**H_∞ FUZZY CONTROL FOR TIME-DELAY AFFINE TAKAGI-SUGENO FUZZY MODELS: AN ILMI APPROACH**

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**ABSTRACT**

This paper deals with the fuzzy based H_∞ control problem for time-delay affine Takagi-Sugeno (T-S) fuzzy models. A class of nonlinear time-delay systems is approximated by a time-delay affine T-S fuzzy model in this paper. Based on Lyapunov-Razumikhin theorem and S-procedure, the stability and stabilization problems are solved by employing a Parallel Distributed Compensation (PDC) type H_∞ fuzzy controller. The synthesis for the time-delay affine T-S fuzzy models is a Bilinear Matrix Inequality (BMI) problem and it can not be solved via a convex optimization algorithm. Hence, an Iterative Linear Matrix Inequality (ILMI) algorithm is used to solve the BMI problems in this paper. Finally, a numerical simulation for a delayed pendulum system is given to show the applications of the present approach.

**Key Words:** Takagi-Sugeno fuzzy model, time-delay system, Lyapunov-Razumikhin theorem, S-procedure.

**I. INTRODUCTION**

Over the past few decades, a typical approach of analysis and synthesis for nonlinear systems with time delays is the well-known local linearization method. Some delay independent/dependent stability conditions and stabilization approaches have been proposed for these linearized time-delay differential/different equations (Mahmoud, 2000; Maun et al., 1987). However, it is known that each local linear model is just valid around the operating point and so these results can only guarantee the local stability of nonlinear time-delay systems. Recently, one solution to this problem has been developed for global stability and stabilization issues in T-S fuzzy models (Cao et al., 2001; Gu et al., 2001; Wang et al., 2004; Lee et al., 2000; Yi et al., 2002). Considering uncertain nonlinear fuzzy systems with time delays in (Wang et al., 2004) and (Lee et al., 2000) discussed the robustness and H_∞ control design for T-S fuzzy models. Moreover, the conditions for global exponential stability of free fuzzy systems with uncertain delays were presented in (Yi et al., 2002).

It has been recognized in (Cao et al., 2001; Wang et al., 2004; Lee et al., 2000; Yi et al., 2002; Tanaka and Wang, 2001; Chang and Chang, 2006) that T-S fuzzy models provide a more useful approach not only in model structuring but also in synthesis for the control of nonlinear time-delay systems. In general, T-S fuzzy models can be divided into two categories: homogeneous T-S fuzzy models (Cao et al., 2001; Gu et al., 2001; Wang et al., 2004; Lee et al., 2000; Yi et al., 2002; Tanaka and Wang, 2001; Chang, 2001; Chang and Sun, 2003, Chang and Shing, 2004; Chang and Wu, 2005) and affine T-S fuzzy models (Chang and Chang, 2006; Hassibi and Boyd, 1998; Johansson et al., 1999; Lee et al., 2000; Kim and Kim, 2002; Kim and Kim, 2001). In the homogeneous T-S fuzzy models, the consequent part of the fuzzy models are linear dynamic systems with no constant bias term. On the other hand, the consequent parts of the linear dynamic subsystems of affine T-S fuzzy models are affined by a constant bias term for each rule. In the aforementioned works for time-delay systems (Cao et al., 2001; Gu et al., 2001; Wang et al., 2004; Lee et al., 2000; Yi et al., 2002), the authors mostly focused on the time-delay homogeneous T-S fuzzy models. However, there are very rare studies (Chang and Chang, 2006) that pay close attention to the
time-delay affine T-S fuzzy models. Therefore, the purpose of this paper is to deal with the control problem for time-delay affine T-S fuzzy models. We will concentrate on the following issues: (a) time-delay affine T-S fuzzy model structuring problems and (b) BMI-based $H_{\infty}$ fuzzy controller design problems.

In this paper, the $H_{\infty}$ control scheme (Maun et al., 1987) is used to deal with the robust performance design problem for time-delay affine T-S fuzzy models. It can provide the guaranteed $H_{\infty}$ performance for the attenuation of the unknown disturbance with a known constant upper bound $v_{ub} \geq ||v(t)||$, where $||\cdot||$ refers to the Euclidean vector norm. $M_{ip}$ is the fuzzy set, $x_{1}(t), \ldots, x_{p}(t)$ are known premise variables, $p$ is the premise variable number and $r$ is the number of fuzzy model rules. $\tau(t) \leq \tau$ is the bounded time delay function in the state and $\tau > 0$ is a constant scalar. $\psi(t)$ is the initial condition of the state defined on $-\tau \leq t \leq 0$. Besides, the region $X_{i} \subseteq \Re^{m}$ is assumed to be a fuzzy subspace and $X_{i}$ is called a cell. The set of cell indices is denoted as $\hat{I}$ and the union of all cells $x(t) = U_{i \in \hat{I}}X_{i}$ is referred to as the whole fuzzy space. Let $\hat{I}_{0} \subseteq \hat{I}$ be the set of indices for the fuzzy rules that contain the origin and $\hat{I}_{1} \subseteq \hat{I}$ be the set of indices for the fuzzy rules that does not contain the origin. The origin is an equilibrium point of the time-delay affine T-S fuzzy models and it is assumed that $a_{i} = 0$ for $i \in \hat{I}_{0}$.

Given a pair of $(x(t), u(t))$, the final outputs of the time-delay affine T-S fuzzy model (1) are inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{p} \omega_{i}(x(t))[A_{i}x(t) + A_{id}x(t - \tau(t)) + B_{i}u(t) + a_{i}] + Ev(t)$$

$$= \sum_{i=1}^{p} h_{i}(x(t))[A_{i}x(t) + A_{id}x(t - \tau(t)) + B_{i}u(t) + a_{i}] + Ev(t),$$

where $A_{i} \in \Re^{m \times n}$, $A_{id} \in \Re^{m \times n}$, $B_{i} \in \Re^{m \times m}$, $E \in \Re^{n}$ and $a_{i} \in \Re^{n}$ are constant matrices. $x(t) \in \Re^{n}$ is the state vector, $u(t) \in \Re^{m}$ is the input vector, $v(t) \in \Re$ denotes the unknown disturbance with a known constant upper bound $v_{ub} \geq ||v(t)||$, where $||\cdot||$ refers to the Euclidean vector norm. $M_{ip}$ is the fuzzy set, $x_{1}(t), \ldots, x_{p}(t)$ are known premise variables, $p$ is the premise variable number and $r$ is the number of fuzzy model rules. $\tau(t) \leq \tau$ is the bounded time delay function in the state and $\tau > 0$ is a constant scalar. $\psi(t)$ is the initial condition of the state defined on $-\tau \leq t \leq 0$. Besides, the region $X_{i} \subseteq \Re^{m}$ is assumed to be a fuzzy subspace and $X_{i}$ is called a cell. The set of cell indices is denoted as $\hat{I}$ and the union of all cells $x(t) = U_{i \in \hat{I}}X_{i}$ is referred to as the whole fuzzy space. Let $\hat{I}_{0} \subseteq \hat{I}$ be the set of indices for the fuzzy rules that contain the origin and $\hat{I}_{1} \subseteq \hat{I}$ be the set of indices for the fuzzy rules that does not contain the origin. The origin is an equilibrium point of the time-delay affine T-S fuzzy models and it is assumed that $a_{i} = 0$ for $i \in \hat{I}_{0}$.

