On the Integrated Production and Distribution Problem with Bidirectional Flows

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The integrated production and distribution problem with bidirectional flows is a complicated optimization problem, usually with large problem sizes when encountered in practice. In this study, we propose a partial linear programming relaxation-based heuristic approach to solve a variation of this problem. The approach is called a partial relaxation in the sense that it relaxes the integer requirements only on selected variables. We also report on the gaps between the optimal solution and the heuristic solution provided by this partial relaxation, including analytical gaps for a special case and empirical gaps for randomly generated test cases. Our study of this problem was motivated by the operational planning problem of a medical equipment leasing network that involves a forward flow for new and refurbished devices and a reverse flow for used devices to be returned to suppliers over a multiple time-period planning horizon.

Key words: partial linear programming relaxation; heuristic; integrated production and distribution; bidirectional flows

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1. Introduction

The integrated production and distribution problem (PDP) with bidirectional flows is concerned with coordinating the operations of manufacturing (for new products) and reprocessing (for reuse of resources) together with scheduling the forward and reverse flows of goods over a network. The forward flow sends finished goods to customers, whereas the reverse flow brings back recyclable materials (e.g., empty soda bottles, waste papers, or unsold periodicals), obsolete products that have to be processed for environmentally safe disposal (e.g., expiring pharmaceutical and chemical products, or contaminated materials), or used products to be refurbished for future reuse (e.g., office equipment or empty containers). In this paper, we consider a mathematical programming-based heuristic search algorithm that solves a variation of this problem with an objective to minimize the production, refurbishment, trucking, and inventory operation costs.

In the literature, there have been several streams of research on problems involving reverse flows: network design for reverse flows, inventory control for returns, closed-loop supply chains, and green supply chains. Many studies have been published along these streams and are reviewed in the Online Supplement (available at http://joc.pubs.informs.org/e companion.html). In contrast, existing results on search algorithms for integrated scheduling of production, inventory, and distribution operations with bidirectional flows are, however, very limited. Distinct from the classical production lot sizing and delivery problems that focus on batch sizing and forward shipping (see Lee and Denardo 1986; Dror and Ball 1987; Dror and Trudeau 1988; Cohen and Lee 1988, 1989; Blumenfeld et al. 1991; Martin et al. 1993; Chandra and Fisher 1994; Chen and Lee 1995; Fumero and Vercellis 1997, 1999; Erengücü et al. 1999; Ozdamar and Yazgac 1999; Brown et al. 2001; Bloomquist et al. 2002; Gupta et al. 2002; Chen 2004; Li et al. 2005; Li and Ou 2005; Chen and Vairakarakis 2005; Lei et al. 2006; Liu et al. 2007; Armstrong et al. 2008; Geismar et al. 2008), the integrated operations scheduling problem with bidirectional flows that we focus on in this study introduces new challenges to...
the optimization and the design of search algorithms. The production lot size is now affected by the timing and the quantity of returned items that can be reused, and the dispatching of delivery vehicles must simultaneously optimize the delivery schedules (for new and refurbished products) and the pickup schedules (for returns). To our knowledge, there are very few published studies that are directly related to this area. One study is by Chen et al. (2007) on container vessel scheduling with bidirectional flows between a single origin and a single destination, where the forward flow sends customer orders packed in cargo containers from the origin port to the destination port, while the reverse flow brings empty containers back to the origin. The assignment of the integer portion (i.e., the multiples of full vessel loads) of the orders to vessels was made by solving a respective linear program (upon the unimodularity of the integer-load problem), and the residual loads of the orders were later assigned to the vessels via a greedy heuristic. Although their proposed approach can achieve near-optimal solutions for the problem in their study, the production issue was not addressed. Another study is by Del Castillo and Cochran (1996) on the production and distribution planning problem for products delivered in reusable containers. Their model was defined for a two-stage network with a forward flow from a plant to a depot and a reverse flow of empty containers back to the plant. The availability of empty containers was modeled as a constraint on the production of the product. A hierarchical solution approach was proposed, by which linear programming models were applied to determine the master production-distribution plan, and then a computer simulation was used to create detailed operations schedules. However, such a hierarchical approach usually reduces the computational effort significantly at the expense of losing the optimality because of the separation of planning and scheduling. In contrast to the approach by Del Castillo and Cochran (1996), the search algorithm that we propose in this study derives simultaneously the production, refurbishment, inventory, and trucking plans and operations schedules for a bidirectional flow network involving not only the manufacturers/suppliers, and the distribution center, but also many customer-demand points. In particular, we propose a partial linear programming (LP) relaxation-based solution approach, and analyze its theoretical and empirical computational performance upon a hypothesized three-stage supply chain network. This proposed search algorithm is very fast, able to find near-optimal solutions for special cases, and has the potential to find good solutions to general PDPs with bidirectional flows.

Our study on this integrated operations scheduling problem was motivated by the practices of a medical equipment leasing and service network that supplies, repairs, inventories, and distributes both new and refurbished medical devices to business customers (e.g., hospitals, diagnostic centers, doctors’ offices, and sales offices serving individual patients, etc.). Because of the cost concern, refurbished devices/components with quality standard and workmanship warranties are usually preferred by customers. Customers at the end of a leasing contract have the option to either purchase or return a device. Various leasing and rental service programs are offered by the network. One of them is called preventive maintenance, by which a leased part (a device or a component) is replaced by either a new or a newly refurbished part periodically during the leasing term. The need to send back used parts to the suppliers creates a reverse flow for the network. All the returned parts must go through a reusability assessment at the supplier’s site for their future lease/rental before the refurbishment. Because only parts that are still within their normal life span qualify for this assessment, the pass rates are usually stable. According to the orders received from regional sales offices, first-tier suppliers (i.e., specialty contractors) may need to purchase new supplies from second-tier suppliers (i.e., wholesalers or original equipment manufacturers) from time to time. For each batch of new supplies purchased for the network, these first-tier suppliers also perform value-added services such as final assembly, testing/adjustment, sterilization, packaging and labeling, and sometimes customization, etc. Because most first-tier suppliers are specialty contractors with technical ability in maintaining/reparing certain types of devices, they usually do not carry an inventory for new devices. From a supply chain point of view, this equipment leasing and service network is only one part of a large, complex medical equipment supply chain with multiple tiers of suppliers.

The operations planning and scheduling issues faced by this service network include when and for how much a (first tier) supplier should place a purchasing order for a new batch of a supply, how often the process for assessing the reusability of returns should be scheduled, how frequently a truck will be traveling with a full (or empty) load in the network, which also affects the operations planning of the contracted trucking company, and which level of inventory at each stage (i.e., suppliers, the DC, and customers) is most desirable. The answers to these questions build guidelines and policies to manage the operations that in turn rely on the solution to the integrated optimization problem. Because most customers are contracted customers, the orders (demands) or flows are usually placed in advance. Also, by the nature of such a preventive maintenance program, the quantity of returns per time period can be reasonably
anticipated. Therefore, the related optimization problem can be treated as a discrete optimization problem, and the issue is how to coordinate the operations schedules for production/repair, inventory, and shipping with the given bidirectional flows over a planning horizon to minimize the total cost.

