The method of normal splines for linear DAEs on the number semi-axis

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Abstract

The method of normal spline-collocation (NSC), applicable to a wide class of ordinary linear singular differential and integral equations, is specified for the boundary value problems for differential-algebraic equations of second order on the number semi-axis. The method consists in minimization of a norm of the collocation systems’ solutions in an appropriate Hilbert–Sobolev space. The NSC method does not use the notion of differentiation index and it is applicable to DAEs of any index as well as to equations not reducible to the normal form. The problems on the infinite interval can be solved in two ways. The first way is based on the use of the original space of functions defined on the semi-axis, and the second way is based on a singular transformation of the semi-axis into the unit segment. A new reproducing kernel, that provides the first way, is presented. An algorithm to create a non-uniform collocation grid is described.

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1. Introduction

An important contemporary problem in numerical mathematics is the construction of methods for ordinary differential and integral equations which are singular or degenerate in various senses. Here we consider boundary value problems for linear differential-algebraic equations (DAEs) of second order on the number semi-axis \([0, \infty)\) with arbitrary degenerate main parts.

It is known that DAEs are more complicated objects than regular ODEs concerning theoretical properties and the numerical solution of initial and boundary value problems. Their solutions (if they exist) can depend non-continuously on the input data, and the structure of the solution set is generally undefined. A system of DAEs can have a unique solution as well as an infinite-dimensional solution set (see [7], the recent investigation [10], and [20]). Thus, in the general case, numerical problems for DAEs are irregular (ill-posed), and the complexity of their numerical solution is defined by the possibility of their regularization.

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Many DAEs can be transformed to the normal Cauchy form by a finite number of differentiations and algebraic transformations. The minimal number of such differentiations is called the differentiation index (DI) [24]. There is a set of effective specialized step-type methods for DAEs on a finite number segment with DI up to three [24]. Equations with higher DI should be stably converted to the lower index ones by analytic differentiation prior to applying such methods [4]. We also note formalized methods of construction of the ‘left regularizing operators’, based on differentiation and the algebraic technique of pseudo-inversion [7,8]. However, differentiation of the given DAE generally leads to a DAE/ODE with a broader solution set. Moreover, methods based on the DI notion are not applicable under variable degeneracy of an equation’s main part, when the system cannot be reduced to the normal form. An example of such a DAE is the Schrödinger equation with the singular point-wise potential [28]. For such a kind of problem we know only theoretical investigations [10,29], and specialized methods which use physical properties of solutions [28].

Thus, the problem of development of numerical methods for DAEs, that are free from the DI identification, should be accounted as actual [20]. We note, there exists a parametrization method (PM) of such type, originally created for optimal control problems [11], that is expanded to a rather wide class of arbitrary degenerate, generally non-linear DAEs [18–20]. Using the linearity of a DAE can help to construct more effective numerical methods.

The method of normal spline-collocation for ordinary linear integral, differential and integro-differential equations, including implicit equations, has been created by Gorbunov in the eighties. Initially, it was meant to regularize linear integral equations of the first kind with disturbed right-hand parts and pointwise error estimations [12,13]. The regularizing problem was the normal solution to an integral inequality, corresponding to the uniform error estimation, in an appropriate Hilbert–Sobolev (HS) space of functions which have the square integrable derivatives up to some order \( l \geq 1 \). We name this order the derivative’s factor of the space. After discretization of the inequality on the external variable, the arisen linear and continuous integral functionals according to the Riesz theorem [6,33] have been transformed to inner products corresponding to the norm of the space. Factors of the products, or canonical images of the linear functionals, have become the basis system for representation of the normal solution. The canonical transformation has allowed to construct rather simple theory and algorithms of the new numerical method of variational type.

Later the idea of partial discretization and canonical representation of linear continuous functionals in an HS space was extended to systems of differential and integro-differential equations with arbitrary degenerate main parts [14,15] in the HS spaces \( W_{l, n}^l(a, b) \) of \( n \)-dimensional functions with the derivative’s factors \( l \geq 2 \). The NSC method for this class of numerical problems consists in minimization of the solution norm on the solution set of a collocation system. According to the embedding theorem of the Sobolev spaces in the Chebyshev ones [31,33], each equation of the collocation system can be considered as an equality to zero of a complex linear continuous functional which consists of elementary functionals, representing values of the required solution and its derivatives in the collocation points, and its weighted integrals. Canonical images of these complex functionals constitute the basis system for representation of the required normal solution. Thus, different from the classical and spline collocation methods for ODEs [26,35], and for DAEs [4], in the NSC method the basis system is not being entered a priori, but it is under automatic construction according to the chosen norm and coefficients of the problem to be solved.

The canonical representation of linear continuous functionals in \( W_{l, n}^l(a, b) \) is the key for construction of effective algorithms. This problem can be solved by constructing a reproducing kernel (RK) [3] corresponding to the norm. It was fulfilled for spaces \( W_{l, n}^l[0, 1] \) of any factor \( l \) [14,21], and in the case of infinite intervals \( (-\infty, \infty) \) and \([0, \infty)\) for \( l = 1 \) and \( l = 2 \) [15].

The problem of the normal solution of a system of linear equations in a Hilbert space can be understood as a particular case of the problem of generalized splines that was introduced in the approximation theory by Atteia [5]. The known realizations of the latter problem are the minimization of the square of semi-norms [27,36]. In our case the functional under minimization is the square of a norm. Correspondingly, our normal solutions were called the normal splines.

The NSC method for problems on finite intervals appeared effective for stiff problems [14], and for linear DAEs with variable degeneration of their main parts [21,22]. Algorithms of creation of adaptive non-uniform grids were constructed for boundary as well as for initial value problems.

In applications, differential equations of second (and up to fourth) order often arise, therefore it is productive to construct numerical methods directly applicable to such equations without their transformation to systems of first order. There is a special algebraic technique for application of step-type methods, developed for regular ODEs of
second order, to DAEs [30]. This technique is limited by the assumption of uniqueness of the solution. The NSC method yields a natural way for solving more general DAEs of arbitrary order provided that the problems are posed in $W^{2,n}_{2,0}(a,b)$ with the factor $l$ larger than the order of derivatives in the equations. A corresponding scheme of the NSC method in the case of a bounded segment $[a,b]$ has been presented in [16,23]. Below this scheme is developed for problems on the semi-axis.

