Circular Mixed Hypergraphs III: $C$–perfection

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Printed on August 1, 1999

Abstract

A mixed hypergraph is a triple $\mathcal{H} = (X, C, D)$, where $X$ is the vertex set and each of $C$, $D$ is a family of subsets of $X$, the $C$-edges and $D$-edges, respectively. A proper $k$-coloring of $\mathcal{H}$ is an injective mapping $c : X \to \{1, \ldots, k\}$ such that each $C$-edge has two vertices with a common color and each $D$-edge has two vertices with distinct colors. Maximum number of colors in a coloring using all the colors is called upper chromatic number $\chi(\mathcal{H})$. Maximum cardinality of subset of vertices which contains no $C$-edge is $C$-stability number $\alpha_C(\mathcal{H})$. A mixed hypergraph is called $C$-perfect if $\chi(\mathcal{H}') = \alpha_C(\mathcal{H}')$ for any induced subhypergraph $\mathcal{H}'$. A mixed hypergraph $\mathcal{H}$ is called circular if there exists a host cycle on the vertex set $X$ such that every edge ($C$- or $D$-) induces a connected subgraph on the host cycle. We investigate the problem of $C$-perfection of circular mixed hypergraphs.

1 Introduction

In the classical theory of coloring for graphs and hypergraphs [2, 4], we ask for colorings of the vertices so that each edge requires at least two vertices of different colors, and ask for the minimum number of colors required. It is natural to ask the dual question to color the vertices so that each edge requires at least two vertices of the same color, and ask for the maximum number of colors needed. It is also natural to ask the combination of the above two questions [21, 22, 25, 19, 20].

In the present paper we deal with such a combination of constraints on colorings and use the terminology of [22] with small modifications.

A mixed hypergraph is a triple $\mathcal{H} = (X, C, D)$, where $X$ is the vertex set and each of $C$, $D$ is a family of subsets of $X$, the $C$-edges and $D$-edges, respectively. A proper $k$-coloring of a mixed hypergraph is a mapping from the vertex set to a set of $k$ colors so that each $C$-edge has two vertices with the same color and each $D$-edge has two vertices with different colors. A mixed hypergraph is $k$-colorable (uncolorable; uniquely colorable) if it
has a proper coloring with at most \( k \) colors (admits no proper colorings; admits precisely one proper coloring apart from permutations of colors). A strict \( k \)-coloring is a proper coloring using all \( k \) colors. The minimum number of colors in a strict coloring of \( H \) is the lower chromatic number \( \chi(H) \); the maximum number is the upper chromatic number \( \bar{\chi}(H) \). We use \( c(x) \) for the color of the vertex \( x \).

If \( H = (X, C, D) \) is a mixed hypergraph, then the subhypergraph induced by \( X' \subseteq X \) is the mixed hypergraph \( H' = (X', C', D') \) defined by \( C' = \{ C \in C : C \subseteq X' \} \) and \( D' = \{ D \in D : D \subseteq X' \} \). A mixed hypergraph \( H' = (X', C', D') \) is a partial mixed hypergraph of \( H = (X, C, D) \) if \( X' \subseteq X, C' \subseteq C, D' \subseteq D \). For the last we use the notation \( H' \subseteq H \). For short, sometimes we write \( (X, D) \) or \( H_D \) instead of \( H = (X, \emptyset, D) \), and write \( (X, C) \) or \( H_C \) instead of \( H = (X, C, \emptyset) \), keeping in mind that the respective coloring restrictions are fulfilled.

For each \( k \), let \( r_k \) be the number of partitions of the vertex set into \( k \) nonempty parts (color classes) such that the coloring constraints are satisfied on each \( C \)- or \( D \)-edge. In fact \( r_k \) is the number of different strict \( k \)-colorings of \( H \) if we disregard permutations of colors. The vector \( R(H) = (r_1, \ldots, r_n) \) is the chromatic spectrum of \( H \).

A mixed hypergraph is reduced if it contains no included \( C \)-edges and no included \( D \)-edges, and moreover, the size of each \( C \)-edges is at least 3, and the size of each \( D \)-edge is at least 2. As it follows from the splitting-contraction algorithm [22], the coloring properties of arbitrary mixed hypergraph can be derived from the respective reduced mixed hypergraph. Therefore, without loss of generality, throughout the paper we consider (unless contrary is stated) the reduced mixed hypergraphs.

A mixed hypergraph \( H \) is called a mixed interval hypergraph [5] if there exists a linear ordering of its vertices such that every edge \((C \text{- or } D\text{-})\) induces an interval in that ordering. Mixed interval hypergraphs have been introduced and investigated in [5].

**Definition 1** A mixed hypergraph \( H \) is called circular if there exists a host cycle \( C \) on the vertex set \( X \) such that every \( C \)-edge and every \( D \)-edge induces a connected subgraph of the cycle \( C \).

In other words, for circular mixed hypergraph there exists always a circular ordering of the vertex set \( X \), say, \( X = \{x_1, x_2, \ldots, x_n, x_1\} \) such that every edge \((C \text{- or } D\text{-})\) induces an interval in this ordering.

Introduced in [21, 22], the theory of mixed hypergraphs is growing rapidly and represents an area with many possible applications. As models they may be applied in list-free modelling of list-colorings of graphs [19], integer programming [19, 13], investigating the coloring properties of block designs [14, 15, 16, 17] with potential applications in coding theory [6], see also [7], resource allocation, Data Base Management, parallel computing, scheduling of systems of power supplies, in the study of heredity in populations with sexual reproduction [23] where the problems have combinatorial nature.


A special case of the mixed hypergraph coloring problem was investigated by Ahlswede e.a. [1] where it was shown that questions about colorings in which the vertices of every
edge receive a certain percentage of distinct colors were equivalent to several multi-user source coding problems.

