Reducibility in Finite Posets

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A notion of reducibility in finite posets is studied. Deletable elements in upper semimodular posets are characterized. Though it is known that the class of upper semimodular lattices is reducible, we construct an example of an upper semimodular poset that is not reducible. Reducibility of pseudocomplemented posets is studied.

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1. Introduction

All the posets/lattices considered here are finite with element 0. An element $x$ of a poset satisfying certain properties is deletable if $P - x$ is a poset satisfying the same properties. A class of posets is reducible if each poset of this class admits at least one deletable element.

When restricted to lattices, a class of lattices is reducible if and only if one can go from any lattice in this class to the trivial lattice by a sequence of lattices of the class obtained by deleting one element in each step. This notion, however, is different from the notion of dismantalability for lattices; see [6].

It is known that the class of distributive lattices need not be reducible and, thus, also the class of modular lattices. However, the class of pseudocomplemented lattices as well as the classes of semimodular and locally distributive lattices are reducible; see [2]. It would be worthwhile to investigate the notion of reducibility for more general structures such as semilattices and/or posets.

In Section 2, we characterize the elements that are deletable in upper semimodular posets. We will show by a counterexample that the class of upper semimodular posets is not reducible. Venkatnarasimhan [7] investigated pseudocomplemented posets. It was then natural to find out whether the class of pseudocomplemented posets also turns out to be reducible or not. In Section 3, we show by a counterexample that this is not so.

For a subset $A$ of a poset $P$, the lower cone $A^\uparrow$ of the set $A$ is the set given by

$$A^\uparrow = \{ x \in P : x \leq b \text{ for all } b \in A \}.$$ 

The upper cone $A^\downarrow$ of $A$ is defined dually.

In a poset $P$, for elements $a, b$ the notation $a \prec b$ denotes $a$ is covered by $b$.

2. Reducibility in Upper Semimodular Posets (USM Poset)

A poset $P$ with element 0 is an upper semimodular (in brief USM) poset if it satisfies the following condition

(USM) : For $a, b, c \in P, (a \neq b)$ with $c \prec a, \ c \prec b$ there is a $d \in P$ such that $a \prec d, \ c \prec d, \ b \prec d$.

Since $P$ is finite, it also has an element 1. The Jordan–Dedekind chain condition is well known. A poset $P$ is said to satisfy the Jordan–Dedekind (in brief JD) condition if all maximal chains between the same endpoints have the same finite length; see Gratzer [4] or Birkhoff [1].
Figure 1. A counterexample for the converse of Lemma 1.

An element $x \in P$ is join-irreducible (respectively meet-irreducible) if either $x$ is 0 (respectively 1) or $x$ covers (respectively is covered by) a single element. $J$ (respectively $M$) shall denote the set of all non-zero join-irreducible (respectively non-unit meet-irreducible) elements of a given poset. An element $x$ of a poset $P$ is called a node if $x$ is comparable with any other element. We also use the following notations: for $x \in P$,

$$\{x\}^- = \{y \in P : y \leq x\}$$

and

$$\{x\}^+ = \{z \in P : x \leq z\}.$$

If $x \in J$, $\{x\}^-$ is a singleton set and so also is $\{x\}^+$ if $x \in M$, and we shall denote these elements by $x^-$ and $x^+$, respectively. For $x \in P$, the depleted poset $P - x$ will be denoted by $P'$ and for $a, b \in P'$, if $a$ is covered by $b$ in $P'$ in the induced partial order, we shall denote the same by $a \rightarrow b$.

We mention the following results.

**Lemma 1.** Let $P$ be a poset, $x \in P$, $P' = P - x$ and $x \rightarrow b$. If $b \in J$, then $y \rightarrow b$ (i.e., $y$ is covered by $b$ in $P'$) for all $y \in \{x\}^-$. 

**Proof.** Obvious. 

**Remark.** It was proved in [2] that the converse of Lemma 1 is true for USM lattices and for $x \in J$. It would be natural to see if this is the case for USM posets. However Figure 1 is a counterexample that states that the converse of Lemma 1 does not hold for USM posets.

The poset depicted in Figure 1 is an USM poset; $x \in J$, $x \rightarrow b$ and $x^- \rightarrow b$ in $P'$ but $b \notin J$.
LEMMA 2. Let $P$ be a poset, $x \in P$, $P' = P - x$ and $a, b \in P'$. Then $a \lhd' b$ if and only if $a \lhd b$ or $a \lhd x \lhd b$ and there does not exist a $t \neq x$ in $P$ such that $a \lhd t \lhd b$.

