WEAKLY CONTINUOUS AND C2-RINGS

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Abstract

A ring R is called right weakly continuous if the right annihilator of each element is essential in a summand of R, and R satisfies the right C2-condition (every right ideal that is isomorphic to a direct summand of R is itself a direct summand). We show that a ring R is right weakly continuous if and only if it is semiregular and $J(R) = Z(R_R)$. Unlike right continuous rings, these right weakly continuous rings form a Morita invariant class. The rings satisfying the right C2-condition are studied and used to investigate two conjectures about strongly right Johns rings and right FGF-rings and their relation to quasi-Frobenius rings.

A ring R is called semiregular if $R/J(R)$ is regular and idempotents lift modulo $J(R)$. A well known result of Utumi [19] asserts that if R is a right selfinjective ring (indeed a right continuous ring) then R is semiregular and $J(R) = Z(R_R)$. In this paper we investigate how much of a converse there is to Utumi's result. We show that a ring R is semiregular with $J(R) = Z(R_R)$ if and only if the right annihilator of every element is essential in a direct summand of R, and every right ideal that is isomorphic to a direct summand of R is itself a summand (R is a right C2-ring). Consequently we call such a ring right weakly continuous, and we show that R is right weakly continuous if and only if it is $Z(R_R)$-semiregular, a natural generalization of semiregularity. Finally, we show that, unlike the right continuous rings, the right weakly continuous rings form a Morita invariant class.

Next, we investigate the right C2-rings and their connection with the FGF-conjecture, which asserts that every right FGF-ring is quasi-Frobenius. Here a ring R is called an FGF-ring if every finitely generated right R-module can be embedded in a free module. We show that the conjecture is true if every right FGF-ring is a right C2-ring. We also show that every right FP-injective, right FGF-ring is quasi-Frobenius. In addition, the right C2-rings are related to the Johns conjecture. A ring R is called strongly right Johns [8] if it is right noetherian and every right ideal in each matrix ring over R is an annihilator, and it is an open question whether every strongly right Johns ring is quasi-Frobenius. We show that every strongly right Johns, right C2-ring is quasi-Frobenius.

Throughout this paper the ring R is always associative with unity and all R-modules are unital. We write $J = J(R)$ for the Jacobson radical of R. If M_R is a right R-module, we write $Z(M)$ and $soc(M)$, respectively, for the singular submodule and the socle of M. For a ring R, we write $soc(R_R) = S_r$, $soc(_RR) = S_l$, $Z(R_R) = Z_r$ and $Z(_RR) = Z_l$. The notations $N \subseteq^{ess} M$ and $N \subseteq^{max} M$ mean that N is an essential, (respectively maximal) submodule of M. We write $M_n(R)$

for the ring of $n \times n$ matrices over R. Right annihilators will be denoted as $\mathtt{r}_X(Y) = \{x \in X \mid yx = 0$ for all $y \in Y\}$, with a similar definition of left annihilators $\mathtt{l}_X(Y)$.

1. I-Semiregular Rings

An element a in a ring R is called regular if $aca = a$ for some $c \in R$, and R itself is called (von Neumann) regular if every element is regular. We begin with a weakening of the regularity condition.

Lemma 1.1. *Let I be an ideal of a ring R. The following are equivalent for $a \in R$:*

(1) *There exists $e^2 = e \in aR$ with $a - ea \in I$.*
(2) *There exists $e^2 = e \in aR$ with $aR \cap (1-e)R \subseteq I$.*
(3) *$aR = eR \oplus S$ where $e^2 = e$ and $S \subseteq I$ is a right ideal.*

Proof. Given (1) we have $aR = eR \oplus [aR \cap (1-e)R]$. If $x \in aR \cap (1-e)R$ then $x = (1-e)x \in (1-e)aR \subseteq I$. This proves (2). If (2) holds, (3) follows with $S = aR \cap (1-e)R$. Finally, given (3) write $a = er + s$. Then $a - ea = s - es \in I$ because $S \subseteq I$, proving (1). □

If I is an ideal of a ring R, an element $a \in R$ is called **right I-semiregular** if the conditions in Lemma 1.1 are satisfied, and R is called a **right I-semiregular ring** if every element is right I-semiregular. **Left I-semiregular** elements and rings are defined analogously. A ring is called *semiregular* [13] if it is right (equivalently left) J-semiregular. In particular, if R is semiregular then R is left and right I-semiregular for every ideal $I \supseteq J$. In general we have

Theorem 1.2. *The following conditions are equivalent for an ideal I of a ring R :*

(1) *R is right I-semiregular.*
(2) *For all finitely generated right ideals $T \subseteq R$, there exists $e^2 = e \in T$ with $T \cap (1-e)R \subseteq I$.*
(3) *For all finitely generated right ideals $T \subseteq R$, $T = eR \oplus S$ where $e^2 = e$ and $S \subseteq I$ is a right ideal.*

When these conditions are satisfied we have:

(i) $J \subseteq I$, $Z_r \subseteq I$ and $Z_l \subseteq I$.
(ii) *R/I is regular and idempotents can be lifted modulo I.*
(iii) *For all $a \in R$, there exists a regular element $d \in R$ with $a - d \in I$.*
(iv) *Every right ideal of R that is not contained in I contains an idempotent not in I.*

Proof. (1)⇒(2). We induct on n where $T = a_1R + \cdots + a_nR$. If $n = 1$ there is nothing to prove by (1). If $n \geq 2$ then (1) gives $f^2 = f \in a_1R$ with $(1-f)a_1R \subseteq I$. Write $K = (1-f)a_2R + \cdots + (1-f)a_nR$. Since $f \in a_1R$ we have $T = a_1R + K$. By induction, choose $g^2 = g \in K$ such that $(1-g)K = K \cap (1-g)R \subseteq I$. Then $fg = 0$ because $g \in K$, so $e = f + g - gf$ is an idempotent and $e \in T$. Thus it remains to verify that $T \cap (1-e)R = (1-e)T \subseteq I$. But $(1-e) = (1-g)(1-f)$ and $(1-f)K = K$, so

$$(1-e)T \subseteq (1-g)(1-f)a_1R + (1-g)(1-f)K \subseteq (1-g)I + (1-g)K \subseteq I.$$

This proves (2).

