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Exact modules *)

By

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1. Introduction. We shall say that a left R-module M is exact if for all $f \in M^d = \operatorname{Hom}({}_RM, {}_RR)$ there are finite sets $\{f_i\} \subseteqq M^d$, $\{m_i\} \subseteqq M$ such that $f = \sum f_i m_i f$ where the maps act on the right. Examples include all finitely generated projective modules (take $\{f_i, m_i\}$ to be a dual basis.) The reason for calling these modules exact is that this condition is necessary and sufficient for a certain functor, denoted by $(\ )^0$, from R-mod to $\operatorname{End}({}_RM)$-mod to be exact. This functor is investigated in [3] and we shall describe some of its properties in Sect. 3. It is of interest to note that a weaker version of this condition, in which the set $\{m_i\}$ is replaced by a subset of M^{dd}, the double dual of M, characterizes locally projective modules (see [6]). In particular, when M is exact, M^d is a locally projective right R-module.

The main aim of this paper is to obtain characterizations for certain classes of rings via this notion of an exact module. This we do in Sect. 2. We show that R is left Noetherian if and only if every projective module is exact; R is a PP-ring if and only if every cyclic module is exact; R is a semihereditary ring if and only if every finitely generated module is exact; and R is hereditary Noetherian if and only if every module is exact.

In Sect. 3 a further property of the functor $(\ )^0$ is developed. Specifically, suppose that ${}_RP$ is a finitely generated projective module, $E = \operatorname{End}({}_RP)$, $N \in E$-mod and $M = P \otimes_E N$. Then we have two functors, both denoted by $(\ )^0$, one from R-mod to $\operatorname{End}({}_RM)$-mod and one from E-mod to $\operatorname{End}({}_EN)$-mod. We show that if A is an arbitrary E-module, so A^0 is and $\operatorname{End}({}_EN)$-module and $(P \otimes_E A)^0$ is an $\operatorname{End}({}_RM)$-module, then $A^0 \cong (P \otimes A)^0$ as $\mathbb{Z}$-modules. As a corollary (and using Sect. 2) we deduce that the properties of being left Noetherian, semihereditary, or hereditary Noetherian pass from R to $\operatorname{End}({}_RP)$, where P is a finitely generated projective module.

Throughout all rings have unity and modules are unital and, unless said otherwise, left sided.

2. Characterizations by exactness. We begin with a characterization of left Noetherian rings by free modules. For any index set I denote the direct sum of $|I|$ copies of R by $R^{(I)}$ and the direct product by R^I. The index set of natural numbers will be denoted by $\mathbb{N}$.

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Theorem 1. *The following are equivalent for a ring R:*

(1) *R is left Noetherian.*
(2) *Every free R-module is exact.*
(3) *The direct sum $R^{(\mathbb{N})}$ is exact.*
(4) *The direct sum $R^{(R)}$ is exact.*
(5) *Every projective R-module is exact.*

Proof. (1) $\Rightarrow$ (2). Let $F = R^{(I)}$ be a free module which we write as rows, so $F^d = R^I$ written as columns. Let $w \in R^I$ with components $w_\lambda \in R$, $\lambda \in I$. Put $L = \sum R w_\lambda$. Then L is finitely generated by say $w_1, \ldots, w_n$ with $\{1, \ldots, n\} \subseteqq I$. Hence $w_\lambda = \sum_{i=1}^{n} r_{\lambda i} w_i$. Denote the column matrix in R^I with λ-entry $r_{\lambda i}$ by f_i, denote by e_λ the row matrix in $R^{(I)}$ with, for a fixed λ, the λ entry 1 and all other entries 0. Then $w - \sum_{i=1}^{n} f_i w_i = \sum_{i=1}^{n} f_i e_i w$ and the module is exact.

(2) $\Rightarrow$ (3), (4). Clear.

(4) $\Rightarrow$ (1). Let L be a left ideal of R. Then $L = \sum R a_r$, $r \in R$. Define

$$f\colon R^{(R)} \to R \quad \text{by} \quad \langle x_r \rangle \to \sum x_r a_r.$$

Then $\operatorname{im} f = L$ and, by exactness, there are

$$f_i \in \operatorname{Hom}(R^{(R)}, R),\ m_i \in R^{(R)}, \quad \text{such that} \quad \sum f_i m_i f = f.$$

There is a finite subset J of R such that if $r \in R \setminus J$ then the r^{th} component of every m_i is zero. Now $(x_r) \sum f_i m_i = (y_r)$ has $y_r = 0$ for all $r \in R \setminus J$ and so $(x_r) \sum f_i m_i f = \sum_{r \in J} y_r a_r$. Hence $L = \operatorname{im} f = \sum_{r \in J} R a_r$, and R is left Noetherian.

