A fast adaptive quadtree scheme for a two-layer shallow water model

Wei-Koon Lee, Alistair G.L. Borthwick, Paul H. Taylor

Faculty of Civil Engineering, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia
Engineering Science Department, University of Oxford, Parks Road, Oxford OX1 3PJ, UK

Abstract

This paper presents a dynamically adaptive quadtree grid generation system for the solution of a two-dimensional two-layer shallow water model. Roe-type two-layer shallow water solvers require numerical approximation of the system eigenvalues as well as numerical balancing, which increase computational cost considerably when a regular grid is used. In order to improve computational efficiency, we consider a dynamically adaptive quadtree grid generation system capable of increasing local resolution where high gradients occur in the physical flow variables. Test results show that satisfactory convergence can be obtained using the present scheme with the adaptive grid generator at a fraction of the cost incurred by a regular grid.

1. Introduction

The past 10 years have seen rapid development in two-layer shallow water models (2L-SWM), aimed at representing near-horizontal flows that have a pronounced two-layer vertical structure with a distinct density difference between the layers. Although there may be no exact separation interface between the layers, the two-layer approximation offers considerable computational advantage in resolving the fluid mechanical behaviour of such flows for practical engineering purposes when compared to numerical solvers based on the full Navier–Stokes equations. Examples of two-layer flow phenomena include fresh water intruding upon denser sea water, mud flow or debris flow beneath water, and circulation in a stratified ocean or lake.

An early attempt to model the two-layer shallow flow problem was reported by Vreugdenhil [1] who formulated a set of differential shallow water equations in weak-interaction form. Ignoring mixing between the layers, the surface and internal waves were solved using separate grids due to the order of magnitude difference in propagation speed between waves on the free surface and at the density interface. Castro et al. [2] showed that, in general, numerical schemes obtained by independent upwindings of each layer of the two-layer shallow water model are unconditionally unstable. The difficulty arises due to the coupling of momentum between the layers, which involves certain derivatives of the dependent variable. Castro et al. [2] introduced a fully coupled two-layer shallow water model for flow in a straight rectangular channel where the flux and coupling terms are treated together by using a generalised Roe linearisation scheme. The scheme was later extended to a straight channel with irregular geometry [3,4], and implemented in two-dimensions by considering a projected Riemann problem along the normal direction at every intercell [5].

Owing to the non-linear coupling between the layers, the system eigenvalues of a two-layer shallow water model have to be evaluated numerically by schemes which utilise characteristic decomposition, such as the Roe-type method.
This increases significantly the computational cost of a two-layer shallow water model. In order to avoid calculation of the system eigenvalues, various strategies have been employed which allow explicit access to the eigenstructure of the modified system, typically using only partial information on the complete wave structure. Examples include operator splitting [6], Strang splitting [7,8], a relaxation method [9], and a Riemann–problem–solver–free Godunov-type scheme [10].

Another problem associated with the system eigenvalues is that, in the event the system approaches a critical point, at least one of the eigenvalues can become very small, thus causing the system to lose accuracy with the error propagating both upstream and downstream. In order to overcome this difficulty, numerical viscosity can be added to the system when close to vanishing internal eigenvalues are detected. Popular choices include Harten regularization [2,3,11,4], and the Harten–Hyman entropy fix [12,13]. Other researchers have used an explicit viscosity term [1,6]. Both Harten regularization [14] and the Harten–Hyman entropy fix [15] involve the use of parameters whose values must be tuned empirically on a case by case basis. Rebollo et al. [16] introduced an alternative entropy-correction-free solver which is based on the construction of an equivalent parabolic system of partial differential equations (PDEs) from the original hyperbolic PDEs. A numerical correction term is added so that all smooth steady solutions of the original system are also solutions of the equivalent system. The scheme is well-balanced and satisfies the C-property such that smooth steady solutions are second order accurate. This approach was successfully applied to a two-layer shallow water model by Chacon–Rebollo et al. [17], and extended to a general non-homogeneous hyperbolic system of equations with non-conservative terms by Castro et al. [18].

The idea of the C-property in the context of SWEs was first introduced by Bermúdez and Vázquez-Cendón [19] for problems where the numerical scheme is required to calculate exactly at grid nodes the stationary solution corresponding to water at rest. For non-homogeneous hyperbolic systems of conservation laws this is of particular importance. Otherwise, large errors in the numerical solution, in particular, to the wave speed, may result in unphysical solutions [20]. An extension of the C-property to a more general condition was introduced by Greenberg and Leroux [21], with the concept of a ‘well-balanced’ scheme which preserves all equilibria of the system at the grid nodes. The challenge is to balance properly the flux gradient and source terms in the hyperbolic equation system. Numerous approaches have since been reported in the literature and continuously improved [22–27]. In particular, the algebraic approach of Rogers et al. [25,26] use a deviatoric form of the shallow water equations that identically balances the set of hyperbolic equations without the need of additional computational effort. This would be appealing for the two-layer shallow water model which already demands higher computational resources due to layer coupling. Unfortunately, it is shown by Lee et al. [28] that the algebraic balancing technique cannot be extended to a two-layer shallow water model in the presence of interface perturbation due to the effect of an additional term associated with the layer coupling which violates the well-balanced property. The motivation of the present study is to implement an adaptive grid algorithm that increases the computational efficiency of a Roe-type two-dimensional shallow water model. Well-balanced schemes have also been presented for applications involving wetting and drying [29–31]. For a two-layer shallow water model, the wet–dry front treatment needs to consider the disappearance of either or both the layers. Castro et al. [11] used an extended C-property to handle this wet–dry transition. The approach was further improved by Castro et al. [32] who considered a non-linear Riemann problem instead of a linear one.

High order solvers of the two-layer shallow water equations were first reported by Castro et al. [12] for well balanced coupled systems of conservation laws with source terms in one-dimension. This approach, which is based on reconstruction of states, has been subsequently extended to two-dimensions using the original Roe scheme [33] and an ADER scheme [34].

In a predominantly horizontal flow, locally high gradients in the flow variables may occur near complicated lateral boundary configurations as well as in the interior domain. The present paper describes a solver of the two-layer shallow water model in the presence of interface perturbation due to the effect of an additional term associated with the layer coupling which violates the well-balanced property. The motivation of the present study is to implement an adaptive grid algorithm that increases the computational efficiency of a Roe-type two-dimensional shallow water model. Well-balanced schemes have also been presented for applications involving wetting and drying [29–31]. For a two-layer shallow water model, the wet–dry front treatment needs to consider the disappearance of either or both the layers. Castro et al. [11] used an extended C-property to handle this wet–dry transition. The approach was further improved by Castro et al. [32] who considered a non-linear Riemann problem instead of a linear one.

High order solvers of the two-layer shallow water equations were first reported by Castro et al. [12] for well balanced coupled systems of conservation laws with source terms in one-dimension. This approach, which is based on reconstruction of states, has been subsequently extended to two-dimensions using the original Roe scheme [33] and an ADER scheme [34].

In a predominantly horizontal flow, locally high gradients in the flow variables may occur near complicated lateral boundary configurations as well as in the interior domain. The present paper describes a solver of the two-layer shallow water equations based on an automated, high performance adaptive grid generator that can provide increased resolution in zones where the physical flow variables have high gradient or may even be discontinuous. The grid generator is essentially that of Rogers et al. [26,35] and Liang et al. [36].

This paper is organised as follows. Section 2 presents the governing equations of the two-dimensional two-layer shallow water model. Section 3 describes the numerical scheme, including treatments for wet–dry transition and the occurrence of complex eigenvalues. The adaptive grid generator is discussed briefly in Section 4, and a number of validation test cases are reported in Section 5. Finally, some concluding remarks are drawn in Section 6.

