An approach to the fuzzification of complete lattices

Wei Yao∗
Department of Mathematics, Hebei University of Science and Technology, Shijiazhuang, P.R. China

Abstract: This paper presents an approach to the fuzzification of complete lattices. This kind of fuzzification is a functor from the category of complete lattices to the category of complete \( L \)-ordered sets. It not only can be considered as an algebraic representation of the Lowen functor \( \omega_L \), but also can characterize the stratification of \( L \)-topological spaces and the underlying set of Hutton unit interval.

Keywords: Stratified \( L \)-topology, complete lattice, frame, (Complete) \( L \)-ordered set, \( L \)-frame, Lowen functor \( \omega_L \).

1. Introduction

Since Zadeh introduced fuzzy set theory in 1965, many people are interested in defining and constructing fuzzy counterparts of crisp structures. Then several new fields are formed, for example Fuzzy Algebra [24–26, 32], Fuzzy Topology [6, 20], Fuzzy Geometry [5], Fuzzy Graphs [16, 27, 33, 42], Fuzzy Measure Theory and Fuzzy Integrals [17, 34, 37, 44], Fuzzy Order Theory [2–4, 7–9, 12, 35, 43], etc.

People working in fuzzy mathematics may have a rule in their minds that a crisp structure can be considered as a fuzzy structure by replacing subsets by their characteristic functions.

When we define some fuzzy structures, this rule should be kept in mind. For example in (fuzzy) topology theory, the characteristic function of any topology is a fuzzifying topology and the family of characteristic functions of all open sets of a topology is a fuzzy topology. These rules also seem to hold for (fuzzy) orders that, for a partial order \( (P, \leq) \) and a lattice \( L \), the characteristic function \( \chi_{\leq} \) of \( \leq \) from \( P \times P \) to \( L \) is an \( L \)-order on \( P \) (Example 3.6(3) in [38]). But when we put our eyes on complete lattices, a problem occurs:

**Proposition 1.1.** Let \( L \) be a frame. Suppose that there is a nontrivial Boolean element \( a \in L \) (this means \( a \in L \backslash \{0, 1\} \) and there exists an element \( \neg a \in L \) such that \( a \lor \neg a = 1, a \land \neg a = 0 \)). The pair \((L, \chi_{\leq})\), regarded as an \( L \)-ordered set, is not complete. (see Subsection 2.3 for definition of completeness of \( L \)-ordered sets).

**Proof.** Let \( a \in L \backslash \{0, 1\} \) be a nontrivial Boolean element and let \( A \) be the constant map from \( L \) to \( L \) with the value \( a \). If \( b \) is the join of \( A \), then

\[
\chi_{\leq}(b, a) = \bigwedge_{c \in L} A(c) \rightarrow \chi_{\leq}(c, a) = \bigwedge_{c \in L} a \rightarrow \chi_{\leq}(c, a) = a \rightarrow \left( \bigwedge_{c \in L} \chi_{\leq}(c, a) \right) = a \rightarrow 0 = \neg a.
\]

Then \( \neg a = 0 \) or \( \neg a = 1 \) and it follows that \( a \in \{0, 1\} \), which is a contradiction. □
There are many lattices, for example every nontrivial Boolean algebra, that satisfy the condition in Proposition 1.1. What is the reason of this problem? We now pay attention to the fuzzification of complete lattices from aspects of logic. Proposition 1.1 tells us that the fuzzification of \((L, \leq)\) as a complete lattice generally could not be \((L, \leq_{x})\). In fact, some mathematical structures related to logic can be considered as two-point lattice-valued fuzzy structures, that is, the truth value table for crisp mathematical structures is the two-point lattice \(2\). For lattice-valued based structure, we consider as two-point lattice \(L\), the truth value table of course is \(L\). So in our opinion, if we define a fuzzification from the complete lattices to complete \(L\)-ordered sets, firstly it should be held that the fuzzification of \(2\) is the lattice \(L\).

Categorical speaking, every fuzzification should be a functor from the category of crisp structures to the category of fuzzy structures, and when the truth value table is \(2\), this functor is an isomorphism (but generally not an equality). For example, the Lowen functor \(\omega_{L}\) is a functor from \(\text{Top} \rightarrow \mathcal{L}\). (Notice that \(\text{Top} \neq 2\).)

The aim of this paper is to introduce a fuzzification, precisely a functor, from the category of complete lattices to the category of complete \(L\)-ordered sets. For the case of \(L\) being a completely distributive lattice, this fuzzification can be considered as a generalization of Liu-Zhang’s functor [21] from the category of frames to the category of their \(L\)-frames, and more interesting, as the algebraic representation of the Lowen functor \(\omega_{L}\).

As applications, this functor can characterize the stratification of \(L\)-topological spaces and the underlying set of the Hutton unit interval.

2. Preliminaries

2.1. Category theory

For category theory, we recommend [1, 23].

Given two functors \(S, T : A \rightarrow B\), a natural transformation \(a : S \Rightarrow T\) is a function which assigns each object \(A\) a \(B\)-morphism \(a_{A} : S(A) \rightarrow T(A)\) such that for every \(A\)-morphism \(f : A \rightarrow A'\), it holds that \(T(f) \circ a_{A} = a_{A'} \circ S(f)\).

2.2. Lattices and fuzzy topology

For lattices and fuzzy topology, we recommend [10, 14].