(3) $\Rightarrow$ (1). As above, every countably generated left ideal is finitely generated and R is left Noetherian.

(5) $\Rightarrow$ (2). Clear.

(2) $\Rightarrow$ (5). Let ${}_R K \subseteqq {}_R M$. If for every $h \in K^d$ there exists $f \in M^d$ such that $f_{|K} = h$ and $\operatorname{im} f = \operatorname{im} h$, we say that K is a quasi-summand of M. This holds, in particular, if K is a summand of M. It is easily seen that quasi-summands of exact modules are exact and so direct summands of exact modules are exact. (We note in passing that conversely, if M is a finite direct sum of exact modules, then M is exact.) $\square$

We are indebted to K. R. Fuller for pointing out that (3) could be inserted in Theorem 1.

We now give a useful sufficient condition for a module to be exact and use it to characterise exact submodules of finitely generated projective modules.

Theorem 2. (a) *A module ${}_R M$ is exact whenever Mf is finitely generated and projective for all $f \in M^d$.*

(b) *A submodule L of a finitely generated projective module ${}_R P$ is exact if and only if L is finitely generated and projective.*

Proof. (a) Given $f \in M^d$ let $\{f_1, \dots, f_n\} \subseteq (Mf)^d$ and $\{r_1, \dots, r_n\} \subseteq Mf$ be a dual basis for Mf, so that $\sum f_i r_i = 1_{Mf}$. Define $g_i = ff_i \in M^d$ and choose $m_i \in M$ such that $m_i f = r_i$ for each i. Then $f = \sum g_i m_i f$, as required.

(b) It was observed at the outset that finitely generated projective modules are exact, so we need only consider the case when L is exact and a submodule of a finitely generated projective module P. Let $\{g_j, p_j\}$, $g_j \in P^d$, $p_j \in P$, be a dual basis for P and denote the restriction of g_j to L by $f_j \in L^d$. By exactness $f_j \in L^d L f_j$ and moreover, by a standard argument (see, for example, [3, Lemma 1]) there is $s \in L^d L$ such that $f_j = s f_j$ for all j. Hence $l s g_j = l g_j$ for all $l \in L$ and so $\sum g_j p_j = 1_P$ implies that $s = 1_L$. Thus L has a dual basis as required. □

Of course the converse to Theorem 2(a) is false. Every finitely generated projective module ${}_R M$ is exact as observed above but, if $M = R$, it is not necessarily the case that all principal left ideals of R are projective. However, we do have

Corollary 3. *Let $\mathcal{M}$ be a homomorphically closed class of R-modules. The following are equivalent:*

(1) *Every left ideal of R in $\mathcal{M}$ is finitely generated and projective.*
(2) *Every module in $\mathcal{M}$ is exact.*
(3) *Every left ideal of R in $\mathcal{M}$ is exact.*

Proof. (1) ⇒ (2) and (3) ⇒ (1) follow from Theorem 2 and (2) ⇒ (3) is immediate. □

If we let $\mathcal{M}$ be various special classes we obtain

Corollary 4. (1) *R is left hereditary and left Noetherian if and only if every left R-module is exact.*

(2) *R is left semihereditary if and only if every finitely generated left R-module is exact.*

(3) *R is a left PP-ring (principal left ideals projective) if and only if every principal left R-module is exact.*

(4) *R has a projective left socle if and only if every simple left R-module is exact.*

In a slightly different direction, let σ be a preradical on R-mod so that $\{M \mid \sigma(M) = M\}$ is homomorphically closed. Then σ is called left exact if $\sigma(K) = K \cap \sigma(M)$ whenever ${}_R K \subseteq {}_R M$. Examples include the singular preradical and the socle. Then

Corollary 5. *Let σ be a left exact preradical for R-mod. The following are equivalent:*

(1) *Every finitely generated module M with $\sigma(M) = M$ is exact.*
(2) *Every finitely generated left ideal contained in $\sigma({}_R R)$ is projective.*

Proof. (1) ⇒ (2). If $L \subseteq \sigma(R)$ is a finitely generated left ideal then $\sigma(L) = L$ because σ is left exact, so L is exact by (1). Then (2) follows by Theorem 4.

(2) $\Rightarrow$ (1). If $_RM$ is as in (1) and $f: M \to R$ is R-linear then $Mf \subseteqq \sigma(R)$ because σ is a preradical. Then (1) follows from Lemma 3. □

Similarly, one can show that every module $_RM$ with $\sigma(M) = M$ is exact if and only if $\sigma_R(R)$ is left Noetherian and every left ideal contained in $\sigma(_RR)$ is projective.