There are several classes of mixed hypergraphs which have been introduced recently; among them are the mixed hypertrees [19, 22], monostars [8], co-perfect [22], mixed interval [5], uncolorable [19], uniquely colorable [20], pseudo-chordal [25], quasi-interval [18] and some other mixed hypergraphs. They represent the generalisations of graphs, hypergraphs and colorings from different points of view. However, only the mixed interval hypergraphs and mixed hypertrees represent the generalisations based on the underlying host graph (for a hypergraph the host graph is the graph on the same vertex set and such that every hyperedge induces connected subgraph of the host graph). Namely in those cases the host graphs are paths and trees respectively, i.e. graphs without cycles. In this sense they express nothing else than the coloring properties of generalized paths and trees with respect to their connected subgraphs.

Every graph \( G \) satisfies \( \chi(G) \geq \omega(G) \), where \( \omega(G) \) is the size of the largest clique. The perfect graphs are the graphs such that \( \chi(G') = \omega(G') \) for every induced subgraph \( G' \). Many subclasses of perfect graphs have beautiful structural properties and admit fast algorithms for many optimization problems.

Voloshin [22] introduced a natural analogue of perfection for the upper chromatic number. In a mixed hypergraph, a set of vertices is \( C \)-stable if it contains no \( C \)-edge. The \( C \)-stability number \( \alpha_C(H) \) is the maximum cardinality of a \( C \)-stable set of \( H \). Always \( \chi(H) \leq \alpha_C(H) \), because a set with more distinct colors than \( \alpha_C(H) \) would assign different colors to all the vertices of some \( C \)-edge. A mixed hypergraph \( H \) is \( C \)-perfect [22] if \( \chi(H') = \alpha_C(H') \) for every induced subhypergraph \( H' \).

The study of \( C \)-perfection is at the very beginning. Several classes of \( C \)-perfect and minimal non-\( C \)-perfect mixed hypergraphs have been found. For mixed hypergraph, a host graph is a graph on the same vertex set and such that every \( C \)- and every \( D \)-edge induces a connected subgraph of the host graph. A cycloid [22] is a \( r \)-uniform \( C \)-hypergraph denoted \( C_n^r \) which has \( n \) \( C \)-edges and admits a simple cycle on \( n \) vertices as a host graph. A polystar is a mixed hypergraph with at least two \( C \)-edges in which the set \( Y \) of vertices common to all \( C \)-edges (center) is nonempty, and every pair in \( Y \) forms a \( D \)-edge. When center consists of one vertex then polystar is also called monostar. Hence, each polystar in \( C \)-hypergraph is a monostar. A bistar (called co-bistar in [22]) is a mixed hypergraph in which there exists a pair of distinct vertices common to all \( C \)-edges and not forming a \( D \)-edge.

Bistars are \( C \)-perfect; polystars are not [22]. Also cycloids of the form \( C_{2r-1} \) are not \( C \)-perfect ([22]). When \( n = 2r - 1 \), we have \( \alpha_C(C_n^r) = 2r - 3 \) and \( \chi(C_n^r) = 2r - 4 \). These cycloids are analogous to the known minimal imperfect graphs. Polystars and cycloids of the form \( C_{2r-1}^r, r \geq 3 \), are minimal non-\( C \)-perfect mixed hypergraphs in the sense that every proper induced subhypergraph of such a cycloid is \( C \)-perfect, and every subhypergraph of a polystar that is not a polystar is \( C \)-perfect.

Voloshin conjectured [22] that a \( r \)-uniform \( C \)-hypergraph is \( C \)-perfect if and only if it has no induced monostar or cycloid of the form \( C_{2r-1}^r, r \geq 3 \). These two natural non-\( C \)-perfect families lead to an analogue of Berge’s Strong Perfect Graph Conjecture, which
states that a graph $G$ is perfect if and only if no odd cycle of length at least 5 occurs as an induced subgraph of $G$ or $\bar{G}$.

Voloshin [22] proved this conjecture for $C$-hypertrees. There are some other classes of $C$-hypergraphs for which the conjecture is true. However, situation is more complex than in case of graphs. Bulgaru and Voloshin [5] proved that a mixed interval hypergraph is $C$-perfect if and only if it has no induced polystars. Voloshin [24] extended this to mixed hypertrees. Prisakaru [18] studied the mixed hypergraphs whose $C$-edges are the maximal cliques of an interval graph; such a quasi-interval $C$-hypergraphs are $C$-perfect if and only if they have no induced polystars.

Polystars generally are not uniform and they already suggested that the family of non-uniform minimal non-$C$-perfect mixed hypergraphs may be complex.

$C$-perfection is closely related to the algorithmic complexity issues with respect to the upper chromatic number. It is well known that perfection of graphs has led to the efficient polynomial and linear time algorithms for solving many (not only coloring) generally $NP$-complete problems. One can already see that the similar situation happens to the $C$-perfection.

All known classes of $C$-perfect mixed hypergraphs can be upper colored efficiently. These include bistars and the computation of upper chromatic number for mixed interval hypergraphs [5] and quasi-interval $C$-hypergraphs [18]. These were the simplest cases of mixed hypergraphs. As we have already mentioned, mixed hypertree is $C$-perfect if and only if it does not contain any monostar as an induced subhypergraph. In this case there is an efficient polynomial algorithm (and it is possible to develop a linear time algorithm) for finding the upper chromatic number and a respective coloring. When monostars are allowed in $C$-hypertrees, the problem is already $NP$-complete. It is $NP$-complete even for monostars themselves [9]. One may expect that $C$-perfection will lead to efficient polynomial time algorithms for finding the upper chromatic number and respective colorings. In addition, $C$-perfection may serve as a hint for search efficient polynomial algorithms for other hard combinatorial problems on discrete structures.