A map \(f : P \rightarrow Q\) between two posets is called monotone if for every \(x, y \in P, x \leq y\) always implies \(f(x) \leq f(y)\). Let \(f : P \rightarrow Q, g : Q \rightarrow P\) be two monotone maps between posets. The pair \((f, g)\) is called an adjunction between \(P\) and \(Q\) if \(f(x) \leq y \Rightarrow x \leq g(y)\) for every \(x \in P, y \in Q\), where \(f\) (resp., \(g\)) is called the left adjoint (resp., right adjoint) of \(g\) (resp., \(f\)). It is well-known that if a map \(f : M \rightarrow N\) between complete lattices preserves arbitrary joins (resp. meets), then it uniquely determines a right (resp. left) adjoint \(f^\star : N \rightarrow M\) (resp. \(f^\flat : N \rightarrow M\)), which preserves arbitrary meets (resp. joins).

For a bounded lattice, we always use \(0, 1\) to denote its least (bottom) element and largest (top) element, called the zero and unit, respectively. Let \(L\) be a lattice. An element \(a \in L\) is called coprime if for every \(b, c \in L\), \(a \leq b \lor c\) always implies \(a \leq b\) or \(a \leq c\). The set of all non-zero coprime elements of \(L\) is denoted by \(J(L)\).

A complete lattice \(L\) is called a frame or a complete Heyting algebra, if the binary meet is distributive over arbitrary joins, that is, for every \(x \in L\) and every \(S \subseteq L\), we have \(x \land \bigvee S = \bigvee_{x \leq s \forall s \in S} x \land s\), or equivalently, there is a derived operation \(\rightarrow\) on \(L\) such that for every \(a, b, c \in L\), \(a \land (b \rightarrow c) \equiv a \land b \land c\). A frame morphism between two complete lattices is a map that preserves finite meets and arbitrary joins. Note that for a frame morphism \(f : A \rightarrow B\), we have \(f(0) = 0\) (since \(1\) is the meet of empty set and \(0\) is the join of empty set). The category of complete lattices with frame morphisms is called the category of semiframe [30], denoted by \(\text{SFrm}\), which firstly appeared in [28] under the symbols CSLF. And \(\text{Frm}\) is its full subcategory of frames. Clearly, for objects a semiframe has no differ from a complete lattice. The opposite category of \(\text{SFrm}\) (resp. \(\text{Frm}\)) is called the category of semilocales (resp., locales), denoted by \(\text{SLoc}\) (resp., \(\text{Loc}\)).

Let \(L\) be a complete lattice. For \(a, b \in L\), \(a\) is called way-below \(b\), in symbols \(a \ll b\), if for every directed subset \(D\) of \(L\), \(a \ll d\) always implies \(a \leq d\) for some \(d \in D\). \(L\) is called a continuous lattice if for every \(b \in L\), \(a \ll b\) where \(\ll b = \{a \in L \mid a \ll b\}\) (\(\ll b\) is always directed since \(L\) is complete).

For \(a, b \in L\), \(a\) is called wedge-below \(b\), in symbols \(a \ll b\), if for every subset \(D\) of \(L\), \(b \leq \bigvee D\) always implies \(a \leq d\) for some \(d \in D\). \(L\) is called completely distributive if for every \(b \in L\), \(b = \bigvee P(b)\), where \(P(b) = \{a \in L \mid a \ll b\}\) [36]. Put \(P(b) = P(b) \cap J(L)\), we have \(L\) is completely distributive then \(b = \bigvee P(b)\) for every \(b \in L\). It is well-known that every completely distributive lattice is simultaneously a continuous lattice and a frame.
For a map \( f : X \to Y \), define \( \beta^{\sigma}_{L}(A) = \bigvee_{x \in A} (A(x) \land \sigma) \) by \( \beta^{\sigma}_{L}(A) = \bigvee_{x \in A} (A(x) \land \sigma) \) and \( \beta^{\sigma}_{L}(B) = \bigvee_{y \in B} (B(y) \land \sigma) \), called the L-valued Zadeh function or L-forward power set operator, and define \( \beta^{\sigma}_{L}(B) = \bigvee_{y \in B} (B(y) \land \sigma) \) by \( \beta^{\sigma}_{L}(B) = \bigvee_{y \in B} (B(y) \land \sigma) \), called the L-backward power set operator [29].

The upper topology \( \sigma(L) \) on a complete lattice is the topology generated by the subbasis \( \{ [a, \infty) | a \in L \} \). The Scott topology \( \sigma(L) \) on a continuous lattice \( L \) is the topology generated by the basis \( \{ [0, a] | a \in L \} \). It is well-known that if \( L \) is completely distributive, then \( \sigma(L) = \sigma(L) \) (Theorem VII.3.4 in [10]).

For a set \( X \) and an element \( a \in L \), we still use \( a \) to denote the constant \( L \)-subset with the value \( a \). An \( L \)-topology on a set \( X \) is a subfamily \( \delta \subseteq L^{X} \) that satisfies (O1) \( 0 \in \delta \), (O2) for any \( A, B \in \delta \), \( A \land B \in \delta \), (O3) for any \([A, e] \subseteq \delta \), \( \bigvee_{e \in A} X \subseteq \delta \). An \( \sigma(L) \)-topology \( \delta \) on \( X \) is called stratified if \( (OS) \subseteq \delta \) for all \( a \in L \).

Let \( (X, \tau) \) be a topological space and \( L \) be a completely distributive lattice. \( A \subseteq L^{X} \) is called a lower semicontinuous function w.r.t. \( \tau \) if for every \( a \in L \), \( \nu(A) = \{ x \in X | A(x) \subseteq a \} \) is in the family of all lower semicontinuous functions w.r.t. \( \tau \). \( \nu(A) \) is denoted by \( \nu(A)(\tau) \), which forms a stratified \( L \)-topology on \( X \). It is easy to see that \( A \subseteq L^{X} \) is lower semicontinuous if \( A : (X, \tau) \to (L, \sigma(L)) \) is continuous. The transformation \( \nu(A) \) forms a functor from \( \mathbf{Top} \) to \( \mathbf{LTop} \); this is the famous Lowen functor.