3. Exactness and Morita contexts. If R is a ring it is well known that R is semisimple and left artinian if and only if every left module V is projective, equivalently if $\mathrm{Hom}_R(V, \_)$ is an exact functor. Similarly R is (von Neumann) regular if $W \otimes_R(\_)$ is exact for all right modules W.

In this section exactness for a module is related to the exactness of a functor associated with a Morita context. Recall that a four-tuple $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is called a Morita context if R and S are rings, $_RV_S$ and $_SW_R$ are bimodules, and there exist multiplications $V \times W \to R$ and $W \times V \to S$, written $(v, w) \to vw$, and $(w, v) \to wv$, which induce bimodule homomorphisms $V \otimes_S W \to R$ and $W \otimes_R V \to S$ and which satisfy $v(wv_1) = (vw)v_1$ and $w(vw_1) = (wv)w_1$ for all $v, v_1 \in V$ and $w, w_1 \in W$. These conditions are equivalent to the requirement that $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be a ring where "matrix" operations are employed.

There are functors $W \otimes_R \_$ and $\mathrm{Hom}_R(V, \_)$ from R-mod to S-mod and a natural transformation $\lambda: W \otimes \_ \to \mathrm{Hom}(V, \_)$ given by, for $A \in R$-mod, $w \otimes a \to \alpha$ where $w \in W$, $a \in A$ and $\alpha: v \to (vw)a$ for all $v \in V$. Denoting the submodule

$$\{\sum w_i \otimes a_i \in W \otimes A \mid \sum (vw_i)a_i = 0 \text{ for all } v \in V\}$$

of $W \otimes A$ by $\mathrm{ann}_A(V)$, we have the factor module $A^0 = W \otimes_R A)/\mathrm{ann}_A V$. There is a natural epimorphism $v_A: W \otimes A \to A^0$ and a natural monomorphism

$$\mu_A: A^0 \to \mathrm{Hom}_R(V, A), \quad \text{given by} \quad [w \otimes a + \mathrm{ann}_A(V)]\,\mu_A = \alpha,$$

so that $\lambda_A = v_A \mu_A$.

The functor $(\ )^0$ has been investigated in [3] where it was shown that $(\ )^0$ is exact if and only if, for all $w \in W$, there exists $s \in WV$ such that $V(w - sw) = 0$ [3, Theorem 1]. For the standard context with $V = M$, $W = M^d$, and $S = \mathrm{End}(_RM)$ this becomes the condition that M be an exact module as defined earlier.

In what follows we shall be concerned with using these results to carry properties of a ring R to the endomorphism ring S of a finitely generated projective module $_RV$. Hence we start with a Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ in which $WV = S$.

Proposition 6. *Let* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *be a Morita context such that* $WV = S$, *let* $_SN$ *be an* S-*module, and write* $_RM = V \otimes_S N$. *Consider the standard contexts*

$$\begin{bmatrix} R & M \\ M^d & E \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} S & N \\ N^d & F \end{bmatrix}$$

and let the functors $(\ )^0$ *associated with these contexts be denoted* $(\ )^{0M}$ *and* $(\ )^{0N}$ *respectively. Give any S-module* ${}_SA$ *there is a* $\mathbb{Z}$*-isomorphism*

$$\tau_A\colon A^{0N} \to (V \otimes {}_SA)^{0M}$$

which is natural in A.

Proof. As noted above there are natural embeddings μ_A and $\mu_{V\otimes A}$ as in the diagram.

$$\begin{array}{ccc} A^{0N} & \xrightarrow{\mu_A} & \mathrm{Hom}_S(N, A) \\ & & \downarrow{\scriptstyle 1\otimes(\ )} \\ (V\otimes A)^{0M} & \xrightarrow{\mu_{V\otimes A}} & \mathrm{Hom}_R(M, V\otimes A). \end{array}$$

The map (actually an isomorphism since $WV = S$) $\alpha \to 1 \otimes \alpha$ from

$$\mathrm{Hom}_S(N, A) \to \mathrm{Hom}_R(M, V\otimes A)$$

induces the natural isomorphism we seek.

More precisely, since $WV = S$ fix a representation $\sum_{i=1}^{n} w_i v_i = 1 \in S$ where $w_i \in W$ and $v_i \in V$. A typical generator of $\mathrm{im}\,\mu_A$ is $\phi \cdot a$ where $\phi \in N^d = \mathrm{Hom}_S(N, S)$. For each $i = 1, 2, \ldots, n$ define $\phi_i \in M^d = \mathrm{Hom}_R(V\otimes N, R)$ by $(v\otimes n)\phi_i = v(n\phi)\,w_i$.