Traditional approaches to problems on the semi-axis $[0, \infty)$ (as well as on the axis $(-\infty, \infty)$) are based on truncation of $[0, \infty)$ to a finite segment $[0, T]$ and on definition of some additional conditions at the end point $T$ [1,34]. The NSC method allows to overcome an infinity problem in two natural ways. The first is based on the use of a solution linear DAEs of second order in the HS spaces for differential and integral equations. It seems that the NSC method, developed below, should be more effective with the PM method and NSC method enhances the value of the variational approach to complex irregular problems. Correspondingly, it bypasses hard problems of the DI identification and decreasing DIs more than 3. Our experience integral equations of Fredholm type. The NSC method, as well as the PM method, does not use the notion of DI. The effective alternative to the simpler step-type ones in the case of boundary value problems for ODEs [32] and framework of step-type [24] and classical projective methods [26], but it is admissible for the NSC method. Such a technique generates the singularity at the end of the segment. It makes this way not applicable within the framework of step-type and classical projective methods, but it is admissible for the NSC method.

Both the PM and NSC methods belong to the projective-variational ones of spline type. Variational methods present the effective alternative to the simpler step-type ones in the case of boundary value problems for ODEs [32] and integral equations of Fredholm type. The NSC method, as well as the PM method, does not use the notion of DI. Correspondingly, it bypasses hard problems of the DI identification and decreasing DIs more than 3. Our experience with the PM method and NSC method enhances the value of the variational approach to complex irregular problems for differential and integral equations. It seems that the NSC method, developed below, should be more effective compared with the PM method in linear cases.

The paper is organized as follows. The second section represents the statement of computational problems for linear DAEs of second order in the HS spaces $W^{2,n}_{2,0}(a,b)$ of sufficiently smooth functions on the segment $[0, 1]$ and on the semi-axis $[0, \infty)$. The third section describes the transformation of the problem on $[0, \infty)$ to one on $[0, 1]$. In the forth section the NSC method for these problems is described. The fifth section is devoted to the problem of the canonical transformation of the point-wise linear continuous functionals. Here a new RK for the space $W^{3, n}_{2,0}[0, \infty)$ is presented. The sixth section represents two theorems of convergence of the NSC method in both cases and discusses the problem of construction of effective grids on the basis of illumination of the equation discrepancy. Two numerical examples are given in the last, seventh, section.

2. Statement of computational problems

Let us consider a DAE of second order

$$\begin{align*}
A(t)\ddot{x}(t) + B(t)\dot{x}(t) + C(t)x(t) = f(t), & \quad a < t < b, \tag{1}
\end{align*}$$

were $x, f \in \mathbb{R}^n$, $A(t), B(t), C(t)$ are square $n$-order matrices, and $a$ and $b$ are given numbers, in particular, $b = \infty$. The matrices, as well as the function $f(t)$, are assumed to be at least continuously differentiable. Smoothness of the coefficients is not required for construction of the NSC algorithm, but it is used in connection with theoretical substantiations. The matrices $A(t), B(t), C(t)$ can be arbitrary degenerate on the interval $(a, b)$, but the matrix $A(t)$ is not trivial.

The main purpose of this work is to consider computational problems for Eqs. (1) on the semi-axes $[0, \infty)$, i.e. under $a = 0$ and $b = \infty$. In this case typical conditions have the form

$$D^0 x(0) + E^0 \dot{x}(0) = g^0, \quad \lim_{t \to \infty} x(t) = 0. \tag{2a}$$

However, one of two ways of solving this infinite boundary value problem, considered below, is its transformation to the finite one on the segment $[0, 1]$ with conditions which are a particular case of the conditions

$$D^0 x(0) + E^0 \dot{x}(0) = g^0, \quad D^1 x(1) + E^1 \dot{x}(1) = g^1. \tag{2b}$$

The vectors $g^0, g^1 \in \mathbb{R}^n$, and the matrices $D^0, E^0, D^1, E^1 \in \mathbb{R}^{n \times n}$. The simpler initial value problem for (1) on a finite segment is being covered by the NSC method presented below with corresponding algorithmic simplification [16].

Each of the systems (1), (2a) and (1), (2b) are assumed solvable in a class of functions that are sufficiently smooth. Below, the notation $x_i^{(r)}(t)$ will be used for derivatives of order $r$, such that $\dot{x}_i = x_i^{(1)}$, $\ddot{x}_i = x_i^{(2)}$. In view of singularity,
the considered solvable problem can have a set of solutions. In view of linearity, this set is a linear manifold in any vector space.

The main assumption lying in the base of the NSC method is the belonging of required solutions to the Hilbert–

Sobolev space $W_{2,n}^l(a,b)$ of vector-functions $x(t) = \{x_1(t), \ldots, x_n(t)\}$, which components have derivatives up to order $l$, and functions $x_i^{(r)}(t)$, $r = 0, 1, \ldots, l$, are square integrable on $(a,b)$.

Two kind of norms in $W_{2,n}^l(a,b)$ are used for the construction of the NSC method depending on the type of intervals. The space multi-dimensionality is counted by the common formula

$$\|x\|_{l,n} = \left[ \sum_{i=1}^{n} \|x_i\|^2_2 \right]^{1/2}. \quad (3)$$

The scalar norms $\| \cdot \|_l$ of the components $x_i(t)$ are:

(a) in the case of the segment $[0, 1]$,

$$\|x_i\|_l = \left[ \sum_{r=0}^{l-1} (x_i^{(r)}(0))^2 + \int_0^1 (x_i^{(l)}(s))^2 \, ds \right]^{1/2}. \quad (4a)$$

(b) in the case of the semi-axis $[0, \infty)$,

$$\|x_i\|_l = \left[ \int_0^\infty (x_i(s))^2 + (x_i^{(l)}(s))^2 \, ds \right]^{1/2}. \quad (4b)$$

The inner products corresponding to these norms are:

(a) in the case of $[0, 1]$,

$$\langle x, y \rangle_{l,n} = \sum_{i=1}^{n} \left[ \sum_{r=0}^{l-1} (x_i^{(r)}(0)y_i^{(r)}(0)) + \int_0^1 (x_i^{(l)}(s)y_i^{(l)}(s)) \, ds \right]. \quad (5a)$$

(b) in the case of $[0, \infty)$,

$$\langle x, y \rangle_{l,n} = \sum_{i=1}^{n} \int_0^\infty (x_i(s)y_i(s) + x_i^{(l)}(s)y_i^{(l)}(s)) \, ds. \quad (5b)$$

The minimal value of the derivative’s factor $l$ is at least 3. This limitation is generated by the fact that the Sobolev space of scalar functions $W_{2,1}^l(a,b)$ is continuously (actually compactly) embedded in the Chebyshev space $C^{l-1}(a,b)$. Correspondingly, convergence in the norm of $W_{2,1}^l(a,b)$ entails uniform convergence of (continuous) derivatives $x_i^{(r)}(t)$ with $r = 0, \ldots, l-1$. It allows to count the values of these derivatives under fixed given $t$ as continuous linear functionals in the space $W_{2,1}^l(a,b)$. Moreover, this property of convergency ensures closure of the solutions manifold.

