ON STICKY MATROIDS

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The "sticky conjecture" states that a geometric lattice is modular if and only if any two of its extensions can be "glued together". It is known to be true as far as rank 3 geometries are concerned. In this paper we show that it is sufficient to consider a very restricted class of rank 4 geometries in order to settle the question. As a corollary we get a characterization of uniform sticky matroids, which has been found by Poljak and Turzik in 1984.

1. Introduction

Let \( L \) be a geometric lattice and denote the set of copoints of \( L \) by \( \mathcal{X} \). A subset \( \mathcal{X} \subset \mathcal{X} \) is called a linear subclass, if for every coline \( c \) of \( L \), either \( c \) is covered by at most one element of \( Z \) or all copoints covering \( c \) are contained in \( \mathcal{X} \) (cf. [5]). The set of all linear subclasses of \( L \) can be ordered by inclusion, giving rise to a lattice with point set equal to \( \mathcal{X} \). This is called the extension-lattice of \( L \) and is denoted by \( \mathcal{E}(L) \) (cf. [2, 3]).

An important question arising from this, is to investigate, how far the geometric lattice \( L \) is determined by its extension lattice \( \mathcal{E}(L) \). For example, one could ask: Given two geometric lattices \( L \) and \( L' \) having isomorphic extension lattices, must \( L \) and \( L' \) be isomorphic. In general, the answer is: No. Consider, e.g., any projective geometry \( L \) of rank at least 4. The removal of a point will not change the incidence relation between the copoints and colines of \( L \), and hence the extension lattice will not be changed. However, if \( L \) is of rank 3, the situation becomes different. Here it is easy to see (cf. Section 2), that \( L \) is modular if and only if \( \mathcal{E}(L) \) is. As we will show in Section 3, this observation is related to another characterization of modular rank 3 geometries, found by Poljak and Turzik (cf. [8]). They proved that a finite rank 3 geometry is modular if and only if it is "sticky", i.e., if any two extensions \( L_1 \) and \( L_2 \) of \( L \) can be "glued together".

In their paper, Poljak and Turzik conjecture that their theorem holds also for geometries of rank \( \geq 4 \). In Section 4, we show that, in order to prove this conjecture, it is sufficient to prove it only for a quite restricted class of rank 4 geometries (which will hopefully some day turn out to be empty). As a simple corollary we will get a characterization of sticky uniform matroids, due to Poljak and Turzik (1984).

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2. Extension lattices

Let $L$ be a geometric lattice (finite or infinite). Then there is a 1–1 relation between the following three sets (cf. [5]):

(a) The set of all point-extensions (single element extensions) of $L$.
(b) The set of all linear subclasses of $L$.
(c) The set of all modular filters of $L$.

Using the relation between (b) and (c) one can prove:

**Proposition 1.** The canonical order-reversing map $\varepsilon : L \to \varepsilon(L)$ has the following properties:

(i) $\varepsilon(x \lor y) = \varepsilon(x) \cap \varepsilon(y)$, for all $x, y \in L$,
(ii) if $y$ covers $x$ in $L$, then $\varepsilon(x)$ covers $\varepsilon(y)$ in $\varepsilon(L)$

**Proof.** $\varepsilon$ maps every $x \in L$ to the corresponding principal class $Z_x := \{ c \in \mathcal{C} \mid c \supseteq x \}$. From this, property (i) follows immediately. Now let $y$ cover $x$ in $L$ and let $Z_x$ and $Z_y$ resp. $\mathcal{F}_x$ and $\mathcal{F}_y$ denote the corresponding principal classes resp. modular filters. Suppose that $Z \in \varepsilon(L)$ such that $Z_x \supseteq Z_y$, i.e., $\mathcal{F}_x \supseteq \mathcal{F}_y \supseteq \mathcal{F}_y$ for the modular filter $\mathcal{F}$ corresponding to $Z$. Let $c \in Z_x \setminus Z_y$, then $(c, y)$ is a modular pair and therefore, $x = c \land y \in \mathcal{F}$. Thus $Z_x \subseteq Z$, implying $Z = Z_x$. Since $Z$ has been arbitrary; this shows that $Z_x$ covers $Z_y$ in $\varepsilon(L)$.

