Optimization of SMES and superconducting magnets with a derivative free deterministic method

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Abstract
This article presents a constrained optimization method, based on the duality theory, which does not need the gradients. The method is used to optimize superconducting devices. In order to reduce the computing effort, the initial optimization problem is divided into two coupled optimization problems. One manages the geometrical parameters, the other finds the best current densities for a given geometrical configuration.

Keywords: Optimization; Augmented Lagrangian; Derivative free; Superconductor; Superconducting magnet; SMES

1. Introduction
This article presents an original methodology of optimization. The currently studied applications are the superconducting magnets and superconducting magnetic energy storage (SMES) devices optimization. For these types of problems the magnetic quantities are evaluated by numerical integration of the Biot–Savart law. Therefore, it is often impossible to obtain the gradients with respect to the geometrical parameters of the device. These problems are known to have a lot of local minima. For this reason probabilistic methods are often used [1–3]. Due to their probabilistic nature these kinds of methods have to explore a large number of device configurations so their convergence, in term of objective function evaluations, is slow. Important efforts are made to improve the convergence rate [2–5]. On the contrary, the deterministic methods [8,9,11] have a better convergence rate and allow a more accurate localization of the minima. The main difficulty is to give a pertinent starting configuration in order to decrease the risk of falling into a local minimum. In this article we introduce a deterministic method which can be used alone or to quickly improve initial approximations given by a probabilistic method. The originality of our approach comes from two points that are usually poorly exploited. In a first time we show how to build non-linear constrained optimization procedures that does not need the use of the gradients. In a

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second time we introduce a decoupling between the geometric parameters and the current densities. This new formulation of the problem leads to two coupled optimization problems. A substantial gain in CPU time is reached exploiting at the best each new geometrical configuration.

2. Magnetic quantities calculations

Most of the times superconducting devices are iron free, thus magnetostatic calculations are done using the Biot–Savart law. For a bounded domain \( \Omega \) supporting the current density \( J(x) \), we have:

\[
A(x_0) = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{J(x_p)}{|x_0 - x_p|} \, dx_p
\]

\[
B(x_0) = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{J(x_p)(x_0 - x_p)}{|x_0 - x_p|} \, dx_p
\]

\[
E_{ab} = \frac{1}{2} \int_{\Omega} A^{(a)}(x_p) J^{(b)}(x_p) \, dx_p
\]

For the solenoid case, even if we use elliptic integrals, only a semi-analytical integration is possible [12]. Therefore, a numerical integration procedure must be used. Like the functions Eqs. (1) and (2) have singularities, an adaptive integration method is used, a Gauss–Kronrod 15–31 points [14] integration rule.

3. Optimization without gradient

3.1. Introduction

Let us define the following non-linear constrained optimization problem:

\[
X = \arg \min_{X \in \mathbb{R}^n} f(X) \quad \text{subject to } g(X) = 0 \quad \text{and } h(X) \leq 0
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \), \( g : \mathbb{R}^n \to \mathbb{R}^p \), \( p \) equality constraints

\( h : \mathbb{R}^n \to \mathbb{R}^q \), \( q \) inequality constraints

In order to solve the problem Eq. (4) with a deterministic method several classical strategies exist:

1. The initial constrained problem is transformed into a problem without constraint by adding a penalty to the objective function each time the constraints are violated. Those simple methods are not used anymore alone [6] because they leads to badly conditioned problems.

2. If the functions \( f, g \) and \( h \) are smooth, the necessary conditions of Kuhn and Tucker can be used. This leads to a non-linear system of equations that can be numerically solved by a Newton like method. Those type of methods are very efficient if the gradients are available [8].

3. The initial problem is transformed into a dual one. The Lagrangian function \( L(\lambda, X) \) is introduced and the resolution of the initial problem consists in finding the saddle points of the Lagrangian. This approach is also very efficient [9,7].