Then

$$1 \otimes \phi \cdot a = \sum_{i=1}^{n} \phi_i \cdot (v_i \otimes a)$$

in $\mathrm{Hom}_R(M, V\otimes A)$, and it follows that $\mathrm{im}\,\mu_A$ is carried into $\mathrm{im}\,(\mu_{V\otimes A})$.

Conversely, each generator of $\mathrm{im}\,(\mu_{V\otimes A})$ has the form $\psi \cdot t$ where

$$\psi \in M^d = \mathrm{Hom}_R(V\otimes N, R) \quad \text{and} \quad t = \sum_{k=1}^{n} v'_k \otimes a_k \in V\otimes A.$$

For each $k = 1, 2, \ldots, m$, define

$$\psi_K\colon N \to S \quad \text{by} \quad n\,\psi_k = \sum_{i=1}^{n} w_i\,[(v_i \otimes n)\,\psi]\,v'_k.$$

Then $\psi_k \in V \otimes N^d = \mathrm{Hom}_S(N, S)$ and so $\sum_k \psi_k \cdot a_k$ lies in $\mathrm{im}\,\mu_A$. Moreover, if $v\otimes n \in V\otimes N$ then

$$\begin{aligned}(v\otimes n)\,[1 \otimes \sum_k \psi_k \cdot a_k] &= v \otimes \sum_k (n\,\psi_k)\,a_k \\ &= \sum_k v\,(n\,\psi_k) \otimes a_k \\ &= \sum_k \sum_i \{v\,w_i\,[(v_i\otimes n)\,\psi]\,v'_k\} \otimes a_k \\ &= \sum_k [(v\otimes n)\,\psi](v'_k \otimes a_k) \\ &= (v\otimes n)\,\psi \cdot t.\end{aligned}$$

This means $\psi \cdot t = \sum (1 \otimes \psi_k \cdot a_k)$ lies in the image of μ_A, and it follows that $A^{0N} \cong (V\otimes A)^{0M}$. □

Our first application of Proposition 6 is a result on preservation of exactness. We shall continue to distinguish the functors $()^{0M}$ and $()^{0N}$.

Theorem 7. *Let* $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *be a Morita context satisfying* $WV = S$. *If* ${}_SN$ *is such that* ${}_RM = V \otimes_S N$ *is an exact R-module then* ${}_SN$ *is an exact S-module.*

Proof. Let $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\alpha} C \longrightarrow 0$ be an exact sequence in S-mod. Then

$$V \otimes A \xrightarrow{1 \otimes \alpha} V \otimes B \xrightarrow{1 \otimes \beta} V \otimes C \longrightarrow 0 \text{ is exact so, by hypothesis}$$

$$(V \otimes A)^{0M} \xrightarrow{(1 \otimes \alpha)^{0M}} (V \otimes B)^{0M} \xrightarrow{(1 \otimes \beta)^{0M}} (V \otimes C)^{0M} \longrightarrow 0$$

is exact. Proposition 6 now implies that

$$A^{0N} \xrightarrow{\alpha^{0N}} B^{0N} \xrightarrow{\beta^{0N}} C^{0N} \longrightarrow 0$$

is also exact and so, since $()^0$ always preserves monomorphisms [3, Proposition 2], it follows that $()^{0N}$ is an exact functor. □

Applying this result to module properties preserved under tensoring with finitely generated projective modules and using Corollary 4 we have, for $S = \operatorname{End}({}_RV)$ where ${}_RV$ is a finitely generated projective module:

(1) If R is left hereditary and left Noetherian, then so also is S (Small [5]).

(2) If R is left semihereditary, then so also is S (Lenzing [2] and Sandomierski [4]).

Similarly using Theorem 1;

(3) If R is left Noetherian, then so also is S.

If we restrict V to be of the form Re, $e = e^2 \in R$, so that tensoring with V preserves the property of being cyclic, we can add:

(4) If R is a left PP-ring, then so also is S (Chase [1]).

A further consequence of Theorem 7 is that being an exact module is preserved under Morita equivalence so giving the Morita invariance of the ring properties in (1), (2), and (3). In addition, since $V \otimes_S N$ is a simple R-module when ${}_RV$ is a progenerator and ${}_SN$ is simple, it follows from Corollary 4(4) that the class of all rings with a projective left socle is a Morita invariant class.

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