A PENALIZED VERSION OF THE LOCAL MINIMIZATION SCHEME FOR RATE-INDEPENDENT SYSTEMS

DOROTHEE KNEES AND VIKTOR SHCHERBAKOV

Abstract. The letter presents a penalized version of the time-discretization local minimization scheme first proposed by Efendiev and Mielke in 2006 to resolve time discontinuities in rate-independent systems with nonconvex energies. In order to penalize inequality constrains enforcing the local minimality, the Moreau–Yosida approximation is employed. We prove the convergence of time-discrete solutions to functions that are parametrized BV solutions of the time-continuous problem (in an abstract infinite-dimensional setting), provided that the discretization and approximation parameters are chosen appropriately. We test our scheme on a one-dimensional example and find a notable improvement compared with the original version.

1. Introduction

In this letter, we propose and analyze a novel time-discretization scheme that approximates a rate-independent evolution $z\colon [0,T] \to \mathcal{Z}$ of a doubly nonlinear differential inclusion of the form

$$0 \in \partial\mathcal{R}(\partial_t z(t)) + \mathrm{D}_z\mathcal{I}(t, z(t)), \quad z(0) = z_0,\ t \in [0,T], \tag{1.1}$$

where $\mathcal{I}$ stands for a possibly nonconvex energy functional and $\mathcal{R}$ is a convex and positively homogeneous of degree one dissipation potential. Rate-independent systems represent a specific case of quasistatic systems, and they occur ubiquitously in solid mechanics and have been extensively studied in the last decades; see the monograph [MR15] and references therein. It is well known that if the energy functional $\mathcal{I}$ is not convex with respect to z , one hardly expects in general the existence of continuous on the whole time interval $[0,T]$ solutions of the differential inclusion (1.1). This circumstance has motivated the development of different concepts of solutions to (1.1) that can handle properly arising jump discontinuities, among them we mention only a couple of the most widely used ones: the global energetic solutions [MR15, Section 1.6] and balanced viscosity solutions (BV solutions) [MR15, Section 3.8]. On the other hand, from the standpoint of numerical analysis, a question of primary importance asks: Which solution concept is approximated by a given time-discretization scheme?

Let us now turn our attention to the local minimization scheme from [EM06], which is the starting point of our study. Fix $\tau > 0$. Setting the initial conditions $z_0^\tau := z_0 \in \mathcal{Z}$ and $t_0^\tau := 0$, for $k \geq 1$, we determine by induction z_k^τ and t_k^τ as follows:

$$z_k^\tau \in \operatorname{Argmin}\left\{ \mathcal{I}(t_{k-1}^\tau, z) + \mathcal{R}(z - z_{k-1}^\tau)\colon z \in \mathcal{Z},\ \left\|z_k^\tau - z_{k-1}^\tau\right\|_{\mathbb{V}} \leq \tau \right\}, \tag{1.2}$$

$$t_k^\tau := \min\left\{ t_{k-1}^\tau + \tau - \left\|z_k^\tau - z_{k-1}^\tau\right\|_{\mathbb{V}},\, T \right\}. \tag{1.3}$$

The norm $\|\cdot\|_{\mathbb{V}}$ is defined in Section 2. One of key features of this procedure is that the time increment is not fixed in advance and is found through the minimization. Thus, the scheme has a time-adaptive character with finer time steps at those points where a solution of the rate-independent system (1.1) might develop a discontinuity.

In a finite-dimensional setting, the convergence of time-discrete interpolants generated by (1.2)–(1.3) to a parametrized BV solution as $\tau \to 0$ was established in [EM06]. However, it was not clarified whether a finite number of minimization steps leads to the final time T and whether the length of parametrization is finite. In an abstract infinite-dimensional setting, a variation of the procedure (1.2)–(1.3) was suggested and studied in [Neg14]. There also, it was not proved that the final time T is reached in a finite number of iterations. A full convergence analysis of the local minimization scheme in the infinite-dimensional setting has been recently carried out in [Kn19, Section 2], and in particular all the aforementioned gaps have been filled. Further, a-priori error estimates for the class of locally/globally uniformly convex energies have been derived in [MS20]. In closing this short literature review, we mention a penalized version of (1.2)–(1.3) proposed in [Kn19, Section 3.1], but it is not adaptive in time.

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With the present letter, we aim at developing a penalized variation of (1.2)–(1.3) that maintains the time-adaptive character of the original procedure. The underlying idea consists in employing the Moreau–Yosida approximation in order to penalize the constraint contained in (1.2). The full proofs of all statements below are almost precisely like those of [Kn19], we thus omit them here, highlighting only the differences.