(2)⇒(3). Given (2) take $S = T \cap (1-e)R$.

(3)⇒(1). This is clear by Lemma 1.1.

Suppose these conditions hold. Then (iv) follows from Lemma 1.1, and (i) follows from (iv). Write $\bar{r} = r + I$ for each $r \in R$. If $a \in R$, (1) gives $e^2 = e \in aR$ such that $a - ea \in I$. If $e = ab$ then $\bar{a} = \bar{e}\bar{a} = \bar{a}\bar{b}\bar{a}$, so R/I is regular. If in addition $a^2 - a \in I$ then $e - ae = ab - a^2b = (a - a^2)b \in I$. If $f = e + ea - eae$ then $f^2 = f \in aR$ and $\bar{f} - \bar{a} = \bar{e} + \bar{e}\bar{a} - \bar{e}(\bar{a}\bar{e}) - \bar{a} = \bar{e} + \bar{e}\bar{a} - \bar{e}^2 - \bar{a} = \bar{e}\bar{a} - \bar{a} = \bar{0}$. This proves (ii). Furthermore, $a - aba = a - ea \in I$ and $(aba)b(aba) = aba$. Hence (iii) follows with $d = aba$. □

Example 1.3. If we take $R = \mathbb{Z}$ and $I = 2\mathbb{Z}$, then R/I is regular (a field) and both idempotents 0 and 1 lift modulo I. Moreover if $n \in R$ then $n - d \in I$ where $d = 0$ or 1 according as n is even or odd. Hence R satisfies (ii) and (iii) in Theorem 1.2, but R is not right I-semiregular since $e \in 3\mathbb{Z}$ and $3 - 3e \in I$ is impossible for $e^2 = e$ because $e = 0$ or $e = 1$. □

The next result shows that if $J \subseteq I$ then (ii) and (iii) are equivalent to right (and left) I-semiregularity. However this implies that $I = J$ and R is semiregular.

Proposition 1.4. *Let I be an ideal of a ring R. If $I \subseteq J$, the following are equivalent*:

(1) *R is right I-semiregular.*

(2) *R is left I-semiregular.*

(3) *For all $a \in R$, there exists a regular element $d \in R$ with $a - d \in I$.*

(4) *R/I is regular and idempotents can be lifted modulo I.*

When this is the case, $I = J$.

Proof. Since (3) and (4) are left-right symmetric, we prove (1)⇔(3) and (1)⇔(4). We have (1)⇒(3) and (1)⇒(4) by Theorem 1.2.

(3)⇒(1). By (3) let $a - d \in I$ where d is regular, say $dcd = d$. Write $f = cd$ so $f^2 = f$ and $df = d$. Then $a - af = a(1 - f) = (a - d)(1 - f) \in I$. Moreover, $f - ca = c(d - a) \in I \subseteq J$ so let $u(1 - f + ca) = 1$ where $u \in R$. This gives $fucaf = f$, whence $af(uc)af = af$. Thus $a - af \in I$ and af is regular, so we may assume that $d \in aR$. Hence let $a - d \in I$ where $d \in aR$ and $drd = d$. Now consider $e = dr$. Then $e^2 = e \in aR$ and, since $ed = d$, we have $a - ea = (1 - e)a = (1 - e)(a - d) \in I$.

(4)⇒(1). If $a \in R$ let $a - aba \in I$ where $b \in R$. Hence $ba - (ba)^2 \in I$ so by (4) choose $f^2 = f$ such that $f - ba \in I \subseteq J$. Thus $1 - f + ba = u$ is a unit, so that $fba = fu$. It follows that $e = au^{-1}fb \in aR$ is an idempotent. Writing $\bar{r} = r + I$ in R/I, we have $\bar{u} = \bar{1}$ and $\bar{a}\bar{f} = \bar{a}\bar{b}\bar{a} = \bar{a}$, so $\bar{a} - \bar{e}\bar{a} = \bar{a} - \bar{a}\bar{f}\bar{b}\bar{a} = \bar{a} - \bar{a}\bar{b}\bar{a} = \bar{0}$. Thus $a - ea \in I$, proving (1).

Finally, since we are assuming that $I \subseteq J$, we have $I = J$ by Theorem 1.2. □

Note that Proposition 1.4 shows that either right or left J-semiregularity is equivalent to semiregularity, as mentioned earlier.

2. Right Weakly Continuous Rings

Utumi identified two conditions enjoyed by any right selfinjective ring R :

C1 Every right ideal is essential in a summand of R_R.

C2 If a right ideal T is isomorphic to a summand of R_R then T is a summand.

A ring R is called right *continuous* if it satisfies C1 and C2. If R satisfies only C1 it is called a right *CS-ring*. Our interest is mainly in the right **C2-rings**, that is rings R where R_R satisfies the C2-condition. We need the following well known fact (see [10, Proposition 1.19]).