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THEN \( u(t) = -\sum_{i=1}^{n} h_{ij}(x(t)) [F_{ij}(x(t)) + \mu_{ij}] \).

Substituting Eq. (7) into Eq. (2), one can obtain the corresponding closed-loop system as

\[
\dot{x}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij}(x(t)) h_{ij}(x(t)) \left[ (A_{ij} - B_{ij}) x(t) + (a_{ij} - B_{ij}) \mu_{ij} \right] + A_{ii} x(t) - \tau(t) + Ev(t)
\]

\[
\dot{x}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij}(x(t)) h_{ij}(x(t)) \left[ \frac{G_{ij} + G_{ji}}{2} x(t) + \frac{G_{ij} + G_{ji}}{2} x(t) - \tau(t) \right] + Ev(t),
\]

where \( G_{ij} = A_{ij} - B_{ij} F_{ij} \) and \( g_{ij} = a_{ij} - B_{ij} \mu_{ij} \).

Based on the PDC type fuzzy controller Eq. (7), a sufficient condition for ensuring delay-independent stability of controlled time-delay affine T-S fuzzy models Eq. (8) is introduced. It is derived by using the Lyapunov-Razumikhin theorem (Mahmoud, 2000; Maun et al., 1987) and \(S\)-procedure (Hassibi and Boyd, 1998; Johansson et al., 1999; Lee et al. 2000; Kim and Kim, 2002; Boyd et al., 2000). Moreover, the \(H_{\infty}\) control performance is provided to guarantee the following attenuation:

\[
\frac{1}{\gamma} \int_{t_{0}}^{t_{f}} x^{T}(t) \dot{x}(t) dt < \int_{t_{0}}^{t_{f}} \gamma^{2} v^{T}(t) v(t) dt, \quad \forall v(t) \neq 0,
\]

with zero initial condition for all \( v(t) \in L_{2}[0, t_f] \), where \( t_f \) is the terminal time of the control. \( \gamma \) is a prescribed value which denotes the worst case effect of \( v(t) \) on \( x(t) \). Besides, \( S = \gamma \tau > 0 \) is a positive definite weighting matrix and \( S = \mathcal{R}^{n \times n} \).

The inequality in Eq. (9) can be seen as bounded disturbance and bounded state but with a prescribed gain. The stability conditions for the time-delay affine T-S fuzzy model in Eq. (8) are described in the following theorem.

**Theorem 1:** Given an \( H_{\infty} \) attenuation parameter \( \gamma > 0 \) and the \( S \)-procedure weighting parameters \( T_{ijq}, n_{ijq}, v_{ijq} \). The time-delay affine T-S fuzzy model described in Eq. (8) is quadratically stable in the large and the \( H_{\infty} \) control performance Eq. (9) is guaranteed for an attenuation \( \gamma \) if there exist positive definite matrices \( P > 0, S > 0, N_{1} > 0 \), control gains \( F_{ij}, \mu_{ij} \) and scalars \( \xi_{ijq} \geq 0 \) such that

\[
\begin{align*}
\Gamma_{ij} & = \left( \begin{array}{c}
\frac{G_{ij} + G_{ji}}{2} \\
\frac{G_{ij} + G_{ji}}{2}
\end{array} \right) P + P \left( \begin{array}{c}
\frac{G_{ij} + G_{ji}}{2} \\
\frac{G_{ij} + G_{ji}}{2}
\end{array} \right), \\
T_{ij} & = \frac{G_{ij} + G_{ji}}{2} P - \frac{G_{ij} + G_{ji}}{2} P - \frac{G_{ij} + G_{ji}}{2} P
\end{align*}
\]

Besides, the symbol * denotes the transposed elements (matrices) for symmetric positions and the \( S \)-procedure is defined such that

\[
\sigma_{ijq}(x(t)) \triangleq x^{T}(t) T_{ijq} x(t) + 2 n_{ijq}^{T} x(t) + v_{ijq} \leq 0,
\]

for all \( x(t) \) which activates rule \( i \) (i.e., \( h_{i}(x(t)) > 0 \)).

**Proof:**

Select a Lyapunov function as

\[
V(x(t)) = x^{T}(t) Px(t).
\]

The derivative of the Lyapunov function \( V(x(t)) \) along the trajectories of time-delay affine T-S fuzzy model in Eq. (8) is
\[
\dot{V}(x(t)) = \sum_{i=1}^{\hat{N}} \sum_{j=1}^{\hat{N}} h_i(x(t)) h_j(x(t)) \\
\times \left\{ x^T(t) \left( \frac{G_{ij} + G_{ji}}{2} \right)^T P + P \left( \frac{G_{ij} + G_{ji}}{2} \right) x(t) \\
+ x^T(t) \left( \frac{G_{ij} + G_{ji}}{2} \right)^T P x(t) \\
+ x^T(t) A^T_{ji} P x(t) + x(t) P A_{ji} x(t - \tau(t)) \\
+ \frac{x^T(t) \left( \frac{G_{ij} + G_{ji}}{2} \right)^T A^T_{ji} P x(t) + x(t) P A_{ji} x(t - \tau(t))}{2} \right\} \\
+ x^T(t) P E v(t) + v^T(t) E^T P x(t) .
\]  

(18)

For any two real matrices \(X\) and \(Y\), one has (Maun et al., 1987)

\[
X^T Y + Y^T X \leq X^T N X + Y^T N^{-1} Y,
\]

(19)

where \(X \in \mathbb{R}^{n \times m}, Y \in \mathbb{R}^{n \times m}\) and \(N > 0\) is a constant matrix (or scalar). From the relationship of Eq. (19), we have

\[
x^T(t - \tau(t)) A^T_{ji} P x(t) + x(t) P A_{ji} x(t - \tau(t)) \\
\leq x^T(t - \tau(t)) N x(t - \tau(t)) \\
+ x^T(t) P A_{ji} N^{-1} A^T_{ji} P x(t), \text{ for } i = 1, \ldots, r
\]

and

\[
x^T(t - \tau(t)) A^T_{ji} P x(t) + x(t) P A_{ji} x(t - \tau(t)) \\
\leq x^T(t - \tau(t)) N x(t - \tau(t)) \\
+ x^T(t) P A_{ji} N^{-1} A^T_{ji} P x(t), \text{ for } j = 1, \ldots, r
\]

and

\[
v^T(t) E^T P x(t) + x^T(t) P E v(t) \\
\leq v^T(t) \left( \gamma^2 I \right) v(t) + x^T(t) P E \left( \frac{1}{\gamma^2} \right) E^T P x(t).
\]

(22)

From Eq. (18) and Eqs. (20-22), it is obvious that

\[
V(x(t)) \leq \sum_{i=1}^{\hat{N}} \sum_{j=1}^{\hat{N}} h_i(x(t)) h_j(x(t)) \\
\times \left\{ -x^T(t) (\Gamma_y^T + P A_{ji} N^{-1} A^T_{ji} P + \left( \frac{G_{ij} + G_{ji}}{2} \right) / 2) \right\} \\
+ PE (\frac{1}{\gamma^2}) E^T P x(t) + x^T(t) P \left( \frac{G_{ij} + G_{ji}}{2} \right) \right\} \\
+ v^T(t) (\gamma^2 I) v(t) ,
\]