2. Basic assumptions and statements of the main results

We begin by describing a standard semilinear system, which is the framework we are working in [MR15, Example 3.8.4]. Let $(\mathcal{Z}, \|\cdot\|_{\mathcal{Z}})$ and $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ be Hilbert spaces and $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space that are densely and continuously embedded in the following way: $\mathcal{Z} \Subset \mathcal{V} \subset \mathcal{X}$, where $\Subset$ denotes compact embedding. We consider the energy functional $\mathcal{I}\colon [0,T] \times \mathcal{Z} \to \mathbb{R}$ of the structure

$$\mathcal{I}(t,z) := \frac{1}{2}\langle Az, z\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \mathcal{F}(z) - \langle \ell(t), z\rangle_{\mathcal{V}^*,\mathcal{V}}. \tag{2.1}$$

Here, $A\colon \mathcal{Z} \to \mathcal{Z}^*$ is a linear bounded symmetric $\mathcal{Z}$-elliptic (with a constant $c_A > 0$) operator, and $\langle\cdot,\cdot\rangle_{\mathcal{Z}^*,\mathcal{Z}}$ stands for the duality pairings between $\mathcal{Z}$ and $\mathcal{Z}^*$ (and similar to $\mathcal{V}$ and $\mathcal{V}^*$). The functional $\mathcal{F}\colon \mathcal{Z} \to [0,+\infty)$ represents a possibly nonconvex lower order term. So we suppose that $\mathcal{F} \in C^2(\mathcal{Z}; [0,+\infty))$, $\mathrm{D}_z\mathcal{F} \in C^1(\mathcal{Z}; \mathcal{V}^*)$, and $\|\mathrm{D}_z^2\mathcal{F}(z)v\|_{\mathcal{V}^*} \le C(1 + \|z\|_{\mathcal{Z}}^q)\,\|v\|_{\mathcal{Z}}$ for some $q \ge 1$ and $C > 0$. In addition, we require that $\mathcal{F}\colon \mathcal{Z} \to [0,+\infty)$ and $\mathrm{D}_z\mathcal{F}\colon \mathcal{Z} \to \mathcal{Z}^*$ are both weak-weak continuous. The function $\ell\colon [0,T] \to \mathcal{V}^*$ is a time-dependent external loading, with $\ell \in C^1([0,T]; \mathcal{V}^*)$. Collecting the above assumptions yields that $\mathcal{I} \in C^1([0,T] \times \mathcal{Z}; \mathbb{R})$ and there are constants $\alpha, \beta > 0$ such that $|\partial_t \mathcal{I}(t,z)| \le \alpha\,(\mathcal{I}(t,z) + \beta)$ for all $z \in \mathcal{Z}$ and all $t \in [0,T]$. Finally, the dissipation functional $\mathcal{R}\colon \mathcal{X} \to [0,+\infty)$ is assumed to be convex, lower semicontinuous, positively homogeneous of degree one, and equivalent to the norm $\|\cdot\|_{\mathcal{X}}$.

In the sequel, the space $\mathcal{V}$ is also considered to be equipped with the norm $\|v\|_{\mathbb{V}} := (\langle \mathbb{V}v, v\rangle_{\mathcal{V}^*,\mathcal{V}})^{\frac{1}{2}}$, where $\mathbb{V}\colon \mathcal{V} \to \mathcal{V}^*$ is a linear bounded symmetric $\mathcal{V}$-elliptic (with a constant $c_{\mathbb{V}} > 0$) operator. It is clear that the norms $\|\cdot\|_{\mathcal{V}}$ and $\|\cdot\|_{\mathbb{V}}$ are equivalent. Without loss of generality, we may as well suppose that $\|z\|_{\mathbb{V}} \le \|z\|_{\mathcal{Z}}$ for all $z \in \mathcal{Z}$.

We are now in a position to give our notion of parametrized solutions to (1.1), utilizing the terminology from [MRS12, Section 5] and adapting Definition 2.1 from [MS19].

Definition 2.1. For $S \in (0,+\infty)$, we say that a pair $(\widehat{t}, \widehat{z})\colon [0,S] \to \mathbb{R} \times \mathcal{Z}$ is a $\mathcal{V}$-parametrized solution of the rate-independent system (1.1) if the following properties are satisfied:

(a) *Regularity*:

$$\widehat{t} \in W^{1,\infty}(0,S;\mathbb{R}),\ \widehat{z} \in W^{1,\infty}((0,S);\mathcal{V}) \cap L^\infty((0,S);\mathcal{Z}),$$
$$\mathrm{D}_z\mathcal{I}(\widehat{t}(s), \widehat{z}(s)) \in \mathcal{V}^* \quad \text{for a.a. } s \in [0,S];$$

(b) *Initial conditions*:

$$\widehat{t}(0) = 0, \quad \widehat{z}(0) = z_0;$$

(c) *"Arc-length" normalization*:

$$\widehat{t}'(s) \ge 0,\ \widehat{t}'(s) + \|\widehat{z}'(s)\|_{\mathbb{V}} \le 1 \quad \text{for a.a. } s \in [0,S], \quad \widehat{t}(S) = T;$$

(d) *Complementarity condition*:

$$\widehat{t}'(s)\, \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\widehat{t}(s), \widehat{z}(s)), \partial\mathcal{R}(0)) = 0 \quad \text{for a.a. } s \in [0,S],$$

where $\mathrm{dist}_{\mathcal{V}^*}(\xi, \partial\mathcal{R}(0)) := \inf\{\, \|\xi - \sigma\|_{\mathbb{V}^{-1}} : \sigma \in \partial\mathcal{R}(0)\,\}$ and $\|\xi\|^2_{\mathbb{V}^{-1}} := \langle \xi, \mathbb{V}^{-1}\xi\rangle_{\mathcal{V}^*,\mathcal{V}}$;