The remaining part of this paper is organized as follows. In §2, we formally define our problem as a mixed-integer program, present the assumptions underlying the model, and discuss the computational difficulties in attempting to solve this model directly. A partial LP relaxation-based heuristic search algorithm is then proposed. In §3, we analyze the error gaps between the optimal and the proposed heuristic solutions to a special case of the problem. This analysis offers an intuitive explanation for how the error gaps are generated by such partial LP relaxation-based heuristics. We also discuss the extensions of this special case analysis to several related but more general situations. The empirical performance of this heuristic approach under various parameter settings is reported in §4. Finally, we discuss future extensions of the study in §5.

2. Problem Definition and Solution Approach

The hypothesized network on which we shall analyze and evaluate the proposed solution approach consists of three stages: the first-tier specialty suppliers (or the suppliers), the central distribution center (the DC), and the first-tier business customers (or the customers). Without loss of generality, we assume that there is a group of distinct first-tier suppliers, denoted by set $I$, $|I| \geq 1$, and each supplier $i \in I$ supplies and repairs one of the $|I|$ products. A supplier purchases new products for the network from outside vendors and maintains/repairs the returns (i.e., used products) for future reuse. For each batch of new products that supplier $i$ purchases, there is a variable cost (i.e., the unit purchasing cost), $a_i$/unit, and a fixed cost, $s_i$/order. Similarly, for each batch of used products to be refurbished at a supplier’s site, there is a fixed cost for setting up the process to assess the reusability, $s_i$/order, regardless of the batch size, and a variable cost, $b_i$/unit, for refurbishment. The DC receives forward flows (of new/refurbished products) from suppliers, sends shipments to the customers, receives reverse flows (of used products) from customers, and dispatches the shipment of returns back to the suppliers. Let $K$ denote the set of customers in the network; $|K| \geq 1$. Each customer’s demand on the forward flows and that on the reverse flows in each time period are known at the beginning of a given multiperiod planning horizon. Let $T$ denote the set of time periods in the planning horizon; $|T| \geq 1$. The demands on forward flows must be met on time and cannot be backlogged, but the shipments may arrive early and can be inventoried at a cost. All the reverse flows originate from customers’ sites and the reverse flow originated in time period $t \in T$ can only be transported back to the DC and then to the suppliers in time period $t$ or after. The objective is to find an integrated operation plan for shipping, purchasing/refurbishing, and inventory so that the total operational cost over the $|T|$-period planning horizon is minimized. For convenience, we shall use production for purchasing in the remaining discussion of the paper.

Figure 1 depicts this hypothesized network, where the forward flows are represented by solid lines ($\rightarrow$), starting from suppliers, passing through the DC, and ending at customers, whereas the reverse flows are denoted by dashed lines ($\rightarrow\rightarrow$), starting at customers, passing through the DC, and ending at suppliers. In addition, we assume the following:

- All the returned products that arrive at a supplier’s site in period $t$ can only be assessed and repaired in period $t+1$ or afterward. That is, there is

![Figure 1](image-url)
a minimum one-period lead time at a supplier’s site before the returned products can be processed (e.g., the time needed to unpack and verify records).

- The recovery rate at supplier $i$ is given as a constant, $\theta_i$, $i \in I$. That is, for each batch of returns to be assessed, only a fixed percentage, $0 < \theta_i < 100\%$, $i \in I$, of the batch is recoverable. Only those that pass the assessment are repaired, and this results in a variable cost for refurbishment. In real life, such recovery rates can sometimes be random variables.

- Direct truck trips are assumed between each pair of locations on the network. All capacitated trucks are DC based (i.e., always departing from and returning to the DC for each trip), and the transportation costs are based on the number of truck trips.

- All the (first-tier) suppliers have sufficient production capacity and do not carry inventory for new/refurbished products, but do carry inventory for returned products waiting to be assessed for refurbishment.

- Both forward and reverse flows require transportation capacity and can be held at either the DC or at a customer’s site subject to a holding cost. However, the forward flows and the reverse flows, if stored at any site, must share the available local inventory capacity.

The following mixed-integer program formally defines the integrated optimization problem that we consider in this study. Let $P$ denote this problem and define the following notation.

### Input Data

- $c$: Loading capacity of a delivery truck;
- $f_i^P, f_i$: Round-trip costs between the DC and supplier $i \in I$, and customer $k \in K$, respectively;
- $g_i$: Refurbishing capacity of supplier $i \in I$ for its used product;
- $w_i^P, w_i^{DC}, w_i$: Inventory capacities of supplier $i$, the DC, and customer $k$, respectively;
- $a_i, b_i$: Variable costs for producing new, and for refurbishing, product $i \in I$;
- $s_i^N, s_i^F$: Fixed costs for producing, and for assessing the reusability of, a batch of product $i \in I$;
- $h_i^P, h_i^{DC}, h_i$; Unit holding costs for product $i$ at the supplier (for used products), the DC, and customer $k$, respectively;
- $d_{it}$: Demand for new/refurbished product $i \in I$ by customer $k \in K$ in $t \in T$;
- $r_{it}$: Quantity of used product $i$ originated at customer $k$ during period $t$, which will be transported back to supplier $i$ via the DC, where $i \in I$;
- $\theta_i$: Recovery rate for used product $i \in I$;
- $V_{it}^P, V_{it}^{DC}, V_{it}$: Initial inventory of used product $i$ at the supplier, the DC, and customer $k$, respectively, at the beginning of the planning horizon;
- $U_{0it}^{DC}, U_{k0i}$: Initial inventory of new/refurbished product $i$ at the DC, and customer $k$, respectively, at the beginning of the planning horizon;
- $M$: A large positive number.

### Decision Variables

- $Q_{it}$: Batch size of new product to be produced by supplier $i$ in time period $t$;
- $R_{it}$: Batch size of used product to be assessed for reusability by supplier $i$ in $t$;
- $V_{it}^{DC}$: Ending inventory of the returns (used products) at supplier $i$ in $t$;
- $U_{0it}^{DC}, U_{k0i}$: Ending inventories of new, and used, product $i$ at the DC in $t$, respectively;
- $U_{k0i}, V_{k0i}$: Ending inventories of new, and used, product $i$ at customer $k$ in $t$;
- $X_{i0i}, X_{k0i}$: Forward flow shipping quantities of new/refurbished product $i$ from the supplier to the DC, and from the DC to customer $k$, in $t$, respectively;
- $Y_{k0i}, Y_{i0i}$: Reverse-flow shipping quantities of used product $i$ from customer $k$ to the DC, and from the DC to supplier $i$, in $t$, respectively;
- $Z_{it}^N, Z_{it}^F$: Total numbers of truck trips between supplier $i$ and the DC, and between the DC and customer $k$ in $t$;
- $x_{it}^N$: 0–1 variable, 1 if any new product is produced by supplier $i$ in $t$;
- $x_{it}^F$: 0–1 variable, 1 if any used product is assessed for reusability by supplier $i$ in $t$.