2. Governing equations

Consider a hyperbolic system with non-conservative products and source terms. The partial differential equations governing the two-dimensional flow of two superposed immiscible layers of shallow water fluids are written as

\[ \mathbf{W}_t + \mathbf{F}_1(\mathbf{W}_x, \mathbf{W}_y) = \mathbf{B}_1(\mathbf{W}) \mathbf{W}_x + \mathbf{B}_2(\mathbf{W}) \mathbf{W}_y + \mathbf{S}_1(x, \mathbf{W}) + \mathbf{S}_2(y, \mathbf{W}), \]

where the subscripts \( x, y \) and \( t \) denote partial derivatives with respect to the \( x \)-direction, \( y \)-direction, and time. The vector of unknowns, \( \mathbf{W} \), the flux function vectors, \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \), and the source term vectors which describe the variable bed topography, \( \mathbf{S}_1 \) and \( \mathbf{S}_2 \) are defined below:

\[ \mathbf{W} = \mathbf{W}(x, y, t) = \begin{bmatrix} W_1(x, y, t) \\ W_2(x, y, t) \end{bmatrix}, \quad \mathbf{W}_j = \begin{bmatrix} h_j \\ q_{xj} \\ q_{yj} \end{bmatrix}, \]

(2)
\[ F_1(W) = \begin{bmatrix} F_{11}(W_1) \\ F_{12}(W_2) \end{bmatrix}, \quad F_{ij}(W_j) = \begin{bmatrix} q_{x_j} \\ q_{y_j} \\ q_{x_j}q_{y_j}/h_j \end{bmatrix}, \]

\[ F_2(W) = \begin{bmatrix} F_{21}(W_1) \\ F_{22}(W_2) \end{bmatrix}, \quad F_{2j}(W_j) = \begin{bmatrix} q_{x_j} \\ q_{y_j} \\ q_{x_j}q_{y_j}/h_j \end{bmatrix}, \]

\[ S_1(x, W) = \begin{bmatrix} S_{11}(x, W_1) \\ S_{12}(x, W_2) \end{bmatrix}, \quad S_{1j}(x_j, W_j) = \begin{bmatrix} 0 \\ -gh_jb(x, y)_x \end{bmatrix}, \]

\[ S_2(y, W) = \begin{bmatrix} S_{21}(y, W_1) \\ S_{22}(y, W_2) \end{bmatrix}, \quad S_{2j}(y_j, W_j) = \begin{bmatrix} 0 \\ 0 \\ -gh_jb(x, y)_y \end{bmatrix}, \]

and the matrices \(B_1\) and \(B_2\) are

\[ B_1(W) = \begin{bmatrix} 0 & B_{11}(W_1) \\ B_{12}(W_2) & 0 \end{bmatrix}, \quad B_2(W) = \begin{bmatrix} 0 & B_{21}(W_1) \\ B_{22}(W_2) & 0 \end{bmatrix}, \]

where

\[ B_{1j}(W_j) = \begin{bmatrix} 0 & 0 \\ -gh_j & 0 \end{bmatrix}, \quad B_{12}(W_2) = \begin{bmatrix} 0 & 0 \\ -rg_h & 0 \end{bmatrix}, \]

and

\[ B_{2j}(W_1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_{22}(W_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \]

The flow rate components \(q_{x_j}(x, y, t)\) and \(q_{y_j}(x, y, t)\) and the layer thickness \(h_j(x, y, t)\) are the dependent variables, \(b(x, y)\) is the bed profile measured from a fixed reference horizontal datum. The subscript \(j\) denotes the layers, where index 1 refers to the upper layer and index 2 refers to the lower layer, such that \(\rho_1\) and \(\rho_2\) are the constant densities of the respective layer and \(r = \rho_1/\rho_2 < 1\), and \(g\) is the gravity acceleration.

The system matrix, to be evaluated at a chosen ‘intermediate state’, is given by

\[ A(W) = \begin{bmatrix} A_1(W_1)n_x + A_2(W_1)n_y & -B_{11}(W_1)n_x - B_{21}(W_1)n_y \\ -B_{12}(W_2)n_x - B_{22}(W_2)n_y & A_1(W_2)n_x + A_2(W_2)n_y \end{bmatrix}, \]

where \(n = (n_x, n_y)\) is the unit vector. Here, \(A_1(W_j)\) and \(A_2(W_j)\) are Jacobian matrices of each respective layer, and defined as

\[ A_1(W_j) = \frac{\partial F_{ij}}{\partial W_j} = \begin{bmatrix} 0 & 1 & 0 \\ c_j^2 - u_j^2 & 2u_j & 0 \\ -u_jv_j & v_j & u_j \end{bmatrix}, \quad A_2(W_j) = \frac{\partial F_{2j}}{\partial W_j} = \begin{bmatrix} 0 & 0 & 1 \\ c_j^2 - v_j^2 & 0 & 2v_j \\ -u_jv_j & v_j & u_j \end{bmatrix}, \]

where \(u_j = q_{x_j}/h_j, v_j = q_{y_j}/h_j\), and \(c_j^2 = gh_j\).

We assume that the system is strictly hyperbolic, i.e. there are six distinct eigenvalues in each of the \(x-\) and \(y-\)directions respectively such that the corresponding eigenvectors are linearly independent. Two of the eigenvalues are given by \(\lambda_{1,2} = U_j\) for \(j = 1, 2\) where \(U_j = u_jn_x + v_jn_y\) is the velocity across the cell face in the respective layer. Another four eigenvalues can be determined from the characteristic equation derived from the matrix \((10)\):

\[ (\lambda^2 - 2U_1\lambda + U_1^2 - gh_1)(\lambda^2 - 2U_2\lambda + U_2^2 - gh_2) = r^2gh_1h_2. \]

For \(r \ll 1\), the eigenvalues of the system \((1)\) approach those of each layer. In particular, when \(r = 0\), the eigenvalues exactly correspond to each layer \((U_j \pm \sqrt{gh_j})\) and thus the coupling terms essentially do not affect the nature of the system. Our interest, however, lies in the case when \(r \approx 1\), which is commonly encountered in coastal flows and estuaries. In this case, the first order approximation of the eigenvalues can be written as \([37]\):

\[ \hat{\lambda}_{3,4} = V_m \pm \sqrt{gh_1 + h_2}. \]
and
\[
\lambda_{5,6} = V_c \pm \sqrt{\frac{g' h_1 h_2}{h_1 + h_2} \left( 1 - \frac{(U_2 - U_1)^2}{g' (h_1 + h_2)^2} \right)},
\]
(14)
where \(g' = g(1 - r)\) is the reduced gravity. The mean velocity \(V_m\) and the convective velocity \(V_c\) are given respectively by
\[
V_m = \frac{h_1 U_1 + h_2 U_2}{h_1 + h_2}, \quad V_c = \frac{h_1 U_2 + h_2 U_1}{h_1 + h_2}.
\]
(15)
The external eigenvalues \(\lambda_{3,4}\) and the internal eigenvalues \(\lambda_{5,6}\) are related respectively to barotropic and baroclinic components of the flow, noting that the internal eigenvalues have much lower values because \(g' \ll 1\) when \(r \approx 1\). If the solution has four different real eigenvalues, the flow is stable and the system is hyperbolic. This requires the following approximated condition to be satisfied
\[
\frac{(U_2 - U_1)^2}{g' (h_1 + h_2)} < 1..
\]
(16)
Violation of condition (16) means that the system loses its hyperbolicity, a situation associated with the appearance of Kelvin–Helmholtz instabilities at the layer interface. In real flow, this corresponds to intense mixing between the layers which dissipates energy and results in an exchange of momentum, a phenomenon which is not compatible with an immiscible-lower layer model. Numerically, such a disturbance will grow and eventually overwhelm the solution unless the viscous effect is added. Note also that the system is not strictly hyperbolic for the limiting case when \(r = 1\), where there are only three real distinct eigenvalues. We note that a two-layer flow of identical densities reduces to a system of three equations which comprise the continuity and momentum equations for a single layer, and a decoupled transport equation for the ratio of the lower layer thickness to the total thickness [8].