In the case of the infinite interval $[0, \infty)$, the square integrability of the continuous functions $x(t)$ entails fulfilling the limiting conditions in (2a) automatically. Correspondingly, we can take into account explicitly only the $n$ non-trivial conditions from (2a):

$$D^0 x(0) + E^0 \dot{x}(0) = g^0. \quad (6)$$

The problem (1) under $a = 0$ and $b = \infty$ with the conditions (6) in the space $W_{2,n}^l[0, \infty)$ can be named quasi-initial value problem.

In view of possible non-uniqueness of solutions to the system (1) with the conditions (2b) or (6), these problems should be predetermined. In applied problems experts usually have some notion about the required solution as some trial function $z(t)$. If such a notion is absent, we count $z(t) \equiv 0$.  

We state the variational problems to construct the solutions $x_z$ to the corresponding DAE (1) with one of the conditions (2b) or (6) which are the nearest ones in the norm of $W^1_{2,n}(a, b)$ to the corresponding function $z(t)$. This means that we have to minimize the functional

$$
\|x - z\|_1
$$

under the constraints (1) and either (2b) or (6). Such solutions $x_z$ are named $z$-normal solutions. They exist and are unique for any problem under consideration which is solvable in $W^1_{2,n}$.

3. Transformation of the problem on the semi-axis to the unit segment

The problem (1), (2a) on the semi-axis $[0, \infty)$ can be reformulated as the boundary value problems on the standard unit segment $[0, 1]$ via the singular and invertible time transformation

$$
\tau = 1 - \exp(-t), \quad \text{or} \quad t = -\ln(1 - \tau),
$$

or another similar transformation $\tau(t)$, or $t(\tau)$, such that $\tau(0) = 0$ and $\tau(\infty) = 1$, correspondingly, $t(0) = 0$ and $t(1) = \infty$.

Let us introduce a new unknown function $y(\tau) = x(t(\tau))$. Its first derivative will be denoted by $y'$ and the second by $y''$. In view of (8) $dt = d\tau/(1 - \tau)$. $\hat{x} = (1 - \tau)y'$, $\hat{\tau} = (1 - \tau)^2y'' - (1 - \tau)y'$. In terms of the new coefficients

$$
\hat{A}(\tau) = A(t(\tau)), \quad \hat{B}(\tau) = B(t(\tau)) - A(t(\tau)), \quad \hat{C}(\tau) = C(t(\tau)), \quad \hat{f}(\tau) = f(t(\tau))
$$

the problem (1), (2a) takes the form

$$
(1 - \tau)^2\hat{A}(\tau)y''(\tau) + (1 - \tau)\hat{B}(\tau)y'(\tau) + \hat{C}(\tau)y(\tau) = \hat{f}(\tau), \quad 0 \leq \tau \leq 1,
$$

with the boundary conditions

$$
D^0y(0) + E^0y'(0) = x^0, \quad y(1) = 0.
$$

The transformation (8) generates or strengthens singularity in the main part of (9) by the multiplier $(1 - \tau)^2$. Hence, the transformed equation has the variable degeneracy of the main part, and the common methods for DAEs of the finite DI are not applicable to the problem (9), (10). But this new problem belongs to the same class of problems (1), (2b) with the arbitrary degenerate matrices $\hat{A}(\tau)$ and $\hat{B}(\tau)$, and it can be solved by the NSC method, developed for the problems on finite segments.

Thus, the problem (1), (2a) on the infinite semi-axis with the help of the transformation (8) can be solved by the NSC method, developed for the problems (1), (2b). However, despite of more simplicity of the latter problems, the amplification of singularity of the equation being solved can degrade the calculation process of the NSC method. Therefore, a direct variant of the NSC method on the semi-axis should be developed and investigated.

4. The NSC method

The method of normal spline collocation for approximation of the $z$-normal solution to the system (1) with conditions (2b) or (6) is described below. On the interval $(a, b)$ a grid

$$
a < t_1 < t_2 < \cdots < t_m \leq b
$$

is introduced, and we change the system of differential equations (1) to the corresponding collocation system

$$
A(t_k)\hat{x}(t_k) + B(t_k)\hat{x}(t_k) + C(t_k)x(t_k) = f(t_k), \quad k = 1, \ldots, m.
$$

The required $z$-normal solution $x_z$ satisfies this finite system, consisting of the $n \times m$ scalar equations, and the corresponding $2n$ conditions (2b) or $n$ conditions (6). Thus, in the case of the finite segment $(a = 0, b = 1)$ we have the $N_a = n(m + 2)$ scalar equations (12), (2b), and in the case of the semi-axis $(a = 0, b = \infty)$ the number of scalar equations (12), (6) equals $N_b = n(m + 1)$.

Each of these systems has a linear manifold of solutions, and for each of them we state the problem of a $z$-normal solution, that is, to minimize the functional (7) under the constraints (12) and either (2b) or (6). We denote the
z-normal solution by \(x^m_z\) taking into account that the solutions are defined by the grid (11) with \(m\) knots. Existence and uniqueness of \(x^m_z\) is explained below. Let us consider this variational problem.

In view of the mentioned embedding theorem the left part of each scalar equation of the collocation system (12) at a given \(i \in \{1, \ldots, n\}\) and a given \(k \in \{1, \ldots, m\}\) is the composite linear continuous functional (LCF)

\[
l_{ik}(x) = \sum_{j=1}^{n} \left[ a_{ij}(t_k) \ddot{x}_j(t_k) + b_{ij}(t_k) \dot{x}_j(t_k) + c_{ij}(t_k) x_j(t_k) \right].
\]

(13)

The boundary/initial conditions also define composite functionals. In the case \([0, 1]\), conditions (2b) define for \(i = 1, \ldots, n\) the LCFs

\[
l_{ik}(x) = \begin{cases} 
\sum_{j=1}^{n} [d^0_{ij} x_j(0) + e^0_{ij} \dot{x}_j(0)], & k = m + 1, \\
\sum_{j=1}^{n} [d^1_{ij} x_j(1) + e^1_{ij} \dot{x}_j(1)], & k = m + 2.
\end{cases}
\]

(14)

In the case \([0, \infty)\), conditions (6) define for each \(i = 1, \ldots, n\) only the first \(m + 1\) LCFs from (14).

According to the Riesz theorem [6,33], the elementary point-wise functionals in the right-hand parts of (13) and (14) \(\{\ddot{x}_i(t_k), \dot{x}_i(t_k), x_i(t_k), x_i(0), x_i(1), \dot{x}_i(0), \dot{x}_i(1)\}\) can be transformed to the canonical form as the inner products (5a) or (5b), corresponding to the norm (4a) or (4b). This problem is described below.