(e) *Energy-dissipation identity*:

$$\mathcal{I}(\widehat{t}(s), \widehat{z}(s)) + \int_0^s \mathcal{R}(\widehat{z}'(\sigma)) + \|\widehat{z}'(\sigma)\|_{\mathbb{V}}\, \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\widehat{t}(\sigma), \widehat{z}(\sigma)), \partial\mathcal{R}(0))\mathrm{d}\sigma$$
$$= \mathcal{I}(0, z_0) + \int_0^s \partial_t\mathcal{I}(\widehat{t}(\sigma), \widehat{z}(\sigma))\widehat{t}'(\sigma)\mathrm{d}\sigma \quad \text{for all } s \in [0,S].$$

Furthermore, a $\mathcal{V}$-parametrized solution $(\widehat{t}, \widehat{z})\colon [0,S] \to \mathbb{R} \times \mathcal{Z}$ of the rate-independent system (1.1) is called nondegenerate if $\widehat{t}'(s) + \|\widehat{z}'(s)\|_{\mathbb{V}} > 0$ for a.a. $s \in [0,S]$.

Before proceeding further, let us introduce some notation. If $z \in \mathcal{V}$ and $r > 0$, the closed ball $B_{\mathcal{V}}(z,r)$ of radius r about z is $B_{\mathcal{V}}(z,r) := \{\, \widetilde{z} \in \mathcal{V}\colon \|\widetilde{z} - z\|_{\mathbb{V}} \le r\,\}$. Denote by $I_{B_{\mathcal{V}}(z,r)}$ and $I^\delta_{B_{\mathcal{V}}(z,r)}$ the indicator function of $B_{\mathcal{V}}(z,r)$ in the sense of convex analysis and its Moreau–Yosida approximation of index $\delta > 0$, respectively. We refer to

[ABM14, Section 17.2.1] for the precise definition and basic properties of the Moreau–Yosida approximation of nonsmooth convex functionals. Simple calculations show that, for all $\overline{z} \in \mathcal{V}$, it holds

$$I^\delta_{B_\mathcal{V}(z,r)}(\overline{z}) = \frac{1}{2\delta}\left(\mathrm{dist}_\mathcal{V}\left(\overline{z}, B_\mathcal{V}(z,r)\right)\right)^2 := \frac{1}{2\delta}\inf\left\{ \|\overline{z}-\widetilde{z}\|^2_\mathbb{V} : \widetilde{z} \in B_\mathcal{V}(z,r) \right\}.$$

Finally, let $P_{B_\mathcal{V}(z,r)}(\overline{z})$ be the metric projection of $\overline{z}$ onto $B_\mathcal{V}(z,r)$.

We have now all the ingredients to state our penalized version of the local minimization scheme (1.2)–(1.3). Let us fix $\tau,\delta>0$. Setting the initial conditions $z_0^{\tau,\delta} := z_0 \in \mathcal{Z}$ and $t_0^{\tau,\delta} := 0$, we define recursively $z_k^{\tau,\delta}$ and $t_k^{\tau,\delta}$ for $k \geq 1$ as

$$z_k^{\tau,\delta} \in \operatorname{Argmin}\left\{ \mathcal{I}(t_{k-1}^{\tau,\delta}, z) + \mathcal{R}(z - z_{k-1}^{\tau,\delta}) + I^\delta_{B_\mathcal{V}(z_{k-1}^{\tau,\delta},\tau)}(z) \colon z \in \mathcal{Z} \right\}, \tag{2.2}$$

$$t_k^{\tau,\delta} := \min\left\{ t_{k-1}^{\tau,\delta} + \max\left\{ \tau - \|z_k^{\tau,\delta} - z_{k-1}^{\tau,\delta}\|_\mathbb{V}, 0 \right\}, T \right\}. \tag{2.3}$$

It is immediate from the direct method of calculus of variations that minimizers exist.

Proposition 2.2. *There is a constant $c>0$, independent of τ, δ, and k, such that*

$$\mathcal{I}(t_k^{\tau,\delta}, z_k^{\tau,\delta}) + \mathcal{R}(z_k^{\tau,\delta} - z_{k-1}^{\tau,\delta}) \leq \mathcal{I}(t_{k-1}^{\tau,\delta}, z_{k-1}^{\tau,\delta}) + \int_{t_{k-1}^{\tau,\delta}}^{t_k^{\tau,\delta}} \partial_t \mathcal{I}(\sigma, z_k^{\tau,\delta})\mathrm{d}\sigma, \tag{2.4}$$

$$\mathcal{I}(t_k^{\tau,\delta}, z_k^{\tau,\delta}) + \sum_{i=1}^k \mathcal{R}(z_i^{\tau,\delta} - z_{i-1}^{\tau,\delta}) \leq (\beta + \mathcal{I}(0,z_0))e^{\alpha T}, \tag{2.5}$$

$$\|z_k^{\tau,\delta}\|_\mathcal{Z} \leq c. \tag{2.6}$$

Proof. Using $z_{k-1}^{\tau,\delta}$ as a competitor in (2.2), and owing to the nonnegativity of $I^\delta_{B_\mathcal{V}(z_{k-1}^{\tau,\delta},\tau)}$, we at once obtain (2.4). The uniform bound (2.5) follows from (2.4) and an adapted version of [MR15, Theorem 2.1.5 (iii)], and the uniform bound (2.6) is a direct consequence of (2.5) and the coercivity of $\mathcal{I}$ (uniformly in t). □