### Objective Function

$$G = \min \sum_{i} \sum_{t} s_i^P \cdot x_{it}^N + \sum_{i} \sum_{t} s_i^F \cdot x_{it}^F + \sum_{i} a_i \cdot Q_{it}$$

subject to

1. Constraints on the refurbishing capacity of supplier $i \in I$:

$$R_{it} \leq g_i, \quad \forall i \in I,$$

$$R_{it} \leq V_{i0, t+1}^P, \quad \forall i \in I, \quad \forall t \in T.$$

2. Inventory capacity constraints at suppliers, the DC, and customer sites:

$$V_{it}^P \leq w_i^P, \quad \forall i \in I, \quad \forall t \in T,$$

$$\sum_i (U_{it}^{DC} + V_{it}^{DC}) \leq w_i^{DC}, \quad \forall t \in T,$$

$$\sum_i (U_{it} + V_{it}) \leq w_k, \quad \forall k \in K, \quad \forall t \in T.$$
3. Flow balance constraints for suppliers:
   \[ Q_{it} + R_{it} \cdot \theta_{i} - X_{it} = 0 \quad \forall i \in I, \forall t \in T, \]
   \[ V_{it} - Y_{it} = V_{it}^p \quad \forall i \in I, \forall t \in T. \]

4. Flow balance constraints for the DC:
   \[ U_{i,t-1}^DC + X_{it}^p - \sum_k X_{it} = U_{i,t}^DC \quad \forall i \in I, \forall t \in T, \]
   \[ V_{i,t-1}^DC + \sum_k Y_{it} - Y_{it}^p = V_{i,t}^DC \quad \forall i \in I, \forall t \in T. \]

5. Flow balance constraints for customer sites:
   \[ U_{ki,t-1} + X_{kt} - d_{kit} = U_{ki,t} \quad \forall k \in K, \forall i \in I, \forall t \in T, \]
   \[ V_{ki,t-1} + r_{kit} - Y_{it} = V_{ki,t} \quad \forall k \in K, \forall i \in I, \forall t \in T. \]

6. Truck capacity constraints:
   \[ X_{it}^p \leq c \cdot Z_{it}^p \quad \forall i \in I, \forall t \in T, \]
   \[ Y_{it} \leq c \cdot Z_{it} \quad \forall i \in I, \forall t \in T, \]
   \[ \sum_i X_{kt} \leq c \cdot Z_{kt} \quad \forall k \in K, \forall t \in T, \]
   \[ \sum_i Y_{it} \leq c \cdot Z_{it} \quad \forall k \in K, \forall t \in T. \]

7. Constraints to ensure the charge of fixed costs:
   \[ Q_{it} \leq M \cdot \pi_{it}^a \quad \forall i \in I, \forall t \in T, \]
   \[ R_{it} \leq M \cdot \pi_{it}^b \quad \forall i \in I, \forall t \in T. \]

8. Others:
   \[ \pi_{it}^a, \pi_{it}^b = \{0, 1\} \quad \forall i \in I, \forall t \in T, \]
   \[ Z_{it}, Z_{kt} \geq 0, \text{ integers} \quad \forall i \in I, \forall k \in K, \forall t \in T. \]

All other variables are nonnegative.

Note that model \( P \) can be extended to cover more general situations such as those with inventory for new/refurbished products at each supplier’s site, multiple products by each supplier, and more than one DC between the suppliers and the customers, etc. Also note that real-life networks can be much more complex than the hypothesized network we use here, and their logistics structures may vary significantly from industry to industry.

Problem \( P \) differentiates itself from classical PDP models (see a comprehensive review by Chen (2004)) in three ways. First, both forward and reverse flows must now be optimized simultaneously, while most existing PDP models focus on forward flows only. Second, because the cost of refurbishing a used product is usually lower than the cost of supplying a new one, the optimization of production plans now becomes inseparable from the optimization of shipping schedules for returning the used products. Third, there are multiple types of integer variables involved in \( P \), whereas most relevant studies deal with only one type of integer variable. Model \( P \) contains two types of yes/no decision variables (i.e., binary) and one type of general integer variable that defines, for each time period, the number of truck trips dispatched.

Solving \( P \) directly as a mixed-integer program is difficult mainly due to the large number of integer variables involved. For example, for a problem instance with two suppliers, 60 customers, and 24 time periods in the planning horizon, the model results in 1,488 general integer variables and 96 binary variables. One possible heuristic approach is to solve \( P \) sequentially. For example, among possible designs of sequential approaches, one is to fix the amount of reverse flow at a certain level and then reduce \( P \) to one with forward flow only. Different levels of reverse flow can be evaluated to determine the best solution. Another possible approach is to partition a given horizon into many subzones and solve \( P \) for each subzone so that the reduced \( P \) contains fewer integer variables. However, either approach would require us to preselect the values of some integer variables. When multiple types of integer variables are present, as in our case, improper preselection of integer values could lead to poor solution quality. In a related empirical study, Zhong (2006) has shown that the empirical gaps between the optimal solution for \( P \) and the best solution by a sequential optimization approach could be beyond 40% in some worst cases.

The idea behind our proposed approach to \( P \) is an intuitive one. We are interested in developing a partial LP relaxation-based heuristic search algorithm for this type of PDP problem. Among various integer variables, we prefer not to relax binary integer variables because the value of binary variables, in our case, is more crucial to the solution quality than that of the general integer variables. Therefore, we relax only a subset of general integer variables (i.e., the truck trips between the DC and customers) during the search process. This is because, in the worst case, such a relaxation will make us pay a full truckload cost for a less-than-truckload trip, which is usually less damaging than setting up the production line or processing a batch of used products in a wrong time period. The resulting solution approach is an iterative procedure (see Figure 2). Each iteration starts with a given planning horizon of \( T' \) time periods, where \( T' \leq |T| \). The \( T' \) time periods are then partitioned (regrouped) into three planning intervals, where each interval consists of consecutive time periods in the given planning horizon. For the time periods in the first interval, all the original constraints in \( P \) and the integer requirements remain unchanged. For the time periods in the second and the third intervals, all the original constraints and binary variables are retained while the
integer requirements on the number of truck trips between the DC and customers is relaxed. In addition, all the forward demand and all the backward demand of the time periods in the third planning interval are consolidated into a single forward demand and a single backward demand, respectively. This relaxed problem, which covers all the regrouped time periods in one optimization model, is called a partial LP relaxation of \( P \) and is solved optimally. At the end of each iteration, the integrated operational plan (including the ending inventories) for the first interval is fixed while all the operational plans for the second and the third intervals (including the linear relaxation solutions) are discarded. The role of the second and the third intervals in each iteration is simply to make sure the data in future time periods are properly incorporated into the optimization for the operational plans for those in the first time interval. The planning horizon in the next iteration then consists of only time periods in the second and the third planning intervals of the current iteration. The iterative process continues until all the time periods in a planning horizon receive a nonrelaxation solution.