If the geometric lattice $L$ is modular, every modular filter is principal, i.e., it equals $\{ y \in L \mid y \supseteq x \}$ for some $x \in L$. Therefore, in this case, the map $\varepsilon$ is surjective, i.e., an anti-isomorphism. This proves

**Proposition 2.** If the geometric lattice $L$ is modular, so is $\varepsilon(L)$.

In general, $\varepsilon(L)$ is not even a geometric lattice, indeed it may not even have a rank function. Note however, that Proposition 1(ii) implies that $\varepsilon(L)$ contains a maximal chain of length $r = \text{rank}(L)$.

**Proposition 3.** If $L$ is geometric and $\varepsilon(L)$ is modular, then $L$ can be embedded into a modular geometric lattice by a rank- and sup-preserving function.

**Proof.** If $\varepsilon(L)$ is modular, its rank is determined by any of its maximal chains, hence it is equal to the rank of $L$. The dual of $\varepsilon(L)$, say $\varepsilon(L)^*$, is also modular and the map $L \to \varepsilon(L) \to \varepsilon(L)^*$ has the desired properties.

As we mentioned already in the introduction, we can not conclude that $L$ is itself a modular lattice, unless $\text{rank}(L) = 3$ (cf. Section 3). We note that, if we consider extension lattices, we may restrict ourselves to connected geometries.
Remark. If \( L_1 \) and \( L_2 \) are geometric lattices, then \( \varepsilon(L_1 \times L_2) \cong \varepsilon(L_1) \times \varepsilon(L_2) \).

Proof. Straightforward, using the fact that, if a coline of \( L_1 \times L_2 \) is covered by more than two copoints, then it must be of the form \((d_1, 1)\) or \((1, d_2)\) where \( d_i \) is a coline of \( L_i \).

3. Rank 3 geometries

Proposition 4. A geometric lattice \( L \) of rank 3 is modular if and only if \( \varepsilon(L) \) is modular.

Proof. The "only if"-part follows from Proposition 2. To prove the converse, suppose that \( \varepsilon(L) \) is modular. As we have seen in Section 2, this implies that \( \varepsilon(L) \) has rank 3. It is easy to see that if \( \varepsilon(L) \) is the direct sum of a point and a line, so is \( L \). Thus suppose \( \varepsilon(L) \) is connected, i.e., it is a projective plane. In particular, every line of \( \varepsilon(L) \) contains at least three points. If \( L \) is not modular, there exist two lines \( l_1 \) and \( l_2 \) in \( L \) which do not intersect. Then \( Z = \{l_1, l_2\} \) is a two-point line in \( \varepsilon(L) \), a contradiction.

Corollary. \( L \) is a projective plane if and only if \( \varepsilon(L) \) is.

In [8] the notion of sticky matroids has been introduced.

Definition. A geometric lattice \( L \) with pointset \( E \) (finite or infinite) is sticky, if for any two extensions \( L_1 \) and \( L_2 \) with pointsets \( E \cup X_1 \) and \( E \cup X_2 \), resp., there exists a geometric lattice \( \tilde{L} \) with pointset \( E \cup X_1 \cup X_2 \) such that \( \tilde{L} \setminus X_2 = L_1 \) and \( \tilde{L} \setminus X_1 = L_2 \).

It is well known that modular lattices are sticky. In [8] the following result has been proved.

Proposition 5. A finite geometric lattice of rank 3 is modular if and only if it is sticky.

The proof given by Poljak and Turzik is based on counting points and lines and does therefore not apply to infinite geometries. (Nevertheless, it is not difficult to prove an infinite version of Proposition 5.)

To reveal the relationship between Propositions 4 and 5, we note that the key result for proving Proposition 5 is

Lemma 5. (cf. [8]) If a finite geometric lattice of rank 3 is nonmodular, then it
contains either
  a) three pairwise nonmodular lines, or
  b) two disjoint pairs of nonmodular lines.