PROOF. Obvious. ∎

The following lemma is essentially due to Haskins and Gudder [5, p. 370, Corollary 3.6].

LEMMA. In an USM poset,

$$z \lhd x, z \lhd y, x \parallel y \Rightarrow \text{there exists a } t \text{ such that } y \lhd t \text{ and } x \lhd t.$$ 

THEOREM 3. Let $P$ be an USM poset, $x \in P$, $P' = P - x$ and $a, b \in P'$.

(i) If $x \in J \cap M$ and $x \lhd b$, then $x^- \lhd' b$ if and only if $b \in J$.

(ii) If $x \in M$ and $a \lhd x$, then $a \lhd' x^+$ if and only if $a \in M$.

PROOF. (i) Let $x \in J \cap M$, $x \lhd b$ and $x^- \lhd' b$ in $P' = P - x$.

Suppose on the contrary that $b \not\in J$. Then $b$ covers at least two elements and since $x \lhd b$, we have $t(\neq x) \in P$ such that $t \lhd b$.

We have $x^- \lhd x \lhd b$ and $t \lhd b$.

If $x^- \leq t$, then $x^- \lhd t \lhd b$. Therefore we obtain, in $P'$, $x^- \lhd' b$, a contradiction.

If $x^- \not\leq t$, consider a maximal element in $\{x^-, t\}$, say $y$ (which exists since $P$ is finite and has an element 0).

Now, using the above lemma repeatedly (initially for $z = y$) and the fact that $x \in M$, we obtain an element $w$ such that $x^- \lhd w \lhd b$ and $w \parallel x$, a contradiction. Therefore we must have $b \in J$.

Conversely, if $b \in J$ and as $x \lhd b$, by Lemma 1, $x^- \lhd' b$.

(ii) Suppose $a \lhd x$ and $a \lhd' x^+$ and assume on the contrary that $a \not\in M$. There exists a $y \neq x$ such that $a \lhd y$. By upper semimodularity we have an element which covers both $x$ and $y$. But there is a unique element $x^+$ covering $x$. Therefore $y \lhd x^+$. This contradicts our assumption $a \lhd' x^+$. ∎

COROLLARY 4. Let $P$ be an USM poset and $x$ be a non-zero non-unit element of $P$. If $x \in J \cap M$, then $x$ is deletable.

PROOF. Assume that $x \in J \cap M$ and $P = P' = P - x$ is not an USM poset. There exist a pair $a_1, b_1$ in $P'$ such that $a_1, b_1$ cover a common element in $P'$ but they are not covered by a common element in $P'$. Certainly $x$ is comparable with at least one of $a_1, b_1$. In fact we have the following two cases.

Case I. $x$ covers both $a_1, b_1$ in $P$ and which is not possible since $x \in J$.

Case II. The common covering of $a_1, b_1$, say $a$ in $P'$ has the property that $a \lhd x^+$ and $a \not\in M$. This is also not possible by Theorem 3(ii). ∎
Figure 2 shows that, in general, the class of USM posets is not reducible. Figure 2 represents the diagram of a poset which is an upper semimodular poset, no element of which is deletable.

We obtain a characterization of a deletable element in an USM poset.

**Theorem 5.** Let \( P \) be an USM poset and \( x \) be a non-zero non-unit element of \( P \). The poset \( P' = P - x \) is USM if and only if \( x \) is a node of \( P \) or \( x \) satisfies the following conditions:

(i) for every pair of elements in \( \{x\}^- \) which covers a common element there exists a \( y_1 \neq x \) which covers that pair;

(ii) for every pair \( x_1 \in \{x\}^- \), \( x_2 \in \{x\}^+ \) there exists a \( y' \neq x \) such that \( x_1 \prec y' \prec x_2 \).

**Proof.** Suppose \( P \) is an USM poset.

To prove that \( P' = P - x \) is an USM poset we show that if \( a \prec b, c \prec' d \) and \( b \neq c \), then there exists a \( d \in P' \) such that \( b \prec' d, c \prec' d \).

Let \( a \prec' b, c \prec' c \) and \( b \neq c \).

If \( x \) is a node then we are through. Thus, suppose that \( x \) is not a node.