3. Numerical scheme

For the ease of presentation, consider the one-dimensional form of the hyperbolic system (1):
\[
\mathbf{W}_t + \mathbf{F}(\mathbf{W})_x = \mathbf{B}(\mathbf{W})\mathbf{W}_x + \mathbf{S}(\mathbf{x}, \mathbf{W}),
\]
(17)
which can be written in quasilinear form:
\[
\mathbf{W}_t + \mathbf{A}(\mathbf{W})\mathbf{W}_x = \mathbf{S}(\mathbf{x}, \mathbf{W}),
\]
(18)
where
\[
\mathbf{A}(\mathbf{W}) = \frac{\partial \mathbf{F}(\mathbf{W})}{\partial \mathbf{W}} - \mathbf{B}(\mathbf{W}).
\]
(19)
Using the original ideas of Chacon-Rebollo et al. [16,17] and Castro et al. [18], a standard upwind finite volume discretization of the two-layer shallow water system (17) which has the form of a general non-homogeneous non-conservative hyperbolic system can be interpreted as being a second-order centred approximation of an equivalent parabolic system:
\[
\mathbf{W}_t + \mathbf{F}(\mathbf{W})_x - \nu(|\mathbf{D}(\mathbf{W})|\mathbf{W}_x)_x = \mathbf{B}(\mathbf{W})\mathbf{W}_x + \mathbf{S}(\mathbf{x}, \mathbf{W}),
\]
(20)
where \(\mathbf{D}\) is the upwind matrix known here as the characteristic diffusion of the scheme, and \(\nu = \Delta x/2\). Chacon-Rebollo et al. [16] proposed to add a numerical source term \(\mathbf{C}(\mathbf{x}, \mathbf{W})\) to (20),
\[
\mathbf{W}_t + \mathbf{F}(\mathbf{W})_x - \nu(|\mathbf{D}(\mathbf{W})|\mathbf{W}_x)_x = \mathbf{B}(\mathbf{W})\mathbf{W}_x + \mathbf{S}(\mathbf{x}, \mathbf{W}) + \mathbf{C}(\mathbf{x}, \mathbf{W}),
\]
(21)
such that the centred approximation (in space) of the terms on the right-hand-side in (21), will solve (up to second order) all equilibria solutions, which are also the solutions of (17), and similarly any higher order approximations.

From (17), a smooth stationary solution \(\mathbf{W}\) verifies
\[
\mathbf{F}(\mathbf{W})_x = \mathbf{B}(\mathbf{W})\mathbf{W}_x + \mathbf{S}(\mathbf{x}, \mathbf{W}).
\]
(22)
Hence, if (21) approximates (17), it is required that the correction term \(\mathbf{C}\) exactly balances the diffusion term in (20):
\[
\mathbf{C}(\mathbf{x}, \mathbf{W}) = -\nu \frac{\partial}{\partial x}(|\mathbf{D}(\mathbf{W})|\mathbf{W}_x).
\]
(23)
Using Eq. (18), and provided that the system matrix \(\mathbf{A}\) is non-singular, we have
\[
\mathbf{W}_x = \mathbf{A}^{-1}\mathbf{W}\mathbf{S}(\mathbf{x}, \mathbf{W}).
\]
(24)
Therefore, the correction term may be expressed as

$$C(x, W) = -v \frac{\partial}{\partial x} \left( D(W) | A^{-1}(W) W S(x, W) \right).$$

(25)

Eq. (21) can now be written, after rearrangement, as

$$W_x + \Phi(x, W)_x = B(W) W_x + S(x, W),$$

(26)

where $\Phi(x, W)$ is the modified flux given by

$$\Phi(x, W) = F(W) - v | D(W) | \left( W_x - A^{-1}(W) W S(x, W) \right).$$

(27)

Eq. (26) can be interpreted as providing a viscous approximation to the modified system (21). The correction term is essentially a modification to the numerical flux and does not affect the diffusion term. In this paper, we extend this numerical viscosity approach originally applied in the one-dimensional two-layer shallow water model by Chacon-Rebollo et al. [16,17] and Castro et al. [18] to the two-dimensional system described in Section 2. The system of equations to solve can thus be written as

$$B_i(W) W_x + B_2(W) W_y + S_1(x, W) + S_2(y, W),$$

(28)

where

$$\Phi_1(x, W) = F_1(W) - v | D_1(W) | \left( W_x - A_1^{-1}(W) S_1(x, W) \right).$$

(29)

and

$$\Phi_2(y, W) = F_2(W) - v | D_2(W) | \left( W_y - A_2^{-1}(W) S_2(y, W) \right).$$

(30)

Chacon-Rebollo et al. [16] proposed that the inverse of matrix $A(W)$ (10) be defined differently in the neighbourhood of a critical point as follows:

$$A^{-1}(W) = X(W) A^{-1}(W) X^{-1}(W),$$

(31)

where $A$ is the diagonal matrix whose coefficients are the system eigenvalues, $X(W)$ is the corresponding matrix whose columns are the system eigenvectors, and

$$A^{-1}(W) = \text{Diag}(\lambda_i^{-1}), \quad \lambda_i^{-1} = \begin{cases} 1/|\lambda_i| & \text{if } |\lambda_i| > \epsilon, \\ 0 & \text{if } |\lambda_i| \leq \epsilon. \end{cases}$$

(32)

Here $\lambda_i$ are the system eigenvalues (numbering 4 and 6 in the one- and two-dimensional model, respectively), and $\epsilon$ is a small parameter. The redefinition of $A^{-1}(W)$ ensures that instability caused by the vanishing eigenvalues of the upwind matrix cannot propagate upstream and thus no specific entropy correction is needed near any critical point.

Chacon-Rebollo et al. [17] described the scheme above as being ‘asymptotically well-balanced’ in the sense that the steady solutions, which need not be known a priori, are computed up to second order throughout the domain except those on a set whose measure tends to zero as $\Delta x$ tends to zero. Specifically, the numerical flux and the numerical diffusion of the scheme are separately balanced by the centred part and the non-centred part of the numerical source term, independent of the discretization parameter (see Section 3.1) appearing in each of them. Computationally, the behaviour of the scheme is similar to that of Roe’s method. The technique can be applied to any general scheme which can be formulated as an artificial viscosity method.

Castro et al. [18] present several choices for the diffusion matrix $D(W)$, which is essentially a free parameter. Hence, the scheme readily admits both flux-splitting and flux-difference or Roe methods as particular cases. Chacon-Rebollo et al. [16] describe a hybrid flux-difference–flux-splitting scheme, which allows supercritical conditions to be captured and avoids the propagation of spurious oscillations without excessive diffusion. In the present work, we choose

$$D(W) = X(W) A(W) X^{-1}(W).$$

(33)

### 3.1. One-parameter discretization of the system

In this section, we present the general discretization of the two-layer shallow water equations by means of a finite volume scheme. The computational domain is divided into $M$ finite volumes, each comprising a square cell where $\Delta y = \Delta x$. By default, each cell $i$ has four cell faces $I_k$, $k = 1, 4$. In this section, we assume a regular grid. In Section 4, the use of a quadtree grid will be discussed. A regular three-point discretization of the system (28) yields:
\[
\mathbf{W}_i^{n+1} = \mathbf{W}_i^n + \frac{\Delta t}{\Delta x} \left\{ \left( \mathbf{\Phi}_i^W - \mathbf{\Phi}_i^E \right) + \left( \mathbf{\Phi}_i^S - \mathbf{\Phi}_i^N \right) + \frac{1}{2} \sum_{k=1}^{4} \left[ \mathbf{B}_{i_k} \Delta \mathbf{W}_{i_k} + \mathbf{S}_{i_k} \right] \right\},
\]

where the subscripts \( N, E, S, W \) (\( \Gamma_1, \ldots, \Gamma_4 \)) indicate approximations at the north, east, south and west faces, and \( C \) represents the centre of the cell under consideration. By \( \mathbf{W}_i^n \) we define an approximation to the average of the solution in the control volume at time \( t_n = n\Delta t \), \( \Delta t \) being the time step.

The discretized modified flux \( \mathbf{\Phi} \) is given by
\[
\mathbf{\Phi}_{i_k} = \mathbf{F}_{i_k} - \nu |\mathbf{D}_{i_k}| \left( \frac{\Delta \mathbf{W}_{i_k}}{\Delta x} - A_{i_k}^{-1} \mathbf{S}_{i_k} \right),
\]

where \( \Delta \mathbf{W}_{i_k} = \mathbf{W}_i - \mathbf{W}_k \). The subscripts \( L \) and \( R \) represent the cell-centred values to the left and right of the cell face respectively, with right being to the north or east side, and where either \( L \) or \( R \) must be located at the centre of the cell under consideration.