**Lemma 2.1**. (Corollary 6 in [22]) Suppose that \( K \) is a completely distributive lattice and \( (X, \mathcal{T}) \) is a topological space. Then the following statements are equivalent:

1. \( \forall t \in L \), \( \nu(t)(\mathcal{T}) \in \mathcal{T} \).
2. \( \forall t \in L \), \( \nu^{\sigma}_{L}(t)(\mathcal{T}) \in \mathcal{T} \).
3. \( \forall t \in L \), \( \nu^{\sigma}_{L}(t)(\mathcal{T}) \in \mathcal{T} \).

**2.3. (Complete) L-ordered sets and L-frames**

For (complete) \( L \)-ordered sets and \( L \)-frames, we refer to [39, 40].

An \( L \)-ordered set is a pair \((X, e)\) of a set \( X \) and a map \( e : X \times X \to L \), satisfying that:

- (E1) \( \forall x \in X, e(x, x) = 1 \);
- (E2) \( \forall x, y, z \in X, e(x, y) \land e(y, z) \leq e(x, z) \);
- (E3) \( \forall x, y \in X, e(x, y) = e(y, x) \) implies \( x = y \).

If \((X, e)\) is an \( L \)-ordered set, then \( \subseteq : \{(x, y) \in X \times X | e(x, y) = 1\} \) is a crisp order \( X \), this poset is denoted by \((X, e)\). For any \( A, B \subseteq L^{X} \), the substraction degree \[11\] of \( A \) in \( B \) is defined by (Theorem 3.5 in [41]) let \( f : X \to Y \) be a L-valued function on \( X \), which forms a \( L \)-frame by means of \( L \)-orders. Let \( (X, e) \) be a complete \( L \)-ordered set. Then \( \mathcal{A} \) is a complete lattice, where \( \mathcal{V} \subseteq L^{X} \) for every \( A \subseteq L \). Let \((A, e)\) be a complete \( L \)-ordered set and let \( \mathcal{A} \) be the meet operation in \( \mathcal{A} \). A complete \( L \)-ordered set \((A, e)\) is called an \( L \)-frame if for every \( a \in L \), the map \( a \wedge - : (A, e) \to (A, e) \) is monotone.
and preserves joins of L-fuzzy subsets, that is for every \( S \in L^A \),
\[
a \land S = \xi(a \land \neg \gamma_S^*(S)),
\]
where \( (a \land \neg \gamma_S^*) \) is the Zadeh forward powerset operator w.r.t. \( a \land \neg \gamma \). A map \( f : (A, e_A) \to (B, e_B) \) is called an L-frame homomorphism if \( f \) preserves finite meets and joins of L-fuzzy subsets. The resulting category is denoted by \( \text{L-Frm} \) and its dual \( \text{L-Loc} \).

Similarly, \( \text{S-Frm} \) and \( \text{S-Loc} \), we define \( \text{S-Loc} \) as the category of complete \( L \)-ordered sets and \( L \)-frame-homomorphisms.

For a topological space \((X, T)\), \( \Omega((X, T)) = (T, \subseteq) \) is a frame and \( \Omega : \text{Top} \to \text{Loc} \) is a functor. Similarly, for a stratified \( L \)-topological space \((X, \delta), \Omega_2((X, \delta)) = (\delta, \triangleq, \lambda) \) is an \( L \)-frame and \( \Omega : \text{SL-Top} \to \text{L-Loc} \) is a functor.

### 3. An approach to the fuzzification of complete lattices: the algebraic representation of the Lowen functor \( sl \)

#### 3.1. An approach to the fuzzification of complete lattices

Suppose that \( C \) is a complete lattice. A map \( \lambda : C \to L \) is called a \( \text{S-Loc} \)-map if for every \( S \in C, \lambda(S) = \bigvee_{s \in S} \lambda(s) \). Let \( C(L) \) be the set of all \( \text{S-Loc} \)-maps from \( C \) to \( L \). It is easily seen that every element in \( C(L) \) is an antitone map and for every \( \lambda \in C(L), \lambda(0) = 1 \). And the map sending 0 to 1 and sending every nonzero element to a fixed element \( a \in L \) is a \( \bigvee-\bigwedge \) map from \( C \) to \( L \), we denote such a map as \( \lambda \).

**Proposition 3.1.** (1) \( C(L) \) is a complete lattice under pointwise order, where for every \( \{\lambda_j\} \in C(L) \) and every \( x \in C \),
\[
\left( \bigwedge_{j} \lambda_j \right)(x) = \bigwedge_{j} \lambda_j(x)
\]
and for \( \{a_j\} \in L \), \( \bigvee_{j} a_j = \bigvee_{a_j \geq x} a_j \).

(2) If \( L \) is completely distributive, then
\[
\left( \bigvee_{j} \lambda_j \right)(x) = \bigvee_{a_j \geq x} \bigwedge_{j} \lambda_j(a_j).
\]

**Proof.** (1) We only need to show that \( \bigwedge_{j} \lambda_j \) computed as the equation \( \left( \bigwedge_{j} \lambda_j \right)(x) = \bigwedge_{j} \lambda_j(x) \) is an element of \( C(L) \). On one hand,
\[
\left( \bigwedge_{j} \lambda_j \right)(0) = \bigwedge_{j} \lambda_j(0) = \bigwedge_{j} 1 = 1.
\]
In fact, on the other hand, for a nonempty subset \( S \subseteq C \),
\[
\left( \bigwedge_{j} \lambda_j \right)(S) = \bigwedge_{j} \lambda_j(\bigvee S) = \bigwedge_{j} \lambda_j \bigwedge_j \lambda_j(k) = \bigwedge \lambda_j(\lambda_j(k)) = \bigwedge \lambda_j(\lambda_j(k)).
\]
The equation \( \bigvee_{a_j \geq x} \bigwedge_{j} \lambda_j(a_j) \) is routine.