If \( L \) is nonmodular, Proposition 4 shows that \( \mathcal{E}(L) \) is nonmodular, too. Hence either
  a') \( \mathcal{E}(L) \) contains a chain of length \( >3 \), or
  b') \( \mathcal{E}(L) \) is geometric, but nonmodular.
It is easy to see that a) \( \Leftrightarrow \) a') and hence b) \( \Leftrightarrow \) b').

4. The "sticky conjecture"

In this section, all lattices considered are assumed to be finite.
In their paper, Poljak and Turzik made the following conjecture:

\[ \text{Stickiness implies modularity.} \quad (S) \]

Let us first consider a class of geometric lattices for which \((S)\) holds.

Definition. Let \( L \) be a geometric lattice and let \( x, y \in L \) be nonmodular. We say that \( x \) and \( y \) can be intersected, if there exists a pointextension \( L' = L \cup p \) such that, in \( L' \), \( p \) lies on \( x \) and \( y \) but not on \( x \wedge y \) (cf. [2] for more detail). We say that \( L \) has the Intersection Property (IP), if for any nonmodular pair \((x, y)\) of elements in \( L \), \( x \) and \( y \) can be intersected.

Example.

a) If \( L \) arises from a linear vectorspace, \( L \) has the IP. More generally, if \( L \) can be embedded (by a rank- and sup-preserving map) into a modular lattice, \( L \) has the IP.

b) If \( L \) contains three colines \( D_1, D_2 \) and \( D_3 \) and a line \( l_4 \) (see Fig. 1) such that \( D_1 \lor D_2 = 1 \) but \( D_1 \) and \( D_3, D_2 \) and \( D_3 \) as well as \( l_4 \) and \( D_i \) \((i = 1, 2, 3)\) all span

![Fig. 1.](image-url)
pairwise different copoints, then $L$ has not the IP. (To get a proof for this, recall how the Vamos matroid is shown to be nonlinear.)

**Lemma 6.** For the class of geometric lattices, which do have the IP, the conjecture (S) holds.

**Proof.** Let $L$ be a geometric lattice of rank $\geq 4$ and suppose it is sticky. If $L$ is modular, there is nothing to prove. If not, there exists a line $l$ and a copoint $c$ which do not intersect. Since $L$ has the IP, $l$ and $c$ can be intersected. Let $L_1 = L \cup p$ be a pointextension of $L$ in which $l$ and $c$ intersect. We are now going to construct an extension $L_2$ of $L$ in which $l$ and $c$ can not be intersected. This will give the desired contradiction. First, choose two co-planes $p_2$ and $p_3$ on $c$ and let $c_2 := p_2 \vee l$, $c_3 := p_3 \vee l$. Extend $L$ according to the two linear subclasses $\mathcal{E}_2 := \{c_2, c\}$ and $\mathcal{E}_3 := \{c_3, c\}$. This gives the configuration of Fig. 1, containing $l_4 := l$, $D_2 = c \land c_2$ and $D_3 = c \land c_3$. Next, we extend $L$ further in an obvious way, until it contains the whole configuration of Fig. 1, and we are done. \(\square\)

**Lemma 7.** If $L$ is sticky, so is every contraction of $L$.

**Proof.** It is sufficient to prove that $L/p$ is sticky for a single point $p \in L$. Let $G := L/p$. We regard $G$ as the interval $[p, 1]$ in $L$, i.e., $G = \{x \in L \mid x \geq p\}$. Suppose that $G_1 = G \cup A$ is an extension of $G$. We are going to show that there exists an extension $L_1 = L \cup A$ such that $G_1 = L_1/p$. Using induction, it suffices to prove this for a pointextension $G_1 = G \cup q$. Let $\mathcal{F} \subset G$ denote the modular filter corresponding to $G_1$. Then $\mathcal{F} \subset L$ is a modular filter of $L$ and thus defines a pointextension $L_1 = L \cup q$. Then $L_1/p = G_1$, which proves the claim.