The cornerstone of our analysis is the following statement, which ensures that the recursive procedure (2.2)–(2.3) leads to $t_{N(\tau,\delta)}^{\tau,\delta} = T$ after a finite number of iteration steps $N(\tau,\delta)$.

Proposition 2.3. *Assume that $z_0 \in \mathcal{Z}$ satisfies $\mathrm{D}_z\mathcal{I}(0,z_0) \in \mathcal{V}^*$. Then, there is an index $N(\tau,\delta) \in \mathbb{N}$ such that $t_{N(\tau,\delta)}^{\tau,\delta} = T$. Moreover, there exist constants $c_1, c_2 > 0$, independent of τ, δ, and $k \leq N(\tau,\delta)$, such that for all $k \leq N(\tau,\delta)$*

$$\begin{aligned}\frac{1}{\delta}\left(1 - \frac{\tau}{\max\{\tau,\, \|z_k^{\tau,\delta} - z_{k-1}^{\tau,\delta}\|_\mathbb{V}\}}\right)\|z_k^{\tau,\delta} - z_{k-1}^{\tau,\delta}\|_\mathbb{V} + c_1 \sum_{i=1}^k \|z_i^{\tau,\delta} - z_{i-1}^{\tau,\delta}\|_\mathcal{Z} \\ \leq c_2\left(t_k^{\tau,\delta} + \|\mathrm{D}_z\mathcal{I}(0,z_0)\|_{\mathcal{V}^*} + \sum_{i=1}^k \mathcal{R}(z_i^{\tau,\delta} - z_{i-1}^{\tau,\delta})\right).\end{aligned} \tag{2.7}$$

Proof. We first note that straightforward calculations reveal the Gâteux differentiability of $I^\delta_{B_\mathcal{V}(z_{k-1}^{\tau,\delta},\tau)}$ with

$$\begin{aligned}\mathrm{D}_z I^\delta_{B_\mathcal{V}(z_{k-1}^{\tau,\delta},\tau)}(z) &= \frac{1}{\delta}\mathbb{V}\left(z - P_{B_\mathcal{V}(z_{k-1}^{\tau,\delta},\tau)}(z)\right) = \frac{1}{\delta}\mathbb{V}\left(z - P_{z_{k-1}^{\tau,\delta} + B_\mathcal{V}(0,\tau)}(z)\right) \\ &= \frac{1}{\delta}\mathbb{V}\left(z - z_{k-1}^{\tau,\delta} - P_{B_\mathcal{V}(0,\tau)}(z - z_{k-1}^{\tau,\delta})\right).\end{aligned}$$

Moreover, the metric projection of $z \in \mathcal{V}$ onto $B_\mathcal{V}(0,\tau)$ can be expressed implicitly as

$$P_{B_\mathcal{V}(0,\tau)}(z) = \frac{\tau}{\max\{\tau,\, \|z\|_\mathbb{V}\}} z. \tag{2.8}$$

Invoking next the necessary optimality condition for $z_k^{\tau,\delta}$ and one-homogeneity of $\mathcal{R}$, we infer

$$-\mathrm{D}_z\mathcal{I}(t_{k-1}^{\tau,\delta}, z_k^{\tau,\delta}) - \mathrm{D}_z I^\delta_{B_\mathcal{V}(z_{k-1}^{\tau},\tau)}(z_k^{\tau,\delta}) \in \partial\mathcal{R}(z_k^{\tau,\delta} - z_{k-1}^{\tau,\delta}) \subset \partial\mathcal{R}(0), \tag{2.9}$$

which in turn implies

$$\begin{aligned}\mathcal{R}\left(z_k^{\tau,\delta} - z_{k-1}^{\tau,\delta}\right) &= \mathcal{R}(z_k^{\tau,\delta} - z_{k-1}^{\tau,\delta}) + \mathcal{R}^*\left(-\mathrm{D}_z\mathcal{I}(t_{k-1}^{\tau,\delta}, z_k^{\tau,\delta}) - \mathrm{D}_z I^\delta_{B_\mathcal{V}(z_{k-1}^{\tau,\delta},\tau)}(z_k^{\tau,\delta})\right) \\ &= -\left\langle \mathrm{D}_z\mathcal{I}(t_{k-1}^{\tau,\delta}, z_k^{\tau,\delta}) + \mathrm{D}_z I^\delta_{B_\mathcal{V}(z_{k-1}^{\tau,\delta},\tau)}(z_k^{\tau,\delta}), z_k^{\tau,\delta} - z_{k-1}^{\tau,\delta} \right\rangle_{\mathcal{Z}^*,\mathcal{Z}}.\end{aligned} \tag{2.10}$$

