CHANGING OF THE NUMBER OF MINIMUM DOMINATING SETS AFTER EDGE ADDITION: CRITICAL EDGES

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Abstract

Let \( \gamma(G) \) and \( \#\gamma(G) \) denote the domination number and the number of all distinct minimum dominating sets of a graph \( G \), respectively. We show that if \( G \) is a graph without isolated vertices then for every edge \( e \in E(G) \), \( \gamma(G+e) < \gamma(G) \) if and only if \( \#\gamma(G+e) < \#\gamma(G) \).

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1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes, et al. [3]. We denote the vertex set and the edge set of a graph \( G \) by \( V(G) \) and \( E(G) \), respectively. For any vertex \( v \) of \( G \), the open neighborhood of \( v \) is the set \( N(v,G) = \{ u \in V(G) : uv \in E(G) \} \), while the closed neighborhood of \( v \) is the set \( N[v,G] = N(v,G) \cup \{v\} \). The degree of \( v \) is defined as \( \deg(v,G) = |N(v,G)| \). For a set of vertices \( S \subseteq V(G) \), \( N(S,G) \) is the union of \( N(x,G) \) for all \( x \in S \), and \( N[S,G] = N(S,G) \cup S \). For \( s \in S \subseteq V(G) \), \( pn_G(s,S) = N[s,G] - N[S - \{s\},G] \) is the private neighborhood of \( s \) relative to \( S \). A dominating set in a graph \( G \) is a set of vertices \( D \subseteq V(G) \) such that every vertex of \( G \) is either in \( D \) or is adjacent to an element of \( D \). A dominating set \( D \) of a graph \( G \) is a minimal dominating set if no set \( D' \subseteq D \) is a dominating set. The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set of \( G \). Any dominating set of cardinality \( \gamma(G) \) is called a \( \gamma(G) \)-set, or just \( \gamma \)-set when the graph \( G \) is clear from the context. The set of all \( \gamma \)-sets of a graph \( G \) is denoted by \( \mathcal{D}(G) \). If \( U \subseteq V(G) \), then denote \( \mathcal{D}(U,G) = \{ M \in \mathcal{D}(G) : U \subseteq M \} \). The number of distinct \( \gamma(G) \)-sets is denoted
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\#\gamma(G) ([6]). The number of all \gamma(G)-sets each of which has \( U \subseteq V(G) \) as a subset is denoted by \#\gamma(U, G).

It is often of interest to known how the value of a graph parameter is affected when a small change is made in a graph, for instance vertex or edge removal, edge addition and edge contraction. In this connection, in this paper we consider this question in the case \#\gamma(G) when an edge from \( G \) is added to \( G \).

If \( e \in G \) and \( \gamma(G + e) < \gamma(G) \) then \( e \) is called a \( \gamma(G) \)-critical edge. A graph \( G \) is \( \gamma \)-edge-addition-critical if all edges of \( G \) are \( \gamma(G) \)-critical. This concept was introduced by Sumner et al. [7]. The study of effects on domination related parameters when a graph is modified by adding an edge is classical; see for instance [2, 4, 5, 10] and for surveys [3, Chapter 5] and [1, 8]. Note that \( \gamma(G + e) + 1 \geq \gamma(G) \geq \gamma(G + e) \) ([3]).

Definition 1.1. Let \( G \) be a graph. An edge \( e \in E(G) \) is \#\gamma(G)-critical if \( \#\gamma(G + e) < \#\gamma(G) \). A graph \( G \) is \#\gamma \)-edge-addition-critical if all edges of \( G \) are \#\gamma(G)-critical.

Our main results are:

Theorem 1.2. Let \( x_1 \) and \( x_2 \) be two distinct, nonadjacent and nonisolated vertices of a graph \( G \). Then \( x_1x_2 \) is \( \gamma(G) \)-critical if and only if \( x_1x_2 \) is \#\gamma(G)-critical.