Following Chacon-Rebollo et al. [16], we use a Godunov-like scheme for which the constant values associated with the Riemann problem at the cell interfaces are approximated by
\begin{align*}
\mathbf{W}_{L,x} &= (1 - \alpha) \mathbf{W}_L + \alpha \mathbf{W}_i, \\
\mathbf{W}_{R,x} &= (1 - \alpha) \mathbf{W}_R + \alpha \mathbf{W}_i,
\end{align*}

where the function of the single parameter \( \alpha \) is identical to flux limiters used in standard shallow water equations. The lower limit corresponds to a stable but more diffusive scheme, similar to a minmod limiter, whereas the upper limit corresponds to a less diffusive scheme which can become unstable in resolving discontinuities, similar to a superbee limiter. The parameter \( \alpha \) has no effect on the smooth regions but becomes important in the transition zones.

The centred approximation of the flux function is thus written as
\[
\mathbf{F}_{i_k} = \frac{1}{2} \left[ \mathbf{F}(\mathbf{W}_{L,x}) + \mathbf{F}(\mathbf{W}_{R,x}) \right].
\]

The matrices \( \mathbf{D}_{i_k} \) (Eq. 33), \( A_{i_k}^{-1} \) (Eq. 31) and the source term \( \mathbf{S}_{i_k}^c \) associated with the numerical diffusion and the non-centred part of the source term are determined using the intermediate states between the two neighbouring cells. In particular, we adopt the following Roe’s averages:
\begin{align*}
\mathbf{u}_j &= \frac{\sqrt{h_{kj} u_{k,j}} + \sqrt{h_{lj} u_{l,j}}}{\sqrt{h_{kj}} + \sqrt{h_{lj}}}, \\
\mathbf{u}_i &= \frac{\sqrt{h_{kj} v_{k,j}} + \sqrt{h_{lj} v_{l,j}}}{\sqrt{h_{kj}} + \sqrt{h_{lj}}},
\end{align*}

and
\[
\mathbf{c}_j = \sqrt{\frac{1}{2} g (h_{kj} + h_{lj})}.
\]

The source term \( \mathbf{S}_{i_k}^c \) in Eq. (35) can then be written as
\begin{align*}
\mathbf{S}_{c,Lw}^c &= \frac{b_k - b_i}{\Delta x} \begin{bmatrix} 0 & -c_1^2 & 0 & 0 & -c_2^2 & 0 \end{bmatrix}^T, \\
\mathbf{S}_{c,Rs}^c &= \frac{b_R - b_i}{\Delta x} \begin{bmatrix} 0 & 0 & -c_1^2 & 0 & 0 & -c_2^2 \end{bmatrix}^T.
\end{align*}

The components of the discretized coupling terms \( \mathbf{B} \) are given by
\begin{align*}
\mathbf{B}_{c,N}(\mathbf{W}) &= \alpha \mathbf{B}(\mathbf{W}_{L,x/2}) + (1 - \alpha) \mathbf{B}(\mathbf{W}_{L,(1-x)/2}), \\
\mathbf{B}_{c,S}(\mathbf{W}) &= \alpha \mathbf{B}(\mathbf{W}_{R,x/2}) + (1 - \alpha) \mathbf{B}(\mathbf{W}_{R,(1-x)/2}).
\end{align*}

The source terms \( \mathbf{S}_{i_k} \) are
\begin{align*}
\mathbf{S}_N &= \left( \frac{b_{i+1/2} - b_i}{\Delta y} \right) \begin{bmatrix} 0 \\ 0 \\ -g h_{1,L+1/2} \\ 0 \\ 0 \end{bmatrix} + \left( \frac{b_{R-L} - b_i}{\Delta y} \right) \begin{bmatrix} 0 \\ 0 \\ -g h_{1,L+(1-x)/2} \\ 0 \\ 0 \end{bmatrix}, \\
\mathbf{S}_W &= \left( \frac{b_{i-1/2} - b_i}{\Delta y} \right) \begin{bmatrix} 0 \\ 0 \\ -g h_{2,L+1/2} \\ 0 \\ 0 \end{bmatrix} + \left( \frac{b_{R-L} - b_i}{\Delta y} \right) \begin{bmatrix} 0 \\ 0 \\ -g h_{2,L+(1-x)/2} \\ 0 \\ 0 \end{bmatrix}.
\end{align*}
both layers, which is commonly referred to as the appearance of wet–dry fronts. Following Castro et al. [11], a numerical scheme for solving system (1) satisfies the C-property if, for each cell datum. Using Roe’s method as in (38), the equality (44) is verified. Nonetheless in the presence of a wet–dry front, the equality (44) holds at the cell interfaces provided the numerical source terms are redefined as follows to avoid the emergence of spurious pressure forces:

\[
S_{h} = \frac{(b_{L,x} - b_{L})}{\Delta x} \begin{bmatrix} 0 \\ -gh_{1,1/2} \\ 0 \end{bmatrix} + \frac{(b_{R,x} - b_{R})}{\Delta x} \begin{bmatrix} 0 \\ -gh_{1,1/2} \\ 0 \end{bmatrix}
\]

\[
bS_{h} = \frac{(b_{R} - b_{R,x})}{\Delta y} \begin{bmatrix} 0 \\ -gh_{1,1/2} \\ 0 \end{bmatrix} + \frac{(b_{R} - b_{L,x})}{\Delta y} \begin{bmatrix} 0 \\ -gh_{1,1/2} \\ 0 \end{bmatrix}
\]

and

\[
S_{w} = \frac{(b_{R} - b_{R,x})}{\Delta x} \begin{bmatrix} 0 \\ -gh_{1,1/2} \\ 0 \end{bmatrix} + \frac{(b_{R} - b_{L,x})}{\Delta x} \begin{bmatrix} 0 \\ -gh_{1,1/2} \\ 0 \end{bmatrix}
\]

Numerical tests [16,18] have demonstrated that a choice of \( x = 1/8 \) gives a satisfactory overall compromise between stability and accuracy, without any need to invoke entropy corrections for the approximate solutions of sonic rarefactions. Nevertheless, \( x = 0 \) appears to be a better choice for stationary shocks [18]. Hence, Castro et al. [18] propose the use of an adaptive discretization function \( \alpha(x) \) where the value of \( \alpha \) used is switched to zero locally, whenever a transition in the sign of the internal eigenvalues \( \lambda_{i} \) is detected at a non-propagating discontinuity:

\[
\alpha_{i} = \begin{cases} 
1/8 & \text{if } \text{sgn}(\lambda_{i}^{b_{L}}) = \text{sgn}(\lambda_{i}^{b_{L,R}}), \\
0 & \text{if } \text{sgn}(\lambda_{i}^{b_{L}} - \lambda_{i}^{b_{L,R}}) = 1.
\end{cases}
\]

3.2. Wet–dry treatment

The solution of two-layer fluids presents another difficulty associated with the vanishing layer thickness of either layer or both layers, which is commonly referred to as the appearance of wet–dry fronts. Following Castro et al. [11], a numerical scheme for solving system (1) satisfies the C-property if, for each cell \( i \), the following equalities hold across any cell interface for any given steady solution:

\[
A_{h}, \Delta W_{h,i} = F(W_{h}) - F(W_{L}) - B_{h}, \Delta W_{h,i},
\]

and

\[
A_{w}, \Delta W_{w,i} = S_{h,i}.
\]