2 Main results

Since in the proofs many cases have to be considered, and for the clarity of the exposition we first formulate the main results of this section. These results are implied by the theorems of the next subsection. Let us strongly obey the agreement that all the vertices are ordered on the host cycle and in all expressions like ‘$(u, v)$-path’ or ‘$D$-distance between $u$ and $v$’ we mean the host-path or the number of vertices minus 1 in this path respectively, when starting at $u$ and ending at $v$ and going according the given cyclic ordering. Moreover, for the vertex $u \in X$ the notation $u^+, u^{++}, \ldots$ or, equivalently, $u^1, u^2, \ldots$ will always mean the first successor, the second successor, $\ldots$ of $u$ in the cyclic ordering.

Definition 2 Let $S = (x_0, C_1, x_1, C_2, x_2, \ldots, x_{t-2}, C_{t-1}, x_{t-1}, C_t, x_t)$ be a sequence of different vertices $x_0, \ldots, x_t$ and different $C$-edges $C_1, \ldots, C_t$ of size 3 such that $x_{t-1}$ is the
first vertex and $x_i$ is the last vertex of $C_i$. The sequence $S$ is called a \textit{triple-chain} if $x_i \neq x_0$ and is called a \textit{triple-circuit (triple-cycle)} if $x_i = x_0$.

Triple-chains in mixed hypergraphs may pass (transport) the colors from the vertex $x_0$ on the vertex $x_i$. Namely, if every two consecutive vertices of the host path corresponding to $S$ form a $D$-edge, then necessarily $c(x_0) = c(x_1) = \ldots = c(x_i)$, though intermediate vertices may receive arbitrary colors different from $c(x)$. Such a triple-chain is said to be strong.

**Definition 3** Let $H = (X, C, D)$ be a \textit{circular mixed hypergraph}. For each $k \geq 3$ we denote $C = C_k$ if and only if every $k$ consecutive vertices of $X$ form a $C$-edge. For each $l \geq 2$ we denote $D = D_l$ if and only if every $l$ consecutive vertices of $X$ form a $D$-edge. The \textit{circular mixed hypergraph} $H = (X, C_k, D_l)$ is called $(k, l)$-uniform.

**Definition 4** A $(3,2)$-uniform circular mixed hypergraph $H = (X, C_3, D_2)$ is called a \textit{complete circular mixed hypergraph} and is denoted by $KC(n, 3, 2)$.

Thus the mixed hypergraph $(X, \emptyset, D_2)$ is a simple classical cycle on $n$ vertices, and the mixed hypergraph $(X, C_k, \emptyset), k \geq 3,$ is a co-cycloid as defined in [22].

**Definition 5** For every odd $n \geq 3$ we define a circular mixed hypergraph $F_n$ such that $F_n = (X, C, D_2)$ where $(X, C, \emptyset)$ is a minimal triple-chain which covers all the vertices of $X$ and $(X, \emptyset, D_2)$ is a classical cycle.

Thus the number of $C$-edges and $D$-edges of $F_n$ is $(n - 1)/2$ or $n$ respectively. $F_3$ consists of a $D$-triangle and of one $C$-edge coinciding with $X$.

### 3 $C$-perfect circular mixed hypergraphs

A set $X'$ of vertices of a mixed hypergraph $H = (X, C, D)$ is said to be $C$-independent, if $X'$ does not contain a $C$-edge.

The cardinality of a largest $C$-independent set of $H$ is called the $C$-stability number $\alpha_C(H)$ of $H$. A mixed hypergraph $H = (X, C, D)$ is said to be $C$-perfect if for each induced subhypergraph $H'$ of $H$ the upper chromatic number $\bar{\chi}(H')$ equals the $C$-stability number $\alpha_C(H')$, i.e., $\bar{\chi}(H') = \alpha_C(H')$.

Let $C$ and $C'$ be two $C$-edges of a circular mixed hypergraph $H = (X, C, D)$ meeting each other either in one vertex or in two vertices joint by a $D$-edge of size 2. Let $C$ and $C'$ not cover $X$, i.e., $C \cup C' \neq X$. The subhypergraph $H'$ induced by the vertex set $C \cup C'$ has only $C$-edges $C^*, C \neq C^* \neq C'$, containing $C \cap C'$. Obviously $\bar{\chi}(H') = \lvert C \cup C' \rvert - 2 < \lvert C \cup C' \rvert - 1 = \alpha_C(H')$. Consequently, if $H$ contains two $C$-edges meeting in a $K_1$ or $K_2$ and not covering $X$, then $H$ is not $C$-perfect.

More general, a mixed hypergraph $H = (X, C, D)$ is called a $C$-monostar or covered $C$-bistar if $\cap \{C \mid C \in C\}$ consists of one vertex or two vertices joint by a $D$-edge of size 2, respectively. A $C$-monostar and a covered $C$-bistar are not perfect. For interval mixed hypergraphs Bulgaru and Voloshin proved:
Theorem 1 (E.Bulgaru and V.I.Voloshin [5]). An interval mixed hypergraph $\mathcal{H}$ is $\mathcal{C}$-perfect if and only if $\mathcal{H}$ does not contain induced $\mathcal{C}$-monostars and induced covered $\mathcal{C}$-bistars.

A mixed hypergraph $\mathcal{H}$ is called critically $\mathcal{C}$-imperfect if $\bar{\chi}(\mathcal{H}) < \alpha_C(\mathcal{H})$, and each proper induced subhypergraph of $\mathcal{H}$ is $\mathcal{C}$-perfect.