Corollary 1.3. Let \( G \) be a graph with no isolated vertex. Then \( G \) is \( \gamma \)-edge-addition-critical if and only if \( G \) is \#\gamma \)-edge-addition-critical.

2. Proofs

We need the following notation and results.

Let \( u \) and \( v \) be nonadjacent vertices of a graph \( G \). We write \( u \mapsto v \) whenever \( \gamma(G - v) < \gamma(G) \) and \( u \) belongs to at least one \( \gamma \)-set of \( G - v \).

Lemma 2.1. Let \( G \) be a graph, \( x_1, x_2 \in V(G) \), \( x_1x_2 \in E(G) \) and let \( G_1 = G + x_1x_2 \).

(i) ([10] Theorem 3; [9] Theorem 2.8) \( \gamma(G_1) < \gamma(G) \) if and only if either \( x_1 \in p_{G_1}(x_2, M) \) or \( x_2 \in p_{G_1}(x_1, M) \) for each \( \gamma \)-set \( M \) of \( G_1 \).

(ii) ([4], Lemma 5(2)) \( \gamma(G_1) < \gamma(G) \) if and only if at least one of \( x_1 \mapsto x_2 \) or \( x_2 \mapsto x_1 \) holds.

(iii) ([4], Theorem 3) If \( \gamma(G_1) < \gamma(G) \) and \( x_2 \nleftrightarrow x_1 \) then \( x_2 \) belongs to no \( \gamma \)-set of \( G_1 \).

Lemma 2.2. Let \( G \) be a graph, \( x_1, x_2 \in V(G) \), \( x_1 \mapsto x_2 \) and let \( G_1 = G + x_1x_2 \). Then:

(i) \( D(G_1) \) is disjoint union of \( D(\{x_1\}, G_1) \) and \( D(\{x_2\}, G_1) \);
(ii) \( D(\{x_1\}, G_1) = D(\{x_1\}, G - x_2) \);

(iii) if \( x_2 \mapsto x_1 \) then \( D(\{x_1\}, G - x_2) \) and \( D(\{x_2\}, G - x_1) \) form a partition of \( D(G_1) \);

(iv) if \( x_2 \not\mapsto x_1 \) then \( D(G_1) = D(\{x_1\}, G - x_2) \);

(v) \( \#\gamma(\{x_1, x_2\}, G) \geq \#\gamma(\{x_1\}, G - x_2) \neq 0 \).

**Proof.** By Lemma 2.1 (ii), \( \gamma(G_1) = \gamma(G) - 1 = \gamma(G - x_2) \).

(i) By Lemma 2.1 (i), \( |\{x_1, x_2\} \cap M| = 1 \) for each \( \gamma \)-set \( M \) of \( G_1 \).

(ii) If \( M \in D(\{x_1\}, G - x_2) \) then \( M \) is a dominating set of \( G_1 \) with \( |M| = \gamma(G - x_2) = \gamma(G_1) \). Hence \( M \) is a \( \gamma \)-set of \( G_1 \) which implies \( D(\{x_1\}, G - x_2) \subseteq D(\{x_1\}, G_1) \).

Now assume \( M \in D(\{x_1\}, G_1) \). By (i), \( x_2 \not\in M \) and then \( M \) is a dominating set of \( G - x_2 \) of cardinality \( \gamma(G_1) = \gamma(G - x_2) \) which implies \( M \) is a \( \gamma \)-set of \( G - x_2 \). Thus \( D(\{x_1\}, G_1) \subseteq D(\{x_1\}, G - x_2) \).

(iii) The result follows immediately by (i) and (ii).

(iv) By Lemma 2.1(iii), \( D(\{x_2\}, G_1) = \emptyset \) and the result follows by (i) and (ii).

