Experimental Analysis of Involutive Criteria

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Abstract

In this paper we present results of our computer experiments to study effectiveness of involutive criteria for avoiding useless prolongations in construction of polynomial Janet bases. These bases are typical representative of involutive bases. Though involutive bases are usually redundant as Gröbner ones, they can be used as well as the reduced Gröbner bases in whose theory Volker Weispfenning made a substantial contribution. Our experimental study shows that the role of criteria in the involutive approach is definitely weaker than that in Buchberger's algorithm.

1 Introduction

The concept of Gröbner bases for polynomial provided with their algorithmic characterization in terms of $S-$polynomials was invented 40 years ago by Buchberger in his PhD thesis [B65]. Since that time Gröbner bases have become the most universal algorithmic tool in commutative algebra [BW93] and found numerous fruitful applications in different areas of science and technology.

The fundamental property of a Gröbner basis of a polynomial ideal is the divisibility of any element in the related initial ideal by the leading monomial of an element in the basis [BW93]. In our paper [GB98] an algorithmic approach was developed based on an axiomatically formulated restriction for the conventional monomial division.

Such a restricted division called in [GB98] involutive, given a finite polynomial set, partitions the variables into two disjoint subsets called multiplicative and non-multiplicative. A slightly different notion of involutive division together with underlying algorithms was introduced in [A98]. In both approaches, however, construction of an involutive basis which is a (generally redundant) Gröbner basis consists in examining the involutive reducibility of non-multiplicative prolongations, i.e., products of the polynomials and their non-multiplicative variables. In doing so, the involutive algorithms also examine all essential $S-$polynomials [GB98, A98] but with rather special reductions based on involutive division.

To locate some zero-redundant and, thus, useless non-multiplicative prolongations, two Buchberger's criteria [BW93] were adopted to the involutive algorithms in the form of four different criteria [GB98, GBY01, AH02]. In this paper we present the effect of these criteria on the benchmarks from [Bench] which are widely used for testing and analysis of the

Gröbner basis software. We used C implementation [GBY01] of the involutive algorithm in its optimized form described in [G04] where some preliminary experimental results on the criteria were given.

2 Involutive criteria

Let $\mathbb{M}$ be the monoid of monomials in $\mathbb{R} = \mathbb{K}[x_1,\dots,x_n]$, $\succ$ be an admissible monomial order, and $\mathscr{L}$ be an involutive division [GB98] on $\mathbb{M}$. Then for any finite set $F \in \mathbb{R}$ and $f \in F$ division $\mathscr{L}$ defines the set $NM_{\mathscr{L}}(f,G)$ of non-multiplicative variables for f.

An ancestor $\mathrm{anc}(f)$ of a polynomial $f \in F \subset \mathbb{R}\setminus\{0\}$ is a polynomial $g \in F$ of the smallest $\deg(\mathrm{lm}(g))$ (total degree of the leading term) among those satisfying $f = g \cdot u$ modulo $\mathrm{Ideal}(F\setminus\{f\})$ with $u \in \mathbb{M}$.

Let G be an intermediate basis in the involutive algorithm [G04] and $p = f \cdot x$ be a non-multiplicative prolongation of $f \in G$ such that $\mathrm{lm}(p)$ is $\mathscr{L}-$reducible by some $g \in G\setminus\{f\}$. Then prolongation $p = f \cdot x$ can be discarded if one of the following criteria is applicable:

$C_1(p,g)$ is true iff $\mathrm{lm}(\mathrm{anc}(p))\cdot\mathrm{lm}(\mathrm{anc}(g)) = \mathrm{lm}(p)$,

$C_2(p,g)$ is true iff $\mathrm{lcm}(\mathrm{lm}(\mathrm{anc}(p)),\mathrm{lm}(\mathrm{anc}(g))) \sqsubset \mathrm{lm}(p)$,

$C_3(p,g)$ is true iff $\exists\, h \in G$ such that
$\mathrm{lcm}(\mathrm{lm}(h),\mathrm{lm}(\mathrm{anc}(p))) \sqsubset \mathrm{lcm}(\mathrm{lm}(\mathrm{anc}(p)),\mathrm{lm}(\mathrm{anc}(g))) \wedge$
$\mathrm{lcm}(\mathrm{lm}(h),\mathrm{lm}(\mathrm{anc}(g))) \sqsubset \mathrm{lcm}(\mathrm{lm}(\mathrm{anc}(p)),\mathrm{lm}(\mathrm{anc}(g)))$,

$C_4(p,g)$ is true iff $\exists\, h \in G \wedge y \in NM_{\mathscr{L}}(h,G)$ such that $\mathrm{lm}(h)\cdot y = \mathrm{lm}(f)\cdot x\ \wedge$
$\mathrm{lcm}(\mathrm{lm}(\mathrm{anc}(h)),\mathrm{lm}(\mathrm{anc}(p))) \sqsubset \mathrm{lm}(p) \wedge \mathrm{idx}(h,G) < \mathrm{idx}(f,G)$.