The implementation of this partial LP relaxation heuristic follows a rolling horizon framework and is defined by the flowchart in Figure 3. As we can see, each iteration starts with a given planning horizon, called \( \Omega \), with \( T' \) time periods, \( T' \leq |T| \), and initially, \( T' = |T| \). This horizon is then partitioned into three planning intervals, \( \tau_1 \), \( \tau_2 \), and \( \tau_3 \). In the particular implementation illustrated in Figure 3, the first interval \( \tau_1 \) contains only one period and thus corresponds to only the first time period in \( \Omega \), for which all the original integer variables and constraints in \( P \) remain unchanged. The second interval, \( \tau_2 \), contains only the second time period in \( \Omega \), whereas the third interval, \( \tau_3 \), corresponds to a consolidation of all the remaining time periods in \( \Omega \). For intervals \( \tau_2 \) and \( \tau_3 \), the integer requirements on all the truck trips (which represent most of the integer variables in \( P \)) between the DC and the customers are relaxed, while those on the other integer variables (e.g., production setups, ...
assessment of reusability, and truck trips between suppliers and the DC are retained. The demand (both forward and reverse) of $\tau_1$ and $\tau_2$ equals exactly that of the respective time periods in the original problem, whereas the demand of interval $\tau_2$ equals the sum of the respective demands of all the time periods being consolidated. The resulting three-period relaxation of $P$ is solved optimally. We then keep the optimal solution to $\tau_1$ and discard all the solutions to $\tau_2$ and $\tau_3$. The ending inventories (at suppliers’ sites, the DC, and customers’ sites) of interval $\tau_1$ are then used as the beginning inventories for the next iteration, which starts with a planning horizon of $|T|-1$ time periods (i.e., $t = 2, 3, \ldots, t = |T|$). The iteration repeats until all the time periods have received a nonrelaxation solution.

The resulting iterative approach terminates in $O(|T|)$ iterations and solves, in each iteration, an optimization problem containing much fewer integer variables. The previous example with two suppliers and 60 customers now has only 66 general integer variables and 12 binary variables in each iteration. Note that the number of time periods included in the first planning interval (i.e., $\tau_1$) in each iteration can be more than one period and is a matter of tradeoff between the solution quality and the search effort. Solving problem $P$ directly can be considered as a special case of this partial relaxation approach, in which all the $|T|$ time periods in the original planning horizon are included in the first planning interval, and the search terminates after the first iteration. The impact of the number of time periods included in $\tau_1$ on the solution quality and the computational time is also empirically investigated in §4.

### 3. Marginal Analysis for the Bound on Error Gaps

In this section, we perform a marginal analysis based on a special case of $P$. Because of the simple structure of this special case, we can show exactly how the error gaps between the optimal solution and the partial relaxation-based heuristic solution are generated and why these error gaps are bounded. This analysis in turn indicates that the proposed partial relaxation-based heuristic is able to find near-optimal solutions to such a special case and has the potential to find good solutions to general PDPs with bidirectional flows.

Let $P_2$ denote this special case. It is a subproblem of $P$ and involves only a single product, a single customer, and a two-period planning horizon ($T = 2$) with bidirectional flows between the customer and the DC. We assume that there is no supplier involved and that the DC has a sufficient inventory for the new and refurbished product for the forward flow with a negligible holding cost. The inventory capacity at the customer site is infinite, but the holding cost ($h$) is nonnegligible.

Let $Z_1$ be the number of truck trips from the DC to the customer in period $t$, and let $U_t$ and $V_t$ be the ending inventory for the forward, and the reverse, flow at the customer’s site, $t = 1, 2$, respectively. Then $P_2$ becomes a two-period bidirectional flow transportation problem with fixed trip costs and customer holding costs, and can be formally defined as follows:

$$
\begin{align*}
(P_2) \quad G &= \min \ f(Z_1 + Z_2) + h(U_1 + V_1) \\
& \quad \text{subject to } X_t - c \cdot Z_t \leq 0; \quad t = 1, 2, \\
& \quad Y_t - c \cdot Z_t \leq 0; \quad t = 1, 2, \\
& \quad U_{t-1} + X_t - \bar{d}_t = U_t; \quad t = 1, 2, \\
& \quad V_{t-1} + \bar{r}_t - Y_t = V_t; \quad t = 1, 2, \\
& \quad Z_1, Z_2 \text{ integers, and } X_1, X_2, Y_1, Y_2, \\
& \quad U_1, V_1 \text{ are nonnegative},
\end{align*}
$$

where $\bar{d}_t$ and $\bar{r}_t$, respectively, are customer demand and returns in period $t$; $U_t$ and $V_t$ are the initial and ending inventories of the planning horizon at the customer site for the respective flows; $f$ stands for the shipping cost per truck trip; and $h$ is the unit holding cost of the customer. Without loss of generality, we assume $f > c \cdot h$, which means that whenever the spare capacity of a truck in a time period can be utilized to carry a less-than-truckload amount $\delta$, $\delta < c$, for the other time period to save a truck trip in that period (by paying a holding cost $h \cdot \delta$), a net savings can be achieved.

Let $Z^*_1$ and $Z^*_2$ be the optimal integer solution to $P_2$. Let $P_2$ be a partial LP relaxation of $P_2$ that keeps the integer requirement on $Z_1$ while allowing a linear relaxation for $Z_2$. Let $Z^*_1$ and $Z^*_2$ be the optimal solution to $P_2$, where $Z^*_1$ is a linear variable but $Z^*_2$ is an integer variable. A recent study by Chen et al. (2004) showed that problem $P_2$ has a close-form solution if $T = 2$. However, this problem becomes NP-hard when $T$ is arbitrary and the less-than-truckload demand is not infinitely divisible.

Because parameters are known constants for any given instance of $P_2$, the net demand for forward flow, and that for reverse flow, are known and can be given as

$$
\begin{align*}
&d_1 = \max\{\bar{d}_1 - U_0, 0\}, \\
&d_2 = \max\{\bar{d}_2 + U_2 - \max\{U_0 - \bar{d}_1, 0\}, 0\}, \\
&r_1 = \bar{r}_1 + V_0 - \max\{V_2 - \bar{r}_2, 0\}, \quad r_2 = \max\{\bar{r}_2 - V_2, 0\}.
\end{align*}
$$

To avoid triviality, we shall assume that the net flows $d_1, d_2, r_1$, and $r_2$ are all positive in the following analysis. Define $[d_1/c]$ as the minimum number of truck trips needed to ship the forward flow to the customer to meet the net demand in period $t = 1$
and define \([r_2/c]\) as the minimum number of truck trips required to ship the reverse flow, which is not available for transportation until \(t = 2\), back to the DC in period \(t = 2\). Then \([d_1/c] \leq Z_1^*\), \([d_2/c] \leq Z_2^*\), \([r_2/c] \leq Z_2^*\), and the following results hold.

**Lemma 1.** If \([d_1/c] \geq [r_1/c]\), then at the optimal solutions to \(P_2\) and \(\Phi_2\), \(Z_1^* = [d_1/c]\), and \(Z_2^0 = [d_1/c]\).

**Proof.** By definition, \(d_1\) stands for the minimum amount of forward flow to be shipped back to the customer site in \(t = 1\), or the solution will be infeasible and a shortage will occur for the customer during \(t = 1\), so \(Z_1^* \geq [d_1/c]\) and \(Z_2^0 = [d_1/c]\). Since \([d_1/c] \geq [r_1/c]\), any additional trip in \(t = 1\) will be either empty or sending extra forward flows for \(t = 2\), which does not save the shipping cost but increases the holding cost, and thus is not optimal. Therefore, \(Z_1^* = [d_1/c]\), and \(Z_2^0 = [d_1/c]\).