Here, $\mathcal{R}^*$ is the Legendre–Fenchel conjugate of $\mathcal{R}$ with respect to the duality between $\mathcal{Z}$ and $\mathcal{Z}^*$. In view of [MR15, Lemma 1.3.1], we discover

$$\mathcal{R}(z_{k+1}^{\tau,\delta} - z_k^{\tau,\delta}) \geq -\left\langle \mathrm{D}_z\mathcal{I}(t_{k-1}^{\tau,\delta}, z_k^{\tau,\delta}) + \mathrm{D}_z I^{\delta}_{B_{\mathcal{V}}(z_{k-1}^{\tau,\delta},\tau)}(z_k^{\tau,\delta}), z_{k+1}^{\tau,\delta} - z_k^{\tau,\delta} \right\rangle_{\mathcal{Z}^*,\mathcal{Z}}. \tag{2.11}$$

With these tools at hand, in order to complete the proof, it is enough to combine relations (2.10) and (2.11) and mimic as much as possible the proof of Proposition 2.3 from [Kn19]. □

Corollary 2.4. *Under the assumptions of Proposition 2.3, there is a constant $c > 0$, independent of τ, δ, and $k \leq N(\tau,\delta)$, such that*

$$\|\mathrm{D}_z\mathcal{I}(t_{k-1}^{\tau,\delta}, z_k^{\tau,\delta})\|_{\mathcal{V}^*} \leq c, \tag{2.12}$$

$$\|z_k^{\tau,\delta} - z_{k-1}^{\tau,\delta}\|_{\mathbb{V}} \leq c\delta + \tau. \tag{2.13}$$

Proof. First observe that the term $-\mathrm{D}_z I^{\delta}_{B_{\mathcal{V}}(z_{k-1}^{\tau},\tau)}(z_k^{\tau,\delta})$ is bounded in $\mathcal{V}^*$ uniformly in τ and δ by virtue of estimates (2.5) and (2.7) and identity (2.8). Since $\partial\mathcal{R}(0)$ is bounded in $\mathcal{V}^*$, we derive the uniform bound (2.12) from (2.9). Next, estimate (2.13) follows from (2.5) and (2.7). □

Let us define the length of parametrization $S_{\tau,\delta} \in (0,+\infty]$ as $S_{\tau,\delta} := T + \sum_{i=1}^{N(\tau,\delta)} \|z_i^{\tau,\delta} - z_{i-1}^{\tau,\delta}\|_{\mathbb{V}}$, and let $s_k^{\tau,\delta} := \sum_{i=1}^{k} \max\left\{\tau, \|z_i^{\tau,\delta} - z_{i-1}^{\tau,\delta}\|_{\mathbb{V}}\right\}$ for $1 \leq k \leq N(\tau,\delta)-1$, $s_{N(\tau,\delta)}^{\tau,\delta} := S_{\tau,\delta}$, and $s_0^{\tau,\delta} := 0$. Proposition 2.2 and Proposition 2.3 force that $S_{\tau,\delta}$ is finite and uniformly bounded with respect to τ and δ. For $s \in [s_{k-1}^{\tau,\delta}, s_k^{\tau,\delta}) \subset [0, S_{\tau,\delta}]$, we define the continuous piecewise affine interpolants

$$\widehat{z}_{\tau,\delta}(s) := z_{k-1}^{\tau,\delta} + \frac{(s - s_{k-1}^{\tau,\delta})}{s_k^{\tau,\delta} - s_{k-1}^{\tau,\delta}}(z_k^{\tau,\delta} - z_{k-1}^{\tau,\delta}), \quad \widehat{t}_{\tau,\delta}(s) := t_{k-1}^{\tau,\delta} + \frac{(s - s_{k-1}^{\tau,\delta})}{s_k^{\tau,\delta} - s_{k-1}^{\tau,\delta}}(t_k^{\tau,\delta} - t_{k-1}^{\tau,\delta}),$$

left and right continuous piecewise constant interpolants

$$\overline{z}_{\tau,\delta}(s) := z_k^{\tau,\delta}, \quad \overline{t}_{\tau,\delta}(s) := t_k^{\tau,\delta}, \quad \underline{z}_{\tau,\delta}(s) := z_{k-1}^{\tau,\delta}, \quad \underline{t}_{\tau,\delta}(s) := t_{k-1}^{\tau,\delta},$$

and increment

$$\Delta_{\tau,\delta}(s) := s_k^{\tau,\delta} - s_{k-1}^{\tau,\delta}.$$

Further consequences of Proposition 2.2 and Proposition 2.3 then read

$$\sup_{\tau,\delta>0}\left(\|\widehat{t}_{\tau,\delta}\|_{W^{1,\infty}((0,S_{\tau,\delta});\mathbb{R})} + \|\widehat{z}_{\tau,\delta}\|_{W^{1,\infty}((0,S_{\tau,\delta});\mathcal{V})} + \|\widehat{z}_{\tau,\delta}\|_{L^{\infty}((0,S_{\tau,\delta});\mathcal{Z})}\right) < +\infty,$$

$$\widehat{t}'_{\tau,\delta}(s) \geq 0,\ \widehat{t}'_{\tau,\delta}(s) + \|\widehat{z}'_{\tau,\delta}(s)\|_{\mathbb{V}} = 1 \quad \text{for a.a. } s \in [0, S_{\tau,\delta}], \quad \widehat{t}_{\tau,\delta}(S_{\tau,\delta}) = \overline{t}_{\tau,\delta}(S_{\tau,\delta}) = T.$$