3.2. Augmented Lagrangian methods

The optimization method developed here is based on the augmented Lagrangian of Rockafellar [9]. The initial problem Eq. (4) is transformed into the following dual problem Eq. (5) where the saddle points
$(\lambda, X)$ have to be found.

$$X = \arg \min_{X \in \mathbb{R}^n} \left( \max_{\lambda \in \mathbb{R}, h \in \mathbb{R}^q} L[\lambda, X] \right)$$

(5)

The augmented Lagrangian of Rockafellar is defined by Eq. (6), the parameter $r$ is the augmentation of the Lagrangian:

$$L[\lambda, X] = f(X) + \lambda g(X) + \frac{1}{2} ||g(X)||^2 + \sum_{i=1}^{q} \psi_i(r, \lambda_h, h, X)$$

(6)

The functions $\psi_i$ are defined by:

$$\psi_i(r, \lambda_h, h, X) = \frac{1}{2r} \left( \max\{\lambda_h i + rh_i(X), 0\}^2 - \lambda_h i^2 \right)$$

(7)

The localization of the saddle points of the Lagrangian Eq. (6) is performed using the Uzawa algorithm [9,7], where each iteration is a minimization without constraint followed by a fixed step of the gradient method in the direction $d = (d_g, d_h)$ to update the Lagrangian multipliers $\lambda$. Introducing the projection operator $P_+: \mathbb{R}^q \rightarrow \mathbb{R}^q$ on $\mathbb{R}^q_+$, the Uzawa algorithm is given in Fig. 1.

The Uzawa algorithm has a modular structure that will allow us to choose the way of:

1. Carrying out the unconstrained minimization, step (2b).
2. Updating the multipliers, step (2b).

The following paragraphs detail these two steps. The important point to notice is the possibility to perform all the calculus without using the gradients of the functions $f, h$ or $g$.

### 3.3. Derivative free minimization, step (2a)

For the step (2a) of the Uzawa algorithm (Fig. 1), we must use a derivative free unconstrained minimization procedure. Several methods exist, we can cite the Powell method [10], the Brent method [10] which add some refinements to the previous method and the Mifflin method [11] which is a quasi-Newton method. We have chosen here to use the Powell method because of its simplicity and the good observed

---

Fig. 1. Uzawa algorithm.
Fig. 2. Powell derivative free minimization algorithm.

1. Let \( X(0) \), \( f(0) = f(X(0)) \) be an initial approximation of \( \text{arg \, min}_{X \in \mathbb{R}^n} f(X) \).
2. Take an initial set \( \{e_i, i = 1..n\} \) of \( n \) linearly independent directions.
3. For \( k = 0, 1, \ldots \) until convergence do
   a. \( X(0)^{k+1} = X(k) \) and \( f(0)^{k+1} = f(X(k)) \).
   b. For \( i = 1, \ldots, n \) do
      i. Unidimensional minimization
         \( \delta_i = \text{arg \, min}_{\delta} f(X(k,\lambda) + \delta e_i) \)
      ii. Update
         \( X(k,\lambda) = X(k,\lambda) + \delta_i \), \( f(k,\lambda) = f(X(k,\lambda)) \)
   c. end for \( i \)
4. Extrapolation
   \( d = X(k,\lambda) - X(0) \)
   \( X_{\text{p}} = X(0) + d \) et \( f_{\text{p}} = f(X_{\text{p}}) \)
5. Highest descend
   \[ \begin{align*}
   j &= \text{arg \, min}_{j \in \{1, \ldots, n\}} (f(k+1)^{j} - f(k)^{j}) \\
   \Delta &= f(k+1)^{j} - f(k)^{j} \end{align*} \]
6. Criterion to try an extrapolation
   \( t_1 = (f(k)^{\text{p}} - 2 f(k)^{j} + f(k)) \) \( t_2 = \frac{1}{2} (f(k)^{\text{p}} - f(k)^{j}) \)
7. If \( t_2 < 0 \) and \( t_1 < 0 \) \( \delta = \frac{\Delta}{t_1} \)
   a. Unidimensional minimization
      \( \delta = \text{arg \, min}_{\delta} f(X(0) + \delta d) \)
   b. Update
      \( X^{(k+1)} = X(0) + \delta d \) et \( f^{(k+1)} = f(X^{(k+1)}) \)
   c. New descend direction \( \theta_j = d \)
8. Else
   a. Update
      \( X^{(k+1)} = X(0) + \delta d \) et \( f^{(k+1)} = f(X^{(k+1)}) \)

Fig. 2. Powell derivative free minimization algorithm.