(23)

where \(\Gamma_y\) is defined in Eq. (14). If there exists a positive definite matrix \(P\) such that \(P \geq N\), one can obtain

\[
V(x(t)) \\
\leq \sum_{i=1}^{\hat{N}} \sum_{j=1}^{\hat{N}} h_i(x(t)) h_j(x(t)) \\
\times \left\{ -x^T(t) (\Gamma_y^T + (PA_{ji} N^{-1} A^T_{ji} P + \left( \frac{G_{ij} + G_{ji}}{2} \right) / 2) \right\} \\
+ PE (\frac{1}{\gamma^2}) E^T P x(t) + x^T(t) P \left( \frac{G_{ij} + G_{ji}}{2} \right) \right\} \\
+ v^T(t) (\gamma^2 I) v(t) ,
\]

(24)

In addition, based on the Razumikhin theorem (Mahmoud, 2000; Maun et al., 1987), if the inequality \(V(x(t)) < V(x(t - \tau))\) holds for all time, the stability condition is undoubtedly asymptotically stable. So, it is necessary to check the stability for the case \(V(x(t - \tau)) < V(x(t))\). If there exists a real number \(\delta > 1\) such that \(V(x(t - \delta)) < \delta V(x(t))\) for \(\theta \in [0, \tau]\), then Eq. (24) can be replaced by

\[
V(x(t)) \\
\leq \sum_{i=1}^{\hat{N}} \sum_{j=1}^{\hat{N}} h_i(x(t)) h_j(x(t)) \\
\times \left\{ -x^T(t) (\Gamma_y^T + (PA_{ji} N^{-1} A^T_{ji} P + \left( \frac{G_{ij} + G_{ji}}{2} \right) / 2) \right\} \\
+ PE (\frac{1}{\gamma^2}) E^T P x(t) + x^T(t) P \left( \frac{G_{ij} + G_{ji}}{2} \right) \right\} \\
+ v^T(t) (\gamma^2 I) v(t) .
\]

(25)

Applying the S-procedure (Hassibi and Boyd, 1998; Johansson et al., 1999; Lee et al., 2000; Kim and Kim, 2002; Boyd et al., 2000), the matrix inequality Eq. (25) becomes

\[
V(x(t)) \\
\leq \sum_{i=1}^{\hat{N}} \sum_{j=1}^{\hat{N}} h_i(x(t)) h_j(x(t)) \\
\times \left\{ -x^T(t) (\Gamma_y^T + (PA_{ji} N^{-1} A^T_{ji} P + \left( \frac{G_{ij} + G_{ji}}{2} \right) / 2) \right\} \\
+ PE (\frac{1}{\gamma^2}) E^T P x(t) + x^T(t) P \left( \frac{G_{ij} + G_{ji}}{2} \right) \right\} \\
+ v^T(t) (\gamma^2 I) v(t) \right\} \\
< \sum_{q=1}^{\hat{N}} \xi_{pq} \sigma_{pq}(x(t)) .
\]

(26)
In terms of $\bar{x}(t) = [x^T(t) \ 1]^T$, Eq. (26) can be represented as
\[ \dot{V}(x(t)) = \sum_{i=1}^{C} \sum_{j=1}^{C} h_i(x(t)) h_j(x(t)) \left( \dot{x}^T(t) A_1 + A_2 x(t) \right) , \] (27)
where
\[ A_1 = T_{ij}^c \]
\[ = \begin{bmatrix} (PA_{ij}N_i^{-1}A_{ij}^T P + PA_{ij}N_i^{-1}A_{ij}^T P/2 + P^* + \delta P^* ) \ni \mbox{0} \\ \mbox{0} \ni \mbox{0} \end{bmatrix}. \]

Let us define
\[ \vartheta(\delta) \]
\[ = \sum_{i=1}^{C} \sum_{j=1}^{C} h_i(x(t)) h_j(x(t)) \left( x^T(t) A_1 x(t) \right) \]
\[ - \sum_{i=1}^{C} \sum_{j=1}^{C} h_i(x(t)) h_j(x(t)) \left[ x^T(t)(-P(\gamma^2)E^T P \right. \]
\[ - Sx(t) \} . \]

If the condition $\vartheta(\delta) < 0$ is held, then Eq. (27) can be rewritten as
\[ \dot{V}(x(t)) = \sum_{i=1}^{C} \sum_{j=1}^{C} h_i(x(t)) h_j(x(t)) \left( \dot{x}^T(t) A_1 x(t) \right) \]
\[ - x^T(t)(-P(\gamma^2)E^T P - Sx(t)) < 0 , \] (36)
or the equivalent stability condition
\[ T_{ij}^c \]
\[ + \begin{bmatrix} (PA_{ij}N_i^{-1}A_{ij}^T P + PA_{ij}N_i^{-1}A_{ij}^T P/2 + P^* + \delta P^* ) \ni \mbox{0} \\ \mbox{0} \ni \mbox{0} \end{bmatrix} = T_{ij}^c + \Theta_{ij}^g < 0 . \] (37)

Thus, $\vartheta(\delta) = 0$ presented in Eq. (11) is satisfied. Then, by continuity there exists a $\delta = 1 + \varepsilon$ with $\varepsilon > 0$ sufficiently small such that $\vartheta(\delta) < 0$ for all $i$. Hence, for $V(x(t)) < 0$, the proof of stability condition Eq. (11) is completed for all $x(t) \in X$, $i \in I$. Moreover, for the case of $x(t) \in X$, $i \in I_0$, the stability condition Eq. (10) can be obtained by setting the term $g_{ij} = 0$ and ignoring the $S$-procedure from the similar proof procedure.

The computation of $T_{ij}, n_{ij}$, and $\varepsilon_{ij}$ for $S$-procedure is explained in (Kim and Kim, 2002; Kim and Kim, 2001). It can be noted that Theorem 1 belongs to the class of BMI for $P, S, N_i, F_i, \mu_i$, and $\xi_{ij}$. Thus, the fuzzy controller synthesis problem cannot be solved by a convex optimization algorithm. Due to this reason,
the ILMI algorithm (Kim and Kim, 2002; Kim and Kim, 2001) is developed in the next section to obtain a feasible solution of the conditions of Theorem 1 for the synthesis of the time-delay affine T-S fuzzy models.