For the two-layer shallow water flow, the family of steady-state solutions is given by

\[
q_{1} = 0, \quad q_{2} = 0,
\]

\[
h_{1}(x) = \begin{cases} 
\eta_{1} - \eta_{2} & \text{if } b(x,y) < \eta_{2} \\
\eta_{1} & \text{if } \eta_{2} \leq b(x,y) < \eta_{1}, \\
0 & \text{if } \eta_{2} \leq b(x,y).
\end{cases}
\]

\[
h_{2}(x) = \begin{cases} 
\eta_{2} - b(x,y) & \text{if } b(x,y) < \eta_{2} \\
0 & \text{if } \eta_{2} \leq b(x,y).
\end{cases}
\]

where \( \eta_{1} \) and \( \eta_{2} \) are the elevation of the upper and lower layers, respectively, measured from a fixed reference horizontal datum. Using Roe’s method as in (38), the equality (44) is verified. Nonetheless in the presence of a wet–dry front, the equality (44) holds at the cell interfaces provided the numerical source terms are redefined as follows to avoid the emergence of spurious pressure forces:

\[
S_{n,s} = \begin{bmatrix} 
0, 0, -c_{1}^{2}(h_{L1} + h_{L2}), 0, 0, -c_{2}^{2}(rh_{L1} + h_{L2}) \\
0, 0, c_{1}^{2}(h_{R1} + h_{R2}), 0, 0, c_{2}^{2}(rh_{R1} + h_{R2}) \\
0, 0, -c_{1}^{2}dz, 0, 0, -c_{2}^{2}(r(dz - h_{L1}) + h_{L2}) \\
0, 0, -c_{1}^{2}dz, 0, 0, c_{2}^{2}(r(dz - h_{R1}) + h_{R2}) \\
0, 0, -c_{1}^{2}dz, 0, 0, -c_{2}^{2}dz
\end{bmatrix}^{T}
\]

\text{if } h_{L1} + h_{L2} < dz,

\text{if } h_{R1} + h_{R2} < -dz,

\text{if } h_{L2} < dz < h_{L1} + h_{L2},

\text{if } h_{R2} < -dz < h_{R1} + h_{R2},

\text{in other cases,}

\text{if } h_{L1} + h_{L2} < dz,

\text{if } h_{R1} + h_{R2} < -dz,

\text{if } h_{L2} < dz < h_{L1} + h_{L2},

\text{if } h_{R2} < -dz < h_{R1} + h_{R2},

\text{in other cases,}

\text{if } h_{L1} + h_{L2} < dz,
and

$$S_{NW} = \begin{cases} 0, & -c_l^2(h_{l,1} + h_{l,2}) \quad 0, & -c_l^2(r h_{l,1} + h_{l,2}), \quad 0 \end{cases}^T$$

if $h_{l,1} + h_{l,2} < dz$, \hspace{1cm}

$$\begin{cases} 0, & c_l^2(h_{k,1} + h_{k,2}), \quad 0, & c_l^2(r h_{k,1} + h_{k,2}), \quad 0 \end{cases}^T$$

if $h_{k,1} + h_{k,2} < -dz$, \hspace{1cm}

$$\begin{cases} 0, & -c_l^2 dz, \quad 0, & -c_l^2 (r dz - h_{l,2} + h_{l,1}), \quad 0 \end{cases}^T$$

if $h_{l,2} < dz < h_{l,1} + h_{l,2}$, \hspace{1cm}

$$\begin{cases} 0, & c_l^2 (r dz - h_{k,2} + h_{k,1}), \quad 0 \end{cases}^T$$

if $h_{k,2} < -dz < h_{k,1} + h_{k,2}$, \hspace{1cm}

$$\begin{cases} 0, & -c_l^2 dz, \quad 0, & -c_l^2 dz \end{cases}^T$$

in other cases,

where $dz = b_k - b_l$.

For fluids in motion, the source terms are calculated as in (44) but with a further modification which treats the front as an internal boundary condition. Suppose at time $t_n$, $l_k$ is a two-layer cell, and $l_k$ is the corresponding neighbouring cell which is dry or single-layered across the cell interface $x_{r_1}$, which is oriented in the $x$-direction. Here, a wet–dry front is identified and we define

$$\mathbb{W} = \begin{cases} \left[ (h_1^n), 0, (q_1^n), (h_2^n), 0, (q_2^n) \right]^T \quad \text{if } l_k \text{ is a dry cell,} \\
\left[ (h_1^n), (q_1^n), (h_2^n), 0, (q_2^n) \right]^T \quad \text{if } l_k \text{ is a 1-layer cell.} \end{cases}$$

The treatment makes sure that the state(s) of the wet cell $l_k$ is(are) modified such that there is no flux across the interface of a wet–dry front. The flux parallel to the said cell interface remains to be defined following the boundary condition. In practice, $\mathbb{W}^{-1}$ is calculated using the numerical fluxes obtained from $\mathbb{W}$ and the modified state $\mathbb{W}$. The approximation given by the treatment is satisfactory provided the mass flow rate in the wet cell is close to zero.

3.3. Treatment of complex eigenvalues

The hyperbolicity criterion (16) is only satisfied provided the velocity difference between the layers ($U_2 - U_1$) is small. When this is not the case, complex eigenvalues associated with the appearance of Kelvin–Helmholtz instabilities will result. Physically, the destabilizing effect of shear could overcome the stratification of the layers, producing fluid roll-up leading to mixing between the layers. Ideally, a numerical solver is sought that is well defined even if complex eigenvalues appear. Nevertheless, such occurrences are usually localised and can be either avoided or accounted for numerically. Castro et al. [3] adopt the following straightforward approach: whenever the internal eigenvalues $\lambda_{in}$ become the same, they are perturbed to give $\lambda_{in} + \epsilon$ where $\epsilon$ is a small parameter. This strategy is used in the present work. In an alternative approach proposed by Castro et al. [5], a linear friction term is added to the momentum equations to simulate the exchange of momentum. Note that the above approach addresses the two internal eigenvalues only ($\lambda_{5,6}$). In the two-dimensional system where another two of the eigenvalues are given by the velocity in the upper and lower layer respectively ($\lambda_{1,2} = U_j$ for $j = 1, 2$), whenever the velocities in both layers are the same, i.e. $U_1 = U_2$, the corresponding eigenvalues $\lambda_1$ and $\lambda_2$ are also perturbed by $\pm \epsilon$, where $\epsilon$ is such that the system always retains six real and distinct eigenvalues.

4. Quadtree grid generation system

This section describes the adaptive quadtree grid used for the two-layer shallow water model. Although quadtree grids appear unstructured, they have a tree-like indexing system and can be readily applied in solving discretised partial differential equations written in Cartesian coordinates. The resulting grid generation system is automatic, capable of high local resolution, and computationally efficient.

The quadtree grid generator referred to herein has undergone considerable development since its inception [38] and is presently known as OxQuad. Yiu et al. [38] presented an automatic mesh generation technique which can accommodate local mesh refinement adaptively. The mesh is produced by recursively subdividing the domain into quadrants using a quadtree to store and manipulate the mesh information. Over the past 15 years or so years the data storage structure, grid adaptability, and internal and external boundary fitting have been enhanced. The quadtree technique was successfully implemented for non-linear wave simulation [39], simulation of separated flows [40], advection of pollutant fronts [41], wave-current interaction [35], solution of shallow water equations [42,43], and wave runup [44]. In particular, the dynamically adaptive quadtree grid Godunov-type shallow water equation solver has been rigorously tested by Rogers et al. [25,26] and Liang et al. [36].

The OxQuad quadtree grid generation algorithm can be summarised as follows:

1. Input seeding points which describe the locations of internal and external boundaries of the physical flow domain, and positions of initial discontinuities (if known) of the flow variables.
2. Rescale the physical flow domain to fit into the root cell (a unit square) and specify the minimum and maximum level of subdivision.
3. Where there is a seeding point, subdivide the cell into four equal-sized quadrants (children) and check each new child-cell in turn for the presence of seeding points.
4. If two or more seeding points are found inside a cell, step (3) is followed by the current step until no further subdivision is required.

5. Perform regularisation so that all cells achieve a prescribed minimum level of subdivision, and the difference in subdivision levels between adjacent cells (including diagonally touching cells) does not differ by more than one level.

Since the boundaries are described by initial seeding points, maximum level of subdivision is adopted by default. However, in order to further reduce computational cost, minimum level of subdivision may be prescribed for straight boundaries. In this case, the boundary is also subjected to adaptation when the relevant criteria are met. Additional seeding points describing initial discontinuities or other features of interest in the domain can only be specified generally using simple geometrical lines or region so that maximum subdivision is applied. The maximum subdivision thus produced may be chosen to be adaptable or non-adaptable throughout the simulation. Where the region of interest is known a priori or cannot be described using simple geometries, a universal fine grid configuration may be used at the beginning of simulation and allow the adaptation procedure to subsequently coarsen regions which do not require fine resolution to the minimum level of subdivision.

The algorithm approaches boundary fitting in a fractal-like manner where a stepped approximation is inherent. The disruptive effect on the flow solution, if any, can be minimised by increasing the grid resolution. Alternatively, the grid may be further trianglerised [38], or represented by cut-cells [45].