(v) Since \( x_1 \mapsto x_2, D(\{x_1\}, G - x_2) \) is not empty and \( M \cup \{x_2\} \in D(\{x_1, x_2\}, G) \) for each \( M \in D(\{x_1\}, G - x_2) \). \( \square \)

**Proposition 2.3.** Let \( G \) be a graph, \( x_1, x_2 \in V(G) \) and let \( G_1 = G + x_1x_2 \).

(i) If \( x_1 \mapsto x_2, x_2 \mapsto x_1 \) and \( \deg(x_2, G) \geq \deg(x_1, G) \geq 1 \) then
\[ \#\gamma(G_1)(\deg(x_1, G) + \#\gamma(\{x_1\}, G_1)) (\deg(x_2, G) - \deg(x_1, G)) + \#\gamma(\{x_1, x_2\}, G) = \#\gamma(\{x_2\}, G - x_1) \deg(x_1, G) + \#\gamma(\{x_1\}, G - x_2) \deg(x_2, G) + \#\gamma(\{x_1, x_2\}, G) \leq \#\gamma(G). \]

(ii) If \( x_1 \mapsto x_2, x_2 \not\mapsto x_1 \) and \( \deg(x_s, G) \geq 1, s = 1, 2 \), then \( \#\gamma(G_1)(\deg(x_2, G) + 1) \leq \#\gamma(\{x_1\}, G) \leq \#\gamma(G). \)

(iii) Let \( x_1 \mapsto x_2, \deg(x_1, G) = 0 \) and \( \deg(x_2, G) \geq 2 \). Then \( \#\gamma(G_1) + \#\gamma(G - x_2)(\deg(x_2, G) - 1) \leq \#\gamma(G). \)

(iv) Let \( x_1 \mapsto x_2, \deg(x_1, G) = 0 \) and \( \deg(x_2, G) = 1 \). Let \( N(x_2, G) = \{x_3\} \) and \( A = \{M: M \text{ is a minimal dominating set of } G - x_2, x_3 \in M \text{ and } |M| = \gamma(G)\} \). Then \( \#\gamma(G) = \#\gamma(G_1) + |A| \).

(v) Let \( x_1 \not\mapsto x_2, x_2 \mapsto x_1 \) and \( \deg(x_1, G) = 0 \). Then \( \#\gamma(G_1) = \#\gamma(\{x_2\}, G) \). If \( x_2 \) is in each \( \gamma(G) \)-set then \( \#\gamma(G_1) = \#\gamma(G) \). If \( x_2 \) belongs to some \( \gamma(G) \)-set but not to all \( \gamma(G) \)-sets then \( \#\gamma(G_1) < \#\gamma(G) \).

(vi) If \( \deg(x_1, G) = \deg(x_2, G) = 0 \) then \( \#\gamma(G_1) = 2\#\gamma(G) \).
Proof. Let \( u, v \in V(G) \) with \( u \leftrightarrow v \) and \( \deg(v, G) \geq 1 \). Then each neighbor of \( v \) belongs to no \( \gamma \)-set of \( G - v \). Define \( B(u, v) = \{ M \cup \{ z \} : M \in D(\{ u \}, G - v) \text{ and } z \in N(v, G) \} \). Hence \( B(u, v) \subseteq D(\{ u \}, G - D(\{ u, v \}, G) \) and \( |B(u, v)| = \#\gamma(\{ u \}, G - v) \).

(i) \( \#\gamma(G) \geq |D(\{ x_1 \}, G) \cup D(\{ x_2 \}, G)| = |D(\{ x_1 \}, G) - D(\{ x_1, x_2 \}, G)| + |D(\{ x_2 \}, G) - D(\{ x_1, x_2 \}, G)| + \#\gamma(\{ x_1, x_2 \}, G) \geq |B(x_1, x_2)| + |B(x_2, x_1)| + \#\gamma(\{ x_1, x_2 \}, G) = \#\gamma(\{ x_1 \}, G - x_2)deg(x_2, G) + \#\gamma(\{ x_2 \}, G - x_1)deg(x_1, G) + \#\gamma(\{ x_1, x_2 \}, G)\). It remains to note that \( \#\gamma(\{ x_2 \}, G - x_1) = \#\gamma(\{ x_1 \}, G - x_2) \) by Lemma 2.2(iii) and \( \#\gamma(\{ x_1 \}, G - x_2) = \#\gamma(\{ x_1 \}, G - x_2) \) by Lemma 2.2(ii).