Now let $G_1$ and $G_2$ be two extensions of $G$. Take any two extensions $L_1$ and $L_2$ of $L$ such that $L_i/p = G_i$ ($i = 1, 2$). Since $L$ is sticky, there exists a common extension $\tilde{L}$ of $L_1$ and $L_2$. Let $\tilde{G} := \tilde{L}/p$. Then the canonical maps $\sigma_i : G_i = L_i/p \rightarrow L_i \rightarrow \tilde{L}/p = \tilde{G}$ ($i = 1, 2$) are rank- and sup-preserving, showing that $\tilde{G}$ is an extension of both $G_1$ and $G_2$. Thus $G = L/p$ is sticky. \(\square\)

**Theorem 8.** If (S) holds for all geometric lattices of rank 4, it holds for all geometric lattices.

**Proof.** Suppose (S) is true for rank($L$) = 4. Let $L$ be a geometric lattice of rank $k > 4$ and suppose it is sticky. By Lemma 7, every rank 4 interval $[x, 1]$ in $L$ is sticky. Thus, by assumption, every rank 4 interval at the "top" of $L$ is modular. This implies (cf. [1, p. 130]) that $L$ can be imbedded into a modular lattice (by a rank- and sup-preserving map). Hence, $L$ has the Intersection Property and the claim follows from Lemma 6. \(\square\)

Theorem 8 together with Lemma 6 shows that it is sufficient to prove (S) for all
rank 4 geometries which fail to have the IP. These can be characterized as follows:

**Definition.** A geometric lattice is said to satisfy the *bundle condition* (cf. [6]), if it does not contain four lines \( l_1, \ldots, l_4 \) as indicated in Fig. 1 (i.e., such that \( l_1 \) and \( l_2 \) are not coplanar, but all other pairs \((i, l'_i)\) span pairwise distinct planes).

**Theorem 9.** If \( L \) is a rank 4 geometry, \( L \) has the IP if and only if it satisfies the bundle condition.

**Proof.** The "only if"-part is clear. Suppose therefore, that \( L \) does not have the IP. Thus there exists a pair of nonmodular flats \( x, y \) which can not be intersected. It is easy to see that the problem of intersecting a line with a plane can be reduced to the problem of intersection two coplanar lines. Thus we may assume that there exist two nonmodular lines \( l_1, l_2 \) which are coplanar, but can not be intersected, i.e., \( \epsilon(l_1) \vee \epsilon(l_2) = 1 \) in \( \epsilon(L) \). Let \( P \) denote the points in \( \epsilon(L) \) and let \( P_0 \subset P \) be the set of points covered by \( \epsilon(l_1) \) or \( \epsilon(l_2) \). In \( \epsilon(L) \), we obtain the element \( \epsilon(l_1) \vee \epsilon(l_2) = 1 \) by successively taking the "lineclosure", starting with \( P_0 \). More precisely, for \( i \geq 1 \), let \( P_i \) be \( P_0 \) plus all points \( p \in P \) such that \( p \) is covered by a "line" in \( \epsilon(L) \), which is spanned by two points of \( P_{i-1} \). (By a "line", we mean an element of \( \epsilon(L) \) which is the image of a line in \( L \) under \( \epsilon \).) Let \( \mathcal{L}_i \) be the set of all lines in \( \epsilon(L) \) spanned by the points of \( P_i \) \((i \geq 0)\). Since \( L \) contains two lines which are not coplanar, \( \epsilon(L) \) contains two which do not intersect. Since \( \bigcup_{i=0}^{\infty} \mathcal{P}_i = P \), the union of the \( \mathcal{L}_i \)'s, \( i \geq 0 \), contains all lines of \( \epsilon(L) \). Let \( i \in \mathbb{N} \) be minimal such that \( \mathcal{L}_i \) contains two lines, say \( \epsilon(h_1) \) and \( \epsilon(h_2) \) which do not intersect in a point of \( P \). If \( i = 0 \), we have the configuration shown in Fig. 2. Then \( l_1, l_2, h_1, h_2 \) are lines in \( L \) such that any two, except \( h_1 \) and \( h_2 \) are coplanar, i.e., \( L \) does not satisfy the bundle condition.