**Case I.** Let \( x \in \{b, c\}^u \).

If \( x \) does not cover \( b \) or \( c \), then there exists a \( d \in P' \) such that \( b \prec d, c \prec d \). Hence \( b \prec' d, c \prec' d \). Suppose \( x \) covers both \( b \) and \( c \). Since \( x \) is not a node there exists a \( y \neq x \) such that \( x_1 \prec y \prec x_2 \) for some \( x_1 \in \{x\}^- \) and \( x_2 \in \{x\}^+ \). Since \( a \prec b, a \prec c \), by (i) there exists some \( y_1 \in P' \) such that \( b \prec' y_1, c \prec' y_1 \) and \( y_1 = d \) is the required element.

**Case II.** \( x \notin \{b, c\}^u \). There are two possibilities.

If \( x \notin [a, b] \) and \( x \notin [a, c] \), then \( a \prec b, a \prec c \). By upper semimodularity of \( P \) we have \( d \in P \) such that \( b \prec d, c \prec d \); and the element \( d \) is the required element.

So suppose that \( x \in [a, b] \); that is, \( a \prec x \prec b \). This case is impossible. Indeed, if \( a \prec x, a \prec c \), by upper semimodularity there exists a \( p \in P' \) such that \( x \prec p, c \prec p \). Thus, we have an element \( c \neq x \) such that \( a \prec c \prec p \). Therefore, by (ii),
there exists a $y \neq x$ such that $a \prec y \prec b$, which is a contradiction to the fact that $a \prec b$. Similarly, the case $x \in [a, c]$ is impossible.

Conversely, suppose that $P' = P - x$ is an USM poset and neither $x$ is a node nor the two conditions (i) and (ii) are true. That is, there exists a $y \neq x$ such that $a \prec y \prec b$, where $a \prec x \prec b$, and:

(I) either there is a pair of $a_2, a_3 \in \{x\}$ which covers a common element $x_0$, but there is no element other than $x$ which covers the pair, or

(II) there is a pair $x_1 \in \{x\}$ and $x_2 \in \{x\}$ such that for any $y \in P'$, $x_1 \prec y \prec x_2$ does not hold.

**Case I.** Since $P$ is an USM poset, any two maximal chains in $[0, b]$ have the same length. In particular, a maximal chain $C_1$ containing $a_1$, $y$ in $[0, b]$ and a maximal chain $C_2$ containing $a_3$, $x$ in $[0, b]$ have the same length. If $a_3 \prec b$, then $C_2 - x$ and $C_1$ are both maximal chains in $[0, b] \cap P'$, and $|C_2 - x| < |C_1|$ contradicting that $P'$ is USM.

If $a_3 \prec b$, then there exists a $y_1 \in P'$ such that $a_3 \prec y_1 \prec b$. Since $P'$ is USM there exists a $x' \in P'$ such that $a_2 \prec x'$ and $a_3 \prec x'$ in $P'$. By our hypothesis $a_2 \prec x' < x'$. As $a_3 \prec x', b$, $x' \neq b$. Therefore there exists a $b_0 \in P$ such that $x' \prec b_0$ and $b \prec b_0$. The chains $a_3 \prec y_1 \prec b_0$ and $a_3 \prec y_1 \prec b_0$ are two maximal chains of different lengths in $[a_3, b_0] \cap P'$, a contradiction.

**Case II.** Let $a_4, b_1$ be a pair of elements satisfying (II).

If $b_1 = b$ then using $a_1 \prec y \prec b$ we obtain two maximal chains in $[0, b] \cap P'$ of different lengths, a contradiction. Now, suppose $b_1 \neq b$. Thus, $a_1 \prec b$, $x \prec b_1$ and by upper semimodularity $b \prec l$, $b_1 \prec l$, for some $l$. A maximal chain $C_1$ in $[0, l]$ containing $a_1$, $y$, $b$ and a maximal chain $C_2$ in $[0, l]$ containing $a_4$, $x$, $b_1$ have the same length. Since in $P'$, $a_1 \prec y \prec b \prec l$, $a_4 \prec b_1 \prec l$, and $C_2 - x$ and $C_1$ are both maximal chains in $[0, l] \cap P'$ with $|C_2 - x| < |C_1|$, a contradiction.