Let us suppose that this transformation has been performed. We shall denote the canonical images of the composite functionals \(l_{ik}(\cdot)\) by \(h^{\mu(i,k)}\), connected, for example, by the correspondence

\[\mu(i,k) = (k-1)n + i.\]

By definition, the canonical images \(h^{\mu(i,k)}\) are elements of the solution space \(W_{l,n}^{2}\), and the equalities

\[
l_{ik}(x) = \langle h^{\mu(i,k)}, x \rangle_{l,n}
\]

hold for any \(x \in W_{l,n}^{2}\).

Thus, we have the system of canonical images

\[\{h^1, \ldots, h^N\}.\]

(16)

In the finite case, the number \(N\) of these images equals \(N_a\), and in the infinite case we have \(N = N_b\).

In view of the equalities (13)–(15), we can rewrite the system (12) with the conditions (2b) or (6) in the canonical form

\[
\langle h^{\mu}, x \rangle_{l,n} = f^{\mu}, \quad \mu = 1, \ldots, N.
\]

(17)

Here the right-hand part corresponding to the system (12), (2b) is the vector

\[
f^{\mu(ik)} = \begin{cases} 
 f_i(t_k), & k = 1, \ldots, m, \\
 g^0_i, & k = m + 1, \\
 g^1_i, & k = m + 2.
\end{cases}
\]

(18)

where \(i = 1, \ldots, n\). In the case of the system (12), (6), the last row in (18) is absent.

Thus, (17) is the system of linear equations in the Hilbert space \(W_{l,n}^{2}\), and the solution \(x^m_z\) is its z-normal solution, i.e. \(x^m_z\) is the minimizer of the functional (7) under constraints (17). The admissible set, defined by (17), is the intersection of \(N\) hyperplanes in \(W_{l,n}^{2}\). Consequently, this intersection is the closed convex set, and the z-normal solution \(x^m_z\) exists and is unique.

The presented NSC problem can be considered as a particular case of the problem of generalized splines that was introduced in the approximation theory by Atteia [5] as a generalization of the known Holladay’s variational problem generating the classical cubic splines [2]. The solution of Atteia’s problem was named the generalized spline. In actual realizations of this abstract scheme in problems of functions’ interpolation and smoothing [27,36] the functionals being minimized had a type of a semi-norm. In our case such a functional is the norm. Correspondingly, the normal solution \(x^m_z\) has been named the normal spline [14].
The problem of minimizing the functional (7) under constraints (17) is the generalized Lagrange problem in the Hilbert space [25,27]. Applying the multipliers method one can obtain the solution of the problem – the normal spline
\[ x^m(t) = z(t) + \sum_{\mu=1}^{N} u_\mu h_\mu(t), \] (19)
where the multipliers \( \{u_\mu\} \) are defined by the system of linear equations
\[ \sum_{\nu=1}^{N} g_{\mu\nu} u_\nu = f_\mu - [h_\mu, z]_{l,n}, \quad \mu = 1, \ldots, N, \] (20)
with the Gram matrix \( G^N = \{g_{\mu\nu}\} \) of the system \( \{h_\mu\} \).

Thus, the general algorithm of the NSC method consists of the following:

1. for the chosen solution space \( W^{l}_{2,n}(a,b) \) with the known reproducing kernel and for the given grid (11), the system of canonical images (16) and their Gram matrix \( G^N \) have to be created;
2. the system of linear algebraic equations (20) has to be solved;
3. the normal spline \( x^m(t) \) can be calculated by (19) at any point \( t \in (a,b) \).

This general scheme should be added with strategies of the grid (11) creation and of solving the system (20). The grid problem is discussed below. The system of linear equations (20) in a simple case, when the matrix \( G^N \) is positive definite and well-conditioned, can be solved by the Choleski method [32]. In the case the matrix is ill-conditioned, one can use the iterative conjugate-gradient method and the singular value decomposition with some stopping-rule [17].

The presentation (19) includes the NSC method to the class of projection methods. Unlike of such known methods, here the basis system (16) is not defined a-priori, but it is generated by the RK of the solution space \( W^{l}_{2,n} \) and by the coefficients of Eq. (1).

5. Reproducing kernels

The notion of a reproducing kernel of a Hilbert space of scalar functions has been introduced by Aronszajn in [3] for developing the theory of integral representation of functions. In the problems under consideration the components \( x_i \) of the solutions belong to the spaces \( W^{l}_{2,1}(a,b) \). The RK of such a space is the function \( G(s,t) \) possessing the following properties:

(i) for each \( t \in (a,b) \), the function \( G(\cdot,t) \) belongs to \( W^{l}_{2,1}(a,b) \);
(ii) for any function \( x_i \in W^{l}_{2,1}(a,b) \) and any \( t \in (a,b) \), the identity
\[ x_i(t) = \langle G(\cdot,t), x_i \rangle_{l,1} \] (21)
holds;
(iii) the function \( G(s,t) \) is symmetric:
\[ G(t,s) = G(s,t). \] (22)

Consider the identity (21). The functions \( x_i \in W^{l}_{2,1}(a,b) \) have at least \( l - 1 \) (absolutely) continuous derivatives. Differentiating (21), we obtain the following identities for the corresponding derivatives:
\[ x_i^{(r)}(t) = \left\langle \frac{\partial^r G(\cdot,t)}{\partial t^r}, x_i \right\rangle_{l,1}, \quad r = 1, \ldots, l-1. \] (23)

Having the RK \( G(s,t) \) of the space \( W^{l}_{2,1}(a,b) \), we obtain the canonical representations for the point-wise functionals
\[ l^r_k(x_i) = x_i^{(r)}(t_k), \quad r = 0, \ldots, l-1. \] (24)
The problem of canonical transformation consists in finding such functions \( h_{r,k} \in W^l_{2,1} \) that the identities

\[
l_k^r(x_i) = \langle h_{r,k}, x_i \rangle_{l,1}, \quad \forall x_i \in W^l_{2,1},
\]

hold. Comparing (23) with (24) and (25), we obtain the required canonical images

\[
h_{r,k}(s) = \frac{\partial^r G(s, t_k)}{\partial t^r}, \quad r = 0, \ldots, l - 1.
\]

Having such a representation of the elementary point-wise functionals, we can construct the system of canonical images (16) for composite functionals (13) and (14) of the corresponding problem of NSC (see in details [21,22,16]).