Suppose now that \( i > 0 \). Since \( i \) is minimal, either \( \epsilon(h_1) \) or \( \epsilon(h_2) \) is in \( \mathcal{L}_i \setminus \mathcal{L}_{i-1} \). Suppose \( \epsilon(h_1) \in \mathcal{L}_i \setminus \mathcal{L}_{i-1} \).

![Fig. 2.](image-url)
Case 1. $\varepsilon(h_2) \in \mathcal{L}_{i-1}$

Let $p_1, p_2 \in \mathcal{P}_i$ be two points on $\varepsilon(h_1)$. Choose any lines $\varepsilon(l_1)$ and $\varepsilon(l_2)$ in $\mathcal{L}_{i-1}$ containing $p_1$ and $p_2$, resp. Since $i$ is minimal, these two lines both intersect each other and the line $\varepsilon(h_2)$. This yields a configuration as indicated in Fig. 2 and the claim follows.

Case 2. $\varepsilon(h_2) \in \mathcal{L}_i \setminus \mathcal{L}_{i-1}$

Let $p_1, p_2 \in \mathcal{P}_i$ be two points on $\varepsilon(h_1)$ and choose $\varepsilon(l_1)$ and $\varepsilon(l_2)$ as in Case 1. Then $\varepsilon(l_1)$ and $\varepsilon(l_2)$ intersect (since $i$ is minimal). If $\varepsilon(l_1)$ or $\varepsilon(l_2)$ does not intersect $\varepsilon(h_2)$, Case 1 applies. Thus assume that $\varepsilon(l_1)$ and $\varepsilon(l_2)$ both intersect $\varepsilon(h_2)$. If they do so in different points, we encounter a configuration as in Fig. 2. Thus suppose that $\varepsilon(l_1)$, $\varepsilon(l_2)$ and $\varepsilon(h_2)$ are concurrent in a single point, say $q_1 \in \mathcal{P}$. Choose any other point $q_2 \in \mathcal{P}_i$ on $\varepsilon(h_2)$ (see Fig. 3). Choose any line $\varepsilon(l_3)$ in $\mathcal{L}_{i-1}$ containing $q_2$. If this does not intersect $\varepsilon(h_1)$, Case 1 applies. Otherwise it either intersects $\varepsilon(l_1)$ in a point different from $p_1$ or it intersects $\varepsilon(l_2)$ in a point different from $p_2$. In any case, a configuration as in Fig. 3 occurs, which proves the claim.

Thus one is left to prove (S) for the class of rank 4 geometries which do not satisfy the bundle condition. By Lemma 7 and Proposition 2, we can further restrict this class to those geometries for which additionally every rank 3 minor $[p, 1]$ is modular (the latter condition says that if any two copoints intersect in a point, they intersect in a line). Finally, we may impose that at least two of any three copoints intersect in a line (otherwise the geometry is easily seen to be nonsticky). We conjecture that this class is empty.

Remark 1. There is a related conjecture due to Kantor [7], claiming that if in a finite rank 4 geometry any two copoints intersect in a line, then it satisfies the bundle condition. (We have tried hard, but so far without success.)

Remark 2. There is an interesting class of geometric lattices with the Intersection Property. It can be defined as follows:

Definition. Let $L$ and $L'$ be geometric lattices of equal rank. Then $L'$ is said to be an adjoint of $L$ if there exists an order-reversing injective map $L \to L'$.

Fig. 3.
mapping the copoints of $L$ onto the points of $L'$ (cf. [4, 2] for further information).

It is easy to see (cf. [2]), that if $L$ has an adjoint, then $L$ has the Intersection Property. For example, if $L(r, n)$ is the uniform geometric lattice on $n$ points of rank $r$, then the Dilworth truncation of the Boolean algebra $B_n$ at level $n - r$ is an adjoint of $L(r, n)$. Thus in particular, $L(r, n)$ has the Intersection Property and hence, by Lemma 6, $L(r, n)$ is sticky if and only if it is modular. From this it is immediate, that $L(r, n)$ is sticky if and only if $r = n$ or $r \leq 2$, which is the main result in [9].

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References