p)e^{-\gamma_t p} < 0,
\]

where the first inequality follows from (A.2) and the second inequality follows from the fact that \( 1 - e^{-\gamma_t p} - \gamma_t p < 0 \) for \( \forall \gamma_t > 0 \). By Lemma 1 we have \( \frac{\partial^2 R^{\text{add}}(p, \gamma_t, \Delta_j^{\text{add}}(x))}{\partial p^2} < 0 \). Therefore \( p^*_\omega(\gamma_t, \Delta_j^{\text{add}}(x)) \) is decreasing in \( \gamma_t \).

**Proof of Lemma 4.** The FOC for (12) is given by

\[
\frac{\partial R^{\text{tamp}}(p, \gamma, \Delta_j^{\text{tamp}}(x))}{\partial p} = \tilde{F}(p) \left[ -\gamma e^{-\gamma(p - \Delta_j^{\text{tamp}}(x))} - (e^{-\gamma(p - \Delta_j^{\text{tamp}}(x))} - 1) \cdot \frac{f(p)}{\tilde{F}(p)} \right] = 0.
\]
The second partial derivative of $R_{\text{atmp}}^\gamma(p, \gamma, \Delta_{t-1}^\text{atmp}(x))$ evaluated at the FOC point is
\[
\frac{\partial^2 R_{\text{atmp}}^\gamma(p, \gamma, \Delta_{t-1}^\text{atmp}(x))}{\partial p^2} = f(p) \left[ \gamma^2 e^{-\gamma(p-\Delta_{t-1}^\text{atmp}(x))} - (e^{-\gamma(p-\Delta_{t-1}^\text{atmp}(x))} - 1) \frac{\partial}{\partial p} \left( \frac{f(p)}{F(p)} \right) \right] + \gamma e^{-\gamma(p-\Delta_{t-1}^\text{atmp}(x))} \frac{\partial}{\partial p} \left( \frac{f(p)}{F(p)} \right) > 0,
\]
where the second term is negative since $e^{-\gamma(p-\Delta_{t-1}^\text{atmp}(x))} - 1 \leq 0$ from (A.3) and IFR of $F(.)$. Therefore, quasiconvexity of $R_{\text{atmp}}^\gamma(p, \gamma, \Delta_{t-1}^\text{atmp}(x))$ in $p$, together with (A.3), leads to the unique optimal solution $p_{\text{atmp}}^\ast(\gamma', \Delta_{t-1}^\text{atmp}(x))$. □

**Proof of Proposition 3.** (a) We prove by induction on $t$. It is trivially true when $t = 0$. Assume $\Delta_{t-1}^\text{atmp}(x) \leq \Delta_{t-1}^\text{atmp}(x - 1)$.

We need to show that $\Delta_t^\text{atmp}(x) \leq \Delta_t^\text{atmp}(x - 1)$ and it is true because
\[
\Delta_t^\text{atmp}(x) - \Delta_t^\text{atmp}(x - 1) = \Delta_{t-1}^\text{atmp}(x) - \Delta_{t-1}^\text{atmp}(x - 1)
\]
\[
- \frac{1}{\gamma} \left[ G(\Delta_{t-1}^\text{atmp}(x)) - G(\Delta_{t-1}^\text{atmp}(x - 1)) \right]
\]
\[
+ \frac{1}{\gamma} \left[ G(\Delta_{t-1}^\text{atmp}(x - 1)) - G(\Delta_{t-1}^\text{atmp}(x)) \right]
\]
\[
\leq 0,
\]
where two inequalities follow by Lemma 5(b), Lemma 5(c) and $\Delta_{t-1}^\text{atmp}(x) \leq \Delta_{t-1}^\text{atmp}(x - 1)$ respectively. Then it follows readily that
\[
\Delta_t^\text{atmp}(x) = \Delta_{t-1}^\text{atmp}(x - 1)
\]
\[
- \frac{1}{\gamma} \left[ G(\Delta_{t-1}^\text{atmp}(x)) - G(\Delta_{t-1}^\text{atmp}(x - 1)) \right]
\]
\[
\geq \Delta_t^\text{atmp}(x - 1).
\]