The main property (21) of the RK helps to calculate entries of the Gram matrix of the system of point-wise functionals (26) [15]:

\[
\langle h_0, i, h_0, j \rangle = G(t_i, t_j), \quad \langle h_0, i, h_1, j \rangle = \frac{\partial G(t_i, t_j)}{\partial t}, \quad \langle h_1, i, h_0, j \rangle = \frac{\partial^2 G(t_i, t_j)}{\partial s \partial t}, \quad \langle h_1, i, h_2, j \rangle = \frac{\partial^3 G(t_i, t_j)}{\partial s^2 \partial t^2}, \quad \langle h_2, i, h_2, j \rangle = \frac{\partial^4 G(t_i, t_j)}{\partial s^2 \partial t^2}.
\]

These formulas are basic to the construction of the Gram matrix of the canonical system (16) (for details, see [21]).

The problem of canonical transformation of integral and point-wise functionals for the spaces \( W^l_{2,1}[0, 1] \) and \( W^{l+1}_{2,1}[0, 1] \) (with norms (4a)) has been solved in [14], and for any integer \( l \) – in [21]. There it has been shown, that

\[
G(s, t) = \sum_{i=0}^{l-1} \frac{s^i}{i!} \left( \frac{1}{l+i} - (-1)^i \frac{s^{2l-i-1}}{(2l-i-1)!} \right), \quad 0 \leq s \leq t \leq 1.
\]

The values of \( G(s, t) \) for the region \( 0 \leq t < s \leq 1 \) are defined by the symmetry (22). Thus, for \( l = 1 \)

\[
G(s, t) = \begin{cases} 
1 + s, & 0 \leq s \leq t \leq 1, \\
1 + t, & 0 \leq t < s \leq 1.
\end{cases}
\]

The formula (27) allows to apply the NSC method for differential and integro-differential equations on finite segments with derivatives of any order, and to construct the solution approximations of any smoothness.

Consider the case of the semi-axis \([0, \infty)\). The identities (21) and (23) for the norm (4b) look like integral representations of values of the function \( x_i(t) \) and its derivatives up to order \( l - 1 \) in any \( t \geq 0 \):

\[
x_i^{(r)}(t) = \int_0^\infty \left[ \frac{\partial^r G(s, t_k)}{\partial t^r} x_i(s) + \frac{\partial^{l+r} G(s, t_k)}{\partial s^l \partial t^r} x_i^{(l)}(s) \right] ds, \quad r = 0, \ldots, l - 1.
\]

In order to construct the RK \( G(s, t) \) let us consider an integral functional

\[
l_\psi = \int_0^\infty \psi(s) x_i(s) ds,
\]

where \( \psi \) is integrable on \([0, \infty)\). Such a functional arises in integral and integro-differential equations. The problem of canonical representation of the functional (29) consists in finding such a function \( \eta \in W^l_{2,1} \) that the variational identity

\[
\int_0^\infty \psi(s) x_i(s) ds = \int_0^\infty \left[ \eta(s) x_i(s) + \eta^{(l)}(s) x_i^{(l)}(s) \right] ds, \quad \forall x_i \in W^l_{2,1}
\]

holds. Here \( \eta \) is the required canonical image of the functional \( l_\psi \), and the right-hand part is the inner product corresponding to the norm (4b). Existence of the function \( \eta \) in \( W^l_{2,1} \) is provided by the above mentioned Riesz theorem.

1 Directly, without explicit usage of the RK notion.
In order to solve the identity (30) with respect to \( \eta \), we assume that this function has derivatives up to the order \( 2l \). Processing the term
\[
\int_{0}^{\infty} \eta^{(l)}(s)x^{(l)}(s) \, ds
\]
with the help of integration by parts allows to obtain from (30) the next boundary value problem [14]:
\[
\begin{cases}
(-1)^{r} \eta^{(2l)}(t) + \eta(t) = \varphi, \\
\eta^{(r)}(0) = \lim_{b \to \infty} \eta^{(r)}(b) = 0, \quad r = l, \ldots, 2l - 1.
\end{cases}
\] (31)
The conditions at infinity are deduced by consideration of the reduced identity (30) with a finite integration area \([0, b] \) and by subsequent passage \( b \to \infty \).

The problem (31) is a self-adjoint semi-homogeneous boundary value problem. In [14] it has been proved, that this problem has a unique solution for any integrable function \( \varphi \) which can be represented with the help of the corresponding Green function [9]. Now we shall show, that this function is the sought RK, and accordingly, we shall denote it by \( G(s, t) \).

The Green function \( G(s, t) \) of the problem (31) exists and is defined by the equation
\[
(-1)^{r} \frac{\partial^{2l} G(s, t)}{\partial s^{2l}} + G(s, t) = 0, \quad s \neq t,
\] (32)
by the boundary conditions
\[
\frac{\partial^{r} G(0, t)}{\partial s^{r}} = 0, \quad \lim_{b \to \infty} \frac{\partial^{r} G(b, t)}{\partial s^{r}} = 0, \quad r = l, \ldots, 2l - 1,
\] (33)
and by the jump condition
\[
\frac{\partial^{2l-1} G(t - 0, t)}{\partial s^{2l-1}} - \frac{\partial^{2l-1} G(t + 0, t)}{\partial s^{2l-1}} = -1.
\] (34)

Having the Green function, we obtain the solution of (31)
\[
\eta(s) = \int_{0}^{\infty} G(s, t) \varphi(t) \, dt.
\] (35)

This function \( \eta(s) \) on construction has derivatives \( \eta^{(r)}(s) \) up to the order \( 2l \). To prove that it is the canonical image of the functional (29), it is necessary to prove that (35) is a function of the space \( W^{l}_{2,1}[0, \infty) \). It can be checked for concrete functions \( G(s, t) \) presented below. These functions exponentially decrease on \( s \) under any \( t \geq 0 \).

Thus, we have solved in terms of the Green function of the boundary value problem (31) the problem of the canonical transformation of the integral functionals (29). This Green function also represents the point-wise functionals (24) in the canonical form (25) via the formula (26). The identity (28) under \( r = 0 \) is easy to check using integration by parts and equalities (32), (33) and (34). This identity is the representation of the central property of the RK (ii), i.e. the identity (21) for the norm (4b), and the equalities (26) are the consequences of it.

The property (i) is the consequence of the above mentioned property that the function \( G(\cdot, t) (t \geq 0) \) exponentially decreases. The property (iii) of symmetry (22) is the consequence of self-adjointness of the problem (31). It has a technical character for the problem under consideration.