3.4. Multipliers update, step (2b)

The multipliers update, step (2b) of the Uzawa algorithm, is a fixed step gradient method. The displacement \( d \) is evaluated using the gradient:

\[ \lambda \rightarrow l(\lambda) = \min_{\lambda \in \mathbb{R}} L(\lambda, X) \]  

This gradient can be calculated easily. A classical result [9,7] shows that if the point \( X^{(k+1)}_\lambda \) is a minimum (or more precisely a stationary point) of

\[ X \rightarrow L(\lambda, X) \]  

The numerical behavior. The detailed algorithm is given Fig. 2, the unidimensional minimizations are performed by the Brent’s line search method [10].
then the gradient \( \nabla (\lambda \rightarrow l(\lambda)) \) at the point \((\lambda, X_{k+1})\) is:

\[
\nabla l|_{(\lambda, X_{k+1})} = (r_g(X_{k+1}), r_h(X_{k+1}))
\]

(10)

So the expression of the displacement \( d \) of the step (2b) is simply:

\[
(2b) \quad d^{k+1} = (r_g(X_{k+1}), r_h(X_{k+1}))
\]

(11)

4. Decoupling between the geometric parameters and the current densities

We can describe a superconducting device by a set of geometrical parameters \( G \) and a set of current densities \( J \) flowing in the coils. A change in the geometric parameters \( G \) induces a large computing effort because the integral expressions Eqs. (1)–(3) have to be revaluated. Whereas, by linearity, a modification of the current densities \( J \) is immediate. In order to exploit at the best each geometrical configuration, we introduce a decoupling between the geometrical parameters and the current densities. Superconducting devices design takes the following form Eq. (12):

\[
(G, J) = \arg \min_{(G, J)} f(G, J)
\]

\[
g(G, J) = 0
\]

\[
h(G, J) \leq 0
\]

(12)

In our approach, this problem is divided into two coupled optimization problems Eq. (13).

External optimization

\[
\begin{align*}
G &= \arg \min_G f(G, \phi(G)) \\
G(G, \phi(G)) &= 0 \\
h(G, \phi(G)) &\leq 0
\end{align*}
\]

Internal optimization

\[
\begin{align*}
\phi(G) &= J = \arg \min_J f_G(J) \\
R_G(J) &= 0 \\
h_G(J) &\leq 0
\end{align*}
\]

(13)

The aim of this decomposition is to exploit at the best each geometrical configuration. The external optimization has in charge the management of the geometric parameters \( G \) evolution while the internal optimization returns the best current densities \( J \) for the given geometrical configuration \( G \).

The function \( \phi, G \rightarrow J = \phi(G) \) is formally introduced to denote the internal optimization procedure calls from the external one.

Guided by physical considerations, the functions \( f_G, G_G \) and \( h_G \) are chosen to exploit at the best the geometrical configuration \( G \). If for certains \((k, l)\):

\[
(k, l) \in \{1, \ldots, p\} \times \{1, \ldots, q\}
\]

we keep:

\[
\begin{align*}
(g_G)_k(J) &= g_k(G, J) \\
(h_G)_l(J) &= h_l(G, J)
\end{align*}
\]

(14)
and if these constraints can be satisfied for any geometry $G$ then the corresponding constraints can be removed from the external optimization problem (the couple $(G, J = \phi(G))$ will respect all the constraints).

To clarify the approach, let us consider a SMES optimization. This superconducting device used to store energy $E = (1/2) J^T L J$ (where $[L]$ is the inductance matrix) is constituted by $N$ coils. Each coil is energized with a current $J_i$ and has a volume $V_i$. The usual problem, in order to reduce the cryogenic costs, is to minimize the volume of the device for a given stored energy $E_0$. The constraints are to do not violate the critical current of the conductor and to keep a low stray field $B_{\text{Stray}}$. Let us denote $B_{\text{Cond}}(i)$ the maximal magnetic induction on the conductor $i$ and by $J_C(B)$ the critical current of the conductor $i$ for a given magnetic induction $B$. Thus, the SMES design is the optimization problem Eq. (16):

$$
\begin{align*}
\begin{cases}
(G, J) = \arg \min_{(G, J)} \sum V_i(G) \\
E(G, J) = E_0 \\
|J_i| \leq J_C^{\text{cond}}(|B_{\text{Cond}}(G, J)|), \quad i = 1, \ldots, N \\
|B_{\text{Stray}}(G, J)| \leq B_{\text{Stray}}^{\text{max}}
\end{cases}
\end{align*}
$$

(16)