(2) Suppose that \( L \) is completely distributive. For \( \{\lambda_j\} \in C(L) \), define \( f : C \to L \) by
\[
f(x) = \bigvee_{a_j \geq x} \bigwedge_{j} \lambda_j(a_j).
\]
We need to show \( f \) is the join of \( \{\lambda_j\} \). In fact,
(i) \( f \in C(L) \). It is easily seen that \( f \) is antitone and \( \bigvee_{a_j \geq x} \bigwedge_{j} \lambda_j(a_j) = \bigwedge_{j} \lambda_j(\bigvee_{a_j \geq x} a_j) = \bigwedge_{j} \lambda_j(1) = 1 \). For every \( \{a_j\} \in L \), let \( \{a_j\} \). Then \( m < f(a) = \bigvee_{a_j \geq x} \bigwedge_{j} \lambda_j(a_j) \) and then there exists \( \{a_j\} \). For every \( k \in K \), \( a_j = \bigwedge_{a_j \geq x} a_j \), then
\[
\bigvee_{a_j \geq x} a_j = \bigvee_{a_j \geq x} \bigwedge_{j} \lambda_j(a_j) \geq \bigvee_{a_j \geq x} a_j.
\]
Therefore
\[
f(\bigvee_{a_j \geq x} a_j) \geq \bigvee_{a_j \geq x} \bigwedge_{j} \lambda_j(a_j)
\]
and the antitone map of \( f \) and the antitone map of \( f \), we have \( f(\bigvee_{a_j \geq x} a_j) = \bigvee_{j} f(a_j) \).

(ii) \( f \in C(L) \). Firstly, for \( j \notin J \), put \( a_j = x \) if \( j \notin J \) and \( a_j = 0 \) otherwise, then \( \bigvee_{a_j = \pi} a_j = x \) and then \( f(x) \geq \bigvee_{a_j = \pi} \lambda_j(x) = \bigvee_{a_j = \pi} \lambda_j(x) \) (notice that \( g(0) = 1 \) for every \( g \in C(L) \)). Hence \( f \) is an upper bound of \( \{\lambda_j\} \). Secondly, suppose that \( g \) is an upper bound of \( \{\lambda_j\} \). For every \( x \in C \),
Proposition 3.2. Let $\mathcal{L}$ be a complete lattice in classical mathematics. Then $\mathcal{L}$ is a frame.

Proof. (1) It is routine to show that $\mathcal{L}$ is a frame. For every $a \in \mathcal{L}$, put $\lambda(a) = \bigvee \{ x \in \mathcal{L} \mid x(a) = 1 \}$. Then $\lambda(\mathcal{L}) = \mathcal{L}$ and $\lambda$ is a frame.

(2) Every $\lambda \in 2(\mathcal{L})$ is a $\lambda$-map, we have $\lambda(0) = 1$ and thus $\lambda$ is exactly uniquely determined by $\lambda(1)$ in $\mathcal{L}$. Hence $2(\mathcal{L}) \cong \mathcal{L}$.

Proposition 3.3. If $L$ is a frame, then $(\mathcal{L}, \text{subc})$ is a complete L-ordered set, where $(\forall a, x) = \bigvee \mathcal{L}(\lambda) \to \lambda(x).

Proof. For all $x \in \mathcal{L}(\lambda)$, define $\lambda(a) : C \to \mathcal{L}$ by $\lambda(a) = \bigvee \mathcal{L}(\lambda) \to \lambda(x)$. Then $\lambda(a) \in \mathcal{L}(\lambda)$. In fact, (1) $\lambda(a) \in \mathcal{L}(\lambda)$. For every $x \subseteq \mathcal{C}$, $\lambda(a) \to \lambda(x)$.

(1) $\lambda(a) \in \mathcal{L}(\lambda)$. For every $x \subseteq \mathcal{C}$, $\lambda(a) \to \lambda(x)$.

(2) For every $\mu \in \mathcal{L}(\lambda)$, $\mu \to \lambda(a)$.

(3) For every $\lambda \in \mathcal{L}(\lambda)$, define $f(\lambda) : C \to \mathcal{L}$ by $f(\lambda)(\mu) = \bigvee \{ x \in \mathcal{L}(\lambda) \mid x \leq f(x) \}$. Then $f(\lambda) \in \mathcal{L}(\lambda)$.

(4) $f$ is a fuzzy left adjoint of $f^*$. $\lambda(a) \to \lambda(x)$.
$(f_1, f^*)$ is an adjunction between $C_2(L)$ and $C_1(L)$. To complete the proof, by Proposition 2.3, we only need to show that both $f_1$ and $f^*$ are fuzzy monotone. In fact, for every $\lambda_1, \lambda_2 \in C_1(L)$,

$$
\begin{align*}
sub_{C_2}(f_1(\lambda_1), f_1(\lambda_2)) &= \bigwedge_{y \in C_2} f_1(\lambda_1)(y) \\
&= \bigwedge_{y \in C_2} (V(y) \wedge y \leq f(\lambda_1)(y)) \\
&= \bigwedge_{y \in C_2} (V(y) \wedge y \leq f(\lambda_2)(y)) \\
&= \bigwedge_{y \in C_2} f_1(\lambda_2)(y) \\
&= \bigwedge_{y \in C_2} f_1(\lambda_2)(y) \\
&= \sub_{C_2}(\lambda_1, \lambda_2).
\end{align*}
$$