Thus, the Green function \( G(s, t) \), uniquely defined by the problem (32), (33) with the condition (34), is the RK of the space \( W^{l}_{2,1}[0, \infty) \). The RK problem for \( W^{l}_{2,1}[0, \infty) \) (as well as for \( W^{l}_{2,1}(-\infty, \infty) \)) with \( l = 1 \) and \( l = 2 \) has been solved in [15]. The RK for \( W^{1}_{2,1}[0, \infty) \) is
\[
G(s, t) = \begin{cases}
\begin{array}{ll}
e^{-t} \cosh(s), & 0 \leq s \leq t, \\
e^{-s} \cosh(t), & 0 \leq t < s < \infty,
\end{array}
\end{cases}
\] (36)
and for $W_{2,1}^2[0, \infty)$

$$G(s, t) = \frac{1}{2\sqrt{2}} \left\{ e^{(s-t)/\sqrt{2}} \left[ \cos \left( \frac{s-t}{\sqrt{2}} \right) - \sin \left( \frac{s-t}{\sqrt{2}} \right) \right], \quad 0 \leq s \leq t, \right.
\left. e^{(t-s)/\sqrt{2}} \left[ \cos \left( \frac{t-s}{\sqrt{2}} \right) - \sin \left( \frac{t-s}{\sqrt{2}} \right) \right], \quad 0 \leq t < s < \infty. \right\}$$ (37)

The corresponding functions of the first variable $G(\cdot, t)$ under any $t \geq 0$ exponentially decrease on the interval $(t, \infty)$, hence, they are elements of $W_{2,1}^l[0, \infty)$ with $l = 1$ or $l = 2$.

The space $W_{2,1}^2[0, \infty)$ with the RK (36) is sufficient for realizing the NSC method for integral equations on the semi-axis. Usage of the space $W_{2,1}^2[0, \infty)$ with the RK (37) provides obtaining a more smooth approximations of the desired solution and increases the possibilities of the NSC method for such problems.

In order to develop the NSC method for differential equations of second order (1) on $[0, \infty)$, we have to construct the RK for the space $W_{2,1}^2[0, \infty)$.

**Theorem 1.** The reproducing kernel of the space $W_{2,1}^3[0, \infty)$ with the norm (4b) is the symmetric function

$$G(s, t) = \sum_{i=1}^{6} c_i(t) y_i(s), \quad 0 \leq s \leq t < \infty,$$ (38)

where

$$\begin{align*}
c_1(t) &= -\frac{1}{6} \exp(-t), \\
c_2(t) &= \frac{1}{3} \exp \left( -\frac{t}{2} \right) \left( \sqrt{3} \sin \left( \frac{\sqrt{3}}{2} t \right) - \cos \left( \frac{\sqrt{3}}{2} t \right) \right) - \frac{1}{2} \exp(-t), \\
c_3(t) &= \frac{1}{2} \exp \left( -\frac{t}{2} \right) \left( \frac{1}{\sqrt{3}} \sin \left( \frac{\sqrt{3}}{2} t \right) - \cos \left( \frac{\sqrt{3}}{2} t \right) \right) - \frac{1}{3} \exp(-t), \\
c_4(t) &= \frac{1}{6} \exp \left( -\frac{t}{2} \right) \left( \sqrt{3} \cos \left( \frac{\sqrt{3}}{2} t \right) - 5 \sin \left( \frac{\sqrt{3}}{2} t \right) \right) + \frac{1}{\sqrt{3}} \exp(-t), \\
c_5(t) &= -\frac{1}{6} \exp \left( -\frac{t}{2} \right) \left( \cos \left( \frac{\sqrt{3}}{2} t \right) - \sqrt{3} \sin \left( \frac{\sqrt{3}}{2} t \right) \right), \\
c_6(t) &= -\frac{1}{6} \exp \left( -\frac{t}{2} \right) \left( \sin \left( \frac{\sqrt{3}}{2} t \right) + \sqrt{3} \cos \left( \frac{\sqrt{3}}{2} t \right) \right); \tag{39}
\end{align*}$$

and

$$\begin{align*}
y_1(s) &= \exp(s), \\
y_2(s) &= \exp(-s), \\
y_3(s) &= \exp \left( -\frac{s}{2} \right) \cos \left( \frac{\sqrt{3}}{2} s \right), \\
y_4(s) &= \exp \left( -\frac{s}{2} \right) \sin \left( \frac{\sqrt{3}}{2} s \right), \\
y_5(s) &= \exp \left( \frac{s}{2} \right) \cos \left( \frac{\sqrt{3}}{2} s \right), \\
y_6(s) &= \exp \left( \frac{s}{2} \right) \sin \left( \frac{\sqrt{3}}{2} s \right). \tag{40}
\end{align*}$$

**Proof.** The RK of the space $W_{2,1}^3[0, \infty)$, as it was shown above, is the symmetric Green function $G(s, t)$ of the boundary value problem (31), such that the function $G(\cdot, t)$ under any $t \geq 0$ belongs to $W_{2,1}^3[0, \infty)$. The Green
function is the fundamental solution to Eq. (32) (influence function) satisfying the boundary conditions (33) [9]. In turn, the fundamental solution satisfies the jump condition (34) and has the form
\[ G(s, t) = G_0(s, t) + \theta(s - t)E(s - t), \] (a)
where \( G_0(s, t) \) is the general solution of (32), \( \theta(t) \) is the Heaviside function (\( \theta(t) = 0 \) if \( t < 0 \), and \( \theta(t) = 1 \) if \( t > 0 \)), and \( E(t) \) is the solution of the initial value problem
\[ -E''(t) + E(t) = 0, \quad E(0) = E'(0) = \cdots = E^{(4)}(0) = 0, \quad E^{(5)}(0) = 1. \] (b)

The general solution of (32) is being represented in the form
\[ G_0(s, t) = c_1(t)y_1(s) + c_2(t)y_2(s) + c_3(t)y_3(s) + c_4(t)y_4(s) + c_5(t)y_5(s) + c_6(t)y_6(s), \] (c)
where \( \{y_1, \ldots, y_6\} \) is the fundamental system of the homogeneous differential equation (32), and \( \{c_1, \ldots, c_6\} \) are arbitrary functions.

The characteristic equation of (32) is \( \lambda^6 = 1 \). This algebraic equation has the roots \( \lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -(1 - i\sqrt{3})/2, \lambda_4 = -(1 + i\sqrt{3})/2, \lambda_5 = (1 - i\sqrt{3})/2, \lambda_6 = (1 + i\sqrt{3})/2 \). Correspondingly, the sought fundamental system has the form (40). Having this system, one can construct the solution to (b):
\[ E(t) = \frac{1}{6} \left( \exp(t) - \exp(-t) - \exp\left(\frac{t}{2}\right) \cos\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3} \exp\left(\frac{t}{2}\right) \sin\left(\frac{\sqrt{3}}{2}t\right) + \exp\left(\frac{t}{2}\right) \cos\left(\frac{\sqrt{3}}{2}t\right) \right). \] (d)

The functions \( c_i(t) \) are being defined by the boundary conditions (33) applied to the function (a) in view of (c) and (d). In this connection, we obtain their form (39).