Following our approach, a possible decomposition is:

$$
\begin{align*}
\begin{cases}
G = \arg \min_G \sum V_i(G) \\
E(G, \phi(J)) = E_0 \\
\phi(G) = J = \arg \min_J -E(G, J) \\
\begin{cases}
|J_i| \leq J_C^{\text{cond}}(|B_{\text{Cond}}(G, J)|), \quad i = 1, \ldots, N \\
|B_{\text{Stray}}(G, J)| \leq B_{\text{Stray}}^{\text{max}}
\end{cases}
\end{cases}
\end{align*}
$$

(17)

Moreover, in practice the internal optimization problem is often linear, quadratic or weakly non-linear. The magnetic induction on the conductors $B_{\text{Cond}}(J)$ and the stray field $B_{\text{Stray}}(J)$ are linear functions of the current, whereas the energy $E(J)$ is a quadratic one. The critical current density curve $J_C^{\text{cond}}(B)$ is generally non-linear but very smooth.

5. Applications

5.1. Introduction

We present here a fully detailed application of our method applied to a simple magnet optimization. The aim of this magnet is to produce a homogeneous field over a sphere. This kinds of problems arise in applications like medical imagery magnets or in accelerator physics magnets. Let us consider solenoids having the same $Oz$-axis. Due to the axisymmetry it is sufficient to consider a disk of radius $r_{\text{obj}}$ centred at the origin. Let us introduce $N$ uniformly distributed points $(r_i, z_i)$ on the centred circle of radius $r_{\text{obj}}$ and denote by $B_i(G, J)$ the magnetic induction at the point $i$. We also consider the point $(r_0, z_0) = (0, 0)$ centred at the origin.
\[ (r_i, z_i) = \begin{cases} (0, 0), & \text{if } i = 0 \\ (r_{\text{obj}} \sin \left( \frac{2\pi i}{N} \right), r_{\text{obj}} \cos \left( \frac{2\pi i}{N} \right)), & \text{if } i \neq 0 \end{cases} \] (18)

In order to obtain a homogeneous field, the objective function is:

\[ f(G, J) = \sum_{i=0}^{N} \| B_i(G, J) - \left( \begin{array}{c} 0 \\ B_{\text{obj}} \end{array} \right) \|_2^2 \] (19)

where \( B_{\text{obj}} \) is the desired magnetic induction over the disk. In the objective function Eq. (19) non-uniformly distributed points or different weights \( w_i \) could have been used for each point \( i \). The treatment of this extended problem would have not changed the presented approach. Due to the inverse nature of this problem (finding \( J \) from \( B \) in the integral equation Eq. (2)) it is often necessary to perform a regularization when the number of coils increases (to avoid oscillating current solutions). Again, in this case the Tikhonov regularization can be used without any modification of the presented approach, simply considering the new objective function:

\[ f(G, J) = \sum_{i=0}^{N} \| B_i(G, J) - \left( \begin{array}{c} 0 \\ B_{\text{obj}} \end{array} \right) \|_2^2 + \alpha \sum_{1 \leq k < l \leq NJ} |J_k - J_l|^2 \] (20)

5.2. Magnet constituted of two solenoids

The first case we are considering is a magnet Fig. 3 composed by two solenoids \( S_a \) and \( S_b \). Their mean radius is \( R_1 \) and the positions of their centers are \(-Z/2, Z/2\). The thickness \( e \), the length \( 2l \) and the current density \( J_a = J_b \) are fixed and are the same for the two solenoids.

The two geometrical parameters to be optimized are:

\[ G = (R_1, Z) \] (21)

![Fig. 3. Geometry of the magnet and numerical values of the constant parameters.](image)
The unconstrained problem is:
\[(R_1, Z) = \arg \min_{(R_1, Z) \in \mathbb{R}^2_+} f(R_1, Z) \quad (22)\]

The solution of the problem Eq. (22) is expected to be close to the Helmholtz configuration for which \(R_1 = Z\). The advantage of this test problem is that the objective function Eq. (22) can be analytically determined when \(N = 2\). In this case the magnetic field is only evaluated on the \(Oz\)-axis at the three following points
\[\{(0, 0), (0, -r_{obj}), (0, r_{obj})\} \quad (23)\]

The magnetic induction generated by a solenoid on its axis \([12]\) is given by Eq. (24) therefore the analytical expression of the objective function \(f\) Eq. (22) can be easily found (Fig. 4).
\[B(0, z) = \frac{J^2}{2} \left( F \left( \frac{r_2 + z}{r_1}, \frac{l + z}{r_1} \right) + F \left( \frac{r_2 - z}{r_1}, \frac{l - z}{r_1} \right) \right) \quad (24)\]
\[F(\alpha, \beta) = \mu_0 \beta \log \left( \frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{1 + \sqrt{1 + \beta^2}} \right) \quad (25)\]

The level sets of the objective function \(f\) are given in Fig. 5.