**Lemma 2.** If \([d_2/c] \leq [r_2/c]\), then at the optimal solution to \(P_2\), \(Z_2^* = [r_2/c]\).

**Proof.** By definition, \(r_2\) stands for the minimum amount of reverse flow to be shipped back to the DC in \(t = 2\), or \(V_2\) will be violated, so \(Z_2^* \geq [r_2/c]\). Since \([d_2/c] \leq [r_2/c]\), any additional trip beyond \([r_2/c]\) results in empty truck trips and thus is not optimal. Therefore, \(Z_2^* = [r_2/c]\).

Given \(d_1\) and \(r_2\), let us define \(S_{d_1} = [d_1/c] \cdot c - d_1\), \(S_{r_2} = [r_2/c] \cdot c - r_2\), as the spare capacity of the truck for transporting \(d_1\) and \(r_2\) in the respective time period, and define \(\Delta_d = d_2 - [d_2/c] \cdot c\), \(\Delta_r = r_1 - r_1/c \cdot c\), as the amount of less-than-truckload (LTL) or the residual load of \(d_2\) and \(r_1\) in the respective flow in \(t = 1\), and \(t = 2\).

**Proposition 1.** For any given instance of \(P_2\) and its associated partial relaxation \(\Phi_2\), the gap between \(Z_1^*\) and \(Z_2^0\) and the gap between \(Z_2^*\) and \(Z_2^0\) are bounded by

\[
|Z_1^* - Z_2^0| \leq 1, \quad |Z_2^* - Z_2^0| \leq 1. \tag{1}
\]

**Proof.** For any given problem with net flows of \(d_1\), \(d_2\), \(r_1\), and \(r_2\), there are four exclusive cases: \(d_1 \geq r_1\) and \(d_2 \leq r_2\); \(d_1 \geq r_1\) and \(d_2 > r_2\); \(d_1 < r_1\) and \(d_2 \leq r_2\); and \(d_1 < r_1\) and \(d_2 > r_2\). We now show that the error bounds given in (1) hold for all these cases.

**Case 1.** \(d_1 \geq r_1\) and \(d_2 \leq r_2\). The inequality \(d_1 \geq r_1\) implies \([d_1/c] \geq [r_1/c]\). Thus, \(Z_1^* = Z_2^0 = [d_1/c]\) by Lemma 1, and \(|Z_1^* - Z_2^0| = 0\). Furthermore, since \(d_2 \leq r_2\), \(Z_2^* = [r_2/c]\) by Lemma 2, and \(Z_2^0 = r_2/c\) since it suffices to transport \(r_2\) under the linear relaxation. Therefore, \(|Z_2^* - Z_2^0| = |r_2/c - r_2/c| < 1\), and Proposition 1 holds.

**Case 2.** \(d_1 \geq r_1\) and \(d_2 > r_2\). Since \(d_1 \geq r_1\), \(Z_1^* = Z_2^0 = [d_1/c]\) by Lemma 1, and \(|Z_1^* - Z_2^0| = 0\). Now we only need to prove that \(|Z_2^0 - Z_2^*| \leq 1\) holds. Since \(Z_2^0\) is a linear variable, its optimal value, when \(d_2 > r_2\), is achieved at

\[
Z_2^0 = \frac{d_2 - \min(S_{d_1}, d_2 - r_2)}{c}. \tag{2}
\]

In this case \((d_2 > r_2)\), however, the optimal value of \(Z_2^0\) depends on the following two exclusive situations (Chen et al. 2004): \([d_2/c] = [r_2/c]\) and \([d_2/c] > [r_2/c]\).

**Case 2(a).** When \([d_2/c] = [r_2/c]\), this is a special case of Lemma 2 and \(Z_2^* = [r_2/c]\). From (2) and the fact that \(d_2 > r_2\), we know that \(Z_2^* = \lfloor d_2 - \min(S_{d_1}, d_2 - r_2) \rfloor / c = [d_2/c] = [r_2/c] = Z_2^0\). Therefore, \(|Z_2^0 - Z_2^*| = |Z_2^0 - Z_2^*| < 1\), and Proposition 1 holds.

**Case 2(b).** When \([d_2/c] > [r_2/c]\), this is the situation where shifting \(\Delta_d\) to period \(t = 1\) may save a trip for the forward flow in \(t = 2\). Since \(f > c \cdot h\), the optimal number of truck trips in \(t = 2\), \(Z_2^*\), is given by

\[
Z_2^* = \begin{cases} 
\left\lfloor \frac{d_2 - \Delta_d}{c} \right\rfloor = \left\lfloor \frac{d_2}{c} \right\rfloor, & \text{if } S_{d_1} \geq \Delta_d, \\
\left\lfloor \frac{d_2}{c} \right\rfloor, & \text{otherwise}. \quad (3a)
\end{cases}
\]

If \(S_{d_1} \geq \Delta_d\), from (2) and (3a), the following inequalities hold:

\[
|Z_2^0 - Z_2^*| = \left| \left\lfloor \frac{d_2}{c} \right\rfloor - \left\lfloor \frac{d_2 - \min(S_{d_1}, d_2 - r_2)}{c} \right\rfloor \right|,
\]

where \(\min(S_{d_1}, d_2 - r_2) < c\),

\[
< \left| \left\lfloor \frac{d_2}{c} \right\rfloor - \left\lfloor \frac{d_2}{c} - 1 \right\rfloor \right| = \left| 1 - \left(\frac{d_2}{c} - \left\lfloor \frac{d_2}{c} \right\rfloor \right) \right| \leq 1.
\]

If \(S_{d_1} < \Delta_d\), from (2) and (3b), the following inequalities hold:

\[
|Z_2^0 - Z_2^*| = \left| \left\lfloor \frac{d_2}{c} \right\rfloor - \left\lfloor \frac{d_2 - \min(S_{d_1}, d_2 - r_2)}{c} \right\rfloor \right|,
\]

where \(\min(S_{d_1}, d_2 - r_2) < \Delta_d\),

\[
< \left| \left\lfloor \frac{d_2}{c} \right\rfloor - \left(\frac{d_2}{c} - \frac{\Delta_d}{c}\right) \right| = \left| \left\lfloor \frac{d_2}{c} \right\rfloor - \frac{d_2}{c} \right| \leq 1.
\]

Clearly, Proposition 1 holds in both cases.

**Case 3.** \(d_1 < r_1\) and \(d_2 \leq r_2\). When \(d_2 \leq r_2\), \(Z_2^0 = [r_2/c]\) by Lemma 2. Note that the reverse flow \(r_2\) is not available for transportation in \(t = 1\) and that \([r_2/c]\) trips are sufficient for shipping \(d_2\) in period \(t = 2\). Since \(d_1 < r_1\), we have \([d_1/c] \leq [r_1/c]\) and consider the following two exclusive cases: \([d_1/c] = [r_1/c]\) and \([d_1/c] < [r_1/c]\).