Dynamical grid adaptation during numerical simulation is carried out using a set of criteria relating to internal flow features, such as the minimum and maximum root mean square values of the free surface gradients (and/or interface gradients in the present two-layer shallow water model) or depth-averaged velocity component gradients. The cells in the original grid may thus be further subdivided (enriched), and the newly created cells subsequently removed (coarsened) if appropriate. Updated cell identification and neighbour-finding information is stored within the same quadtree grid structure.

Grid connectivity is stored in a simple hierarchical tree structure. Each cell has a cell number, a parent pointer and four child pointers (if they exist). The root cell (level 0) is numbered 1 with its parent numbered 0 and the four quadrants numbered 2–5 (level 1), corresponding to the northwest, northeast, southwest, and southeast positions respectively. These four quadrants are given 2-digit reference numbers defined as 11, 21, 12, and 22, respectively, applicable locally at all levels of subdivision, and the sequence adopted in the cell-checking procedure described in step (3). Cell referencing is achieved using a unique identification number which comprises a concatenation of the local reference numbers at each successive level starting from level 1. Since not all cells have the same subdivision level, trailing zeros are added to give cell identification numbers of the same length. For any cell, its level of subdivision is equal to the number of pairs of non-zero digits in its identification number. The root cell has an all-zero cell identification number.

The non-uniform nature of the mesh generated means that hanging nodes are inevitably present when adjacent cells are of different sizes. A hanging node refers to where the vertex of one cell coincides with the middle of an edge of its neighbour. This demands careful treatment during discretization of the partial differential equations in order not to violate the conservation laws. Hanging nodes are minimised by virtue of the regularisation step (5) where the maximum adjacent cell edge length ratio for cells located inside the flow domain is constrained to a factor of two. Despite the presence of different neighbouring mesh sizes, the discretized equations are solved on a uniform grid template, such that whenever values are needed at grid locations where the information is not stored, appropriate interpolation routines must be employed to approximate the required values. In this paper, we employ linear interpolation as a compromise between efficiency and accuracy.

For a full description of the quadtree grid generator, the readers are referred to Borthwick et al. [41], Rogers et al. [25, 26] and Liang et al. [36].

5. Numerical tests

The two-dimensional two-layer shallow water model has been subjected to several validation tests using an explicit second-order Adams–Bashforth scheme. The CFL-condition is defined as

$$\max \left\{ \frac{|U_{i+1,j}|}{\Delta x}, \quad i = 1, M; \quad k = 1, 4; \quad t = 1, 6 \right\} \Delta t \leq \gamma,$$

where $i$, $k$ and $t$ are the number of computational cells, the number of faces of each (square) cell, and the number of eigenvalues to each face, respectively. In all the tests reported below, $\gamma = 0.9 \text{ (0 < } \gamma \text{ < 1)}$ is used. Unless otherwise stated, a density ratio $r$ of 0.98 is used, which corresponds to densities of typical stratified coastal water, and $g = 9.81 \text{m/s}^2$.

Grid adaptation is performed based on the maximum R.M.S of the surface and interface gradients:

$$\Phi = \max(\Phi_1, \Phi_2), \quad \Phi_{1,2} = \sqrt{\left(\frac{\partial H_{1,2}}{\partial x}\right)^2 + \left(\frac{\partial H_{1,2}}{\partial y}\right)^2},$$

where grid refinement and coarsening are performed when $\Phi$ exceeds 0.05 and falls below 0.02 (i.e. 5% and 2% gradient), respectively. Rogers et al. [35] showed that there is minimal gain by not adapting every time step. Hence, we perform the calculation of $\Phi$ and adaptation at every time step.
The mesh sizes $\Delta x$ at various levels of cell subdivision in the unit square grid domain $((x,y) \in [0,1])$ are given by $2^{-n}$ where $n$ is the level of subdivision. For the purpose of grid comparison, the Grid Convergence Index ($\text{GCI}$) [46,47] is evaluated at all grid points:

$$\text{GCI} = \frac{3|e|}{r_0^p - 1}, \quad e = 100\frac{\eta^p - \eta^0}{\eta^p},$$

where $e$ is the percentage relative error, $\eta^0$ and $\eta$ are the elevations of the surface or interface from a fixed datum, computed using the finer and the coarser grid, respectively. We note that in our results, the $\text{GCI}$ values computed using the surface and interface elevations have the same order of accuracy as those obtained using the velocity fields. For the purpose of discussion, we show only the former in the test cases presented. The grid refinement ratio $r_0$ is given by $\Delta x/\Delta x^0$, and the formal order of accuracy of the numerical scheme $p$ is 2. The various grid configurations used are represented in the following way: for instance, $\Psi_{8-8}$ means a regular grid with level 8 subdivision, and $\Psi_{8-8}$ means an adaptive grid with minimum and maximum level of subdivision of 6 and 8, respectively. Asymptotic range is considered to be achieved when $\text{GCI}_{23} = r_0^6 \text{GCI}_{12}$, where $\text{GCI}_{12}$ compares the finer grid to the intermediate grid, and $\text{GCI}_{23}$ compares the intermediate grid to the coarser grid [47].

5.1. Propagation of interface perturbation over flat bed

This is a benchmark test to examine the coupling term in the two-layer shallow water scheme, originally proposed by Castro et al. [2] for the one-dimensional case, and later adapted for a two-dimensional problem by Kurganov and Petrova [10]. A round-shaped interface perturbation is advected by a uniform flow in both $x$- and $y$-directions (in the north-east direction) over a flat bed. The initial conditions are given by:

$$(h_1, u_1, v_1, h_2, u_2, v_2) = \begin{cases} 
(0.50, 2.5, 2.5, 0.50, 2.5, 2.5) & (x + 1)^2 + (y + 1)^2 \leq 1, \\
(0.45, 2.5, 2.5, 0.55, 2.5, 2.5) & \text{otherwise},
\end{cases}$$

in a domain given by $(x,y) \in [-1,1]$. Transmissive boundary conditions are applied on domain boundaries. In addition to the domain boundary, seeding points are also applied along the initial discontinuity so that the grids are refined in its vicinity. The simulation is performed until $t = 0.1$. Fig. 1 shows the free surface and interface at the end of the simulation using grid $\Psi_{9-9}$.

Fig. 2 illustrates the $\text{GCI}$ contours of the free surface and interface when different grid configurations are adopted. Overall, the results indicate that the computations are convergent to the asymptotic value, with slightly higher differences at the interface but still well within the acceptable range. Some differences due to small oscillations can be observed in Fig. 2(ii), but disappear at finer resolutions (Fig. 2(i)). In general, solutions of the free surface are unlikely to contain marked features of interest for two-layer problems with level initial free surface condition. Note that the staircase feature at the edges are a consequence of adaptation at the boundaries. Examining Fig. 2(iii) and (iv), we note that when compared to their respective regular grid configurations at the highest level of subdivision, the $\Psi_{6-6}$ configuration is marginally poorer than the $\Psi_{6-8}$ configuration, i.e. with larger difference between the minimum and maximum level of subdivisions, the result may be partially affected by the region of poorer resolution. Fig. 3 plots the adapted grid of both configurations at $t = 0.1$.

Fig. 4 compares the computational efficiency of the various grid configurations. It is evident from the figure that the CPU time increases linearly when the level of subdivision of the regular grids is increased. For the adaptive grid configurations, the normalised CPU time is plotted against the maximum number of cells at the end of the simulations. Note that the time used for adaptation varies throughout the simulations. For the adaptive grids $\Psi_{6-8}$ and $\Psi_{5-9}$, the maximum time used for

![Fig. 1. Propagation of interface perturbation over flat bed at t = 0.1: $\Psi_{9-9}$](image-url)
the purpose of adaptation at any time interval are 0.0119 and 0.0318, respectively, normalised by the time of simulation of the said time step. Longer time is needed for $W_5/C_0$ compared to $W_6/C_0$ because refinement in the region of large gradient creates more cells than the reduction offered by lower resolution in the rest of the domain. The negligible adaptation cost and the accuracy of the results obtained suggest that the adaptive quadtree grid is an attractive choice for the present scheme.