And for every $\mu_1, \mu_2 \in C_2(L)$,

$$
\begin{align*}
\sub_{C_1}(f^*(\mu_1), f^*(\mu_2)) &= \bigwedge_{x \in C_1} f^*(\mu_1)(x) \\
&= \bigwedge_{x \in C_1} f^*(\mu_1)(x) \\
&= \bigwedge_{x \in C_1} \mu_1(f(x)) \\
&= \bigwedge_{x \in C_1} \mu_1(f(x)) \\
&= \sub_{C_1}(\mu_1, \mu_2).
\end{align*}
$$

Proposition 3.5. Suppose that $L$ is a completely distributive lattice. Then there is a functor $\mathcal{W}_L$ of $\text{SFrm}$ which sends every $\text{SFrm}$-morphism $f : C_1 \rightarrow C_2$ to $f_1 : C_1(L) \rightarrow C_2(L), \lambda \mapsto f(\lambda)$.

Proof. Firstly, the largest element and the least element of $C_1(L)$ (resp., $C_2(L)$) are $\bar{1}_L$ and $\bar{0}_L$ (resp., $\bar{1}_C$ and $\bar{0}_C$), respectively. It is easily checked that $f_1(\bar{1}_L) = \bar{1}_C$ and $f_1(\bar{0}_L) = 0_C$.

Secondly, for $\lambda_1, \lambda_2 \in C_1(L)$, for every $\lambda \in C_2$,

$$
\begin{align*}
& f_1(\lambda_1) \wedge f_1(\lambda_2) \\
& \leq f_1(\lambda_1) \wedge f_1(\lambda_2) \\
& = f_1(\lambda_1) \wedge f_1(\lambda_2) \\
& = f_1(\lambda_1) \wedge f_1(\lambda_2) \\
& \leq f_1(\lambda_1) \wedge f_1(\lambda_2) \\
& = f_1(\lambda_1) \wedge f_1(\lambda_2). \quad \text{It follows that } f_1(\lambda_1) \wedge f_1(\lambda_2) \leq f_1(\lambda_1 \wedge \lambda_2) \text{ and then } f_1(\lambda_1 \wedge \lambda_2) = f_1(\lambda_1 \wedge \lambda_2) \text{ since } f_1 \text{ is monotone.}
\end{align*}
$$

Finally, since $(f_1, f^*)$ is a fuzzy adjunction between $(C_2(L), \sub_{C_2})$ and $(C_1(L), \sub_{C_1})$, by Proposition 2.4, we have $f_1$ preserves joins of $L$-fuzzy subsets.

Hence, $f_1 : C_1(L) \rightarrow C_2(L)$ is a morphism in $\text{SL- Frm}$.

3.2. $\mathcal{W}_L$ as an algebraic representation of the Lowen functor $\omega_L$

In 1995 for $L$ a frame, Zhang and Liu [45] defined a category of $L$-frames as a comma category ($L \downarrow \text{Frm}$), i.e., the category of objects in $\text{Frm}$ under a fixed object $L$. A Zhang-Liu-$L$-frame [45] is a pair $(\lambda, A)$, where $A$ is a frame and $\lambda : L \rightarrow A$ is a $\text{Frm}$-morphism. A homomorphism $f : (\lambda_A, A) \rightarrow (\lambda_B, B)$ between Zhang-Liu-$L$-frames is a $\text{Frm}$-morphism $f : A \rightarrow B$ such that $f \circ \lambda = \lambda_B$. It is shown in [40] that $L$-frames in sense of Yao and that in sense of Zhang-Liu are categorical isomorphic to each other (Proposition 3.11).

In the following, we will study when $C(L)$ becomes an $L$-frame and $\mathcal{W}_L$ can be restricted as a functor from $\text{Frm}$ to $L \text{ Frm}$ by means of Zhang-Liu-$L$-frames. The contents in this subsection are a modification of that in [21].

Proposition 3.6. (Lemma 3.5 [21]) If both $L$ and $C$ are frames, then $C(L)$ is an $L$-frame.

Proof. Step 1. $C(L)$ is an frame. Let $f, g : C(L) \rightarrow C(L)$ (i.e., $\omega_L$). We need to show that for every $\lambda \in C(L)$, $\lambda \in f(\lambda)$ and $g(\lambda)$. In fact,

$$
\begin{align*}
[f \wedge (V f \wedge g)](\lambda) &= f(\lambda) \wedge (V f \wedge g)(\lambda) \\
&= f(\lambda) \wedge \bigvee_{s \in L} \Lambda f(\lambda) \\
&= \bigvee_{s \in L} \Lambda f(\lambda) \wedge g(\lambda)) \\
&= \bigvee_{s \in L} \Lambda(f(\lambda) \wedge g(\lambda)).
\end{align*}
$$

and

$$
\begin{align*}
[V f \wedge g](\lambda) &= \bigvee_{s \in L} \Lambda(f(\lambda) \wedge g(\lambda)) \\
&= \bigvee_{s \in L} \Lambda(f(\lambda) \wedge g(\lambda)) \\
&= \bigvee_{s \in L} \Lambda(f(\lambda) \wedge g(\lambda)).
\end{align*}
$$

For every $\lambda \in C(L)$, $\lambda \in f(\lambda)$ and $\lambda \in g(\lambda)$.