Lemma 2.1. *Let $K_R \subseteq P_R$ be modules where P is projective. Then $K \subseteq^{ess} P$ if and only if P/K is singular. In particular, if P_R is both projective and singular, then $P = 0$.*

Proposition 2.2. *The following are equivalent for an element $a \in R$:*
(1) *$\mathtt{r}(a) \subseteq^{ess} fR$ for some $f^2 = f \in R$.*
(2) *$aR = P \oplus S$ where P_R is a projective right ideal and S_R is a singular right ideal.*

Proof. (1)⇒(2). Let $\mathtt{r}(a) \subseteq^{ess} (1-e)R$ where $e^2 = e \in R$.
Claim. $aR = aeR \oplus a(1-e)R$.
Proof. Clearly $aR = aeR + a(1-e)R$. If $x \in aeR \cap a(1-e)R$ write $x = aer = a(1-e)s$ where $r, s \in R$. Then $er - (1-e)s \in \mathtt{r}(a) \subseteq (1-e)R$, so $er = 0$. Hence $x = aer = 0$, proving the Claim.
Now $aeR \cong eR$ because the multiplication map $a\cdot : eR \to aeR$ has kernel $\{er \mid aer = 0\} = eR \cap \mathtt{r}(a) = 0$. Hence aeR is projective. Finally, $a\cdot : (1-e)R \to a(1-e)R$ has kernel $(1-e)R \cap \mathtt{r}(a) = \mathtt{r}(a)$. Hence $a(1-e)R \cong (1-e)R/\mathtt{r}(a)$, so $a(1-e)R$ is singular by Lemma 2.1 because $\mathtt{r}(a) \subseteq^{ess} (1-e)R$.

(2)⇒(1). Suppose that $aR = P \oplus S$ as in (2), and let $\pi : aR \to P$ be the projection with $ker(\pi) = S$. Then define $\gamma : R \to P$ by $\gamma(r) = \pi(ar)$, and write $K = ker(\gamma)$. Then γ is onto so, as P is projective, $K = fR$ for some $f^2 = f \in R$. Clearly $\mathtt{r}(a) \subseteq fR$; it remains to verify that $\mathtt{r}(a) \subseteq^{ess} fR$. If $k \in K$ then $ak \in S$ because $\pi(ak) = \gamma(k) = 0$. Hence we have a map $\theta : K \to S$ defined by $\theta(k) = ak$. Then $ker(\theta) = K \cap \mathtt{r}(a) = \mathtt{r}(a)$ so $K/\mathtt{r}(a) \cong im(\theta) \subseteq S$. Thus $K/\mathtt{r}(a)$ is singular so, since K is projective, it follows that $\mathtt{r}(a) \subseteq^{ess} K$ by Lemma 2.1. □

Remark. In fact θ is epic in (2)⇒(1) so $K/\mathtt{r}(a) \cong S$. Indeed, if $s \in S$ let $s = ar$. Then $\gamma(r) = \pi(ar) = 0$ because $S = ker(\pi)$, so $r \in ker(\gamma) = K$. Thus $s = ar = \theta(r)$.

Since a ring R is called a right CS-ring if every right ideal is essential in a summand of R, we call a ring R a right **ACS-ring** (for annihilator-CS) if every element $a \in R$ satisfies the conditions in Proposition 2.2. (Note that we are employing the convention that $0 \subseteq^{ess} 0$.) In order to prove our main theorem, we need the following observation.

Lemma 2.3. *If R is a right C2-ring then $Z_r \subseteq J$.*

Proof. If $a \in Z_r$ then, $\mathtt{r}(1-a) = 0$ because $\mathtt{r}(\ a) \cap \mathtt{r}(1-a) = 0$, so $(1-a)R \cong R$. Hence $(1-a)R$ is a direct summand of R_R by hypothesis, whence $R(1-a)$ is a summand, say $R(1-a) = Rf$, $f^2 = f$. But then $(1-f) \in \mathtt{r}(1-a) = 0$, so $R(1-a) = R$. Since this holds for every $a \in Z_r$, we have $Z_r \subseteq J$. □

Note that the converse of Lemma 2.3 is false: Consider the localization $\mathbb{Z}_{(p)}$ of the ring of integers at the prime p.

Theorem 2.4. *The following are equivalent for a ring R:*
(1) *R is semiregular and $J = Z_r$.*
(2) *R is right Z_r-semiregular.*
(3) *If T is a finitely generated* (*respectively principal*) *right ideal, then $T = eR \oplus S$ where $e^2 = e \in R$ and S is a singular right ideal.*
(4) *R is a right ACS-ring and every finitely generated* (*respectively principal*) *projective right ideal is a summand.*

(5) *R is a right ACS-ring which is also a right C2-ring.*

Proof. (1)⇒(2)⇒(3). These follow from Theorem 1.2.

(3)⇒(4). If $a \in R$, taking $T = aR$ in (3) shows that R is a right ACS-ring by Proposition 2.2. If T is a finitely generated (principal), projective right ideal of R, write $T = eR \oplus S$ as in (3). Then S_R is both singular and projective, so $S = 0$ by Lemma 2.1.

(4)⇒(5). To verify the right C2-condition, let T be a right ideal of R which is isomorphic to a summand of R_R. Then T is projective and principal, so T is a summand by (4), as required.

(5)⇒(1). Let $a \in R$. Since R is a right ACS-ring let $aR = P \oplus S$ where P is projective and S is singular. Thus P is isomorphic to a summand of R_R (being projective) so the C2-condition ensures that $P = eR$ where $e^2 = e$. Since S is singular we have $S \subseteq Z_r$, and $Z_r \subseteq J$ by the C2-condition (Lemma 2.3). Thus $S \subseteq J$, proving that R is semiregular. Finally, if $a \in J$ then $e^2 = e \in J$, so $e = 0$ and $aR = S$ is singular. Hence $a \in Z_r$, proving that $J \subseteq Z_r$. This proves (1). □

We call a ring right **weakly continuous** if it satisfies the conditions in Theorem 2.4. The name comes from the fact that Condition (5) in Theorem 2.4 is a weakening of continuity.

Examples. (1) Utumi [19] proved that every right continuous ring (and hence every right selfinjective ring) is right weakly continuous.