Fig. 2. Interface propagation over flat bed: GCI contours at the free surface (left) and interface (right), comparing grid configurations (from top to bottom) (i) $W_{S,9}$ and $W_{S,8}$, (ii) $W_{S,8}$ and $W_{S,7}$, (iii) $W_{S,9}$ and $W_{S,9}$, and (iv) $W_{S,8}$ and $W_{S,8}$. 

5.2. Propagation of interface perturbation over a hump

The test is adapted from Kurganov and Petrova [10]. The initial conditions are the same as the previous test (51). The round-shaped interface perturbation is advected in the north-east direction by a uniform flow of magnitude 2.5 m/s in both directions towards a hump described by

\[ b(x, y) = 0.05 \exp(-100(x^2 + y^2)). \]  

(52)

The domain is given by \((x, y) \in [-1, 1]\) and the boundary condition is transmissive. Again, a second set of seeding points is applied to the initial discontinuity and the simulation is performed until \(t = 0.1\). Fig. 5 shows the free surface and interface at the end of the simulation using grid \(W_{9,9}\).

As the interface perturbation approaches the hump, complicated wave structure develops at both the interface and the free surface. Owing to the presence of the uneven bed, additional grid adaptation is performed in the vicinity of the hump (Fig. 6), resulting in higher numbers of cells for both \(W_{6,8}\) and \(W_{5,9}\) configurations and hence, slightly longer computation times, whereas the numbers of cells and computation times for the regular grids remain the same as for the flat bed case. The normalised adaptive times per time step for \(W_{6,8}\) and \(W_{5,9}\) are 0.0148 and 0.0328, respectively, which are slightly higher values than obtained from the flat bed case (Fig. 9).

Using the solutions obtained from grid configuration \(W_{9,9}\) as a basis for comparison, Fig. 7 shows that the scheme exceeds second order accuracy when using regular grid configurations (e.g., \(W_{7,7}\)). At coarser grid \(W_{5,5}\), the numerical results predict an L1-error that is smaller than expected and the solution fails to converge. At higher resolution of \(W_{8,8}\), the order of accuracy reduces to just above one (\(\approx 1.3\)) owing to the fact that the round-off error becomes dominant with smaller grid size. In addition, higher resolution in the refinement results in a globally smaller time step which increases the computational speed considerably. Hence, the choice of the minimum level of grid refinement is governed by accuracy whereas the choice of the maximum level of subdivision both accuracy and speed. The \(W_{6,8}\) and \(W_{5,9}\) configurations has similar
L1-error. Comparing both with the $W_{6/C0}^6$ configuration, for example, suggests that $W_{6/C0}^8$ produces relatively higher order of accuracy ($\sim 2$) compared to $W_{5/C0}^9$ ($1.3$). Fig. 8 shows the correlation between the L1-error and the normalised CPU time for the different grid configurations. The adaptive configuration $W_{6/C0}^8$ is more cost efficient compared to $W_{5/C0}^9$ whilst the L1-errors are in the same order and just slightly higher than the regular grid $W_{8/C0}^8$ which takes considerable longer CPU time. Hence we conclude that the adaptive grid $W_{6/C0}^8$ is the more attractive choice compared to $W_{5/C0}^9$ considering the compromise between accuracy and the cost of computation.
Fig. 10 presents the GCI contours of the free surface and interface when different grid configurations are adopted. Here, the results are inferior overall to those of the flat bed case (Section 5.1), but maintain the trend where better convergence is observed at the surface. The GCI values, though higher in the vicinity of the hump, are still well within the acceptable range except for the case comparing the regular grid $W_{9}/C_{0}$ and the adaptive grid $W_{5}/C_{0}$, where $GCI > 4.0$ near the interface. Comparing the regular grid $W_{8}/C_{0}$ and the adaptive grid $W_{6}/C_{0}$ shows better convergence (as also found for the flat bed case).

5.3. Internal circular dam break

We next consider an internal circular dam break with large discontinuity. The initial condition is given by:

$$h_1 = \begin{cases} 1.8 & \text{if } x^2 + y^2 > 4, \\ 0.2 & \text{if } x^2 + y^2 \leq 4, \end{cases} \quad h_2 = 2 - h_1. \quad (53)$$

where the fluids are initially at rest. The domain is given by $(x, y) \in [-5, 5]$ and boundary conditions are reflective. Following the results in Sections 5.1 and 5.2, we use adaptive quadtree grid $W_{6}/C_{0}$ as a compromise between speed and higher resolution in the region of interest. Fig. 11 shows the grid evolution in the early stage of the dam break. A set of seeding points is applied along the initial discontinuity as shown at $t = 0$. As the front propagates further, a large proportion of the domain is subdivided to the maximum level of division due to the presence of large free surface and interface gradients (as shown in Fig. 12).

Our results are in excellent agreement with those reported by Castro et al. [33].
5.4. Partial dam break

In the following test, we reproduce the partial dam break simulation of Paz et al. [48], where the variables and parameters are made dimensionless by using $h_1 = 1$, $r = 0.25$, and $g = 1$. The initial condition is such that the fluids in both layers are at rest and

![Figure 10](image_url)
The domain is given by \((x, y) \in [-10, 10]\) with reflective lateral boundary conditions. A levee is located at \(x = 0\) with a symmetrical gap of size 10. The gap, being also the location of the initial discontinuity, is seeded for cell refinement prior to the beginning of simulation. Adaptive grid \(\Psi_{6-8}\) is used.

Figs. 14 and 15 show the sequential development of the partial dam break at the free surface and interface, respectively. Selected snapshots of the free surface and interface are featured in Fig. 13. Propagation of fluid masses from each layer is evident across the gap and eventually both the free surface and interface approach horizontal (not shown). As the fluid masses propagate across the gap into the downstream end, two vortices are generated in the lee of the levee. Later, the direction of net mass flow is reversed. Consequently, another two vortices develop at the upstream side of the levee. Note that the vortices on the lee side of the lower layer are stronger than those in the upper layer. Moreover, the positions of the vortices in each layer do not exactly coincide, with those of the lower layer being closer to the tip of the gap. These features are well captured by the adaptive grid as shown in Fig. 16. We compare the results obtained from the adaptive grid \(\Psi_{6-8}\) with that of regular grid \(\Psi_{8-8}\) (Fig. 17). The maximum GCI approaches 3.5 at the interface by the time \(t = 50\), which is reasonably good.

5.5. Tidal channel with a hump and a constriction

In this section, the model is used to simulate a simple tidal flow in a channel characterised by a hump and a constriction. The aim is to examine the dynamics of tidal effects on the behaviour of the two-layer flow. The problem is inspired by the works of [3,4] where tidal and exchange flow in the Straits of Gibraltar was studied using a two-layer one-dimensional model with width function.

We consider a narrow channel of length \(x \in [-10, 10]\), with a sill placed at \(x = 0\) and a contraction at \(x = 1\). The bed \(b(x)\) and width \(\sigma(x)\) functions are given by:
Fig. 13. Partial dam break over a flat bed: Free surface at times $t = 0, 10, 15, 20$. Initial elevations to the left and to the right of the levee (and gap) are 2.0 and 1.50, respectively.

Fig. 14. Partial dam break over a flat bed: Free surface at times $t = 0, 10, 20, 30, 40, 50$. Initial elevations to the left and to the right of the levee (and gap) are 2.0 and 1.50, respectively.