As a preliminary step towards passing to the limit as $(\tau,\delta) \to (0,0)$, we establish discrete counterparts of the energy-dissipation identity and complementarity condition.

Proposition 2.5. *For every $s_1, s_2 \in [0, S_{\tau,\delta}]$ with $s_1 \leq s_2$, the following discrete energy-dissipation identity holds true*

$$\begin{aligned}\mathcal{I}(\widehat{t}_{\tau,\delta}(s_2), \widehat{z}_{\tau,\delta}(s_2)) + \int_{s_1}^{s_2}\left(\mathcal{R}_{\frac{\Delta_{\tau,\delta}(\sigma)}{\delta}}(\widehat{z}'_{\tau,\delta}(\sigma)) + \mathcal{R}^*_{\frac{\Delta_{\tau,\delta}(\sigma)}{\delta}}\left(-\mathrm{D}_z\mathcal{I}(\underline{t}_{\tau,\delta}(\sigma), \overline{z}_{\tau,\delta}(\sigma)) + \frac{\tau}{\delta}\mathbb{V}(\widehat{z}'_{\tau,\delta}(\sigma))\right)\right)\mathrm{d}\sigma \\ = \mathcal{I}(\widehat{t}_{\tau,\delta}(s_1), \widehat{z}_{\tau,\delta}(s_1)) + \int_{s_1}^{s_2} \partial_t\mathcal{I}(\widehat{t}_{\tau,\delta}(\sigma), \widehat{z}_{\tau,\delta}(\sigma))\widehat{t}'_{\tau,\delta}(\sigma)\mathrm{d}\sigma + \int_{s_1}^{s_2} r_{\tau,\delta}(\sigma)\mathrm{d}\sigma,\end{aligned} \tag{2.14}$$

where $\mathcal{R}_\eta(v) := \mathcal{R}(v) + \frac{\eta}{2}\|v\|_{\mathbb{V}}^2$ and $r_{\tau,\delta}(s) := \langle \mathrm{D}_z\mathcal{I}(\widehat{t}_{\tau,\delta}(s), \widehat{z}_{\tau,\delta}(s)) - \mathrm{D}_z\mathcal{I}(\underline{t}_{\tau,\delta}(s), \overline{z}_{\tau,\delta}(s)), \widehat{z}'_{\tau,\delta}(s)\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \frac{\tau}{\delta}\|\widehat{z}'_{\tau,\delta}(s)\|_{\mathbb{V}}^2$. Moreover, for every $s_1, s_2 \in [0, S_{\tau,\delta}]$ with $s_1 < s_2$, the remainder $r_{\tau,\delta}$ satisfies the estimate

$$\int_{s_1}^{s_2} r_{\tau,\delta}(\sigma)\mathrm{d}\sigma \leq c(\tau + \delta) + \frac{\tau}{\delta}, \tag{2.15}$$

the constant $c > 0$ being independent of τ and δ. Finally, the discrete complementarity condition

$$\widehat{t}'_{\tau,\delta}(s)\,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\underline{t}_{\tau,\delta}(s), \overline{z}_{\tau,\delta}(s)), \partial\mathcal{R}(0)) = 0 \quad \text{for a.a. } s \in [0, S_{\tau,\delta}] \tag{2.16}$$

is fulfilled.

Proof. Arguing as in the proofs of Proposition 2.4 and Proposition 3.5 in [Kn19], we can deduce (2.14) and (2.15). So we need only to verify (2.16). To do this, we rewrite (2.9) as

$$\begin{cases} -\mathrm{D}_z\mathcal{I}(\underline{t}_{\tau,\delta}(s),\overline{z}_{\tau,\delta}(s)) - \frac{1}{\delta}(\Delta_{\tau,\delta}(s)-\tau)\mathbb{V}(\widehat{z}'_{\tau,\delta}(s)) \in \partial\mathcal{R}(0) & \text{for a.a. } s\in[0,s^{\tau,\delta}_{N(\tau,\delta)-1}), \\ -\mathrm{D}_z\mathcal{I}(\underline{t}_{\tau,\delta}(s),\overline{z}_{\tau,\delta}(s)) \in \partial\mathcal{R}(0) & \text{for a.a. } s\in[s^{\tau,\delta}_{N(\tau,\delta)-1},S_{\tau,\delta}], \end{cases} \tag{2.17}$$