(2) Every regular ring is right and left weakly continuous (with $J = Z_r = Z_l = 0$). Moreover, it is easy to see that, for a right weakly continuous ring R, R is regular if and only if R is a right semihereditary, if and only if R is a right PP-ring (every principal right ideal is projective).

(3) A ring R is called left *Kasch* if every simple left R-module embeds in R, and these rings satisfies the right C2-condition by [20, Lemma 1.15]. Hence every left Kasch, right ACS-ring is right weakly continuous by Theorem 2.4. It is easy to see that left Kasch, left PP-rings are semisimple; this conclusion remains true if we replace left PP by right PP. Indeed, such a ring is right weakly continuous. But $Z_r = 0$ because R is a right PP-ring, so $J = 0$ and R is regular by Theorem 2.4. Hence $Z_l = 0$ so R is semisimple.

(4) If 0 and 1 are the only idempotents in R, then R is right weakly continuous if and only if it is local with $J = Z_r$. Call a ring *I-finite* if it contains no infinite set of orthogonal idempotents. Then the I-finite, right weakly continuous rings are precisely the semiperfect rings with $J = Z_r$.

(5) The ring $\mathbb{Z}$ of integers is an example of a commutative, noetherian ACS-ring which is not weakly continuous, while the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at the prime p is an example of a commutative, local (hence semiregular) ACS-ring which is not weakly continuous.

(6) A direct product $R = \Pi_{i \in I} R_i$ of rings R_i is right weakly continuous if and only if each R_i is right weakly continuous.

(7) A ring R is called right *principally injective* (right *P-injective*) if, for each $a \in R$, every R-linear map $aR \to R$ extends to R (equivalently if $\mathtt{lr}(a) = Ra$). These rings are right C2-rings by [15, Theorem 1.2] and satisfy $J = Z_r$ by [15, Theorem 2.1]. Hence every right P-injective, right ACS-ring is right weakly continuous by Theorem 2.4. Note, however, that if F is a field then $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ is a right and left artinian, right and left ACS ring which is neither right nor left weakly continuous because $Z_r = 0 = Z_l$ while $J \neq 0$.

(8) If ${}_RV_R$ is a bimodule over a ring R, the *trivial extension* of R by V is the direct sum $T(R, V) = R \oplus V$ with multiplication $(r + v)(r' + v') = rr' + (rv' + vr')$. The trivial extension

$R = T(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is a commutative P-injective ring, and so is a C2-ring with $J = Z_r$, but it is not weakly continuous because it is not semiregular (in fact $R/J \cong \mathbb{Z}$.)

Example 2.5. The following ring R is a right and left artinian, left continuous, right weakly continuous ring which is not right continuous. The example is due essentially to Björk [2, Page 70].

Given a field F and an isomorphism $a \mapsto \bar{a}$ from $F \to \bar{F} \subseteq F$, let R be the left F-space on basis $\{1, t\}$ with multiplication given by $t^2 = 0$ and $ta = \bar{a}t$ for all $a \in F$. Then R is a local ring, and the only left ideals are 0, $J = Ft$ and R. Hence R is left artinian, left continuous and right principally injective, and so has $J = Z_r$ by [14, Theorem 2.1]. Thus R is right weakly continuous and we claim that it is not right continuous if $dim_{\bar{F}}(F) \geq 2$. Indeed, if R were right continuous then, being local, it would be right uniform. But if X and Y are nonzero $\bar{F}$-subspaces of F with $X \cap Y = 0$ then $P = Xt$ and $Q = Yt$ are nonzero right ideals with $P \cap Q = 0$. □

Right continuity is not a Morita invariant; in fact [19] the matrix ring $M_2(R)$ is right continuous if and only if R is right selfinjective. However, semiregularity is a Morita invariant by [13, Corollary 2.8] as is the condition that $J = Z_r$ by [21, Lemma 1]. Hence we have

Theorem 2.6. *Being right weakly continuous is a Morita invariant property of rings.*

Corollary 2.7. *The following are equivalent for a ring R :*

(1) *R is right selfinjective.*

(2) *R is right weakly continuous and $R \oplus R$ is CS as a right module.*

(3) *R is a right C2-ring and $R \oplus R$ is CS as a right module.*

Proof. (1)⇒(3) is clear, and (3)⇒(2) because summands of CS-modules are CS.

(2)⇒(1). If the R is right weakly continuous, so also is $M_2(R) \cong end(R \oplus R)$ by Theorem 2.6. In particular $end(R \oplus R)$ is a right C2-ring, and we will show in Theorem 3.8 below that this implies that $R \oplus R$ has the right C2-condition. Hence $R \oplus R$ is continuous, and this implies that R is right selfinjective by a theorem of Utumi [19]. □

We conclude this section with a brief discussion of right ACS-rings. This class of rings includes all domains, right uniform rings, right CS-rings; moreover every regular ring is a right and left ACS-ring. If 0 and 1 are the only idempotents in R, then R is a right ACS-ring if and only if every element $a \notin Z_r$ satisfies $r(a) = 0$. In particular, the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at the prime p is a commutative, local (hence semiregular) ACS-ring in which $Z_r \neq J$.

A direct product $R = \Pi_{i \in I} R_i$ of rings is a right ACS-ring if and only if each R_i is a right ACS-ring.

A ring R is a right *PP-ring* if and only if $\mathbf{r}(a)$ is a direct summand of R for every $a \in R$. Hence the right PP-rings are precisely the right nonsingular, right ACS-rings. A result of Small [18] shows that an I-finite, right PP-ring R is a Baer ring, that is every left (equivalently right) annihilator is generated by an idempotent. (In particular R is left PP and has ACC and DCC on right and left annihilators.) Small's Theorem is the nonsingular case of the next result.