## 3. Reducibility in Pseudocomplemented Posets

Let $P$ be a poset with element 0 and $a, b \in P$. If $[a, b]^l = \{0\}$ and whenever $[a, c]^l = \{0\}$ then $c \leq b$, we shall denote $b$ by $a^*$ and we call it the pseudocomplement of $a$. A pseudocomplemented poset $P$ is one in which for any element $a \in P$, the pseudocomplement $a^*$ exists in $P$. (See [3, 7].)

We introduce the concept of $u$-prime element in an arbitrary poset.

**u-prime element.** An element $x$ of a poset $P$ is called $u$-prime if for all $a, b \in P$, for all $y \in [a, b]^u$, $x \leq y$ implies $x \leq a$ or $x \leq b$.

We prove the following lemma.

**Lemma 6.** Let $P$ be a poset with element 0. An atom $p \in P$ is $u$-prime iff $p$ has a pseudocomplement.

**Proof.** Suppose that an atom $p$ is $u$-prime and $p$ has no pseudocomplement. Therefore there exist $x_1, x_2 \in P$ such that $x_1$ and $x_2$ are incomparable, $\{p, x_1\}^l = \{0\}$, $\{p, x_2\}^l = \{0\}$ and
there is no \( y \in \{x_1, x_2\}^w \) such that \( \{y, p\}^w = \{0\} \). We must have \( p \leq z \), for all \( z \in \{x_1, x_2\}^w \).

But \( p \nleq x_1, p \nleq x_2 \), which is a contradiction with \( p \) \( u \)-prime. Consequently, every atom has a pseudocomplement.

Conversely, let \( p \) be an atom of \( P \) with pseudocomplement \( p^* \) and \( p \leq y \), for all \( y \in \{a, b\}^w \). Suppose \( p \nleq a \) and \( p \nleq b \). Therefore \( \{p, a\}^w = \{0\} \), \( \{p, b\}^w = \{0\} \). This implies that \( a \leq p^*, b \leq p^* \).

Hence \( p^* \in \{a, b\}^w \) leading to \( p \leq p^* \), a contradiction.

\[ \square \]

**Remark.** It was proved in [2] that the following three statements are equivalent for a lattice \( L \).

1. \( L \) is a pseudocomplemented lattice.
2. Each atom of \( L \) has a pseudocomplement.
3. Each atom is \( u \)-prime.

We have proved by the above lemma that (2) and (3) are also equivalent in a poset with element 0, (1) \( \Rightarrow \) (2) is trivial. However, (2) \( \Rightarrow \) (1) (and therefore (3) \( \Rightarrow \) (1)) does not hold in a poset with element 0.

We provide an example of a bounded poset in which every atom has a pseudocomplement but the poset is not pseudocomplemented. In the poset depicted in Figure 3, the element \( x_1 \) has no pseudocomplement.

**Theorem 7.** Let \( x \) be a non-zero non-unit element of a pseudocomplemented poset \( P \). That \( x \) is deletable implies that \( x \) does not satisfy any of the following two properties.

1. \( x \) is an atom \( a \) of \( P \) and there exists a join-irreducible element \( j \) of \( P \) such that:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Non-pseudocomplemented poset having pseudocomplement to every atom.}
\end{figure}
Reducibility in finite posets

(i) $a < j$,
(ii) the set $S = \{t \in P - a \mid t \not\geq j, \ a \in \{ t, j \} \}$ is non-empty and
(iii) $S$ has no greatest element or has a greatest element $\not\geq y$ for all $y \in \{ a^* \}^+$.

(II) $x$ is the pseudocomplement $a^*$ of an atom $a$ and $x$ is not a join-irreducible element of $P$.

PROOF. Suppose (I) holds.

Let $0 < a < j, j \in J, S = \{ t \in P - a \mid t \not\geq j, a \in \{ t, j \} \} \neq \emptyset$.

Let $a^*$ be the pseudocomplement of $a$.

If $\{ j, a^* \}^l \neq \{ 0 \}$ then $a \in \{ j, a^* \}^l$, since $a < j$ and $j \in J$. Hence we have $a \leq a^*$, a contradiction. Therefore, $\{ j, a^* \}^l = \{ 0 \}$. That is

$$j^* \geq a^*. \tag{1}$$

Since $j \in J$, we have $\{ t, j \}^l = \{ 0 \}$ in $P' = P - a$, for all $t \in S$.