The monostar and the covered $\mathcal{C}$-bistar are critically $\mathcal{C}$-imperfect. Let $\mathcal{S}_n$ denote the class of all circular mixed hypergraphs of order $n$ containing no induced monostar and no induced covered $\mathcal{C}$-bistar with the following property: there is an ordering of the $\mathcal{C}$-edges $C_0, C_1, C_2, \ldots, C_{s-1}$ so that $C_j \cup C_{j+1} = X$ and $C_j \cap C_{j+1}$ induces a $K_1$ or a $K_2$, $0 \leq j \leq s - 1$ (induces modulo $s$). If $\mathcal{H}$ contains no $\mathcal{D}$-edges of size 2 then $\mathcal{S}_n$ is empty for even $n$, and $\mathcal{S}_n$ is the cycloid $C_2$, for odd $n = 2r - 1$. An example for even $n = 2s$ is $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ with $X = \{0, 1, \ldots, 2s - 1\}$, $(X, \mathcal{D})$ is the $2s$-cycle, and $\mathcal{C} = \{\{0, 1, \ldots, s\} + 2t | 0 \leq t \leq s - 1\}$, indices taken modulo $n$.

Proposition 1 The hypergraph $\mathcal{H}$ of $\mathcal{S}_n$ are critically $\mathcal{C}$-imperfect.

Proof. Each proper induced subhypergraph of $\mathcal{H}$ is the union of interval mixed hypergraphs containing no induced monostar and no induced covered $\mathcal{C}$-bistar. Then by Bulgaru and Voloshin [5] each proper induced subhypergraph of $\mathcal{H}$ is $\mathcal{C}$-perfect. So we have only to show that $\bar{\chi}(\mathcal{H}) < \alpha_C(\mathcal{H})$.

Obviously $\mathcal{H}$ contains two vertices covering all $\mathcal{C}$-edges (take two opposite vertices of the host cycle), and $\alpha_C(\mathcal{H}) = n - 2$. The vertices of the host cycle can be partitioned into $M_0, M_1, \ldots, M_{s-1}$ being in this cyclic order on the host cycle such that $M_t \in \{K_1, K_2\}$, and there exist $C_j, C_{j+1} + 1$ with $C_j \cap C_{j+1} + 1 = M_t$. Suppose $\mathcal{H}$ has a strict $(n - 2)$-coloring.

We consider two cases.

Case 1. Three vertices $x, y, z$ have the same color, say 1, and the remaining $n - 3$ vertices have pairwise different colors 2, 3, $\ldots, n - 2$. The vertices $x \in M_l, y \in M_{l+1}$, and $z \in M_{l+1}, l \neq l_1 \neq l_2 \neq l_3$ since $n \geq 7$ there is a vertex $p \in M_t, l \notin \{l_1, l_2, l_3\}$, and no vertex of $M_t$ is colored 1. Let $M_t = C_j \cap C_{j+1}$ for one $j, 0 \leq j \leq s - 1$. Then $C_j \{p\} \cap C_{j+1} \{p\} = \emptyset$, and $C_j$ or $C_{j+1}$ is not suitably colored, a contradiction!

Case 2. Two vertices $x$ and $y$ have the same color, say 1, and two other vertices $p$ and $q$ have the same color, say 2, and the remaining $n - 4$ vertices have pairwise different colors 3, 4, $\ldots, n - 2$. The vertices $x \in M_l, y \in M_{l+1}$, $l \neq l_1$. If $l_1 - 1 \neq l_2 \neq l_1 + 1$ then there exists an $M_l = C_k \cap C_{k+1}$ so that $x \in C_k \cap M_l$ and $y \in C_{k+1} \cap M_l$. Since $(C_j \setminus M_t) \cap (C_{j+1} \setminus M_t) = \emptyset$ at least one of $C_j$ or $C_{j+1}$ has no two vertices of the same color, and this $\mathcal{C}$-edge is not suitably colored.

Next suppose $l_2 = l_1 + 1$. Then there are a $k$ and an $l, 0 \leq k, l \leq s - 1$, such that $M_l = C_k \cap C_{k+1}$ and $C_k \supseteq M_l$ and $C_{k+1} \supseteq M_l$. At least one of $C_k$ and $C_{k+1}$ has no two vertices of the same color, and this $\mathcal{C}$-edge is not suitable colored. This contradiction proves the proposition.

We remark, that if any $\mathcal{C}$-edge $C$ is added to $\mathcal{H}$ in a suitable way then the intersection $C_j \cap C$ is not an interval for some $\mathcal{C}$-edge $C_j$.  

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Theorem 2 A circular mixed hypergraph $H$ is $C$-perfect if and only if $H \notin S_n$ and $H$ does not contain induced $C$-monostars and induced covered $C$-bistars.

A set $T$ of vertices is a $C$-transversal (represents all $C$-edges), if each $C$-edge contains a vertex of $T$. If $X'$ is a $C$-independent set of vertices then $X \setminus X'$ is a $C$-transversal of $H$.

Proof. $\Rightarrow$ If $H$ is $C$-perfect then $H \notin S_n$ and $H$ does not contain induced $C$-monostars and induced covered $C$-bistars.

$\Leftarrow$ Suppose that $H \notin S_n$ and $H$ does not contain induced $C$-monostars and induced covered $C$-bistars.

There are three cases.

Case 1. Let no two $C$-edges cover all vertices of $H$. Then they meet in the empty set or in an interval of the host cycle. Let $\{v_1, v_2, \ldots, v_t\}$ be a smallest $C$-transversal of $H$. Hence $X \setminus \{v_1, v_2, \ldots, v_t\}$ is a maximum $C$-independent set with $\alpha_C(H) = n - t$ vertices. Obviously, the upper chromatic number $\chi(H) \leq \alpha_C(H) = n - t$. We have to show that $\chi(H) = n - t$, i.e., $H$ has a strict $(n - t)$-coloring.