Fig. 15. Partial dam break over a flat bed: Interface at times $t = 0, 10, 20, 30, 40, 50$. Initial elevations to the left and to the right of the levee (and gap) are 1.0 and 0.75, respectively.
\[
 b(x) = \begin{cases}
 1.2 \exp(-x^2)/3 & \text{if } x \in [-4, 4], \\
 0.5 \exp(-(x+10)^2) & \text{if } x \leq -8, \\
 0 & \text{otherwise},
\end{cases}
\]  

(55)

and

\[
 \sigma(x) = \begin{cases}
 2.5 + 1.5(1 - \exp(-a^2(x-1)^2)) & \text{if } x \in [-1, 2], \\
 3.7 & \text{otherwise},
\end{cases}
\]  

(56)
respectively, the latter of which is adapted from [3], where

\[
a = \begin{cases} 
0.637 & \text{if } -1 \leq x \leq 1, \\
1.273 & \text{if } 1 < x \leq 2.
\end{cases}
\]  

Consider two fluids with constant densities of ratio \( r = 0.98 \), the lock exchange experiment is conducted using the initial conditions:

\[
(h_1, h_2) = \begin{cases} 
(1.5, 0.5) & \text{if } x \leq 0 \\
(0.2, 1.8) & \text{if } x > 0
\end{cases}
\]  

and water at rest. Physically, this is akin to having a vertical barrier located at \( x = 0 \) which is instantaneously removed at \( t = 0 \). We include the effect of bottom friction \( \tau_b \) and interfacial friction \( \tau_f \) into the model as additional source terms in the form of:

\[
\begin{aligned}
\tau_{bx} &= -\rho_b C_b u_j \sqrt{u_j^2 + v_j^2}, \\
\tau_{by} &= -\rho_b C_b v_j \sqrt{u_j^2 + v_j^2}, \\
\text{and}
\tau_{fx} &= -\rho_f C_f u_r \sqrt{u_r^2 + v_r^2}, \\
\tau_{fy} &= -\rho_f C_f v_r \sqrt{u_r^2 + v_r^2},
\end{aligned}
\]

where \((u_r, v_r) = (u_1 - u_2, v_1 - v_2)\). Following [49], these additional source terms are treated in the same manner as the bed topography in order to maintain overall well-balanced property for steady state with non-zero velocity. The dimensionless bottom friction coefficient \( C_b \) and interface friction coefficient \( C_f \) are arbitrarily taken as \( 5 \times 10^{-3} \) and \( 5 \times 10^{-4} \), respectively.

The model is integrated until steady state is reached using exchange flow boundary conditions, i.e. \( Q_1 = -Q_2 \). Fig. 18 shows the evolution of the interface and the grid adaptation during the initial stage of simulation. The surface profile remains near horizontal throughout and hence is omitted for clarity. The plots show that after the initial transient, the interface at both upstream and downstream boundaries approaches horizontal and the coarser grid is adopted. Finer resolution is maintained at the vicinity of the constriction and the hump where there is considerable interface gradient. Fig. 19 shows the

Fig. 18. Lock exchange in a narrow channel with a hump and a constriction: plots showing evolution of interface with reference to the bed (left column) and grid adaptation (right column) at times (i–v) \( t = 0, 5, 10, 15, 30 \).
comparing the steady state solutions of grid configurations $W_8/C_0$ and $W_6/C_0$. Here, the GCI values of the free surface and interface are very small except a narrow band of the latter which approaches 2.5 at the jump location. The L1-errors based on simulated velocities are 0.0043 and 0.0023 for the upper and lower layer, respectively; the L1-errors based on simulated depths are 0.0039 for both layers.

Using the above steady state solutions as the initial condition, we proceed to apply tidal forcing at the downstream open boundary while maintaining exchange flow boundary condition at the upstream end. This initial condition describes an inflow of the denser fluid and an outflow of the less dense fluid at the downstream boundary, and the opposite at the upstream boundary. Physically, the downstream boundary represents connection to an open sea which acts as a reservoir of denser fluid, while the upstream represents consistent supply of riverine inflow. In the discussions that follow, we shall also refer to the region between the humps as the basin.

The downstream tidal boundary condition associated with the variation of total water depth is given as a function of time by

$$h_1(x_B, t) + h_2(x_B, t) = h_B - A \sin \left( \frac{2\pi t}{T} \right),$$

where the subscript $B$ denotes specified values at the boundary. Here, $h_B = 2$ is the total water depth, $A = 0.01$ is the surface elevation amplitude, $T$ is the period of the tidal wave. We use the dimensionless tidal time $T \sqrt{gL_v/L_h}$ where $L_v$ and $L_h$ are the vertical and horizontal scale respectively and $g = 9.81$ m/s$^2$ is the gravitational acceleration. Consider varied horizontal and vertical scale of 1:1000 and 1:10, respectively, the problem thus represents a 20 km long channel with a minimum and maximum width of 2.5 km and 3.7 km, respectively. The quiescent water depth is 20 m and the tidal range is 2 m. At the middle of the channel, the hump rises up to a height of 12 m above the horizontal bed. For $S_2$ tide ($T = 60$ min), the equivalent tidal period in the model is thus 136.6. At the boundaries, the depth and velocities of both layers are determined using Riemann invariants specified according to the internal Froude number $Fr_j^2$ of the respective layer, which is defined as

$$Fr_j^2 = U_j^2/(g'h_j), \quad j = 1, 2,$$

and the composite Froude number is given by

$$G^2 = Fr_1^2 + Fr_2^2 - (1 - r)Fr_1^2Fr_2^2.$$
Fig. 20 shows the free surface and interface during the $S_2$ tidal cycle. At $t/T = 0.5$ when the downstream elevations are rising, strong currents flow into the basin in both the layers as observed from the plot of velocity vectors (not shown), and the flow approaches critical (Fig. 21(i)) condition. At $t/T = 1.0$ when the downstream elevations are reducing, the current is reversed with net flow out of the basin. As shown in Fig. 21(ii), the flow reaches supercritical condition resulting in a weak internal jump. Fig. 22 shows the GCI comparing the solutions at $t/T = 1.0$ of grid configurations $W_8/C_0$ and $W_6/C_0$. Here, the GCI values for both the free surface and interface are generally small but approaches a maximum of up to 3.0 at the jump location. The L1-errors based on simulated velocities are 0.0038 and 0.0036 for the upper and lower layer, respectively; the L1-errors based on simulated depths are 0.0044 and 0.0043 for these layers.

The results above demonstrated the robustness of the present model in solving a two-dimensional tidal-exchange flow. However, we note that the problem is a greatly simplified representation of an actual tidal channel, and the solutions obtained is highly dependent on the geometry of the domain, the scaling of the tidal period, as well as the bed friction and interfacial friction coefficients. For the purpose of full scale study, a realistic geometry and tidal signal should be employed (e.g., [3,4]).

6. Conclusions

The entropy-correction-free solver of Chacon-Rebollo et al. [16,17], previously applied to one-dimensional non-conservative non-homogenous hyperbolic systems [18], is extended to two-dimensions on a dynamically adaptive quadtree grid generation system. A number of benchmark tests involving two superposed immiscible layers of shallow water fluids are considered. Grid convergence is evaluated by means of the Grid Convergence Index (GCI) of Roache and Knupp [46] and Roache [47]. We show that the results obtained are in excellent agreement with those produced on regular grids, and conclude that the use of adaptive quadtree grids offers considerable computational advantages for Roe-type two-layer shallow water models. The numerical model can be used for the simulation of two-layer flow in a domain with complex boundaries and realistic bathymetry at a fraction of the cost incurred by a scheme applied on a regular grid. The model is applied on a tidally-forced narrow channel with a hump and a constriction successfully. The fractal-like stepped approximation to boundary fitting in the present grid system may be better replaced by Cartesian cut cells to improve the fit to irregular boundaries. The present model requires the minimum and maximum level of subdivision to be specified, which nevertheless can only be determined through intelligent guesses and possibly trial and error. A model capable of automated selection of minimum and maximum grid level subdivision based on a given parameter such as error tolerance would be highly attractive. Another challenge would be to allow independent adaptation at the two layers which may offer substantial improvement in speed as the surface gradient in the two-layer flow in most cases is relatively small and hence further grid refinement is not usually needed. This requires a robust interpolation routine, and can be highly attractive when applied to a multi-layered depth-averaged model.

Acknowledgments

The authors are grateful to Ben Rogers and Qiuhua Liang who contributed to the development of the OxQuad grid generator. The first author would like to thank Professor Manuel J. Castro Díaz of University of Málaga, and Professor Rodrigo R. Paz of National University of the Littoral for the stimulating discussions. The first author also acknowledges the scholarship awarded by Universiti Teknologi MARA and the Ministry of Higher Education Malaysia. The second and third authors are grateful for support from the UK EPSRC Grant EP/F020511, funded as part of the Flood Risk Management Research Consor-
tium FRMRC-II. Part of the computation has been carried out using resources provided by the Oxford Supercomputing Centre.

References