We see a deep valley like in the Rosenbrock function. These kinds of shapes often lead to badly conditioned optimization problems. This example Eq. (22) is thus a good test example for checking the validity of our approach. The existence of an analytical expression for the objective function (and for its gradient) has allowed us to compare our results to the built in procedures of the program Mathematica.

The search for the minimum is performed using three different approaches:
(a) Our procedure based on the derivative free Powell algorithm.
(b) Mathematica procedure FindMinimum.
(c) Mathematica procedure FindRoot applied to \(\nabla f = 0\).

We also present the results obtained for \(N = 12\). But for this case, the analytical expression of the gradient is not available, so we only give the results obtained by our method. In both cases (\(N = 2\) and 12), the three initial points used to initialize the optimization procedures are:
\[A = (0.1, 0.1), \quad B = (0.2, 0.2), \quad C = (0.3, 0.3) \quad (26)\]
The obtained results for $N = 12$ are:

(a)  $R_1$ $Z$ $f$

$X^{(0)} = A$  0.0653159  0.0761841  $3.1436 \times 10^{-5}$

$X^{(0)} = B$  0.0653301  0.0761767  $3.14377 \times 10^{-5}$

$X^{(0)} = C$  0.0653301  0.0761767  $3.14377 \times 10^{-5}$

The obtained results for $N = 2$ are:

<table>
<thead>
<tr>
<th>$R_1$</th>
<th>$Z$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X^{(0)} = A$</td>
<td>0.0679144</td>
<td>0.0744834</td>
</tr>
<tr>
<td>(a)</td>
<td>0.0679144</td>
<td>0.0744834</td>
</tr>
<tr>
<td>(b)</td>
<td>0.0679144</td>
<td>0.0744834</td>
</tr>
</tbody>
</table>

Fig. 5. Level sets of the objective function $f(R_1, Z)$ for $N = 2$. 
We see that our numerical procedure has a good behavior. The obtained solution is similar to the one obtained using the gradient information using a quasi-Newton method (BFGS update of the hessian) (b). For the initial points \(B\) and \(C\), the approach (c) consisting of solving the Euler’s equation \(\nabla f = 0\) converges to some stationary points which are not the expected minimum.

### 5.3. Magnet constituted of three solenoids, free currents

A third solenoid \(S_c\) is added, centred at the origin \(z = 0\) with a mean radius of \(R_2\). The other geometrical parameters \(e\) and \(2l\) are still the previously defined dimensions given in Fig. 6.

<table>
<thead>
<tr>
<th>(X^{(0)} = B)</th>
<th>(X^{(0)} = C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_1)</td>
<td>(Z)</td>
</tr>
<tr>
<td>(a) 0.0679144</td>
<td>0.0744834</td>
</tr>
<tr>
<td>(b) 0.0679144</td>
<td>0.0744834</td>
</tr>
<tr>
<td>(c) 113.053</td>
<td>111.471</td>
</tr>
</tbody>
</table>

![Fig. 6. Magnet constituted of three solenoids.](image)
The three free geometric parameters of the device are:

\[ G = (R_1, R_2, Z) \]  

(27)

Now, we also search for the current densities. Because of the magnet symmetry we take \( J_a = J_b \) thus there are two free current densities:

\[ J = (J_a, J_c) \]  

(28)

These current densities are constrained to stay below the critical current density of the conductor. For our example we have used a typical NbTi industrial conductor critical curve. This critical curve is plotted in Fig. 7 where the filling and security factors of the winding have been taken into account.