the second line being from the fact that $\max\left\{\tau, \|z^{\tau,\delta}_{N(\tau,\delta)} - z^{\tau,\delta}_{N(\tau,\delta)-1}\|_{\mathbb{V}}\right\} = \tau$, since otherwise $t_{N(\tau,\delta)} = t_{N(\tau,\delta)-1} < T$. In light of Corollary 2.4, we conclude, using the first line of (2.17), that

$$0 \le \widehat{t}'_{\tau,\delta}(s)\,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\underline{t}_{\tau,\delta}(s),\overline{z}_{\tau,\delta}(s)),\partial\mathcal{R}(0)) \le \frac{\widehat{t}'_{\tau,\delta}(s)}{\delta}|\Delta_{\tau,\delta}(s)-\tau|\|\widehat{z}'_{\tau,\delta}(s)\|_{\mathbb{V}} \quad \text{for a.a. } s\in[0,s^{\tau,\delta}_{N(\tau,\delta)-1}).$$

We claim that $\widehat{t}'_{\tau,\delta}(s)|\Delta_{\tau,\delta}(s)-\tau| = 0$ for a.a. $s\in[0,s^{\tau,\delta}_{N(\tau,\delta)-1})$. Indeed, for $s\in[s^{\tau,\delta}_{k-1},s^{\tau,\delta}_k)$ and $1\le k\le N(\tau,\delta)-1$, we have $\Delta_{\tau,\delta}(s)-\tau = \max\left\{\|z^{\tau,\delta}_k - z^{\tau,\delta}_{k-1}\|_{\mathbb{V}} - \tau, 0\right\}$. If $\|z^{\tau,\delta}_k - z^{\tau,\delta}_{k-1}\|_{\mathbb{V}} \le \tau$, the desired equality is obvious. Otherwise, from (2.3) it follows easily that $t^{\tau,\delta}_k = t^{\tau,\delta}_{k-1}$, which means $\widehat{t}'_{\tau,\delta}(s) = 0$. This proves (2.16), and we are done. □

Now we can prove the main convergence result of our study. For this purpose, it is first convenient to consider all the time-discrete interpolants to be defined on a single parametrization interval, say $[0,S]$. To be specific, thanks to the uniform boundedness of $S_{\tau,\delta}$ with respect to τ and δ, we set $S := \liminf_{\tau,\delta} S_{\tau,\delta} \in (0,+\infty)$, extending the time-discrete interpolants to the interval $(S_{\tau,\delta},S]$ by their value at $S_{\tau,\delta}$ if $S_{\tau,\delta} < S$.

Theorem 2.6. *Assume that $z_0\in\mathcal{Z}$ satisfies $\mathrm{D}_z\mathcal{I}(0,z_0)\in\mathcal{V}^*$. For every sequence $(\tau,\delta)\to(0,0)$ with $\tau/\delta\to 0$, there exist a subsequence $(\tau_n,\delta_n)_{n\in\mathbb{N}}$ and functions $\widehat{t}\in W^{1,\infty}((0,S);\mathbb{R})$ and $\widehat{z}\in W^{1,\infty}((0,S);\mathcal{V})\cap L^\infty((0,S);\mathcal{Z})$ such that for $n\to\infty$*

$$\widehat{t}_{\tau_n,\delta_n} \overset{*}{\rightharpoonup} \widehat{t} \text{ weakly* in } W^{1,\infty}((0,S);\mathbb{R}), \quad \widehat{t}_{\tau_n,\delta_n}(s)\to\widehat{t}(s) \text{ for every } s\in[0,S],$$
$$\widehat{z}_{\tau_n,\delta_n} \overset{*}{\rightharpoonup} \widehat{z} \text{ weakly* in } W^{1,\infty}((0,S);\mathcal{V})\cap L^\infty((0,S);\mathcal{Z}),$$
$$\widehat{z}_{\tau_n,\delta_n}(s) \rightharpoonup \widehat{z}(s) \text{ weakly in } \mathcal{Z} \text{ for every } s\in[0,S].$$

The limit pair $(\widehat{t},\widehat{z})$ is a $\mathcal{V}$-parametrized solution of the rate-independent system (1.1) *in the sense of Definition 2.1.*

Proof. The proof is nearly word for word identical to that of Theorem 3.7 in [Kn19]. □

Remark 2.7. The above proof does not guarantee that the limit pair $(\widehat{t},\widehat{z})$ is a nondegenerate $\mathcal{V}$-parametrized solution of the rate-independent system (1.1).