Proposition 2.8. *Let R be a right ACS-ring.*

(1) *Every left annihilator $L \nsubseteq Z_r$ contains a nonzero idempotent.*

(2) *If R is I-finite, every left annihilator L has the form $L = Re \oplus S$ where $e^2 = e$ and ${}_RS \subseteq Z_r$.*

Proof. (1). If $L = \mathtt{l}(X)$, choose $a \in L$, $a \notin Z_r$. By hypothesis, $\mathtt{r}(a) \subseteq^{ess} eR$ where $e^2 = e$, and $e \neq 1$ because $a \notin Z_r$. Hence $X \subseteq \mathtt{r}(a) \subseteq eR$, so $0 \neq (1-e) \in \mathtt{l}(X) = L$.

(2). If $L \subseteq Z_r$ take $e = 0$ and $S = L$. Otherwise use (1) and the I-finite hypothesis to choose e maximal in $\{e \mid 0 \neq e^2 = e \in L\}$, where $e \leq f$ means $e \in fRf$. Then $L = Re \oplus [L \cap R(1-e)]$ so it suffices to show that $L \cap R(1-e) \subseteq Z_r$. If not let $0 \neq f^2 = f \in L \cap R(1-e)$ by (1). Then $fe = 0$ so $g = e + f - ef$ satisfies $g^2 = g \in L$ and $g \geq e$. Thus $g = e$ by the choice of e, so $f = ef$ and $f = f^2 = f(ef) = 0$, a contradiction. □

The proof of Proposition 2.2 goes through as written to prove the following module theoretic version.

Lemma 2.9. *If M_R is a module, the following conditions are equivalent for $m \in M$:*
(1) *$\mathtt{r}(m) \subseteq^{ess} eR$ for some $e^2 = e \in R$.*
(2) *$mR = P \oplus S$ where P_R is projective and S_R is singular.*

If R is a ring, we say that a right R-module M_R is an **ACS-module** if the conditions in Lemma 2.9 are satisfied for every element $m \in M$. Hence a ring R is a right ACS-ring if and only if R_R is an ACS-module. The next result gives a similar characterization of the right CS-rings.

Proposition 2.10. *A ring R is a right CS-ring if and only if every principal right R-module is an ACS-module.*

Proof. If R is a right CS-ring, let $m \in M_R$. Then $\mathtt{r}(m)$ is a right ideal of R so $\mathtt{r}(m) \subseteq^{ess} eR$ for some $e^2 = e \in R$ by the CS-condition. Conversely, let T be a right ideal of R and write $M = R/T = mR$ where $m = 1 + T$. By hypothesis $M = P \oplus S$ where P_R is projective and S_R is singular. Hence Lemma 2.9 gives $\mathtt{r}(m) \subseteq^{ess} eR$ for some $e^2 = e \in R$, and we are done because $\mathtt{r}(m) = T$. □

3. C2-Rings

In this section we study the C2-rings and use the results to make some progress on two conjectures about quasi-Frobenius rings. Recall that a ring R is a right C2-ring if every right ideal that is isomorphic to a direct summand of R_R is itself a direct summand.

Examples. (1) Every right weakly continuous ring is a right C2-ring; every regular ring is a right and left C2-ring.

(2) If 0 and 1 are the only idempotents in R then R is a right C2-ring if and only if every monomorphism $R_R \to R_R$ is epic, equivalently $\mathtt{r}(a) = 0$, $a \in R$, implies that a is a unit. In particular, the only C2-domains are the division rings.

(3) If R is an I-finite C2-ring then every monomorphism $R_R \to R_R$ is epic. Indeed, if $\mathtt{r}(a) = 0$, $a \in R$ then $R \supseteq aR \supseteq a^2R \supseteq a^3R \supseteq \cdots$ and each $a^kR = e_kR$ for some $e_k^2 = e_k$ because $a^kR \cong R$. Hence $a^kR = a^{k+1}R$ by the I-finite hypothesis, whence $R = aR$ because $\mathtt{r}(a) = 0$. Hence monomorphisms $R_R \to R_R$ are epic. (A similar argument shows that if M_R is a C2-module and

$end(M)$ is I-finite, then monomorphisms $M \to M$ are epic.) Note that any regular ring is a right C2-ring and monomorphisms $R_R \to R_R$ are epic, but it need not be I-finite. However, we do not know if I-finite rings in which monomorphisms $R_R \to R_R$ are epic must be right C2-rings.

(4) Every right P-injective ring is a right C2-ring by [15, Theorem 1.2]. The converse is false: If V is a two-dimensional vector space over a field F, the trivial extension $R = T(F, V) = F \oplus V$ is a commutative, local, artinian C2-ring (see Corollary 3.5) with $J^2 = 0$ and $J = Z_r$, but R is not P-injective. Indeed, if $V = vF \oplus wF$, let $\theta : V \to V$ be a linear transformation with $\theta(v) = w$. Then $(0, x) \mapsto (0, \theta(x))$ is an R-linear map from $(0, v)R \to R$ which does not extend to $R \to R$ because $w \notin vF$.

(5) Every left Kasch ring is a right C2-ring by [20, Lemma 1.15], but the converse is not true (consider any regular, right selfinjective ring that is not semisimple).

Example 3.1. There exists a left C2-ring R that is not a right C2-ring, and hence not left Kasch. Faith and Menal [7] give an example of a right noetherian ring R in which every right ideal is an annihilator, but which is not right artinian. Thus R is left P-injective ring, and hence is a left C2-ring. But R is not right C2 because it would then be right artinian by Theorem 4.5 below. □

Example 3.2. The trivial extension $R = T(\mathbb{Z}, \mathbb{Z}_{2^\infty})$ is a commutative CS-ring with $Z_r = J \neq 0$ which does not satisfy the C2-condition. In fact, R has simple essential socle. □

We begin by deriving some of the basic characterizations of the right C2-rings.