**Case 3(a).** When \([d_1/c] = [r_1/c]\) and \(d_2 \leq r_2\), this is a special case of Case 1, so Proposition 1 holds.
Case 3(b). When \([d_1/c] < \lfloor r_1/c \rfloor\) and \(d_2 \leq r_2\), this is the case where shifting \(\Delta_n\) to period \(t = 2\) may save a trip in period \(t = 1\). According to Chen et al. (2004),

\[
Z_t^1 = \begin{cases} 
\left\lceil \frac{r_1 - \Delta_n}{c} \right\rceil, & \text{if } \Delta_n \leq S_n, \\
\left\lceil \frac{r_1}{c} \right\rceil, & \text{otherwise}.
\end{cases} \tag{4a}
\]

Since \(Z_t^1\) is a linear variable, the optimal values for \(Z_t^1\) and \(Z_t^2\) are achieved at

\[
Z_t^0 = \left\lceil \frac{r_1}{c} \right\rceil, \quad Z_t^2 = \frac{r_2 + \Delta_n}{c}, \quad \text{if } f > \Delta_n - (h + f/c), \tag{5a}
\]

\[
Z_t^0 = \left\lceil \frac{r_1}{c} \right\rceil, \quad Z_t^2 = \frac{r_2}{c}, \quad \text{otherwise}. \tag{5b}
\]

As \(\Delta_n < c\), we have \(|Z_t^1 - Z_t^0| \leq 1\). Since \(Z_t^2 = \lfloor r_2/c \rfloor\), we see that \(|Z_t - Z_t^2| \leq 1\) from (5a) and (5b). Therefore, Proposition 1 holds.

Case 4. \(d_1 < r_1\) and \(d_2 > r_2\). We now have the following four exclusive cases:

4(a). \([d_1/c] = \lfloor r_1/c \rfloor\) and \([d_2/c] = \lfloor r_2/c \rfloor\),

4(b). \([d_1/c] = \lfloor r_1/c \rfloor\) and \([d_2/c] > \lfloor r_2/c \rfloor\),

4(c). \([d_1/c] < \lfloor r_1/c \rfloor\) and \([d_2/c] = \lfloor r_2/c \rfloor\),

4(d). \([d_1/c] < \lfloor r_1/c \rfloor\) and \([d_2/c] > \lfloor r_2/c \rfloor\).

Since Cases 4(a), 4(b), and 4(c) are special cases of Cases 1, 2(b), and 3(b), respectively, their proofs are omitted. We now proceed to prove that Proposition 1 holds for Case 4(d). Define \(K_1 = \lfloor r_1/c \rfloor - \lfloor d_1/c \rfloor\) as the difference between the numbers of truckloads needed for transporting \(d_1\) and \(r_1\) and define \(K_2 = \lfloor d_2/c \rfloor - \lfloor r_2/c \rfloor\) as the difference between the numbers of truck trips needed for transporting \(d_2\) and \(r_2\). If \(K_1 > K_2\) (see Figure 4(a)), we shift quantity \(\delta_1 = d_2 - \lfloor r_2/c \rfloor \cdot c\) out of \(d_2\) to period \(t = 1\). This leads to the situation with \([d_2/c] = \lfloor r_2/c \rfloor\), where \(d_2 = d_2 - \delta_1\), which reduces to a special case of Case 3(b). The proof is then similar and thus omitted. If \(K_1 \leq K_2\) (see Figure 4(b)), on the other hand, we shift quantity \(\delta_2 = r_1 - \lfloor d_1/c \rfloor \cdot c\) out of \(r_1\) to period \(t = 2\). This leads to the situation \([d_1/c] = \lfloor r_1/c \rfloor\), where \(r_1 = r_1 - \delta_2\), which is a special case of Case 2(b).

The proof is then similar and thus omitted.

The marginal analysis on the error gap between the optimal solutions to \(P_2\) and its partial relaxation \(\Phi_2\) can be extended to few related but more general situations as well. For example, we can extend this analysis to the problem with a nonzero holding cost, \(h_{DC}\), at the DC. In this case, the holding cost of the customer, \(h\), in the objective function of \(P_2\) should be replaced by the net holding cost \((h - h_{DC})\). This net holding cost represents the cost of repositioning one unit of inventory from the DC to the customer. In spite of the change in the objective function, the results given in (1)–(5) continue to hold and the constraints in \(P_2\) remain the same. Therefore, Proposition 1 can be applied to this case as well. Another extension is the case with a limited supply at the DC in each time period. This refers to a situation where the inventory supply at the DC in \(t = 1\) may not be sufficient to satisfy both \(d_1\) and \(\Delta_{d_2}\) (i.e., the amount of less-than-truckload that should be shifted from \(t = 2\) to \(t = 1\)). The decision on shifting \(\Delta_{d_2}\) to \(t = 1\) now must depend on two conditions:

- \(\Delta_{d_2} \leq S_{d_1}\), and
- the DC supply in \(t = 1\) is sufficient to cover \(d_1 + \Delta_{d_2}\). This requires us to revise the conditions stated in Equations (2) and (3a). The marginal analysis can also be applied to an extended case with \(K\) customers \((K > 1)\) and sufficient supplies at the DC. If the supply at the DC during \(t = 1\) is sufficient to cover \(\sum_{b=1,2,\ldots,k} (d_{1,b} + \Delta_{d_{1,b}})\), then the problem can be decomposed into \(K\) independent single-DC single-customer problems \((P_{2,b})\), for which Proposition 1 continues to hold.

Although \(P_2\) represents only a subproblem of model \(P\), it covers the relaxation by the proposed heuristic, which relaxes only the integer requirements on the number of truck trips between the DC and the customers. Also note that \(P_2\) is the closest one we can work on without losing too much intuition and clarity in the analysis. Nevertheless, the error bound analysis here reveals the potential promise of the proposed partial LP relaxation-based heuristic for solving general PDPs with bidirectional flows. We shall investigate this issue empirically in the following section.

4. **Empirical Study**

To observe the performance of the proposed partial LP relaxation-based heuristic in a more general setting, we randomly generated test cases based on a hypothetical network with two plants (i.e., two suppliers, one for each product), a single DC with homogeneous trucks, and a planning horizon with 12 time periods \((|T| = 12)\). Five sets of experiments, each investigating the impact of a particular problem parameter on the computational performance.
of the proposed partial relaxation approach, were conducted. The five parameters in the experiments include:

- Size of the network, in terms of the number of customers: \(|K|\);
- Ratio of the unit production cost to the unit refurbishing cost: \(a/b\);
- Ratio of fixed costs for producing new and for refurbishing used products: \(\bar{S}/\bar{S}\);
- Recovery rate: \(\theta\);
- Ratio of the average reverse flow to the average forward flow: \(\bar{r}/\bar{d}\).

Each set of parameter values defines a problem setting (i.e., a basic instance) on which 10 test cases were randomly generated. The uniform distributions \(d_{kt} = \text{UNIF}(0.2c, 2.4c)\), \(r_{kt} = \text{UNIF}(0.2c, 2.4c)\), \(f_{1} = \text{UNIF}(400, 8,000)\), and \(f_{2} = \text{UNIF}(400, 8,000)\) were used, where \(c\) stands for the truck-loading capacity, \(f_{1}\) and \(f_{2}\) stands for the round-trip cost between supplier \(i\) and the DC, and between the DC and customer \(k\), respectively, \(\forall k \in K, i = 1, 2, \forall t \in T\). All initial inventories were set to zero.