III. ILMI SOLUTIONS

In this section, an ILMI algorithm (Kim and Kim, 2002; Kim and Kim, 2001) is provided to get a suitable condition for the conditions of Theorem 1. The decay rate \( \alpha \) can be considered in the stability condition in order to relax the LMI search procedure and make it feasible. For convenience, we define the operation in this paper according to the Schur-Complements (Boyd et al., 2000):

\[
\text{Schur}^{{\circ}}(C(x)) = C_{11}(x) - C_{21}(x)^{T}C_{22}(x)^{-1}C_{21}(x),
\]

(38)

where \( C_{11}(x) = C_{11}(x)^{T}, C_{22}(x) = C_{22}(x)^{T}, C_{21}(x) \) depend affinely on \( x \) and \( \text{Schur}^{{\circ}}(C(x)) \) is called the Schur-Complements of \( C_{22}(x) \) in \( C(x) \). A relaxed stability condition for the time-delay affine T-S fuzzy model is given as follows:

**Lemma 1:**

For the matrices \( \Gamma_{ij}^{c} \) and \( T_{ij}^{c} \) defined in Theorem 1, we have the following relationship:

\[
x^{T}(t)[\Gamma_{ij}^{c}]x(t) \leq \frac{1}{2}x^{T}(t)[\Psi_{ij}^{c}](x(t), \quad (39)
\]

\[
x^{T}(t)[T_{ij}^{c}]x(t) \leq \frac{1}{2}x^{T}(t)[\Phi_{ij}^{c}](x(t), \quad (40)
\]

where \( \Phi(t) = [\Phi^{T}(t) \ 1]^{T} \) and

\[
\Psi_{ij}^{c} = A_{ij}^{T}P + PA_{ij} + A_{ij}^{T}P + PA_{ij} + Y_{ij}^{T}Y_{ij}^{f} + Y_{ij}^{f}Y_{ij}^{T} + [F_{ij}^{T} F_{ij}^{T} F_{ij}^{T} F_{ij}^{T} F_{ij}^{T} F_{ij}^{T} F_{ij}^{T}]2P[B_{j} B_{j} B_{j} B_{j}]\]

\[
- Y_{ij}^{f}(B_{j}^{T}P + F_{ij}) - (PB_{j} + F_{ij})Y_{ij}^{f} - Y_{ij}^{T}(B_{j}^{T}P + F_{ij}) - (PB_{j} + F_{ij})Y_{ij} + z_{i}^{T}z_{i}^{f} - z_{i}^{T}B_{j}^{T}P - PB_{j}z_{i} - z_{i}^{T}B_{j}^{T}P - PB_{j}z_{i},
\]

(41)

\[
\Phi_{ij}^{c} = \begin{bmatrix} A_{11} & \ast \\ A_{21} & A_{22} \end{bmatrix}^{T}.
\]

(42)

\[
A_{11} = \Psi_{ij}^{c} - \frac{1}{2} \sum_{q=1}^{N} \xi_{iq}^{T} \xi_{ij}^{T},
\]

(43)

\[
A_{21} = (a_{i} - B_{j}y_{i})^{T}P + (a_{j} - B_{j}y_{j})^{T}P + (y_{j} - \mu_{j})^{T}z_{i} + (y_{i} - \mu_{i})^{T}z_{j} - \frac{1}{2} \sum_{q=1}^{N} \xi_{iq}^{T} \xi_{ij}^{T},
\]

(44)

\[
A_{22} = \mu_{i}^{T} \mu_{i} + \mu_{j}^{T} \mu_{j} + y_{ij}^{T}y_{ij} + y_{ij}^{T}y_{ij} - \mu_{i}^{T} \mu_{i} - \mu_{j}^{T} \mu_{j} - \xi_{ij}^{T}n_{ij}^{T} - \xi_{ij}^{T}n_{ij},
\]

(45)

in which

\[
Y_{ij} = B_{j}^{T}P + F_{ij}, \quad z_{i} = B_{j}^{T}P, \quad y_{i} = \mu_{i}.
\]

(46)

**Proof:** see Appendix for the proof.

**Theorem 2:** Given an \( H_{\infty} \) attenuation parameter \( \gamma > 0 \), the \( S \)-procedure weighting parameters \( T_{ij}, n_{ij}, v_{ij} \) and constant matrices (vectors) \( Y_{ij}, z_{i} \) and scalars \( y_{i} \). The stability conditions (10-11) described in Theorem 1 are held, the continuous time-delay affine T-S fuzzy model in Eq. (8) is quadratically stable in the large and the \( H_{\infty} \) control performance Eq. (9) is guaranteed for an attenuation \( \gamma \), if there exist a decay rate \( \alpha < 0 \), positive definite matrices \( P > 0, S > 0, N_{1} > 0 \), control gains \( F_{ij}, \mu_{i} \) and scalars \( \xi_{ij} \geq 0 \) such that

\[
\begin{align*}
\Gamma_{ij}^{\text{rd}} & < 0 & \text{for } x(t) \in X_{i}, i \in I_{0}, \\
P & \geq N_{1},
\end{align*}
\]

(47)

\[
T_{ij}^{\text{rd}} = 0 & \text{for } x(t) \in X_{i}, i \in I_{1},
\]

(48)

where

\[
\begin{bmatrix}
\sigma_{ij} - \alpha P & \ast & \ast & \ast & \ast \\
L_{2ij}^{T} & -I & \ast & \ast & \ast \\
L_{3ij}^{T}P & 0 & -I/2 & \ast & \ast \\
L_{4ij}^{T}P & 0 & 0 & -N_{1} & \ast \\
E^{T}P & 0 & 0 & 0 & -\gamma^{2}I/2
\end{bmatrix}
\]

(49)

\[
\begin{bmatrix}
\sigma_{ij} - \alpha P & \ast & \ast & \ast & \ast \\
L_{2ij}^{T} & -I & \ast & \ast & \ast \\
L_{3ij}^{T}P & 0 & -I/2 & \ast & \ast \\
L_{4ij}^{T}P & 0 & 0 & -N_{1} & \ast \\
E^{T}P & 0 & 0 & 0 & -\gamma^{2}I/2
\end{bmatrix}
\]

(50)
and $L_{1ij} = [\mu_i^T \mu_j^T]$, $L_{2ij} = [F_i^T F_j^T]$, $L_{3ij} = [B_i \ B_j]$, $L_{4ij} = [A_{id} \ A_{jd}]$.

$$
\mathbf{R}_i = 2P + 2S + A_i^T P + PA_i + A_i^T P + PA_i + Y_j^T Y_{ji} - Y_j^T (B_i^T P + F_i) + (PB_i + F_i^T Y_{ji} - Y_i^T B_i^T P + F_i) + (PB_j + F_j^T Y_{ji} - z_i^* B_i^T P - PB_i z_i) - z_i^* B_i^T P - PB_i z_j + z_i^* z_j + z_j^T z_j, \quad (51)
$$

$$
\mathbf{R}_{R11} = \mathbf{R}_i - \sum_{q=1}^{p} 2 \xi_{ijq} T_{ijq}, \quad (52)
$$

$$
\mathbf{R}_{R21} = (a_i - B_i y_i)^T P + (a_j - B_i y_j)^T P + (y_j - \mu_j)^T z_i + (y_i - \mu_i)^T z_j - \sum_{q=1}^{p} 2 \xi_{ijq} n_{iq}, \quad (53)
$$

$$
\mathbf{R}_{R22} = y_i^T y + y_j^T y - \mu_i^T y_i - \mu_j^T y_j - \mu_i^T y_j - \mu_j^T y_j - 2 \xi_{ijq} v_{ijq}, \quad (54)
$$

**Proof:**

As the first step in our proof, we have to transfer two matrices $\mathbf{P}_{ij}^{cd}$ and $\mathbf{T}_{ij}^{cd}$. It can be helpful to achieve the proof. First, multiplying the $\mathbf{P}_{ij}^{cd}$ from left and right by $X(t)$ and $Y(t)$ one obtains

$$
X^T(t) Y_{ij}^{cd} X(t) = X^T(t) (T_{ij}^{cd} + \Theta_{ij}^0) X(t). \quad (55)
$$