Proposition 3.3. *The following conditions are equivalent for a ring R :*

(1) *R is a right C2-ring.*
(2) *Every R-isomorphism $aR \to eR$, $a \in R$, $e^2 = e \in R$, extends to $R \to R$.*
(3) *If $\mathtt{r}(a) = \mathtt{r}(e)$, $a \in R$, $e^2 = e \in R$, then $e \in Ra$.*
(4) *If $\mathtt{r}(a) = \mathtt{r}(e)$, $a \in R$, $e^2 = e \in R$, then $Re = Ra$.*
(5) *If $Ra \subseteq Re \subseteq \mathtt{lr}(a)$, $a \in R$, $e^2 = e \in R$, then $Re = Ra$.*
(6) *If aR is projective, $a \in R$, then aR is a direct summand of R_R.*

Proof. (6)⇒(1)⇒(2)⇒(3)⇒(4)⇒(5) are routine computations. Assume that (5) holds. If aR is projective then $\mathtt{r}(a)$ is a direct summand of R, say $\mathtt{r}(a) = \mathtt{r}(e)$ for $e^2 = e$. Thus $a = ae$, so $Ra \subseteq Re$. But $e \in \mathtt{lr}(a)$ (because $\mathtt{r}(a) \subseteq \mathtt{r}(e)$) so we have $Ra \subseteq Re \subseteq \mathtt{lr}(a)$. Thus $Ra = Re$ by (5), so Ra is a direct summand of R, whence aR is a summand, proving (6). □

Condition (3) of Proposition 3.3 gives

Corollary 3.4. *The direct product $\Pi_i R_i$ of rings R_i is a right C2-ring if and only if each R_i is a right C2-ring.*

Corollary 3.5. *The following conditions are equivalent for a local ring R :*

(1) *R is a right C2-ring.*
(2) *Every monomorphism $R_R \to R_R$ is epic.*
(3) *$J = \{a \in R \mid \mathtt{r}(a) \neq 0\}$.*

In particular, any local ring with nil radical is a right and left C2-ring.

Proof. We have already observed that (1)⇒(2). Given (2), it is clear that $J \subseteq \{a \in R \mid \mathtt{r}(a) \neq 0\}$; this is equality in a local ring. Hence (2)⇒(3). Finally, if (3) holds, suppose $\mathtt{r}(a) = \mathtt{r}(e)$, $a \in R$, $e^2 = e \in R$. By Proposition 3.3 we must show that $e \in Ra$. This is clear if $e = 0$. If $e = 1$ then $\mathtt{r}(a) = 0$ so $a \notin J$ by (3). Hence $Ra = R$ because R is local, and so $e \in Ra$ as required. This proves that (3)⇒(1). Finally the last statement follows from (3) because R is local. □

Corollary 3.6. *If R is a right C2-ring, so is fRf for any $f^2 = f \in R$ such that $RfR = R$.*

Proof. Write $S = fRf$ and suppose that $\mathtt{r}_S(a) = \mathtt{r}_S(e)$, $a \in S$, $e^2 = e \in S$. We must show that $e \in Sa$. It suffices to show that $e \in Ra$ so (by hypothesis) we show that $\mathtt{r}_R(a) = \mathtt{r}_R(e)$. If $r \in \mathtt{r}_R(a)$ then, for all $x \in R$, $a(frxf) = arxf = 0$ so $frxf \in \mathtt{r}_S(a) = \mathtt{r}_S(e)$. Thus $erxf = 0$ for all $x \in R$ so, as $RfR = R$, $er = 0$. Thus $\mathtt{r}_R(a) \subseteq \mathtt{r}_R(e)$; the other inclusion is proved in the same way. □

Proposition 3.6 is half of the proof that "right C2-ring" is a Morita invariant. We will characterize when this is true in Corollary 3.11 below.

We now turn our attention to C2-modules and their relationship to their endomorphism ring. We write $dim(M)$ for the uniform dimension of a module M.

Proposition 3.7. *Let M_R be a finite-dimensional module.*
(1) *If M has the C2-condition then monomorphisms in $end(M)$ are isomorphisms.*
(2) *In this case $end(M)$ is semilocal.*

Proof. If $\sigma : M \to M$ is monic, the C2-condition gives $M = \sigma(M) \oplus K$, $K \subseteq M$. If $K \neq 0$ then

$$dim(M) \geq dim[\sigma(M)] + dim(K) > dim[\sigma(M)] = dim(M)$$

a contradiction. Hence $K = 0$, and so σ is an isomorphism. This proves (1), and then (2) follows from a result of Camps and Dicks [5] because M is finite-dimensional. □

Proposition 3.8. *The following conditions are equivalent for a module M_R with $E = end(M_R)$.*
(1) *M_R has the C2-condition.*
(2) *If $\sigma : N \to P$ is an R-isomorphism where $N \subseteq M$ and P is a direct summand of M, then σ extends to some $\beta \in E$.*
(3) *If $\alpha : P \to M$ is R-monic where P is a direct summand of M, there exists $\beta \in E$ with $\beta \circ \alpha = \iota$, where $\iota : P \to M$ is the inclusion.*
(4) *If $\alpha : P \to M$ is R-monic where P is a direct summand of M, and if $\pi^2 = \pi \in E$ satisfies $\pi(M) = P$, there exists $\beta \in E$ with $\pi \circ \beta \circ \alpha = 1_P$.*

Proof. (1)⇒(2). If σ is as in (2), let $M = N \oplus N'$ by (1). Then $(n + n') \mapsto \sigma(n)$ extends σ.