Here $\mathrm{lcm}(u,v)$ denotes the least common multiple of monomials u and v, $u \sqsubset v$ means that monomial u is a proper conventional divisor of v, $\mathrm{idx}(h,G)$ enumerates position of polynomial h in set G.

Criterion C_1 is Buchberger's co-prime criterion [BW93] in its involutive form [GBY01]. Criterion C_2 was derived in [GB98] as a consequence of the another Buchberger's (chain) criterion. Criteria C_3 and C_4 were derived in paper [AH02] and complement criterion C_2 to the full equivalence to Buchberger's chain criterion.

3 Benchmarking

The below table contains the timings (in seconds) for our C code [GBY01] implementing the involutive algorithm [G04] for Janet division and running on an 4 Gb Opteron-242 computer under SUSE Linux 9.1. Computation were performed over the ring of integers and for the degree-reverse-lexicographical order. The benchmarks taken from [Bench], and we show here only those of them which take more than 10 seconds for computation without criteria. If several criteria were applied, then criterion C_1 was checked first, then criterion C_2, etc.

Example	C1+C2+C3+C4	C1+C2+C3	C1+C2	C1	Without criteria
assur44	10.35	10.29	11.95	12.28	12.39
chemkin	17.84	15.00	15.22	15.6	28.56
cohn3	76.72	76.05	87.66	90.32	90.20
cyclic7	58.72	58.14	58.61	63.82	85.18
cyclic8	12056.24	12109.57	11975.60	20196.00	31751.60
discret3	23322.79	23293.4	23235.9	35282.8	41620.9
dl	270.17	305.47	335.25	394.17	475.22
eco10	52.56	56.35	56.21	58.66	76.99
eco11	765.98	826.12	659.76	677.09	697.44
eco12	4083.00	4474.48	8018.84	8333.48	9786.17
extcyc6	324.70	322.62	330.50	412.47	468.69
f744	4.88	3.86	4.54	5.00	11.92
f855	132.97	75.59	89.57	94.45	374.92
fabrice24	108.52	109.13	109.12	109.18	114.8
filter9	20.97	7.94	9.92	13.82	11.4
hairer2	62.91	70.02	104.80	246.90	2107.24
hcyclic7	64.17	56.73	72.67	145.41	221.27
hcyclic8	6024.97	4701.70	6815.82	23956.50	43647.40
hf744	22.17	17.03	17.05	23.53	52.65
hf855	2157.88	1687.12	1761.97	3466.05	11334.10
i1	98.24	100.08	113.03	127.84	152.72
ilias13	1167.18	4026.63	4087.10	4622.29	4698.26
ilias_k_2	323.59	326.51	387.85	412.33	420.87
ilias_k_3	452.32	458.70	526.69	616.01	652.98
jcf26	224.96	223.77	223.79	223.90	235.79
katsura8	27.48	27.08	26.92	27.59	32.16
katsura9	337.52	335.69	332.94	335.50	402.38
katsura10	4790.55	5089.11	5070.59	5052.38	6129.48
kin1	15.18	14.83	14.86	14.88	20.74
noon7	28.87	26.46	26.03	28.40	27.56
noon8	1552.26	1341.59	1334.02	1458.84	1628.16
noon9	83229	73106.00	71844.90	76369.60	99166.90
pinchon1	10.37	9.09	9.06	10.80	83.83
rbpl	210.94	212.24	211.63	318.33	539.05
rbpl24	108.78	109.12	109.17	109.49	113.94
redcyc7	913.75	916.23	937.82	962.13	1151.04
redeco10	18.52	18.21	17.92	18.14	22.77
redeco11	178.32	172	170.02	172.17	214.34
redeco12	1735.95	1654.52	1662.96	1686.78	2092.96
reimer6	9.69	9.42	21.93	38.56	35.49
reimer7	719.37	714.08	3290.06	9817.16	9385.17

4 Conclusion

Our computer experiments summarized in the above table show that even without criteria our code is reasonably fast, contrary to any known implementation of Buchberger's algorithm which without criteria becomes unpractical even for much smaller problems than those in the table. This is because many useless critical pairs (S-polynomials) are automatically avoided in the involutive algorithm [G04].

As we observed, application of criterion C_4 to some benchmarks leads to notable slow down the calculation. This is because the check of conditions for applicability of criterion C_4 is more expensive from the computational point of view than that for other three criteria.

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