According to Lemma 1, we have $X^T(t) [T_{ij}^{cd}] X(t) \leq \frac{1}{2} \sum_{i=1}^{N_i} X^T(t) [\Psi_{ij}] X(t)$ and then Eq. (55) can be represented as

$$
X^T(t) Y_{ij}^{cd} X(t) \leq \frac{1}{2} X^T(t) [\Psi_{ij}] X(t) + 2 \Theta_{ij}^0 X(t). \quad (56)
$$

Second, according to the Schur-Complements, $\mathbf{T}_{ij}^{cd}$ can be rewritten as

$$
\text{Schur}^{-1}(\mathbf{T}_{ij}^{cd}) = \Psi_{ij}^{0} + 2 \Theta_{ij}^{0} \begin{bmatrix} \alpha P & 0 \\ 0 & 0 \end{bmatrix}. \quad (57)
$$

Hence, if there exist $P > 0$, $S > 0$, $N_1 > 0$, $F_i$, $\mu_i$, $\xi_{ijq} \geq 0$ and $\alpha < 0$ such that $\mathbf{T}_{ij}^{cd} < 0$, $P \geq N_1$ and Eq. (48) are satisfied, then according to Schur-Complements one has $\text{Schur}^{-1}(\mathbf{T}_{ij}^{cd}) < 0$. From Eq. (57), it is obvious that $\text{Schur}^{-1}(\mathbf{T}_{ij}^{cd}) < 0$ implies $\Psi_{ij}^{0} + 2 \Theta_{ij}^{0} < 0$. Thus, if $\Psi_{ij}^{0} + 2 \Theta_{ij}^{0} < 0$ is satisfied, then Eq. (56) implies

$$
X^T(t) Y_{ij}^{cd} X(t) \leq \frac{1}{2} X^T(t) [\Psi_{ij}] X(t) + 2 \Theta_{ij}^{0} X(t) < 0. \quad (58)
$$

Consequently, $\mathbf{P}_{ij}^{cd}$ is negative definite for all $x(t) \in X_i$ and $i \in \hat{I}_i$. Eq. (11) holds. The proof is completed. Similarly, the proof of Eq. (10) for all $x(t) \in X_i$, $i \in \hat{I}_i$ is obtained by setting the bias term $a_i = B_i = 0$ and by removing the $S$-procedure term $\sigma_{ijq}(x(t))$ from the above process.

By using Schur-Complemet, Theorem 2 provides more relaxed and suitable stability conditions than Theorem 1. Based on the conditions of Theorem 2, an ILMI algorithm (Kim and Kim, 2002; Kim and Kim, 2001) can be proposed to find the feasible solutions for the stability conditions of Theorem 1. The purpose of this algorithm is to interactively search for $P$, $S$, $N_1$, $F_i$, $\mu_i$, $\xi_{ijq}$, $\alpha$ and to update the auxiliary variables until $\alpha$ becomes negative.

**<ILMI Algorithm>**

**Step 1.** Define the iterative auxiliary variables as follows:

$$
Y_{ij}^{(k)} = B_i^T P(k-1) + F_j^{(k-1)}; y_i^{(k)} = \mu_i^{(k-1)}; z_i^{(k)} = B_i^T P(k-1), \quad (59)
$$

where $k$ denotes the iteration index. Set $k = 1$ and the initial conditions of $Y_{ij}^{(1)}$, $y_i^{(1)}$ and $z_i^{(1)}$ can be obtained as follows:

$$
Y_{ij}^{(1)} = B_i^T P(0) + F_j^{(0)}; y_i^{(1)} = \mu_i^{(0)}; z_i^{(1)} = B_i^T P(0),
$$

where $P(0)$, $F_j^{(0)}$ and $\mu_i^{(0)}$ are given as follows:

- As for the initial $P(0)$, one can solve it from the following Ricatti equation.

$$
\hat{A}^T P(0) + P(0) \hat{A} - P(0) \hat{B} \hat{B}^T P(0) + Q = 0, \quad (60)
$$

where $\hat{A} = \frac{1}{2} \sum_i A_i$, $\hat{B} = \frac{1}{2} \sum_i B_i$ and $Q > 0$. The matrix $Q$ is assigned by the designers.

- Choose the eigenvalues of $A_i - B_i P(0)$ and solve the initial $P(0)$ by standard pole placement technique.

- Set the initial bias term of the fuzzy controller Eq. (7) such as $\mu_i^{(0)} = 0$.

**Step 2.** Using the auxiliary variables $Y_{ij}^{(k)}$, $y_i^{(k)}$ and $z_i^{(k)}$ to solve the optimization problem for $P(k)$, $S(k)$, $N_i^{(k)}$, $F_i^{(k)}$, $\mu_i^{(k)}$, $\xi_{ijq}^{(k)}$ from Eqs. (47-48) subject to minimizing $\alpha^{(k)}$.

If $\alpha^{(k)} < 0$ then $P(k)$, $S(k)$, $N_i^{(k)}$, $F_i^{(k)}$, $\mu_i^{(k)}$ and $\xi_{ijq}^{(k)}$ obtained in Step 2 are a feasible solution for Theorem 2 and stop the iterative manner. Otherwise, if $\alpha^{(k)} \geq 0$ then go to Step 3.

**Step 3.** Resolve the optimization problem for $P^{(k)}$, $S^{(k)}$, $N_i^{(k)}$, $F_i^{(k)}$, $\mu_i^{(k)}$ and $\xi_{ijq}^{(k)}$ from Eqs. (47-48) subject to minimizing $trace(P^{(k)})$ by
using \( \alpha^{(k)} \) and the corresponding auxiliary variables \( Y^{(k)}_i, y^{(k)}_i \) and \( z^{(k)}_i \) obtained in Step 2. Given a predetermined small value \( \nu \). If
\[
\sum_i \left( \| Y^{(k)}_i \| \right) - \left( B^T P^{(k)} + F^{(k)}_j \right) + \| P^{(k)} - B^T P^{(k)} \|^2 + \| y^{(k)}_i \| - \mu^{(k)}_i \| < \nu, \text{then the conditions Eqs.} (47-48) \text{may not be feasible and stop the iterative manner. Otherwise, we can set} \ k = k + 1 \text{and go back to Step 2 to update the auxiliary variables} \ Y^{(k)}_i, y^{(k)}_i \text{and} \ z^{(k)}_i \text{using} \ P^{(k-1)}, F^{(k-1)}, \mu^{(k-1)}_i, \text{where} \ P^{(k-1)}, F^{(k-1)} \text{and} \ \mu^{(k-1)}_i \text{are determined in Step 3.}
\]