(2)⇒(3). If α is as in (3) then $\sigma : \alpha(P) \to P$ is an R-isomorphism if we define $\sigma[\alpha(p)] = p$ for all $p \in P$. By (2) let $\beta \in E$ extend σ. Then $\beta \circ \alpha = \iota$.

(3)⇒(4). If α is as in (4), let $\beta \circ \alpha = \iota$ by (3) where $\beta \in E$. Then $\pi \circ \beta \circ \alpha = 1_P$.

(4)⇒(1). Suppose a submodule $N \subseteq M$ is isomorphic to P where P is a direct summand of M, say $\alpha : P \to N$ is an R-isomorphism. We must show that N is a direct summand of M. If $\pi^2 = \pi \in E$ satisfies $\pi(M) = P$, (4) provides $\beta \in E$ such that $\pi \circ \beta \circ \alpha = 1_P$. Define $\theta = \alpha \circ \pi \circ \beta \in E$.

Then $\theta^2 = \theta$ and $\theta(M) \subseteq N$, so we are done if we can show that $N \subseteq \theta(M)$. But $\theta \circ \alpha = \alpha$ so $N = \alpha(P) = \theta[\alpha(P)] \subseteq \theta(M)$, as required. $\square$

It is easy to verify that direct summands of a C2-module are again C2-modules. But the direct sum of C2-modules need not be a C2-module. If $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, $A = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ where F is a field, then $R_R = A \oplus B$ is not a C2-module because $B \cong J = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$, but both A_R and B_R are C2-modules (B_R is simple, and A_R has exactly one proper submodule $J \ncong A$).

Theorem 3.9. *Let M_R be a module and write $E = end(M_R)$. Then*:

(1) *If E is a right C2-ring then M_R has the C2-condition.*

(2) *The converse in* (1) *holds if $ker(\alpha)$ is generated by M whenever $\alpha \in E$ is such that $\mathtt{r}_E(\alpha)$ is a direct summand of E_E.*

Proof. (1). Let $\alpha : P \to M$ be R-monic where P is a direct summand of M, let $\pi^2 = \pi \in E$ satisfy $\pi(M) = P$, and write $ker(\pi) = Q$. Hence $M = P \oplus Q$ and we extend α to $\bar{\alpha} \in E$ by defining $\bar{\alpha}(p + q) = \alpha(p)$. Since α is monic, $ker(\bar{\alpha}) = Q = ker(\pi)$. It follows that

$$\mathtt{r}_E(\bar{\alpha}) = \{\lambda \in E \mid \lambda(M) \subseteq Q\} = \mathtt{r}_E(\pi)$$

Since E is a right C2-ring, Proposition 3.3 gives $\pi \in E\bar{\alpha}$, say $\pi = \beta \circ \bar{\alpha}$ with $\beta \in E$. Then $\pi \circ \beta \circ \alpha = 1_P$, and so M_R has the C2-property by Proposition 3.8.

(2) Let $\mathtt{r}_E(\alpha) = \mathtt{r}_E(\pi)$ where α and $\pi^2 = \pi$ are in E. By Proposition 3.3, we must show that $\pi \in E\alpha$.

Claim. $ker(\alpha) = ker(\pi)$.

Proof. $1 - \pi \in \mathtt{r}_E(\pi) = \mathtt{r}_E(\alpha)$, so $\alpha = \alpha \circ \pi$, whence $ker(\pi) \subseteq ker(\alpha)$. On the other hand, our hypothesis gives $ker(\alpha) = \Sigma\{\theta(M) \mid \theta \in E,\ \theta(M) \subseteq ker(\alpha)\}$. Since $\theta(M) \subseteq ker(\alpha)$ implies $\theta \in \mathtt{r}_E(\alpha) = \mathtt{r}_E(\pi)$, it follows that $\theta(M) \subseteq ker(\pi)$. This means $ker(\alpha) \subseteq ker(\pi)$, proving the Claim.

Now write $\pi(M) = P$ and $ker(\pi) = Q$. Then $P \cap ker(\alpha) = 0$ by the Claim so $\alpha_{|P}$ is monic. Since M has the C2-condition, Proposition 3.8 provides $\beta \in E$ such that $\beta \circ (\alpha_{|P}) = \iota$ where $\iota : P \to M$ is the inclusion. We claim that $\beta \circ \alpha = \pi$, which proves (2). If $q \in Q$ then $(\beta \circ \alpha)(q) = 0 = \pi(q)$ by the Claim; if $p \in P$ then $(\beta \circ \alpha)(p) = p = \pi(p)$. As $M = P \oplus Q$ this shows that $\pi = \beta \circ \alpha \in E\alpha$, as required. $\square$

Since a free module generates all of its submodules, we obtain

Corollary 3.10. *If M_R is free then M has the C2-condition if and only if $end(M_R)$ is a right C2-ring. In particular R^n has the right C2-condition if and only if $M_n(R)$ is a right C2-ring*

Question. Is "right C2-ring" a Morita invariant?

By Corollary 3.6, the answer is "yes" if we can show that $M_2(R)$ is a right C2-ring whenever R is a right C2-ring. Hence Corollary 3.10 gives

Corollary 3.11. *The following conditions are equivalent*:

(1) "*Right C2-ring" is a Morita invariant.*

(2) *If R is a right C2-ring then $(R \oplus R)_R$ has the C2-condition.*

Call a ring R a **strongly right C2-ring** if $M_n(R)$ is a right C2-ring for every $n \geq 1$, equivalently (by Corollary 3.10) if $(R^n)_R$ has the C2-condition for every $n \geq 1$. Every right weakly continuous ring is a strongly right C2-ring by Theorem 2.6. It is not difficult to check (using Corollary 3.6) that "strongly right C2-ring" is a Morita invariant.