(ii) \( \#\gamma(G) \geq \#\gamma(\{ x_1 \}, G) \geq \#\gamma(\{ x_1, x_2 \}, G) + |D(\{ x_1 \}, G) - D(\{ x_1, x_2 \}, G)| \geq \#\gamma(\{ x_1, x_2 \}, G) + |B(x_1, x_2)| = \#\gamma(\{ x_1, x_2 \}, G) + \#\gamma(\{ x_1 \}, G - x_2)deg(x_2, G)\). Since \( \#\gamma(\{ x_1, x_2 \}, G) \geq \#\gamma(\{ x_1 \}, G - x_2) = \#\gamma(\{ x_1 \}, G - x_2) \) (by Lemma 2.2(v) and Lemma 2.2(iv), respectively), we have the result.

(iii) Clearly, \( x_2 \leftrightarrow x_1 \). We have, \( \#\gamma(G) = \#\gamma(\{ x_1 \}, G) + |D(\{ x_1 \}, G) - D(\{ x_1, x_2 \}, G)| \geq \#\gamma(\{ x_1, x_2 \}, G) + |B(x_1, x_2)| = \#\gamma(\{ x_1, x_2 \}, G) + \#\gamma(\{ x_1 \}, G - x_2)deg(x_2, G) = \#\gamma(\{ x_2 \}, G - x_1) + \#\gamma(\{ x_1 \}, G - x_2)deg(x_2, G)\). It remains to note that by Lemma 2.2(iii), \( \#\gamma(\{ x_2 \}, G - x_1) = \#\gamma(G_1) - \#\gamma(\{ x_1 \}, G - x_2) = \#\gamma(G_1) - \#\gamma(G - x_2) \).

(iv) Clearly \( D(\{ x_3 \}, G) \) is disjoint union of \( A \) and \( \{ M \cup \{ x_3 \} : M \text{ is a } \gamma \text{-set of } G - x_2 \} \). Hence \( \#\gamma(\{ x_3 \}, G) = |A| + \#\gamma(G - x_2) = |A| + \#\gamma(\{ x_1 \}, G - x_2) \). Since \( D(G) \) is disjoint union of \( D(\{ x_2 \}, G) \) and \( D(\{ x_3 \}, G) \), we have \( \#\gamma(G) = \#\gamma(\{ x_2 \}, G) + \#\gamma(\{ x_3 \}, G) = \#\gamma(\{ x_2 \}, G - x_1) + |A| + \#\gamma(\{ x_1 \}, G - x_2) \) and the result now follows by Lemma 2.2(iii).

(v) By Lemma 2.2(iv), \( \#\gamma(G_1) = \#\gamma(\{ x_2 \}, G - x_1) = \#\gamma(\{ x_2 \}, G) \).

(vi) Obvious.

Proof of Theorem 1.2.

Sufficiency: Let \( x_1x_2 \) be a \#\gamma(G)-critical edge. Assume \( \gamma(G) = \gamma(G + x_1x_2) \). Then each \( \gamma(G) \)-set is a \( \gamma(G + x_1x_2) \)-set. This implies \( \#\gamma(G) \leq \#\gamma(G + x_1x_2) \), a contradiction.

Necessity: Let \( x_1x_2 \) be a \( \gamma(G) \)-critical edge. By Lemma 2.1(ii), at least one of \( x_1 \leftrightarrow x_2 \) or \( x_2 \leftrightarrow x_1 \) holds. Since \( \gamma(\{ x_1, x_2 \}, G) \neq 0 \) (by Lemma 2.2(v)), it follows from Proposition 2.3(i)-(ii) that \( \#\gamma(G + x_1x_2) < \#\gamma(G) \).

Proof of Corollary 1.3. The result immediately follows by Theorem 1.2.
References