IV. NUMERICAL SIMULATIONS FOR A NONLINEAR PENDULUM SYSTEM

Consider a pendulum system as follows (Yi and Heng, 2002):
\[
\ddot{\theta}(t) = -a \sin(\theta(t)) + cT(t) + ev(t),
\]
where \( a > 0, c > 0, e > 0 \) is constant, \( \theta(t) \) is the angle of the rod to the vertical axis, \( T(t) \) is the torque applied to the pendulum and \( v(t) \) denotes the disturbances. The torque is the control input and it is assumed that the control object is to maintain a constant angle \( \dot{\theta}(t) = \dot{\beta} \). In order to maintain \( \dot{\theta}(t) = \dot{\beta} \), the torque must have a steady-state component \( T_{ss} \) that satisfies \(-\sin(\dot{\beta}) + c T_{ss} = 0 \) or \( T_{ss} = \sin(\dot{\beta})/c \). Choose the state variables as \( x_1(t) = \theta(t) - \dot{\beta}, x_2(t) = \dot{\theta}(t) \), and the control variable as \( u(t) = T(t) - T_{ss} \). Then, the new equilibrium point is \( x_1(t) = 0, x_2(t) = 0 \) and \( u(t) = 0 \). The pendulum Eq. (61) can be thus represented as
\[
\dot{x}_1(t) = x_2(t), \quad (62a)
\]
\[
\dot{x}_2(t) = -a \{ \sin(x_1(t) + \dot{\beta}) - \sin(\dot{\beta}) \} + cu(t) + ev(t), \quad (62b)
\]
To consider the time delay effect in the actual situation, it is assumed that the sensor to explore \( x_1(t) = \dot{\theta}(t) \) is perturbed by time delay and the modified pendulum system is given as:
\[
\dot{x}_1(t) = \varphi(t), \quad (63a)
\]
\[
\dot{x}_2(t) = -a \{ \sin(x_1(t) + \dot{\beta}) - \sin(\dot{\beta}) \} + cu(t) + ev(t), \quad (63b)
\]
\[
\varphi(t) = \rho x_2(t) + (1 - \rho)x_2(t - \tau(t)), \quad (63c)
\]
where \( \varphi(t) \in \mathcal{F}^\nu \) is a time-delay weighting function and \( \rho \in [0,1] \) is the weighting coefficient. The limits 1 and 0 of \( \rho \) are corresponding to no delay term and to a completed delay term, respectively. If \( \rho = 1 \), then \( \varphi(t) = x(t) \), i.e., the time delay term \( x(t - \tau(t)) \) is not considered.

In this example, we assume that \( a = c = 10, e = 0.5, \rho = 0.85 \) and \( \dot{\beta} = \pi/4 \).

To obtain the time-delay affine T-S fuzzy model of the pendulum system Eq. (63), it is necessary to apply the linearization technique. Let us choose three operating points as follows:
\[
(x^+, x^+_d, u^+)_{oper1} = (43^\circ, 0^\circ, 0^\circ),
\]
\[
(x, x_d, u)_{oper2} = (0^\circ, 0^\circ, 0^\circ),
\]
\[
(x^-, x^-_d, u^-)_{oper2} = (-133^\circ, 0^\circ, 0^\circ). \quad (64)
\]

Then, three linear subsystems can be constructed by these three operating points. In which, \( (x, x_d, u)_{oper2} \) is the maintain equilibrium point and the others are the off-equilibrium points. Through the above three linear subsystems and defining membership functions as Fig. 1, one can obtain the time-delay affine T-S fuzzy model, which is composed by three rules as follows:

**Rule i:** IF \( x_1(t) \) is \( M_{11} \)

THEN \( \dot{x}(t) = A_1 x(t) + A_{1d} x(t - \tau(t)) \)
\[
+ B_1 u(t) + a_1 + E v(t),
\]
\[
x(t) = \psi(t), \quad t \in [-\tau, 0], i = 1, 2, 3,
\]
\[
x(t) \in X_i, i \in \hat{I}, \quad (65)
\]
where
\[
A_1 = A_3 = \begin{bmatrix} 0 & 0.85 \\ -0.349 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0.85 \\ -7.0711 & 0 \end{bmatrix},
\]
\[
A_{1d} = A_{2d} = A_{3d} = \begin{bmatrix} 0 & 0.15 \\ 0 & 0 \end{bmatrix}, \quad B_1 = B_2 = B_3 = \begin{bmatrix} 0 \\ 10 \end{bmatrix},
\]
\[
a_1 = \begin{bmatrix} 0 \\ -2.6609 \end{bmatrix}, a_3 = \begin{bmatrix} 0 \\ 16.2549 \end{bmatrix}, \text{and} \ E = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}. \quad (66)
\]

According to the membership functions defined in Fig. 1, the \( S \)-procedure is presented as follows. For **Rule i**, 11, i.e., \( 3^\circ \leq x_1(t) \leq 45^\circ \), the matrices of \( S \)-procedure
are given as follows:
\[
T_{111} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},
\]
\[
n_{111} = \begin{bmatrix} -\frac{1}{2} (3\pi/180 + 45\pi/180) \\ 0 \end{bmatrix}
\]
and \( v_{111} = (3\pi/180) \times (45\pi/180). \) (67)

For Rule \( i, j \), i.e., \(-135^\circ \leq x_i(t) \leq -3^\circ\), the matrices of \( S \)-procedure are given as follows:
\[
T_{331} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},
\]
\[
n_{331} = \begin{bmatrix} -\frac{1}{2} (-3\pi/180 - 135\pi/180) \\ 0 \end{bmatrix}
\]
and \( v_{331} = (-3\pi/180) \times (-135\pi/180). \) (68)

Note that the notation Rule \( i, j \) means the correlation between Rule \( i \) and Rule \( j \) of the plant part bounding region.

For the above time-delay affine T-S fuzzy model (65), the fuzzy controller can be designed by applying Theorem 2 and the ILMI algorithm. In Step 1 of the controller design procedure, the eigenvalues of \( A_i - B_i F_i^{(0)} \) are chosen as \((-1, 0)\). Then, the initial \( F_i^{(0)} \) can be obtained by applying standard pole placement technique as \( F_i^{(0)} = [-0.7071 0.1000] \) for \( i = 1, 2, 3 \). Let us assign the matrix \( \Theta \) as \( 2 \times 2 \) identity matrix. Then, the matrix \( P^{(0)} \) can be obtained as follows by solving a Riccati Eq. (60):
\[
P^{(0)} = \begin{bmatrix} 1.2928 & 0.0774 \\ 0.0774 & 0.1064 \end{bmatrix}. \] (69)

Set the initial bias term of the fuzzy controller such as \( \mu_i^{(0)} = 0 \) for \( i = 1, 2, 3 \). Thus, the initial auxiliary parameters can be obtained from definition (59) as follows:
\[
Y_i^{(1)} = B_i^T P^{(0)} + F_i^{(0)}, \quad y_i^{(1)} = M_i^{(0)}
\]
\[
z_i^{(1)} = B_i^T P^{(0)} \quad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2, 3. \] (70)

Using the initial auxiliary parameters in Eq. (70), the minimum decay rate \( \alpha \) can be obtained in Step 2. If the decay rate \( \alpha < 0 \), then one can obtain the feasible solution for system (65). On the other hand, if \( \alpha \geq 0 \), one needs to get a new matrix \( P^{(1)}, F^{(1)}, \mu^{(1)} \) from Step 3 and then substitute it into Step 1 to produce a new auxiliary parameter \( Y_i^{(2)}, y_i^{(2)} \) and \( z_i^{(2)} \). Repeat the fuzzy controller design procedure until the decay rate \( \alpha < 0 \). In this example, it is assumed that the \( H_\infty \) control performance is guaranteed for an attenuation \( \gamma^2 = 0.01 \). Then, we can get a feasible solution after four iterations of the fuzzy controller design procedure. The final decay rate \( \alpha \) is \(-0.3085 \) and the feasible solutions are obtained as follows:

Fig. 2 Responses of pendulum system (61) with no disturbance (\( v(t) = 0 \)) via linear controller developed in (Yi and Heng, 2002)

\[
P = \begin{bmatrix} 0.5656 & 0.0353 \\ 0.0353 & 0.0068 \end{bmatrix}, \quad S = \begin{bmatrix} 0.0022 & 0.0004 \\ 0.0004 & 0.0001 \end{bmatrix},
\]
\[
N_1 = \begin{bmatrix} 0.4948 & 0.0309 \\ 0.0309 & 0.0065 \end{bmatrix},
\]
\[
\xi_{111} = 0.3313, \quad \xi_{331} = 0.2028. \] (71)

And, the fuzzy controller has the following form:
\[
F_1 = [3.6131 \ 1.8581], \quad F_2 = [3.6070 \ 1.8534],
\]
\[
F_3 = [3.6248 \ 1.8637],
\]
\[
\mu_1 = 0.1951, \quad \mu_3 = 0.6566. \] (72)

According to the controller \( u(t) = -\sum_{i=1}^3 h_i(x(t)) \{ F_i x(t) + \mu_i \} \) and definition \( u(t) = T(t) - T_{ss} \), one has the torque
\[
T(t) = u(t) + T_{ss} = -\sum_{i=1}^3 h_i(x(t)) \{ F_i x(t) + \mu_i \} + T_{ss}
\]
\[
= -\sum_{i=1}^3 h_i(\theta(t) - \pi/4) \{ F_i \times \left[ \frac{\theta(t) - \pi/4}{\theta(t)} \right] + \mu_i \} + \sin(\pi/4). \] (73)

In the simulation, the initial conditions of \( \{ \theta(0), \dot{\theta}(0) \} \) are chosen as several points as \(-88^\circ, -3^\circ, -40^\circ, -3^\circ, -10^\circ, -3^\circ, 60^\circ, -3^\circ, 88^\circ, -2^\circ, [88^\circ, -1.5^\circ, 88^\circ, -0.9^\circ, 88^\circ, -0.5^\circ] \) with \( \psi(t) = 0 \) for \(-\tau \leq t < 0 \). Note that Fig. 2 does not consider the time-delay effect and disturbances. It is obvious that the local linear controller is just valid around the equilibrium point and so these results can
only guarantee the local stability of a nonlinear pendulum system. Compared with Fig. 2, Fig. 3 shows the simulated responses for the nonlinear pendulum system Eq. (63), which is driven by the fuzzy controller Eq. (73) with the time-delay effect and disturbances. In addition, the time-delay term is assumed as \( \pi(t) = |\sin(t)| \), and it is bounded in \( 0 \leq \pi(t) \leq \tau \) and \( 0 \leq \pi(t) \leq \varepsilon \), where \( \tau = 1 \), \( \varepsilon = 0.5 \). Besides, \( v(t) = 0.2\sin(t) \) is the disturbance. From the simulated results of Fig. 3, one can find that the controlled nonlinear time-delay pendulum system Eq. (63) is globally stable for the angle \( \lim_{t \to \infty} \theta(t) = \pi/4 \).

V. CONCLUSIONS

The main contribution of this paper is to extend the stability analysis and stabilization issues of time-delay homogeneous T-S fuzzy models to time-delay affine T-S fuzzy models. Moreover, the \( H_\infty \) control performance is guaranteed for the worst case effect of disturbance on system states. The idea of ILMI algorithms has been applied to implement a fuzzy controller for time-delay affine T-S fuzzy models. First, the Lyapunov-Razumikhin theorem was applied to analyze the stability and stabilization problems for the time-delay affine T-S fuzzy models, and then the derived condition was recast into an LMI problem by using S-procedure. At last, a numerical simulation for synthesis of time-delay affine T-S fuzzy models was provided to show the availability and reliability of the proposed controller design procedure.

**NOMENCLATURE**

- \( A^T \) (resp. \( x \)) transpose of matrix \( A \) (resp. vector \( x \))
- \( A \geq 0 \) (resp. \( A < 0 \)) \( A \) is a positive (resp. negative) semi-definite matrix
- \( A \geq B \) (resp. \( A \leq B \)) matrix \( A - B \) is a positive (resp. negative) semi-definite matrix
- \( I \) identity matrix
- \( L_2 [0, t_f] \) is the Lebesgue space consists of square-integrable function which between 0 and \( t_f \)
- \( X \subseteq \mathbb{R}^n \) \( X \) belongs of \( \mathbb{R} \)
- \( \mathbb{R}^m \) real vector with \( m \)-dimension
- \( \mathbb{R}^{m \times n} \) real matrix with \( m \times n \)-dimension