Let $S_i$ denote the set of all $C$-edges containing $v_i$. Then $T_i := \cap \{C| C \in S_i\}$ is an interval of the host cycle containing $v_i$ with endvertices $y_i$ and $z_i$ such that $y_i$ precedes $z_i$. Since two different $C$-edges which meet have a "bad" intersection (different from $K_1$ and $K_2$) $\{z_i^-, z_i\} \subseteq T_i$, if $z_i^- z_i \notin D$, or $\{z_i^-, z_i^-, z_i\} \subseteq T_i$, if $z_i^- z_i \in D$.

If $S_i \cap S_{i+1} \neq \emptyset$ with $w \in S_i \cap S_{i+1}$ then already $\{v_1, v_2, \ldots, v_{i-1}, w, v_{i+2}, \ldots, v_t\}$ is a $C$-transversal with $t - 1$ vertices, a contradiction to the minimality of $t$. Hence $S_i \cap S_{i+1} = \emptyset$ for all $1 \leq i \leq t$ (indices mod $t$). Color $z_i, z_i^-$ with the color $i$, if $z_i^- z_i \notin D$ or $z_i^- z_i \in D$, respectively. All other vertices are colored with the colors $t + 1, \ldots, n - t$ so that any two of the remaining vertices have different colors. So, $H$ has a strict $(n - t)$-coloring, and $\alpha_C(H) = n - t = \chi(H)$. Each induced subhypergraph $H' = (X', C', D')$ of $H$ with $X' \neq X$ is the union of interval graphs; Theorem 1 implies $\chi(H') = \alpha_C(H')$. Consequently, $H$ is $C$-perfect.

Case 2. Let $H$ contain two $C$-edges $A$ and $B$ meeting in two intervals $U$ and $V$, which do not form a common interval. Then the union $A \cup B = X$ covers all vertices of $X$. If $S := \cap \{C| C \in S_i\} \neq \emptyset$ and $S \notin \{K_1, K_2\}$ then two vertices of $S$ can be colored with the same color, say $1$. Give all other vertices pairwise different colors $2, 3, \ldots, n - 1$. The circular mixed hypergraph $H$ has a strict $(n - 1)$-coloring and $\chi(H) = n - 1 = \alpha_C(H)$. Thus $H$ is $C$-perfect.

If $S := \cap \{C| C \in S_i\} \neq \emptyset$ and $S \in \{K_1, K_2\}$ then $H$ contains two $C$-edges $C_1$ and $C_2$ with $C_1 \cap C_2 = S$. Since $H$ does not contain a $C$-monostar or a covered $C$-bistar the union $C_1 \cup C_2 = X$. The conditions $S \subseteq A$, $S \subseteq B$, and $A \cup B = X$ imply that $C_1 \subseteq A$ or $C_1 \subseteq B$ or $C_2 \subseteq A$ or $C_2 \subseteq B$, a contradiction!

In the rest of the proof let $\cap \{C| C \in S_i\} = \emptyset$, and $\alpha_C(H) \leq n - 2$. Each $C$-edge $C, C \notin \{A, B\}$, contains $U$ and the vertices immediately preceding and succeeding $U$, or $C$ contains $V$ and the vertices immediately preceding and succeeding $V$. Hence a vertex of $U$ and a vertex of $V$ form a transversal of $H$. Thus $\alpha_C(H) = n - 2$.

In the following five subcases we shall show that $\chi(H) = n - 2$. Consequently, $H$ is $C$-perfect.

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Case 2.1. Let \( U, V \not\in \{K_1, K_2\} \). Then color the endvertices of the interval \( U \) by 1, the endvertices of the interval \( V \) by color 2, and give all other vertices the colors 3, 4, \ldots, \( n-2 \). The circular mixed hypergraph \( H \) has a strict \((n-2)\)-coloring, and \( H \) is \( C \)-perfect.

Case 2.2. Let \( U \cong K_1 \) (or \( U \cong K_2 \)), \( V \not\in \{K_1, K_2\} \). Let \( u \) be the only vertex (or let \( u, u^+ \) be the vertices and \( uu^+ \) the edge) of \( U \). Then color \( u^- \), \( u^+ \) by color 1, the endvertices of the interval \( V \) by color 2, and give all other vertices pairwise different colors 3, 4, \ldots, \( n-2 \). The circular mixed hypergraph \( H \) has a strict \((n-2)\)-coloring, and \( H \) is \( C \)-perfect.

Case 2.3. Let \( U \cong K_1 \cong V \) with \( U = \{u\} \) and \( V = \{v\} \). Each \( C \)-edge \( C \) contains \( u, u^+ \), if \( uu^+ \not\in \mathcal{D} \), and \( u, u^+ \), \( u^{++} \), if \( uu^+ \in \mathcal{D} \), or \( C \) contains \( v, v^+ \), if \( vv^+ \not\in \mathcal{D} \), and \( v, v^+, v^{++} \), if \( vv^+ \in \mathcal{D} \). If \( uu^+ \not\in \mathcal{D} \), then color \( u \), \( u^+ \) by 1, and, if \( uu^+ \in \mathcal{D} \) and \( u^{++} \not\equiv v \), then color \( u \), \( u^{++} \) by 1. If \( vv^+ \not\in \mathcal{D} \), then color \( v \), \( v^+ \) by 2, and, if \( vv^+ \in \mathcal{D} \) and \( v^{++} \not\equiv u \), then color \( v \), \( v^{++} \) by 2. Give all other vertices pairwise different colors 3, 4, \ldots, \( n-2 \). The circular mixed hypergraph \( H \) has a strict \((n-2)\)-coloring, and \( H \) is \( C \)-perfect.