Step 2. Define $i_\lambda : L \rightarrow C(L)$ by $i_\lambda(\lambda) = \lambda$, then $i_\lambda$ is a $\text{Frm}$-morphism. In fact, firstly, $i_\lambda(0) = 0 = 0_C(L)$. 

and \( i_L(t) = 1 = 1_{W_L} \). Secondly, for every \( a, b \in L \),

\[
i_L(a \sqcap b) = a \sqcap b = a \sqcap b = i_L(a) \sqcap i_L(b).
\]

Finally, for every \( j \in J \subseteq L \),

\[
i_L \left( \bigvee_j a_j \right) = \bigvee_j a_j = \bigvee_j a_j = \bigvee_j i_L(a_j).
\]

\[\square\]

**Proposition 3.7.** (Theorem 3.4 in [21]) If \( L \) is a completely distributive lattice, then \( W_L : \text{Frm} \to L \text{- Frm} \) is a functor from \( \text{Frm} \) to \( L \text{- Frm} \) by restriction.

Suppose that \( (X, T) \) be a topological space.

**Proposition 3.8.** Suppose that \( f : T \to L \) is a \( \bigcup \cdots \bigvee \text{-map} \), i.e., \( f \in T \) \((L) \). Define \( U_f = \bigvee_{x \in V \in T} f(V) \). Then \( U_f \in W_L(T) \).

\[\text{Proof.}\]

For every \( a \in L \),

\[\epsilon_a(U_f) = \{ x \in X \mid \bigvee_{x \in V \in T} f(V) \nleq a \}
= \{ x \in X \mid \exists x \in V \in T, f(V) \nleq a \}
= \{ \lambda \in T \mid f(V) \nleq a \}
\in T.\]

\[\square\]

**Proposition 3.9.** Let \( U \in W_L(T) \). Define \( f : T \to L \) by \( f_U(V) = \bigvee_{x \in U} U(x) \). Then \( f_U : T \to L \) is a \( \bigcup \cdots \bigvee \text{-map} \).

**Proof.** For \( \{ V_j, j \in J \} \subseteq T \), if \( J = \emptyset \), then \( f_U(\emptyset) = 1 \); otherwise, \( f_U(\bigcup_{j \in J} V_j) = \bigvee_{x \in \bigcup_{j \in J} V_j} U(x) = \bigvee_{x \in \bigcup_{j \in J} V_j} \bigvee_{x \in V} f_U(V) \).

\[\square\]

**Proposition 3.10.** Let \( L \) be a completely distributive lattice. Suppose that \( f : T \to L \) is a \( \bigcup \cdots \bigvee \text{-map} \) and \( U \in W_L(T) \). Then \( f \) is \( f_U \).

\[\text{Proof.}\]

(1) For every \( V \in T \),

\[
f_U(V) = \bigvee_{x \in V} U_f(x) = \bigvee_{x \in V} \bigvee_{x \in W \in T} f(W).
\]

Firstly, it is easy to see that \( f_U(V) \geq f(V) \). Secondly, for every \( a \in \bigvee_{x \in V} \bigvee_{x \in W \in T} f(W) \), we have \( a \leq \bigvee_{x \in W \in T} f(W) \) for every \( V \in T \) and then \( a \leq f(W) \) for every \( V \in T \) such that \( x \in W \in T \). Put \( W = \bigcup_{x \in V} W_x \).

Then \( V \subseteq W \) and

\[
f(V) \leq f(W) = f \left( \bigcup_{x \in V} W_x \right) = \bigwedge_{x \in V} f(W_x) \geq a.
\]

Hence, \( f(V) \geq f(U) \).

(2) For every \( x \in X \),

\[
U_f(x) = \bigvee_{x \in V \in T} f_U(V) = \bigvee_{x \in V \in T} U(V).
\]

Firstly, it is easy to see that \( U_f(x) \leq U(x) \). Secondly, for every \( U \in U(x) \), we have \( x \in U \in T \) and

\[
\bigvee_{x \in V \in T} U(V) \geq \bigvee_{x \in V \in T} U(V) \geq 1.
\]

Hence \( U(x) \leq U_f(x) \).

\[\square\]

**Proposition 3.11.** If \( L \) is a complete distributive lattice, then \( U_L(T) \) is a \( \bigcup \cdots \bigvee \text{-map} \).

The opposite functors \( \text{P}3 \) in [23] of \( W_L : \text{Frm} \to L \text{- Frm} \) is a functor \( W_L^\text{op} : L \text{- Loc} \to L \) such that \( A \mapsto \text{W}_L(A) \) and \( f^\text{op} \mapsto \text{W}_L(f)^\text{op} \) for every object \( L \) and every morphism \( f \) in \( \text{Frm} \).

**Theorem 3.12.** If \( L \) is a completely distributive lattice, then the following diagram is commutative up to natural transformation, that is \( \Omega_L \circ u_{\Omega_L} \cong \Omega_L^{(f)} \circ \Omega_L \).

Therefore, \( W_L \) can be considered as an algebraic representation of Lowen functor \( u_{\Omega_L} \).

**Proof.** Suppose that \( (X, T) \) is an object in \( \text{Top} \). Define \( \alpha_L : \{ (W_L)^{(f)} \circ \Omega_L(X, T) \to (\Omega_L \circ u_{\Omega_L})(X, T) = T(L) \to u_{\Omega_L}(T) \) by

\[
\alpha_L(x)(a) = \bigvee_{x \in V \in T} \lambda(V).
\]

Suppose that \( f : (X, T) \to (Y, T) \) is a morphism in \( \text{Top} \). Firstly,

\[
(\Omega_L(f))^\text{op} = f^\text{op} : (T_2, \subseteq) \to (T_1, \subseteq)
\]

and then \( W_L(\Omega_L(f))^\text{op} = W_L[f(\Omega_L(f))^\text{op}] = W_L \).

\[
(f^\text{op})^\text{op} = f^\text{op} : T_2 \to T_1 \text{ is computed as }
\]

\[
[(W_L)^{(f)} \circ \Omega_L(f)]^\text{op}(x) = (f^\text{op})^\text{op}(x).
\]
Proof. Since \( f \) is meet-preserving, \( f^{-1} \) is join-preserving and for every \( \lambda \in \mathcal{C} \), \( f^{-1}(\lambda) = \lambda \circ f^{-1} \) is a \( \vee \) - \( \wedge \) map. Therefore, \( f^{-1} \) is a map.