3. Numerical illustration

We now illustrate the theoretical results by conducting numerical experiments for a one-dimensional example proposed in [Kn19, Section 5.2]. Set $\mathcal{Z}=\mathcal{V}=\mathcal{X}=\mathbb{R}$ as well as:

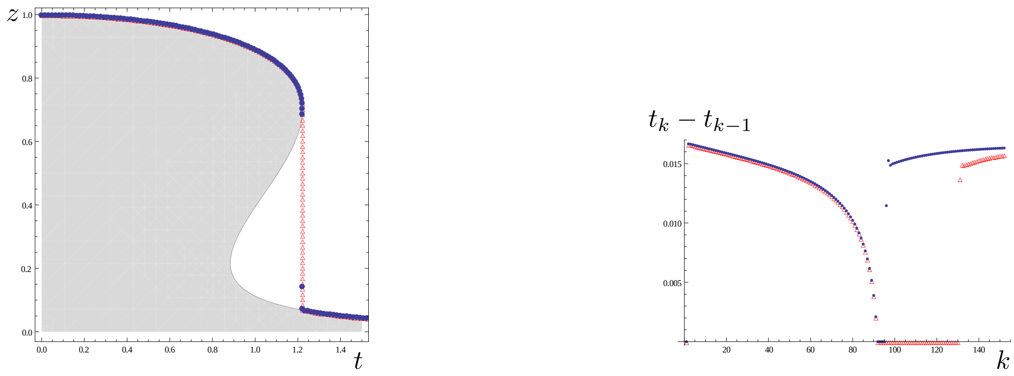
$$\mathcal{I}(t,z) = 5z^2 - \frac{t^2}{2(0.1+z^2)}, \quad \mathcal{R}(v) = 10\,|v|, \quad z_0 = 1, \quad T = 1.5.$$

Note that the energy $\mathcal{I}$ is not exactly of the structure (2.1). Clearly, $\mathrm{D}_z\mathcal{I}(t,z) = \left(10 + \frac{t^2}{(0.1+z^2)^2}\right)z$, and $\mathrm{D}_z\mathcal{I}(t,z)$ is positive if and only if z is positive. Hence, $z(t)>0$ implies $\partial_t z(t)\le 0$. Moreover,

$$\inf\left\{\mathrm{D}^2_z\mathcal{I}(t,z)\colon 0\le t\le T,\ 0\le z\le 1\right\} = \inf\left\{\mathrm{D}^2_z\mathcal{I}(T,z)\colon 0\le z\le 1\right\} \ge -46.3.$$

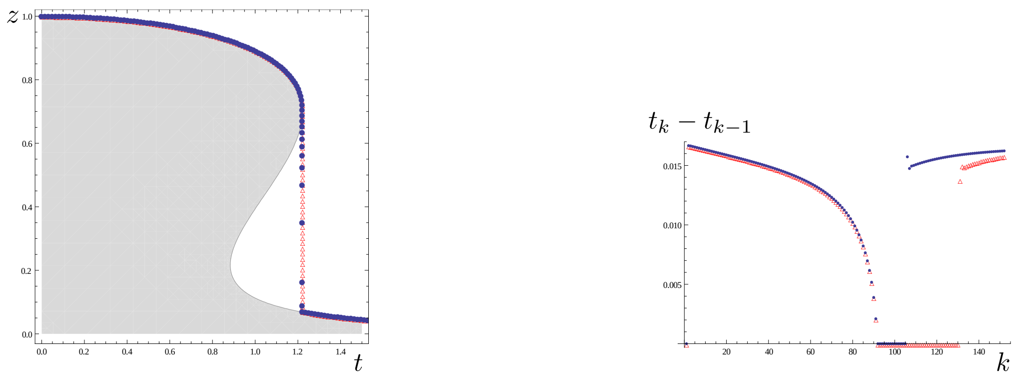
The gray region in Figures 1a and 2a refers to points (t,z) with $-\mathrm{D}_z\mathcal{I}(t,z)\in\partial\mathcal{R}(0)$.

Let $\tau = T/90 = 0.01\overline{6}$. In Figures 1a and 2a, the "red-empty-triangles" curve corresponds to the discrete solutions obtained by the original local minimization scheme (1.2)–(1.3), while the "blue-discs" curve is the discrete solutions generated by the penalized counterpart (2.2)–(2.3) with $\delta=\tau^{0.6}$ and $\delta=\tau^{0.9}$, respectively. In all cases, the total number of minimization steps to reach the final time T is 150. The corresponding time increments are depicted in Figures 1b and 2b. The numerical experiments suggest that the Moreau–Yosida approximation parameter δ can be utilized to reduce the number of minimization steps during the jumps.



(A) (B)

FIGURE 1. Test 1 ($\delta = \tau^{0.6}$): Comparison of the discrete solutions (left) and time increments (right) generated by the original local minimization scheme (1.2)–(1.3) (red empty triangles) and by the penalized counterpart (2.2)–(2.3) (blue discs).



(A) (B)

FIGURE 2. Test 2 ($\delta = \tau^{0.9}$): Comparison of the discrete solutions (left) and time increments (right) generated by the original local minimization scheme (1.2)–(1.3) (red empty triangles) and by the penalized counterpart (2.2)–(2.3) (blue discs).

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Dorothee Knees, Institute of Mathematics, University of Kassel, Heinrich-Plett-Str. 40, 34132 Kassel, Germany
E-mail address: dknees@mathematik.uni-kassel.de

Viktor Shcherbakov, Institute of Mathematics, University of Kassel, Heinrich-Plett-Str. 40, 34132 Kassel, Germany
E-mail address: shcherbakov@mathematik.uni-kassel.de

Viktor Shcherbakov, Lavrentyev Institute of Hydrodynamics, Lavrentyev Ave. 15, 630090 Novosibirsk, Russia