Table 1 shows the performance of the proposed partial LP relaxation-based approach under networks with different sizes, defined by the number of customers \(|K|\). Fixed parameter values in this experiment were \(c = 2,500, \bar{S}_{1} = 500,000, \bar{S}_{2} = 450,000, w_{1} = 300,000, w_{2} = 250,000, \bar{w}_{bc} = 5,000,000, \omega_{k} = 8,000, a_{1} = 150, a_{2} = 120, b_{1} = 15, b_{2} = 12, s_{1}^{l} = 30,000, s_{2}^{l} = 25,000, s_{1}^{h} = 25,000, s_{2}^{h} = 20,000, h_{1} = 8, h_{2} = 6, h_{1}^{bc} = 12, h_{2}^{bc} = 10, h_{ki} = a_{i} \cdot 0.05, \theta_{1} = 0.8, \text{ and } \theta_{2} = 0.6\). Since the CPLEX solver frequently failed to find the optimal solution to problem \(P\) (within the one-hour CPU time limit) as the size of the problems, \(|K|\), increased to a certain level, we used the best feasible CPLEX solutions obtained within the time limit as a surrogate for the optimal solution in such cases. For each test case, three types of empirical observations were recorded with respect to each solution approach: the total operation cost (i.e., the sum of production, inventory, and truck-trip costs), the logistics cost, which includes the truck-trip and inventory costs only, and the required computation time. The reason for itemizing the logistics cost is that the proposed partial LP relaxation heuristic relaxes the integer requirements on selected truck trips, so the logistics cost is more vulnerable to the suboptimality of the relaxation.

As we can see, the partial LP relaxation-based heuristic achieved a promising performance in all 70 test cases. The results summarized in Table 1 show the largest error gap is 0.16% in terms of total operation cost and 0.54% in terms of the logistics cost. While the CPU time (in seconds) of this partial LP relaxation-based heuristic approach also increases as the problem size \(|K|\) increases, the required search effort is much less significant compared with that required for solving \(P\) directly using CPLEX. In all the cases reported in Table 1, the CPU time for the proposed heuristic is less than five seconds. Table 2 reports on the impact of network size \(|K|\) on the performance of the partial relaxation-based heuristic under a slightly different algorithm implementation. In each iteration, we included two time periods in the first planning interval \((\tau_{1})\), for which all the constraints in \(P\) remain unchanged, and two time periods in the second planning interval \((\tau_{2})\), for which all the constraints in \(P\) are retained except the integer requirements on the truck trips between the DC and the customers. All remaining time periods are consolidated into the third planning interval \((\tau_{3})\). The purpose of this implementation is to examine the tradeoff between additional search efforts and improvement on solution quality. As we can see from the second set of 70 randomly generated cases (Table 2), the required computation time increased significantly after the number of customers reached the level of 80. As the number of time periods included in the first interval increased, the number of integer (both binary and general) variables increased and so did the search effort to verify the optimality of a partial relaxation solution in each iteration.

| Table 1 | Empirical Performance of Different Approaches (with One Period in Planning Interval 1, \(|\tau_{1}| = 1\)) |
|---------|---------------------------------|
|         |                  CPLEX optimizer | Partial LP relaxation |
|         |       Average    |         Maximum     |       Average    |         Maximum     |
| \(|K|\) |        (seconds) |              |        (seconds) |              |
| 10     |   613.92   |        3,600*   |   0.68   |        0.77   |
| 20     |   23.28    |        68.81    |   0.80   |        0.88   |
| 40     |   203.97   |       1,109.94  |   1.07   |        1.23   |
| 60     |   362.14   |        3,600*   |   1.35   |        1.88   |
| 70     |   372.79   |        3,600*   |   1.46   |        1.58   |
| 75     |   170.56   |       810.17    |   1.61   |        1.91   |
| 80     |   3,212.62 |       3,600*   |   2.53   |        4.45   |

*The error gap was based on the best feasible solution found by CPLEX within the CPU time limit.
Nevertheless, the solution quality also improved consistently. All the error gaps for the total operation costs were within 0.2%, and the largest error gap for the logistics cost, which is more sensitive than the total operation cost, was well within 0.6% of the optimal or the surrogates of the optimal solutions. Again, the CPU time required by the partial LP relaxation-based heuristic is much less than that required by CPLEX to solve P directly.

Table 3 reports on the impact of the ratio of the unit production cost to the unit refurbishment cost, \( \bar{a}/b \), where \( \bar{a} = (a_1 + a_2)/2 \) and \( b = (b_1 + b_2)/2 \), on the empirical performance of the partial LP relaxation-based heuristic. We varied the values of \( b_1 \) and \( b_2 \) while fixing the values of \( a_1 \) at 200 and \( a_2 \) at 160 to generate different ratios from \( \bar{a}/b = 4 \) to \( \bar{a}/b = 18 \). The number of customers was fixed at \(|K| = 20\), and the values of all other parameters remained the same as that for Table 1. Each performance observation reported here was based on the average of 10 observations from the experiment. As we can see, the partial LP relaxation-based approach showed a very consistent and promising empirical performance in terms of both the CPU time required and its deviation from the optimal solutions.

The results summarized in Table 4 were collected from a related experiment that was designed to evaluate the impact of the relative unit refurbishing cost, \( \bar{b}/\bar{a} \), on the operation cost of a network with \(|K| = 50\), instead of that on the algorithm performance. As we can see, as the value of \( \bar{b}/\bar{a} \) increased, the holding cost at the customer sites increased tremendously (under the operations schedule by the partial relaxation-based heuristic). The reason is that as the unit refurbishing cost \( b \) increases, it becomes less and less attractive to promptly bring back the used products. Meanwhile, the shipping cost decreased relatively as it is no longer worthwhile to send designated truck trips for reverse flow only. This observation also implies that, in a practical situation, if the unit refurbishing cost is relatively high, then the inventory capacity at the customer sites should be large enough to accommodate the minimum-cost operation schedules, and it is likely the reverse flows may only take the “free ride” of truck trips for the forward flows.

Table 4 reports on the impact of the ratio of fixed costs, \( S^a/S^b \), on the empirical performance of the partial LP relaxation-based heuristic, where \( S^a = (S^a_1 + S^a_2)/2 \) stands for the average fixed cost for setting up a production run and \( S^b = (S^b_1 + S^b_2)/2 \) stands for the average fixed cost for assessing a batch of used products to be refurbished. We fixed \( S^a_1 = $30,000 \) and \( S^a_2 = $25,000 \) while varying the values of \( S^b_1 \) and \( S^b_2 \) to have \( S^a/S^b = 1, 1.5, 2, \) and \( 2.5 \) in the experiment.
For each given value of \( S^a/S^b \), 10 test cases were randomly generated. Customer size was fixed at \(|K| = 20\), and the values of all other parameters remained the same as those for Table 1. In addition to a consistent performance of the proposed partial relaxation-based approach, we also notice that the CPU time required by CPLEX that solves \( P \) directly increased significantly when the value of ratio \( S^a/S^b \) increased. One reason for this is that when the fixed cost for assessing the reusability of a batch of returns is relatively low (as indicated by the larger values of ratio \( S^a/S^b \)), it is more profitable to bring back the reverse flow in a timely manner, which in turn imposes a more challenging scheduling problem to find the best trade-off between production and shipping costs. Nevertheless, the computational effort required by the partial relaxation approach is again minimal, mainly because the heuristic relaxes integrality on the truck trips and thus requires a minimal truck scheduling effort.