Case 2.4. Let \( U \cong K_1 \) and \( V \cong K_2 \) with \( U = \{u\} \) and \( V = \{v^-, v, v^-\} \).

Let \( S := \{u^-, u^+, \ldots, v^-\} \) and \( T := \{v^+, \ldots, u^-, u^+, u^+\} \).

Case 2.4.1. Let \( S \not\in \mathcal{C} \) or \( T \not\in \mathcal{C} \). By symmetry reasons we can choose: \( T \not\in \mathcal{C} \). Each \( \mathcal{C} \)-edge \( C \) contains \( u, u^+ \), if \( uu^+ \not\in \mathcal{D} \), and \( u, u^+ \), \( u^{++} \), if \( uu^+ \in \mathcal{D} \), or \( C \) contains \( v, v^+ \), if \( vv^+ \not\in \mathcal{D} \), and \( v, v^+, v^{++} \), if \( vv^+ \in \mathcal{D} \). If \( uu^+ \not\in \mathcal{D} \), then color \( u \), \( u^+ \) by 1, and, if \( uu^+ \in \mathcal{D} \), \( u^{++} \not\equiv v \), then color \( u \), \( u^{++} \) by 1; further, color \( v^- \), \( v^+ \) by 2. Give all other vertices pairwise different colors 3, 4, \ldots, \( n-2 \). If \( uu^+ \in \mathcal{D} \), \( u^{++} = v^+ \), then color \( u \), \( u^{++} = v^+, v^+ \) by 1. Give all other vertices pairwise different colors 3, 4, \ldots, \( n-2 \). If \( uu^+ \in \mathcal{D} \), \( u^{++} = v^- \), then color \( u \), \( u^{++} = v^-, v^- \) by 1. In all other vertices pairwise different colors 2, 3, 4, \ldots, \( n-2 \). In all cases the circular mixed hypergraph \( H \) has a strict \((n-2)\)-coloring, and \( H \) is \( C \)-perfect.

Case 2.4.2. Let \( S, T \in \mathcal{C} \). Each \( \mathcal{C} \)-edge \( C \) contains \( u, u^+ \), if \( uu^+ \not\in \mathcal{D} \), and \( u, u^+ \), \( u^{++} \), if \( uu^+ \in \mathcal{D} \), or \( C \) contains \( v, v^+ \), if \( vv^+ \not\in \mathcal{D} \), and \( v, v^+, v^{++} \), if \( vv^+ \in \mathcal{D} \). (The latter assertion comes from the fact that each \( \mathcal{C} \)-edge \( C' \) meeting \( v \), \( C' \not\in \{A, B, S, T\} \), has no: "good" intersection with \( T \).) Obviously, \( u^{++} \not\equiv v \). If \( uu^+ \not\in \mathcal{D} \), the color \( u \), \( u^+ \) by 1, and, if \( uu^+ \in \mathcal{D} \), then color \( u \), \( u^{++} \) by 1. If \( vv^+ \not\in \mathcal{D} \), then color \( v \), \( v^+ \) by 2, and, if \( vv^+ \in \mathcal{D} \), \( v^{++} \not\equiv u \), then color \( v \), \( v^{++} \) by 2. Give all other vertices pairwise different colors 3, 4, \ldots, \( n-2 \). If \( vv^+ \in \mathcal{D} \), \( v^{++} = u \), then color \( v \), \( v^+ = u, u^+ \) or \( v \), \( v^+ = u, u^{++} \) by 1. Give all other vertices pairwise different colors 2, 3, 4, \ldots, \( n-2 \). In all cases the circular mixed hypergraph \( H \) has a strict \((n-2)\)-coloring, and \( H \) is \( C \)-perfect.

Case 2.5. Let \( U \cong K_2 \cong V \) with \( U = \{v^-, u, u^-u\} \) and \( V = \{v^-, v, v^-\} \). Each \( \mathcal{C} \)-edge contains \( u^-, u^+, v^-, v^+ \). Color \( u^-, u^+ \) by 1 and \( v^- \), \( v^- \) by 2. Give all other vertices pairwise different colors 3, 4, \ldots, \( n-2 \). The circular mixed hypergraph \( H \) has a strict
Case 3. Let $\mathcal{H}$ not satisfy the conditions of Cases 1 or 2. Hence $\mathcal{H}$ contains two $C$-edges $A$ and $B$ meeting in $K_1$ or $K_2$ and covering all vertices of $X$. Thus $(X, \{A, B\}, D)$ is a monostar or a covered $C$-bistar, but it is no induced subhypergraph. Firstly we prove

Lemma 1 Let $\mathcal{H} = (X, C, D)$ contain two $C$-edges $C_1 = \{x_1^1, x_2^1, \ldots, x_{k_1}^1\}$ and $C_2 = \{x_1^2, x_2^2, \ldots, x_{k_2}^2\}$ with $x_{k_1}^1 \in C_1 \cap C_2 \in \{K_1, K_2\}$ and $C_1 \cup C_2 = X$. Let each $C$-edge $C \notin \{C_1, C_2\}$ containing $x_{k_2}^2$ have a bad intersection with $C_2$, i.e., $C$ also contains $x_{k_2}^2$, if $(x_{k_2}^2, x_{k_2}^1) \notin D$, or $x_{k_2}^2$, if $(x_{k_2}^2, x_{k_2}^1) \in D$. Then $\bar{\chi}(\mathcal{H}) = n - 2$.