In view of the symmetry, it is sufficient to represent the Green function only in the region \( 0 \leq s \leq t \), where \( G(s, t) = G_0(s, t) \). Hence, we obtain the formulas (38), (39) and (39).

Finally, we have to check whether the function \( G(\cdot, t) \) under any \( t \geq 0 \) belongs to \( W^{3,1}_2[0, \infty) \). This is readily fulfilled, because
\[ G(s, t) = \sum_{i=1}^{6} c_i(s)y_i(t), \quad t < s, \]
and in view of (39) all the functions \( c_i(s) \) exponentially decrease, and they and all their derivatives are square integrable. \( \square \)

6. Convergence of the NSC method and effective grids

Convergence of the NSC method has been investigated in [14] for systems of linear integro-differential equations of first order with arbitrary matrices on a finite segment \([a, b]\) as well as on the infinite intervals \((−\infty, \infty)\) and \([0, \infty)\). However, a system of second order can be reduced to a system of first order, and corresponding convergence results simply repeat.

**Theorem 2.** (See [14].) Let the interval of the system (1) be a finite segment \([a, b]\), and let the matrices \( A(t), B(t), C(t) \) and right-hand function \( f(t) \) be continuously differentiable. If the diameter of the grid (11) \( \tau_m \to 0 \) for \( m \to \infty \), then the z-normal solution \( x^m_z \) of the corresponding system (17), i.e. the solution (19), (20), approximates in the norm (3), (4a) the z-normal solution of (1), (2b) \( x_z \):
\[ \lim_{m \to \infty} \| x^m_z - x_z \|_{l, n} = 0. \] (41)

It was noted above, that convergence in the norm of \( W^{l,1}_2(a, b) \) entails the uniform convergence of the (continuous) derivatives \( x^{(r)}_z(t) \) with \( r = 0, 1, \ldots, l - 1 \) on the segment \([a, b]\). In turn, it provides the pointwise convergence of the derivatives
\[ \frac{d^r x^m_z}{dt^r}(t) \to \frac{d^r x_z}{dt^r}(t), \quad \text{when} \ m \to \infty, \ r = 0, 1, \ldots, l - 1, \] (42)
for any \( t \in [a, b] \). Thus, one can use the space \( W^{l}_{2,n} \) with a high factor \( l \) in order to obtain the uniform approximation of the required solution with a corresponding smoothness.

Unlike of the classical [26] collocation method, the NSC method converges under arbitrarily condensed collocation grids, and unlike the classical and spline [35] collocation methods, the NSC method is applicable to the singular equations (1).

Now let us consider the problem (1), (6) on the semi-axis \([0, \infty)\) (accounting in (1) \( a = 0, b = \infty \)). It is impossible to cover the whole infinite interval by the collocation grid sufficiently densely. Correspondingly, we introduce the quasi-initial value problem consisting in constructing an element of \( W^{l}_{2,n}[0, \infty) \) which satisfies the reduced equation

\[
A(t)\ddot{x}(t) + B(t)\dot{x}(t) + C(t)x(t) = f(t), \quad 0 \leq t \leq T,
\]

(43)
on some sufficiently large segment \([0, T] \) and the initial conditions (6). Actually any solution to (43) in \( W^{l}_{2,n}[0, \infty) \) also satisfies the limiting condition in (2a), i.e. \( x(\infty) = 0 \). As the sought function \( x(t) \) is defined on all \([0, \infty)\) but satisfies the original equation (1) only on \([0, T] \), we shall name the problem (43), (6) the semi-reduced one.

It is obvious, that the solution to the original problem (1), (2a) entails existence of the solution to the semi-reduced problem (43), (6) in the same space \( W^{l}_{2,n}[0, \infty) \). Reducing the original interval \([0, \infty) \), where Eq. (1) should be satisfied, to the finite segment \([0, T] \) should lead to an extension of the solution set. We denote the \( z \)-normal solution to the reduced problem by \( x_{z,T} \). It is natural to expect, that \( x_{z,T} \) approximates the \( z \)-normal solution to the original one \( x_{z} \) when \( T \to \infty \).

Now we introduce the collocation grid on the reduced segment \([0, T] \)

\[
0 < t_1 < t_2 < \cdots < t_m = T,
\]

(44)
and denote the \( z \)-normal solution to the collocation system (12), corresponding to this grid, by \( x^{m}_{z,T} \). This solution is also an element of \( W^{l}_{2,n}[0, \infty) \), correspondingly, it is defined on \([0, \infty)\), and the limiting condition \( x(\infty) = 0 \) is fulfilled.

**Theorem 3.** (See [14].) Let the interval of the system (1) be the semi-axis \([0, \infty)\), and let the matrices \( A(t), B(t), C(t) \) and right-hand function \( f(t) \) are continuously differentiable. If the diameters of the grids (44) \( \tau_m \to 0 \) for \( m \to \infty \), then the NSC solutions \( x^{m}_{z,T} \) to the semi-reduced quasi-initial value problems (43), (6) approximate in the norm (3), (4b) the \( z \)-normal solution to this problem \( x_{z,T} \):

\[
\lim_{m \to \infty} \| x^{m}_{z,T} - x_{z,T} \|_{l,n} = 0.
\]

These theoretical results prompt strategies of forming the effective grids (11) or (44). Basic to the proofs of the presented theorems is the fact that the sequence of the discrepancies of the corresponding collocation systems (12)

\[
\varphi^{m}(t) = A(t)x^{m}_{z}(t) + B(t)x^{m}_{z}(t) + C(t)x^{m}_{z}(t) - f(t)
\]
tends to null point-wise when the grid diameters \( \tau_m \to 0 \). In this connection the strategy of forming the effective grids is based on the fast vanishing of the discrepancy.

Consider the case of a finite segment \([a, b] \). The simplest uniform grid can be effective in problems with coefficients and solutions changing slowly. For more complicated boundary value problems an effective adaptive strategy of the sequentially vanishing discrepancy was created in [14]. Initial value problems can be solved sequentially on small segments changing slowly. For more complicated boundary value problems an effective adaptive strategy of the sequentially vanishing discrepancy was created in [14]. This strategy was developed in [16] as an adaptive strategy providing a given level \( \varepsilon > 0 \) of the integral of a square of the discrepancy, i.e. the condition

\[
\int_{a}^{b} \| \varphi^{m}(t) \|^{2} dt \leq \varepsilon.
\]

A similar strategy of the grid formation is also developed in frame of the parametrization method [20]. Obviously, this adaptive strategy can be also applied to the quasi-initial value problems.