Table 6 reports on the empirical performance of the partial LP relaxation-based heuristic under different levels of the product recovery rate \( \theta \). For this experiment, both products were assumed to have the same recovery rate, \( \theta_1 = \theta_2 \), which ranged from \( \theta = 0.6 \) to \( \theta = 0.9 \). Customer size was again fixed at \(|K| = 20\), and the values of all other parameters remained the same as those for Table 1. As we can see, the required search effort by both CPLEX and the partial LP relaxation approach increased significantly as the recovery rate increased. This is because when the recovery rate is high, the timely availability of reverse flows at the

### Table 5: Empirical Performance vs. Setup Cost Ratio

<table>
<thead>
<tr>
<th>( S^a/S^b )</th>
<th>CPLEX optimizer</th>
<th>Partial LP relaxation</th>
<th>Performance gap (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>Maximum</td>
<td>Average</td>
<td>Maximum</td>
</tr>
<tr>
<td>1</td>
<td>23.43</td>
<td>91.21+</td>
<td>0.95</td>
</tr>
<tr>
<td>1.5</td>
<td>405.89</td>
<td>1,949.87+</td>
<td>0.92</td>
</tr>
<tr>
<td>2</td>
<td>394.07</td>
<td>3,600.00+</td>
<td>0.84</td>
</tr>
<tr>
<td>2.5</td>
<td>439.36</td>
<td>3,600.00+</td>
<td>0.88</td>
</tr>
</tbody>
</table>

*Search stopped as the one-hour CPU time limit was exceeded.
+ The error gap was based on the best feasible solution found by CPLEX within the CPU time limit.

### Table 6: Empirical Performance vs. Recovery Rate

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>CPLEX optimizer</th>
<th>Partial LP relaxation</th>
<th>Performance gap (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>Maximum</td>
<td>Average</td>
<td>Maximum</td>
</tr>
<tr>
<td>0.6</td>
<td>61.03</td>
<td>556.42+</td>
<td>0.73</td>
</tr>
<tr>
<td>0.7</td>
<td>159.85</td>
<td>1,329.02+</td>
<td>0.91</td>
</tr>
<tr>
<td>0.8</td>
<td>1,714.02</td>
<td>3,600.00+</td>
<td>1.32</td>
</tr>
<tr>
<td>0.9</td>
<td>3,600.00+</td>
<td>3,600.00+</td>
<td>1.81</td>
</tr>
</tbody>
</table>

*Search stopped as the one-hour CPU time limit was exceeded.
+ The error gap was based on the best feasible solution found by CPLEX within the CPU time limit.

The error gap was based on the best feasible solution found by CPLEX within the CPU time limit.

suppliers’ sites offers opportunities for cost savings, since repairing used ones costs much less than producing new ones. This, however, in turn increased the effort of coordinating the forward and backward flows.

Table 7 reports on the empirical performance of the partial LP relaxation-based heuristic approach versus the ratio between the average level of reverse flows and the average level of forward flows. The forward demands \( d_{kit} \) were generated by UNIF(0.2c, 2.4c), and the reverse demands \( r_{kit} \) were generated by UNIF(0.2c \( \delta \), 2.4c \( \delta \)), where parameter \( \delta \) ranged from 0.2 to 1.5 to make the demand ratio, \( \bar{r}/\bar{d} \), vary from 20% to 150%. Note that in practice the total amount of reverse flows is usually less than the total amount of forward flows over a planning horizon. We allowed the value of \( \bar{r}/\bar{d} \) to be more than 100% here merely to test the robustness of the algorithm. In this experiment, customer size was fixed at \(|K| = 20\), and the values of all other parameters remained the same as that for Table 1. Again, the proposed partial LP relaxation-based heuristic demonstrated a promising and consistent performance in this case as well.

Although the empirical results reported in this section were all based on our hypothesized network, it should be pointed out that, as long as the mixed-integer programming model for a real-life network with bidirectional flow over a multiperiod planning horizon can be constructed, this proposed heuristic can be implemented/applied in a similar way and has a potential to find a good-quality solution because of the power of mathematical programming. Of course, real-life supply chain networks may have a much more complex logistics structure than what we modeled here. Nevertheless, the proposed heuristic can still be used for solving the part of the problem that
can be mathematically modeled and can provide support to the decisions that must also consider the factors that cannot be easily modeled.

5. Concluding Remarks
We have proposed a partial LP relaxation-based heuristic approach to solve a variation of the integrated production and distribution problem with bidirectional flows. We have also reported on the gaps between the optimal solution and the heuristic solution provided by this partial relaxation, including analytical gaps for a special case and empirical gaps observed from 480 randomly generated test cases.

The integrated production and distribution problem with bidirectional flows is a complicated but interesting and challenging optimization problem. There are many extensions that can be generated from this study. First, we assumed a direct shipment policy between the locations on the given network. However, when the truck routing must be taken into consideration, which is common in many applications where each customer requests a small delivery or pickup quantity (i.e., much smaller than the truck capacity) in each time period and where the inventory holding costs are relatively high, the approach proposed in this study may no longer be effective. Solution methods that can handle the integrated production and inventory together with the truck routing issues will be more appealing. Second, although we assumed in this study that the demand on reverse flow and the product recovery rate are both given constants, such assumptions may not hold for some general cases. For example, when the products to be returned via reverse flow are from a retailer who receives the returns from customers randomly, the respective integrated PDP becomes a stochastic programming problem which leads to another challenging study. Third, we have assumed here the product recovery rate, \( \theta \), to be a given constant. However, in many applications, such a recovery rate is a decision variable whose value is affected by the disposal cost. It would be interesting to extend this into the model. Also, we have only considered an objective function to minimize the total operation cost in this study. Many other objective functions can be considered as well, such as minimizing the fleet size to satisfy certain performance measures or customer service levels. Additional constraints can also be considered to make the problem more interesting, such as customer delivery time windows.

From both academic and practical points of view, integrated production and distribution problems involving bidirectional flows have the potential to receive significant attention in the future. This is because more and more original equipment manufacturers are now engaged in the implementation of business policies of remanufacturing and refurbishment. In their business processes, production defects, by-products, and products with short life cycles will have to be brought back to the manufacturing sites. Production planning and control of such integrated processes impose a real challenge because the coordination and management activities are much more complex than with traditional production flows. The results reported in this study contribute toward the development of solution approaches as a decision support tool in operations scheduling of bidirectional flow networks such as closed-loop supply chains.

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References