Proof. We color the mixed hypergraph in the following way: $x_{k_1-1}^1, x_{k_1}^1, x_{k_1-2}^1, x_{k_1}^1$ are colored by 1, if $(x_{k_1-1}^1, x_{k_1}^1) \notin D$ or $(x_{k_1-1}^1, x_{k_1}^1) \notin D$, respectively; $x_{k_2-1}^1, x_{k_2}^1$ or $x_{k_2-2}^1, x_{k_2}^1$ are colored by 2, if $(x_{k_2-1}^2, x_{k_2}^2) \notin D$ or $(x_{k_2-1}^2, x_{k_2}^1) \notin D$, respectively; all remaining vertices are now colored by pairwise different colors $3, 4, \ldots, n - 2$. Each $C$-edge $C, C \notin \{C_1, C_2\}$, contains $x_{k_1}^1$ or $x_{k_2}^2$ (because otherwise $C \subseteq C_1$ or $C \subseteq C_2$). If $x_{k_2}^2 \in C$ then by hypothesis $x_{k_2-1}^2 \in C$ for $(x_{k_2-1}, x_{k_2}^2) \notin D$ and $x_{k_2-2} \in C$ for $(x_{k_2-1}, x_{k_2}^2) \in D$. Since $\mathcal{H}$ contains no $C$-edges with a good intersection not covering $X$ each $C$-edge $C$ with $x_{k_1} \in C$ also contains $x_{k_1-1}$, if $(x_{k_1-1}, x_{k_1}) \notin D$ and $x_{k_1-2} \in C$, if $(x_{k_1-1}, x_{k_1}) \notin D$. Hence $\mathcal{H}$ has a strict coloring by $n - 2$ colors, and the proof of the lemma is complete. □

In continuing the proof of Case 3 we shall show either $\bar{\chi}(\mathcal{H}) = n - 2$ or the $C$-edges can be ordered $C_0, C_1, \ldots, C_{s-1}$ so that $C_j \cup C_{j+1} = X$ and $M_j := C_j \cap C_{j+1}$ induces a $K_1$ or a $K_2$, $0 \leq j \leq s - 1$ (induces modulo $s$), and $\mathcal{H} \in S_n$. Let $C_0 = A$, $C_1 = B$ and $M_1 = C_0 \cap C_1 \notin \{K_1, K_2\}$. Let $x_{k_1}^1$ be the end vertex of $C_1$ not in $M_1$. If each $C$-edge $C \notin C_1$ containing $x_{k_1}^1$ has a bad intersection with $C_1$ then the Lemma implies $\bar{\chi}(\mathcal{H}) = n - 2$ and $\mathcal{H}$ is $C$-perfect. Next let there be an $C$-edge $C_2$ with $M_2 := C_1 \cap C_2 \in \{K_1, K_2\}$. Since $\mathcal{H}$ has no induced monostar and no induced covered $C$-bistar the $C$-edge $C_2$ is unique. Thus a chain $C_0, C_1, C_2$ is uniquely constructed. By induction the existence of the ordered sequence $C_0, C_1, C_2, \ldots, C_{s-1}$ can be proved, or $\bar{\chi}(\mathcal{H}) = n - 2$ and $\mathcal{H}$ is $C$-perfect. In the first case $\mathcal{H}$ cannot have additional $C$-edges, and $\mathcal{H} \in S_n$. □

4 Definition of the class $T_n$

We start our construction by partitioning a host cycle on a vertex set $X$ of $n \geq 8$ vertices into $2k \geq 8$ subsets of cardinality $\leq 2$. Add to each 2-element subset a $D$-edge of size 2. Then the host cycle is partitioned into $2k$ subgraphs $U_1, U_2, U_3, \ldots, U_k$, which are in this cyclic order on the host cycle and induce a $K_1$ or a $K_2$ of $(X, D)$. $U_j (U_j^*)$ is said to be opposite to $U_j^* (U_j)$ on the host cycle. For each pair $U_j, U_j^*$ we introduce two $C$-edges $C_j, C_j^*$ with $C_j \cap C_j^* = U_j \cup U_j^*$. Add further $D$-edges of size 2. Let $D$ denote the set of all $D$-edges so introduced, and $C := \{C_1, C_2, \ldots, C_k, C_1^*, C_2^*, \ldots, C_k^*\}, k \geq 4$. Then $T_n$ consists of all circular mixed hypergraphs so obtained.
**Proposition 2** The circular mixed hypergraph $\mathcal{H} \in T_n$ is $\mathcal{C}$-perfect if and only if there exists an index $j$ so that

1. $U_j$ and $U_j^*$ induce a $K_2$; or
2. $U_j$ and $U_{j+1}$ induce a $K_1$ and a $K_2$, and $U_j^*, U_{j+1}^*$ do the same; or
3. $U_j = \{u_j\}, U_{j+1} = \{u_{j+1}\}, U_j^* = \{u_j^*\}, U_{j+1}^* = \{u_{j+1}^*\}$ and $(u_j, u_{j+1}), (u_j^*, u_{j+1}^*) \notin D$; or
4. $U_j = \{u_j\}, U_{j+1} = \{u_{j+1}\}, (u_j, u_{j+1}) \notin D$ and $U_j^*$ or $U_{j+1}^*$ induces a $K_2$.

**Corollary 1** The circular mixed hypergraph $\mathcal{H} \in T_n$ is $\mathcal{C}$-imperfect if and only if the following condition is satisfied:

If $|U_j| = 2 = |U_{j+1}|$, or $|U_j| = 2, |U_{j+1}| = 1$, or $|U_j| = 1, |U_{j+1}| = 2$, or $|U_j| = |U_{j+1}| = 1$ and $U_j, U_{j+1}$ are not joint by an $D$-edge, then $|U_j^*| = |U_{j+1}^*| = 1$ and $U_j^*, U_{j+1}^*$ are joint by an $D$-edge.