4. Applications

If a ring R has the property that every right module can be embedded in a free right module, then R is quasi-Frobenius by a theorem of Faith and Walker (see [1, Theorem 31.9]). A ring R is called a right *FGF-ring* if every finitely generated right R-module can be embedded in a free module, and it is an open question whether every right FGF-ring is quasi-Frobenius (the *FGF-Conjecture*). The conjecture is known to be true if the ring is either right selfinjective [3] or left Kasch [12]. Since left Kasch rings are right C2-rings [20, Lemma 1.15], and since "left Kasch" is a Morita invariant, the following theorem extends both these results.

Theorem 4.1. *Suppose that R is a strongly right C2-ring and that every 2-generated right R-module embeds in a free module. Then R is quasi-Frobenius.*

Proof. Let $a \in E(R_R)$, where $E(M)$ denotes the injective hull of a module M. Since R is right FGF, let $\sigma : R + aR \to (R^n)_R$ be monic. Then $\sigma(R)$ is a summand of R^n by hypothesis because $\sigma(R) \cong R_R$, and so $\sigma(R)$ is a summand of $\sigma(R + aR)$. But $\sigma(R) \subseteq^{ess} \sigma(R + aR)$ because $R \subseteq^{ess} R + aR$. This implies that $a \in R$, and hence that $R = E(R_R)$. Hence R is right selfinjective so, since every principal right module embeds in a free module, R is quasi-Frobenius by [3, Theorem 2.5], using results of Osofsky [17]. □

Corollary 4.2. *Suppose that R is right weakly continuous and every 2-generated right R-module embeds in a free module. Then R is quasi-Frobenius.*

Proof. R is a strongly right C2-ring by Theorem 2.6. □

Since being an FGF-ring is a Morita invariant, Theorem 4.1 immediately gives the following simplification of what is required to prove the FGF-conjecture.

Theorem 4.3. *The following statements are equivalent:*

(1) *Every right FGF-ring is a right C2-ring.*

(2) *Every right FGF-ring is quasi-Frobenius.*

A module Q_R is called *FP-injective* if, whenever K is a finitely generated submodule of a free right R-module F, every R-linear mapping $K \to Q$ extends to F. Our interest is in the *right FP-injective rings,* that is the rings R for which R_R is FP-injective. Examples include regular and right selfinjective rings. The next result was formerly known only when R is a right selfinjective, right FGF-ring.

Theorem 4.4. *If R is a right FP-injective ring for which every 2-generated right module embeds in a free module, then R is quasi-Frobenius.*

Proof. Since FP-injectivity is a Morita invariant, each matrix ring $M_n(R)$ is right FP-injective, and hence is a right C2-ring by [15, Theorem 1.2]. Hence R is a strongly right C2-ring, and Theorem 4.1 completes the proof. □

A ring R is called a right *Johns ring* [7] if it is right noetherian and every right ideal is an annihilator, and R is called *strongly right Johns* [8] if the matrix ring $M_n(R)$ is right Johns for every $n \geq 1$. It is an open question whether or not strongly right Johns rings are quasi-Frobenius. A ring R is called a right *CEP-ring* if every cyclic right R-module can be essentially embedded in a projective module. These rings are known [9, Corollary 2.9] to be right artinian. In the next proposition we will show that right Johns rings and right CEP-rings are closely related. Moreover, we show in Theorem 4.6 that strongly right Johns, right C2-rings are quasi-Frobenius.

Theorem 4.5. *The following are equivalent for a ring R :*

(1) *R is a right CEP-ring.*

(2) *R is a right Johns, right C2-ring.*

Proof. (1)⇒(2). Given (1), R is right artinian by [9, Corollary 2.9] and so is right noetherian, and R satisfies the right C2-condition by [20, Proposition 1.10]. The right CEP-condition implies that $\mathtt{rl}(T) = T$ for every right ideal T of R.

(2)⇒(1). Given (2), R is a right noetherian, right C2-ring. Since R is I-finite, every monomorphism $\alpha : R_R \to R_R$ is epic. Since R is right finite dimensional, it follows from a theorem of Camps and Dicks [5] that R is semilocal. But J is nilpotent by [7, Lemma 2.2], so R is semiprimary, and hence right artinian by Hopkins' Theorem. Since R is right Johns we have $\mathtt{rl}(T) = T$ for every right ideal T of R, and hence R is a right CEP-ring by [14, Proposition 3.3]. □

A ring R is called right *mininjective* [16] if every R-linear map $K \to R_R$, where K is a simple right ideal of R, extends to $R \to R$.

Theorem 4.6. *The following are equivalent*:

(1) *R is a strongly right Johns, right C2-ring.*

(2) *R is a right Johns, right mininjective ring.*

(3) *R is quasi-Frobenius.*

Proof. (1)⇔(3). Given (1), R is a right CEP-ring by Theorem 4.5, and hence is right artinian by [9, Corollary 2.9]. Thus R is quasi-Frobenius by [8, Corollary 1.3]. Hence (1)⇒(3); the converse is obvious.

(2)⇔(3). Assume (2). Since R is right Johns, it is right noetherian and $\mathtt{rl}(T) = T$ for all right ideals T of R. In particular, R is left mininjective by [15, Lemma 1.1], so $S_r = S_l$. By [7, Lemma 2.2] $S_r \subseteq^{ess} R_R$, J is nilpotent and $J = \mathtt{l}(S_r)$. If $S_r = k_1R \oplus \cdots \oplus k_nR$ where each k_iR is simple, then $J = \mathtt{l}(S_r) = \cap_{i=1}^{n} \mathtt{l}(k_i)$. But each Rk_i is simple because R is right mininjective [16, Theorem 1.14], so R is semilocal, and hence semiprimary. By Hopkins' Theorem R is right artinian, and hence quasi-Frobenius by [16, Corollary 4.8]. This proves (2)⇒(3); the converse is clear. □

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