Case (a). Suppose $S$ has no greatest element. Therefore, there exist incomparable elements $t_1, t_2 \in S$ such that no element of $S$ is strictly greater than either $t_1$ or $t_2$. Hence $\{ j, t_1 \}^l = \{ 0 \}, \ \{ j, t_2 \}^l = \{ 0 \}$ in $P'$.
Claim. The element \( j \) has no pseudocomplement in \( P' \).

Suppose \( j \) has pseudocomplement in \( P' \), say \( w \). By (1),

\[
w \geq a^*.
\]

In addition, in \( P' \)

\[
\{ j, t_1 \} = [0] \Rightarrow w \geq t_1,
\]
\[
\{ j, t_2 \} = [0] \Rightarrow w \geq t_2.
\]

By our choice of \( t_1, t_2 \) we have \( w \notin S \). Since \( \{ w, j \} = [0] \) in \( P' \), we obtain \( w \neq j \). This together with \( w \notin S \) implies that \( a \notin \{ w, j \} \), i.e., \( a \neq w \). Now, \( a \) is an atom, therefore

\[
\{ a, w \} = [0] \Rightarrow w \leq a^*.
\]

From (2), (3) and (4) we find that \( w = a^* \geq t_1 \). Since \( t_1 \in S, a \leq t_1 \leq a^* \Rightarrow a \leq a^* \), a contradiction. Therefore \( P' \) is not pseudocomplemented. Hence \( x \) is not deletable.

Case (\( \beta \)). Suppose \( S \) has a greatest element, say \( t \), with \( t \nleq y \) for all \( y \in \{ a^* \}^+ \). Assume that \( j \) has pseudocomplement, say \( w \) in \( P' \).

Therefore \( \{ j, w \} = [0] \) in \( P' \) and \( w \geq a^* \). Since \( \{ t, j \} = [0] \) in \( P' \), we have \( w \geq t \). Now, \( a \leq t, t \leq w \Rightarrow a \leq w \). In addition, \( a \leq w, a \leq j, w \nleq j \Rightarrow w \in S \). We have \( w = t \), since \( t \) is the largest element of \( S \) and \( w \geq t \). Therefore, \( w = t \geq a^* \). If \( t > a^* \) then \( t \geq y \) for some \( y \in \{ a^* \}^+ \), a contradiction. Therefore \( t = a^* \) and \( a \leq t \Rightarrow a \leq a^* \), a contradiction. Hence in this case \( x \) is also not deletable.

(II). Suppose (II) holds, i.e., \( x \) is a pseudocomplement \( a^* \) of an atom \( a \) and \( x \notin J \). Let \( x_1, x_2 \) be two elements such that \( x_1 \lhd x, x_2 \lhd x \).

Assume that \( a \) has a pseudocomplement, say \( w \), in \( P' = P - x \). Observe that \( \{ w, a \} \neq [0] \) in \( P \) also. Indeed, if \( \{ w, a \} \neq [0] \) in \( P \) then \( a \leq w \) a contradiction. Now \( w \leq x \) since \( x = a^* \).

As \( \{ a, x_1 \} = [0] \), both in \( P \) and \( P' \), we have \( x_1 \leq w \leq x \Rightarrow x_1 = x = w \) or \( w = x \), since \( x_1 \lhd x \). But \( w = x \) is not possible because \( w \in P' = P - x \). So we have \( x_1 = w \). Similarly \( \{ a, x_2 \} = [0] \) implies that \( x_2 = w \), a contradiction to the choice of \( x_1, x_2 \). Hence \( a \) has no pseudocomplement in \( P' \). Therefore \( x \) is not deletable.
REMARKS. (1) However, the converse of the above theorem is not true. (2) It has been proved in [2] that the class of pseudocomplemented lattices is reducible. But this is not true for posets. The poset depicted in Figure 4 is pseudocomplemented and has no deletable element.

CONCLUDING REMARK. The referee suggested the definition (more general) of a join-irreducible element \( j \) as an element not obtained as a join of elements different from \( j \). The referee posed the question: are the results using this definition of join-irreducible element still true?

The answer to this question is no. We give some counterexamples.

In the USM poset depicted in Figure 5, the element \( x \) satisfies the condition of Corollary 4 by using the referee’s definition, but \( x \) is not deletable.

In the pseudocomplemented poset depicted in Figure 6, the element \( x \) is deletable and also satisfies condition I of Theorem 7 by using the referee’s definition.

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