**Remark 1** If $|U_j| = |U_{j+1}| = 1$ and $U_j, U_{j+1}$ are joint by an $D$-edge then $U_j^*, U_{j+1}^*$ can be freely chosen.

**Proof of Proposition 2.** For each index $j$ holds: if $P \in U_j$ and $Q \in U^*_j$ then $P, Q$ cover all $\mathcal{C}$-edges of $\mathcal{H}$, i.e., $X \setminus \{P, Q\}$ is a maximum $\mathcal{C}$-independent set, and the $\mathcal{C}$-stability number $\alpha_\mathcal{C}(\mathcal{H}) = n - 2$.

(1) The one-vertex-deleted subhypergraphs of $\mathcal{H}$ are $\mathcal{C}$-perfect.

**Proof.** Delete any vertex $P \in U_j$ from $\mathcal{H}$. Then in $\mathcal{H} \setminus \{P\}$ the $\mathcal{C}$-edges $C_j = U_{j+1} \cup \ldots \cup U_{j+1}^*$, $C_{j+2} = U_{j+2} \cup \ldots \cup U_{j+2}^*$, ..., $C_{j-1} = U_{j-1}^* \cup \ldots \cup U_{j-1}$ remain. Hence the intersection of all these $\mathcal{C}$-edges is $U_{j-1}^* \cup U_j^* \cup U_{j+1}^*$. Then color a vertex $P \in U_{j-1}^*$ and a vertex $Q \in U_{j+1}^*$ by color 1 and color the remaining $n - 2$ vertices pairwise differently with the colors 2, ..., $n - 1$. Hence $\bar{\chi}(\mathcal{H} \setminus \{P\}) = 1$. Obviously, the vertex $P$ covers all $\mathcal{C}$-edges of $\mathcal{H} \setminus \{P\}$, and $\alpha_\mathcal{C}(\mathcal{H} \setminus \{P\}) = \bar{\chi}(\mathcal{H} \setminus \{P\}) = 1$.

Condition (1) implies that $\mathcal{H}$ contains no $\mathcal{C}$-monostar and no covered $\mathcal{C}$-bistar; and $\mathcal{H}$ is $\mathcal{C}$-perfect if and only if $\bar{\chi}(\mathcal{H}) = n - 2 = \alpha_\mathcal{C}(\mathcal{H})$.

**Case A.** Let $\mathcal{H}$ meet the requirements of Proposition 2, i.e., there exists an index $j$ so that $U_j, U_{j+1}, U_j^*, U_{j+1}^*$ satisfy one of the condition 1. - 4. Then it is possible to chose vertices $P_j \in U_j$, $P_{j+1} \in U_{j+1}$, $Q_j \in U_j^*$, $Q_{j+1} \in U_{j+1}^*$ so that $P_j$ and $P_{j+1}$ can be properly colored 1, $Q_j$ and $Q_{j+1}$ can be properly colored 2, and the remaining $n - 4$ vertices obtain the pairwise different colors 3, 4, ..., $n - 2$. So $\mathcal{H}$ has a strict $(n - 2)$-coloring and $\mathcal{H}$ is $\mathcal{C}$-perfect.

**Case B.** Let $\mathcal{H}$ do not meet the requirements of Proposition 2. Then the requirements of Corollary 1 are fulfilled. We shall show that $\bar{\chi}(\mathcal{H}) \leq n - 3$.

Suppose $\mathcal{H}$ has a strict $(n - 2)$-coloring.

There are two cases.

**Case 1.** There are three vertices of color 1; all other $n - 3$ vertices have pairwise different colors 2, 3, ..., $n - 2$. Since $k \geq 4$ there is an index $j$ such that no vertex of $U_j \cup U_j^*$ is
colored 1. Then \( C_j = U_j \cup U_{j+1} \cup \ldots \cup U_j^* \) or \( C_j^* = U_j^* \cup U_{j+1}^* \cup \ldots \cup U_j \) has at most one vertex of color 1, and \( \mathcal{H} \) is not properly colored.

**Case 2.** There is a pair \( P, Q \) of vertices of color 1 and a pair \( R, S \) of vertices of color 2, and all other \( n - 4 \) vertices have pairwise different colors 3, 4, \ldots , \( n - 2 \). Obviously, \( P, Q \) are not in the same \( U_j \). W.l.o.g. let \( P \in U_1 \) and \( Q \in U_j \), \( 2 \leq j \leq k+1 \), where \( U_{k+1} = U_1^* \).

**Subcase 2.1.** Let \( j \geq 3 \). If \( \{ R, S \} \not\subseteq U_2 \cup U_2^* \) then \( C_2 \) or \( C_2^* \) is not properly colored. If \( \{ R, S \} \subseteq U_2 \cup U_2^* \) then \( C_3 \) or \( C_3^* \) is not properly colored.

**Subcase 2.2.** Let \( j = 2 \). Then \( U_1, U_2 \in \{ K_1, K_2 \} \) and if \( U_1 = \{ u_1 \} \) and \( U_2 = \{ u_2 \} \) then \( (u_1, u_2) \not\in D \).

The reformulation of requirements 1. – 4. in Corollary 1 implies that \( U_1^* = \{ u_1^* \} \), \( U_2^* = \{ u_2^* \} \), and \( (u_1^*, u_2^*) \in D \). Hence \( u_1^* \) and \( u_2^* \) are not both colored 2, say \( u_1^* \) is not colored 2. Then \( C_2 = U_2 \cup U_3 \cup \ldots \cup U_2^* \) or \( C_1^* = U_1^* \cup U_2^* \cup \ldots \cup U_1 \) is not properly colored.

**References**