Suppose that \( f: \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) is a morphism in \( \text{CMSL} \). Define \( \tilde{f}: \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) by \( \tilde{f}(\lambda) = \lambda \circ f^{-1} \). Then \( \tilde{f} : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \), \( f \rightarrow \tilde{f} \) defines a functor.

Proposition 3.13. Suppose that \( f : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) is a morphism in \( \text{CMSL} \). Define \( \tilde{f} : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) by \( \tilde{f}(\lambda) = \lambda \circ f^{-1} \). Then \( \tilde{f} : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \), \( f \rightarrow \tilde{f} \) defines a functor.

Proof. Since \( f \) is meet-preserving, \( f^{-1} \) is join-preserving and for every \( \lambda \in \mathcal{C}_1 \), \( f^{-1}(\lambda) = \lambda \circ f^{-1} \) is a \( \vee \) - \( \wedge \) map. Therefore, \( f^{-1} \) is a map.

Suppose that \( f: \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) is a morphism in \( \text{CMSL} \). For every \( A \in \mathcal{L} \) and every \( \gamma \in \mathcal{C}_2 \),

\[
F(f)(\gamma A)(y) = (\gamma A)^{f^{-1}}(y)
\]

Thus, \( \tilde{f} : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \), \( f \rightarrow \tilde{f} \) defines a functor. \( \square \)

4. Some applications

4.1. Stratification of \( \mathcal{L} \)-topology

Suppose that \( L \) is a frame. For an \( \mathcal{L} \)-topology \( \mathcal{L} \), put \( G_1^L(\delta) = \delta \cup \{ a \in L \} \), i.e., \( G_1^L(\delta) \) is the \( \mathcal{L} \)-topology generated by the subbasis \( \{ b \} \cup \{ a \} \). If \( \delta \) is the least-stratified \( \mathcal{L} \)-topology which is greater than \( \lambda \) and further \( G_1^L(\delta) \) becomes a concrete functor from \( \mathcal{L} \)-Top to \( \mathcal{L}_+\text{-Top} \), called the functor of stratification [31].

Proposition 4.1. The family \( \{ a \wedge A \mid a \in L, A \in \mathcal{L} \} \) is a basis of \( G_1^L(\delta) \).

Proof. It is straightforward. \( \square \)

Suppose that \( \delta \) is an \( \mathcal{L} \)-topology on \( X \). For every \( \lambda \in \delta \), define \( F(\lambda) = \bigvee_{A \in \delta} (\lambda \wedge A) \). For every \( \lambda \in G_1^L(\delta) \), define \( G(\lambda) : \lambda \rightarrow \mathcal{L} \), by \( G(\lambda)(V) = \bigwedge_{A \in \lambda} (\lambda \wedge A) \).

Proposition 4.2. \((F, G)\) is an adjunction between \( (\delta, \mathcal{L}) \) and \( (G_1^L(\delta), \mathcal{L}) \).

Proof. Step 1. It is routine to show that \( F : \delta \rightarrow (G_1^L(\delta), \mathcal{L}) \) are well-defined maps.

Step 2. \((F, G)\) is an adjunction. For \( U \in G_1^L(\delta) \),

\[
F(G(U)) = \bigvee_{A \in \delta} (G(U)(A) \wedge A) = \bigwedge_{A \in \delta} (G(U)(A) \wedge A) = U.
\]
Proposition 4.5. For every \( \lambda \in \mathcal{L}(L) \) and every \( V \subseteq G(L) \),
\[
G(F(\lambda))(V) = \bigvee_{x \in x} \left( \bigvee_{y \in y} \lambda(A) \land A(x) \right)
\]
\[
\geq \bigvee_{x \in x} \left( \lambda(V) \land V(x) \right)
\]
\[
\geq \lambda(V).
\]
\[\square\]

Proposition 4.3. For every \( U \subseteq G(L) \), we have \( U = \bigvee_{A \in \mathcal{A}} A \land \text{sub}_X(A, U) \).

Proof. Firstly, it is obvious that \( U \geq \bigvee_{A \in \mathcal{A}} A \land \text{sub}_X(A, U) \). Secondly, by Proposition 4.1, we can suppose that \( U = \bigvee_{A \in \mathcal{A}} A \land \text{sub}_X(A, U) \). Then for any \( j \) in \( J \), we have
\[
\text{sub}_X(A_j, U) \geq \bigvee_{A \in \mathcal{A}} A_j(x) \rightarrow (A_j \land A)(x) \rightarrow (A_j \land A)(x) \geq A_j.
\]
It follows that
\[
\bigvee_{A \in \mathcal{A}} A \land \text{sub}_X(A, U) \geq \bigvee_{A \in \mathcal{A}} A_j \land \text{sub}_X(A_j, U)
\]
\[
\geq \bigvee_{A \in \mathcal{A}} A_j \land A_j \geq U.
\]
\[\square\]

Corollary 4.4. \( F \circ G = \text{id}_{G(L)} \) and \( F \) is surjective. \( G \) is injective.

We now define an equivalence \( \sim \) on \( \mathcal{L}(L) \) as \( \lambda \sim \mu \) iff \( \bigvee_{A \in \mathcal{A}} A \land \lambda(A) = \bigvee_{A \in \mathcal{A}} A \land \mu(A) \) (i.e., \( F(\lambda) = F(\mu) \)). Then we have

Proposition 4.5. \( G(L) \cong \mathcal{H}(L) \).