This example illustrates the decomposition of the original problem into two coupled optimization problems. The original optimization problem is given by Eq. (29) where \( f \) is still the objective function defined in Eq. (19).

\[
\begin{align*}
\left\{ \begin{array}{l}
(R_1, R_2, Z, J_a, J_c) = \arg \min_{(J_a, J_c)} f(R_1, R_2, Z, J_a, J_c) \\
|J_a| \leq J_{C}(\|B_{Cond}^{\alpha}\|_{(R_1, R_2, Z, J_a, J_c)}) \\
|J_c| \leq J_{C}(\|B_{Cond}^{\beta}\|_{(R_1, R_2, Z, J_a, J_c)})
\end{array} \right.
\]

(29)

This optimization problem is decomposed into the two following problems:

1. External optimization problem, geometrical parameters:

\[
\begin{align*}
\left\{ \begin{array}{l}
(R_1, R_2, Z) = \arg \min_{(R_1, R_2, Z)} f(R_1, R_2, Z, \phi(R_1, R_2, Z)) \\
\phi(R_1, R_2) = (J_a, J_c)
\end{array} \right.
\]

(30)

2. External optimization problem, current densities:

\[
\begin{align*}
\left\{ \begin{array}{l}
\phi(R_1, R_2, Z) = (J_a, J_c) = \arg \min_{(J_a, J_c)} f(R_1, R_2, Z, J_a, J_c) \\
\text{Def} \\
|J_a| \leq J_{C}(\|B_{Cond}^{\alpha}\|_{(R_1, R_2, Z, J_a, J_c)}) \\
|J_c| \leq J_{C}(\|B_{Cond}^{\beta}\|_{(R_1, R_2, Z, J_a, J_c)})
\end{array} \right.
\]

(31)
The three initial points \((R_1, R_2, Z, J_a, J_c)\) are:

\[
A = (0.1, 0.1, 0.1, 200, 200) \\
B = (0.2, 0.2, 0.2, 200, 200) \\
C = (0.3, 0.3, 0.3, 200, 200) 
\]  

The obtained solutions for \(N = 12\) with our method (a) are:

\[
(a) \quad R_1, R_2, Z, J_a, J_c, f
\]

\[
A = 0.214772, 0.0688586, 0.0611983, 493.483, -33.2161, 1.94959 \times 10^{-8} \\
B = 0.0937718, 0.200116, 0.353462, 461.169, 593.225, 4.3593 \times 10^{-8} \\
C = 0.261143, 0.393782, 0.351784, 524.897, 663.458, 2.301 \times 10^{-8} 
\]

We see that the objective function \(f\) has several local minima. The convergence to one of this local minimum depends on the initial point. A possible approach to reduce this phenomenon is to initialize the search with a probabilistic method and then use the presented deterministic method to achieve a fast and accurate localization of the expected global minimum. In this example, the obtained solutions are large magnets, the physical interpretation is that homogeneity is better with distant coils. More compact magnets could have been achieved adding geometric constraints (but in this case there is a degradation of the magnetic homogeneity). The solution \((a, A)\) exhibits an oscillation of the current densities. In practice, this is generally not a good situation because for these configurations, the field homogeneity quality is more sensitive to the current values flowing in the coils. In these cases, the use of the Tikhonov regularization [13] gives more robust configurations. Let us give here details concerning the solution \((a, B)\). The first figure (Fig. 8) shows the magnetic field homogeneity:

\[
\left\| B(r, z) - \left( \frac{0}{2} \right) \right\| \times 10^3
\]  

\[
\text{Fig. 8. Solution (a, B): magnetic field plot } \left\| B(r, z) - (0, 2) \right\| \times 10^3.
\]
Due to the symmetry of the problem it is sufficient to plot the field over the region:

\[(r, z) \in [0, 0.1\text{m}] \times [0, 0.1\text{m}]\]  \hspace{1cm} (34)

For this simple magnet (three identical solenoids and two current densities) a homogeneity of 200 ppm has been achieved.

The next figure (Fig. 9) gives the magnetic field and field lines over a larger region.

6. Conclusions

A general constrained optimization procedure has been presented. This method, based on the duality theory, uses the augmented Lagrangian of Rockafellar and does not need the gradients of the involved
functions. This deterministic method can be used as an alternative or as a complement to the probabilistic ones. A decoupling between the geometric parameters and the current densities has been introduced. With this decomposition each geometrical configuration is exploited at the best and a reduction of the computing effort is achieved. The future investigation is the introduction of an initial probabilistic search for the external problem (geometry). While the internal optimization problem is always solved by a deterministic method, an initial probabilistic search will be performed for the external problem. The fully (geometry and currents) deterministic approach is switched on to achieve a fast and precise localization of the minimum as soon as a potentially good starting point is detected by the probabilistic method.

References