**REFERENCES**


\[ x^T(t) \mathbf{T}_{ij}^c x(t) \]
\[ = x^T(t) \left[ \begin{array}{c} \Gamma_{ij}^c \sum_{q=1}^{\tilde{p}} \xi_{ijq} T_{ijq}^c \quad * \end{array} \right] x(t) \]
\[ = x^T(t) \left( \left( \begin{array}{c} \left( \begin{array}{c} G_{ij} + G_{ji} \end{array} \right)^T \right) \frac{P - \sum_{q=1}^{\tilde{p}} \xi_{ijq} T_{ijq}^c}{2} - \sum_{q=1}^{\tilde{p}} \xi_{ijq} \Gamma_{ijq} \end{array} \right) x(t) \right. \]
\[ + x^T(t) P (G_{ij} + G_{ji})^T \left( \begin{array}{c} \left( \begin{array}{c} G_{ij} + G_{ji} \end{array} \right)^T \frac{P - \sum_{q=1}^{\tilde{p}} \xi_{ijq} T_{ijq}^c}{2} - \sum_{q=1}^{\tilde{p}} \xi_{ijq} \Gamma_{ijq} \end{array} \right) x(t) \right) \]
\[ - \sum_{q=1}^{\tilde{p}} \xi_{ijq} \sigma_{ijq} x(t) \]
\[ = \mathbf{x}^T(t) \left( \begin{array}{c} \left( \begin{array}{c} \left\{ \begin{array}{c} \left( \begin{array}{c} A_j - B_j F_j \end{array} \right)^T P + \frac{P (G_{ij} + G_{ji})}{2} \right) x(t) \end{array} \right) \left( \begin{array}{c} \left( \begin{array}{c} A_j - B_j F_j \end{array} \right)^T P + \frac{P (G_{ij} + G_{ji})}{2} \right) x(t) \right) \right) \right) \]
\[ + \mathbf{x}^T(t) P (G_{ij} + G_{ji})^T \left( \begin{array}{c} \left( \begin{array}{c} A_j - B_j F_j \end{array} \right)^T P + \frac{P (G_{ij} + G_{ji})}{2} \right) x(t) \right) \]
\[ - \sum_{q=1}^{\tilde{p}} \xi_{ijq} \sigma_{ijq} x(t) \]
\[ = \mathbf{x}^T(t) \left( \begin{array}{c} \left( \begin{array}{c} A_j^T P + PA_j + A_j^T P + PA_j \end{array} \right) x(t) \end{array} \right) \]
\[ = \mathbf{x}^T(t) \left( \begin{array}{c} \left( \begin{array}{c} A_j^T P + \sum_{i=1}^{m} \left( a_i + a_j \right)^T P x(t) \end{array} \right) \left( \begin{array}{c} \left( \begin{array}{c} A_j^T P + \sum_{i=1}^{m} \left( a_i + a_j \right)^T P x(t) \end{array} \right) \right) \right) \right) \]
\[ + \mathbf{x}^T(t) \left( \begin{array}{c} \left( \begin{array}{c} B_j^T P + \sum_{i=1}^{m} \left( a_i + a_j \right)^T P x(t) \end{array} \right) \left( \begin{array}{c} \left( \begin{array}{c} B_j^T P + \sum_{i=1}^{m} \left( a_i + a_j \right)^T P x(t) \end{array} \right) \right) \right) \right) \]
\[ + \mathbf{x}^T(t) \left( \begin{array}{c} \left( \begin{array}{c} \sum_{i=1}^{m} \left( a_i + a_j \right)^T P x(t) \end{array} \right) \left( \begin{array}{c} \left( \begin{array}{c} \sum_{i=1}^{m} \left( a_i + a_j \right)^T P x(t) \end{array} \right) \right) \right) \right) \]
\[ - \sum_{q=1}^{\tilde{p}} \xi_{ijq} \left\{ \sigma_{ijq} x(t) \right\} \]
\[ = \mathbf{x}^T(t) \left( \begin{array}{c} \left( \begin{array}{c} \left( \begin{array}{c} A_j^T P + \sum_{i=1}^{m} \left( a_i + a_j \right)^T P x(t) \end{array} \right) \left( \begin{array}{c} \left( \begin{array}{c} A_j^T P + \sum_{i=1}^{m} \left( a_i + a_j \right)^T P x(t) \end{array} \right) \right) \right) \right) \right) \]
\[ - \sum_{q=1}^{\tilde{p}} \xi_{ijq} \left\{ \sigma_{ijq} x(t) \right\} \]
\[ = \mathbf{x}^T(t) \left( \begin{array}{c} \left( \begin{array}{c} (F_j + B_j^T P)^T (F_j + B_j^T P) \end{array} \right) \left( \begin{array}{c} \left( \begin{array}{c} (F_j + B_j^T P)^T (F_j + B_j^T P) \end{array} \right) \right) \right) \right) \]
\[ + \left( F_j + B_j^T P \right)^T F_j + B_j^T P) x(t) \]
\[ - \left( \begin{array}{c} \mu_j + B_j^T P x(t) \end{array} \right)^T (\mu_j + B_j^T P x(t)) \]
\[ - \left( \begin{array}{c} \mu_j + B_j^T P x(t) \end{array} \right)^T (\mu_j + B_j^T P x(t)) \]
\[ = \mathbf{x}^T(t) \left( \begin{array}{c} \left( \begin{array}{c} (F_j + B_j^T P)^T (F_j + B_j^T P) \end{array} \right) \left( \begin{array}{c} \left( \begin{array}{c} (F_j + B_j^T P)^T (F_j + B_j^T P) \end{array} \right) \right) \right) \right) \]
\[ + \left( F_j + B_j^T P \right)^T F_j + B_j^T P) x(t) \]
\[ - \left( \begin{array}{c} \mu_j + B_j^T P x(t) \end{array} \right)^T (\mu_j + B_j^T P x(t)) \]
\[ - \left( \begin{array}{c} \mu_j + B_j^T P x(t) \end{array} \right)^T (\mu_j + B_j^T P x(t)) \]
Then, according to the relationship Eq. (19) and Eq. (46), Eq. (A2) becomes

\[
2\dot{x}^T(t) \mathbf{T}_{ij} \dot{x}(t)
\]

\[
\leq x^T(t) \{ A_i^* P + PA_i + A_j^* P + PA_j + 2PB_B^i P \\
+ 2PB_B^j P + F_i^* F_i + F_j^* F_j + Y_{ij}^* Y_{ij} \\
- Y_{ij}^* (F_j + B_i^* P) - (F_j + B_i^* P)^T Y_{ij} + Y_{ji}^* Y_{ji} \\
- Y_{ji}^* (F_i + B_j^* P) - (F_i + B_j^* P)^T Y_{ji} + Y_{ij}^* Y_{ij} \\
- PB z_i + z_j^2 z_i - z_i^2 B_i^* P - PB_z_j x(t) \\
+ x^T(t) (P a_i + P a_j + z_i^2 y_j - z_j^2 q_i - PB_y_j + z_i^2 y_i) \\
- z_i^2 \mu_i - PB_y_j + (a_i^* P + a_j^* P + y_i^2 z_j - y_i B_i^* P) \\
- \mu_j^2 z_i + y_j^2 B_i^* P - \mu_j z_j \} x(t) + \{ \mu_i \mu_j^T + \mu_j \mu_i^T \\
y_j \mu_i - \mu_i y_j + y_i \mu_j - \mu_j y_i \} \\
- \sum_{q=1}^p 2 z_{jq} \{ \sigma_{jq}(x(t)) \}
\]

\[
= \dot{x}^T(t) \mathbf{\Psi}_{ij} \dot{x}(t) \quad (A3)
\]

in which \(\dot{x}^T(t) \mathbf{\Psi}_{ij} \dot{x}(t) = \dot{x}^T(t) \left[ \begin{array}{c} \mathbf{A}_{11} \\ \mathbf{A}_{21}^* \\ \mathbf{A}_{22} \end{array} \right] \dot{x}(t) \) and

\[
\mathbf{\Psi}_{ij} = A_i^* P + PA_i + A_j^* P + PA_j + Y_{ij}^* Y_{ij} + Y_{ji}^* Y_{ji} \\
+ (F_i^* F_j^*) + 2PB_B^i P \\
- Y_{ij}^* (B_i^* P + F_j) - (PB_B + F_j^*) Y_{ij} \\
- Y_{ji}^* (B_j^* P + F_i) - (PB_B + F_i^*) Y_{ji} \\
+ z_i^2 z_i + z_j^2 z_j - z_i^2 B_i^* P - PB_B z_j P \\
\]

\[
(A4)
\]

\[
\mathbf{A}_{11} = \mathbf{\Psi}_{ij} - \sum_{q=1}^p 2 z_{ij} T_{ij} \\
\mathbf{A}_{21} = (a_i - B_i y_j)^T P + (a_j - B_j y_i)^T P + (y_j - \mu_j)^T z_i \\
+ (y_i - \mu_i)^T z_j - \sum_{q=1}^p 2 z_{ij} n_{ij}^T \\
\mathbf{A}_{22} = \mu_j^2 \mu_i + \mu_i \mu_j^2 + y_j \mu_i + y_i \mu_j - \mu_i y_j \\
- y_j \mu_i - \mu_j y_j - \sum_{q=1}^p 2 z_{ij} v_{ij} \\
(A5)
\]

It is obvious that \(\dot{x}^T(t) \mathbf{T}_{ij} \dot{x}(t) \leq \frac{1}{2} \dot{x}^T(t) \mathbf{\Psi}_{ij} \dot{x}(t)\) and the proof of the condition Eq. (40) is completed. Similarly, the proof of Eq. (39) is obtained by setting the bias term \(a_i = \mu_i = 0\) and by removing the S-procedure term \(\sigma_{ij}(x(t))\) from the above process.