4.2. Hutton unit interval as fuzzification of ordinary unit interval (Notes 3.10 in [21]).

Let \( \mathbb{R} \) be the real line and let \( \Sigma \) be the set of all antitone maps \( \lambda : \mathbb{R} \rightarrow L \) satisfying that \( \lambda(t) = 1 \) for all \( t < 0 \) and \( \lambda(0) = 0 \) for all \( t > 1 \). Let \( \lambda^+ = \lambda \) for \( t > 1 \) and \( \lambda^+ = \lambda \) for \( t > 1 \). The underlying set \( \Sigma \) of Hutton unit interval \( H(\mathbb{L}) \) [15] is a quotient set of \( \Sigma \) with respect to the equivalence \( \sim \) defined by \( \lambda \sim \mu \iff \lambda^+ = \mu^+ \).

Proposition 4.6. (Remark 1.3.1 in [18]) (1) \( \lambda^+ = \mu^+ \iff \lambda^- = \mu^- \).

\[\triangleleft\]

Proposition 4.7. (1) \( \lambda \) is a \( \mathbb{V} \)-map if \( \lambda = \lambda^- \).

\[\triangleleft\]

Proof. We only prove (1). Suppose that \( \lambda = \lambda^- \). Let \( \lambda, \mu \in \mathcal{L}(L) \). Put \( t \in \mathbb{R} \), then \( \lambda(t) = \lambda^- (t) = \lambda^+ (t) = \lambda^+ (t) = \lambda\). Similarly, if \( \lambda \) is a \( \mathbb{V} \)-map, then for every \( t \in \mathbb{R} \), \( \lambda^+ (t) = \lambda^+ (t) = \lambda^+ (t) = \lambda^+ (t) = \lambda\). □

Corollary 4.8. \( \lambda^- \) (resp., \( \lambda^+ \)) is the unique \( \mathbb{V} \)-map (resp., \( \mathbb{V} \)-map) of the every equivalence class \( [\lambda] \) in \( \mathcal{L}(L) \).

Corollary 4.9. \( H(\mathbb{L}) \cong \mathcal{H}(L) \), here \( \mathcal{H}(L) \) is the set of all \( \mathbb{V} \)-map from \( I \) to \( L \).

5. Some related potential approaches and further work

5.1. For fuzzification of complete lattices

Proposition 3.2 shows that for a complete lattice \( C \), \( C(2) \cong \{ \tau \} \cong C \), where \( C(2) \) is the set of all \( \mathbb{V} \)-maps from \( C \) to \( 2 \). Then Section 3 constructs a functor from the category of semilocales to the category of \( L \)-semilattices which transforms every complete lattice \( C \) to \( L \). The set of all \( \mathbb{V} \)-maps from \( C \) to \( L \) is considered as algebraic representation of the Lowen functor \( \mathbb{A}(C) \).

If we only pay attention to the fuzzification of complete lattice, then the following four potential approaches could be used. Suppose that \( C \) is a complete lattice. Let \( C(\mathbb{L}) \) be the set of all \( \mathbb{V} \)-maps from \( C \) to \( L \) (exactly is \( C(L) \) in Section 3); \( C(\mathbb{L}) \) be the set of all \( \mathbb{V} \)-maps from \( C \) to \( L \); \( C(\mathbb{L}) \) be the set of all \( \mathbb{V} \)-maps from \( C \) to \( L \); \( C(\mathbb{L}) \) be the set of all \( \mathbb{V} \)-maps from \( C \) to \( L \).
Proposition 5.1. (1) \((C_v'(2), \leq) \cong ([x] a \in C), \subseteq) \cong (C, \leq);
(2) \((C_v(2), \geq) \cong ([x] a \in C), \supseteq) \cong (C, \leq);
(3) \((C_v(2), \geq) \cong ([L] a \in C), \supseteq) \cong (C, \leq);
(4) \((C_v(2), \leq) \cong ([L] a \in C), \subseteq) \cong (C, \leq).

Proposition 5.2. Let \(L\) be a frame. Then

\((1) (C_v(2), \subseteq)\) is a complete \(L\)-ordered set, where

\((\land \lambda)(x) = \bigwedge_{\lambda \in C_v(2)} \lambda(x).

\((2) (C_v'(2), \supseteq)\) is a complete \(L\)-ordered set, where

\((\land \lambda)(x) = \bigwedge_{\lambda \in C_v'(2)} \lambda(x).

\((3) (C_v(2), \supseteq)\) is a complete \(L\)-ordered set, where

\((\land \lambda)(x) = \bigwedge_{\lambda \in C_v(2)} \lambda(x).

\((4) (C_v(2), \subseteq)\) is a complete \(L\)-ordered set, where

\((\land \lambda)(x) = \bigwedge_{\lambda \in C_v(2)} \lambda(x).

5.1.1. For the isomorphism \(T(L) \cong \omega_L(T)\)
If \(L\) is a completely distributive lattice with an order-reserving involution \(\iota\), then we have an alternative way to prove the isomorphism \(T(L) \cong \omega_L(T)\). We here leave the proof to the readers.

Theorem 5.3. If \(L\) is a completely distributive lattice, then \(\omega_L(T) \cong \omega_L(T)\).

5.2. Interval topology
Let \(C\) be a complete lattice and \(x \in C\). Then it is easy to show that the form \(C_v(1, x)\) (resp. \(C_v'(1, x)\)) exactly is a \(V\)-\(\iota\)-map (resp. \(A\)-\(\iota\)-map) from \(C_v(1)\) (resp. \(C_v'(1)\)) to \(U_v(1)\), called the \(\iota\)-lower topology on \(C\). The \(L\)-interval topology \(I_v(L)\) is generated by family \(U_v(1) \cup \iota U_v(1)\) as a subbasis.

References