Here we also suggest a simpler non-adaptive strategy of the grid formation for the problems on the semi-axis. Such problems, as a rule, are connected to transient processes being established. Therefore, the solution varies more in the beginning than at the large values of an independent variable. Correspondingly, the effective grids for the problem (43),
Table 1
Results of the NSC for (e1)

<table>
<thead>
<tr>
<th>$B$</th>
<th>$m$</th>
<th>sequential</th>
<th>uniform</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$8 \times 4 = 32$</td>
<td>3.43E−03</td>
<td>1.16E−02</td>
</tr>
<tr>
<td>100</td>
<td>$8 \times 14 = 112$</td>
<td>1.38E−03</td>
<td>2.10E−01</td>
</tr>
<tr>
<td>1000</td>
<td>$8 \times 42 = 336$</td>
<td>8.58E−04</td>
<td>4.47E+00</td>
</tr>
</tbody>
</table>

(6) should be more dense in the beginning. An example of the sequence of grids $\{t^m_k\}$ that are being condensed on the expanding segments $[0, t^m_m]$, when $m \to \infty$, is the grids

$$t^m_k = \frac{k^p}{m}, \quad k = 0, \ldots, m, \quad p > 1.$$  \hspace{1cm} (46)

Here $t^m_1 = 1/m \to 0$ and $t^m_m = m^{p−1} \to \infty$ if $m \to \infty$. We also note the equality $t^{2m}_m = t^m_m/2$. It means, if the number of knots is doubled, the first half of knots is being placed on the half of the original interval.

7. Numerical examples

**Example 1.** (See [16].) Let us consider the second-order initial value problem with variable degeneracy of the main part on the segment $[0, 1]$

$$-\mu \ddot{x}(t) + \dot{x}(t) + x(t) - 1 = 0, \quad 0 \leq t \leq 1,$$

$$x(0) = 1, \quad \dot{x}(0) = 1 - B,$$

where $p(t) = \sin(10t)$, $q(t) = 10 \cos(10t)$, $r(t) = -1$, the function $f(t)$ corresponds to the solution $x(t) = te^{-t} + e^{-Bt}$, $B > 1$.

The coefficients $p(t)$, $q(t)$ have zero values on $[0, 1]$, and the solution is stiff for large $B$-values. In view of the variable degeneracy, Eq. (e1) has no a finite DI and difficulties are encountered if the known difference methods for this problem are used.

We state the problem of the normal solution to (e1) (with the trivial trial function $z \equiv 0$) in the space $W^2_2[0, 1]$. The problem has been successfully solved by the two variants of the NSC method: the sequential scheme ‘step-by-step’ with eight added knots on each subinterval, and the scheme with the uniform grid. In the sequential scheme the level of the integral discrepancy (45) was $\varepsilon = 0.01$. Table 1 presents the maximal deviations of the normal splines $x^m_N$ from the exact solution $x$ on the doubled grids. The number $m$ equals the number of the grid knots in the both cases. Here effectiveness of the adaptive sequential strategy is evident.

**Example 2.** (See [34].) Here we consider the stiff boundary value problem for the linear ODE of second order with a small parameter in front of the highest derivative on the semi-axes:

$$-\mu \ddot{x}(t) + \dot{x}(t) + x(t) - 1 = 0, \quad x(0) = 0, \quad \lim_{t \to \infty} x(t) = 1.$$  \hspace{1cm} (e2)

The exact solution to this problem is

$$x(t) = 1 - \exp\left(\frac{1 - \sqrt{1 + 4\mu}}{2\mu}t\right).$$

In [34] this problem has been solved on $[0, T]$ by some special difference method by introducing the condition $\dot{x}(T) = 0$. The corresponding maximal deviations of the approximate solution from the exact solution are presented in Table 2 in the columns Zd. Here we also state the problem of the normal solution to (e2). It was solved by three variants of the NSC method and the corresponding results (the maximal deviations of the splines from the exact solution on the doubled grids) are also presented in Table 2. In the first and second variants the problem was reformulated with respect to the function $\ddot{x}(t) = x(t) - 1$ in the space $W^2_2[0, \infty)$. The collocation grid in the first variant (the column NS1) was uniform on the reduced segment $[0, T]$ with the step 0.1, and in the second (NS2) we used the condensing–expanding grid (46) with the degree $p = 3/2$. In the third variant (NS3) the transformation (8) of the original problem on $[0, \infty)$ to the
Table 2
Results for \((e2)\)

<table>
<thead>
<tr>
<th>(\mu)</th>
<th>(T = 5, m = 50)</th>
<th>(T = 10, m = 100)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(Z_d) NS1 NS2 NS3 (Z_d) NS1 NS2 NS3</td>
<td>(Z_d) NS1 NS2 NS3 (Z_d) NS1 NS2 NS3</td>
</tr>
<tr>
<td>1.0</td>
<td>5.20E−02 2.72E−03 8.23E−04 5.13E−01</td>
<td>2.30E−03 1.42E−04 1.37E−04 5.20E−01</td>
</tr>
<tr>
<td>0.1</td>
<td>2.60E−03 2.63E−05 6.86E−06 3.55E−02</td>
<td>2.60E−05 2.62E−05 2.77E−06 1.54E−01</td>
</tr>
<tr>
<td>0.01</td>
<td>2.10E−04 2.61E−05 2.42E−07 1.86E−05</td>
<td>1.50E−06 2.61E−05 1.03E−07 7.50E−05</td>
</tr>
<tr>
<td>0.001</td>
<td>2.00E−05 2.80E−05 5.65E−08 4.20E−06</td>
<td>1.40E−07 2.80E−05 8.30E−09 2.74E−06</td>
</tr>
</tbody>
</table>

problem on \([0, 1]\) was realized. Here the condensing–expanding grid (46) of the variant NS2 was transformed to the grid \(\tau_k = 1 − \exp (−t_k)\). The advantage of the NS2 variant is clearly observed.

8. Conclusion

The represented numerical variational method of normal spline-collocation is intended for ordinary linear differential and differential-algebraic equations of second order on a finite segment as well as on the number semi-axis. The system, which is solved, can have any DI or not have a finite DI (the system with variable degeneracy). Thus, it allows to bypass problems of revealing the DI and diminishing the DI if it is greater than three. The NSC method allows to solve problems on the semi-axis in the class of functions defined on the whole infinite region, bypassing the non-trivial problem of definition of a boundary condition on the right-hand side of a truncated segment.

We intend to expand the NSC method to ordinary difference-differential and to partial differential equations.

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References


